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Convex radiant costarshaped sets and the least sublinear gauge

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Abstract

The paper studies convex radiant sets (i.e. containing the origin) of a linear normed space X and their representation by means of a gauge. By gauge of a convex radiant set $C \subseteq X$ we mean a sublinear function $p : X \to \overline{\mathbb{R}}$ such that $C = [p \leq 1]$. Besides the most important instance, namely the Minkowski gauge $\mu_C(x) = \inf\{\lambda > 0 : x \in \lambda C\}$, the set C may have other gauges, which are necessarily lower than μ_C. We characterize the class of convex radiant sets which admit a gauge different from μ_C in two different way: they are contained in a translate of their recession cone or, equivalently, they are costarshaped, that is complement of a starshaped set. We prove that the family of all sublinear gauges of a convex radiant set admits a least element and characterize its support set in terms of polar sets. The key concept for this study is the outer kernel of C, that is the kernel (in the sense of Starshaped Analysis) of the complement of C. We also devote some attention to the relation between costarshaped and hyperbolic convex sets.

Keywords: Convex sets, Minkowski gauge, sublinear gauge, radiant sets, costarshaped sets, kernel, outer kernel, polar set, reverse polar, hyperbolic convex sets
AMS Classification: 52A07, 46A55, 46B20

1 Introduction

The main aim of this study is to understand how a closed, convex, radiant set C of a normed real vector space X can be described as a sublevel set $[p \leq 1]$ of a sublinear function $p : X \to \overline{\mathbb{R}}$, which we call gauge of C. Thus we only deal with well-known topics in Convex Analysis. Nevertheless the origin and motivation of this study stem from an important topic in Starshaped Analysis, namely the (nonlinear) separation of a point from a radiant or a coradiant set. Moreover, as we will see, a relevant part of the answer is based upon some notions which are typical of Starshaped Analysis (see [22], and references therein).

Our starting point is a separation result, proved in [24], which essentially says that convex coradiant sets can be used to separate points from a radiant set and convex radiant sets can be used to separate points from a coradiant set. By a radiant set (i.e. starshaped at the origin) we mean a set A such that the closed segment $[0, x]$ is contained in A whenever $x \in A$, and we call a set B coradiant if its complement $B^C = X \backslash B$ is radiant. We can reformulate with a similar language the classical (linear) separation results, by saying that coradiant (resp. radiant) halfspaces can be used to separate points from a convex radiant (resp. convex coradiant) set.

As the separation by means of halfspaces can be given an analytic description by means of linear functionals, more precisely by means of lower and upper 1-level sets of linear functionals, the nonlinear separation outlined above would be more efficiently described in analytical terms. Our idea is that a closed, convex, radiant separating set C should be described as the lower level set $[p \leq 1]$ for some continuous and sublinear function $p : X \to \mathbb{R}$, while a closed, convex and coradiant separating set G should be described as $G = [q \geq 1]$ for some continuous and superlinear function $q : X \to \mathbb{R}$. But, while for any closed halfspace H, for which the origin is not a boundary point, there exists exactly one linear and continuous functional $\ell \in X^*$ such that either $H = \{x \in X : \langle x, \ell \rangle \leq 1\}$ or $H = \{x \in X : \langle x, \ell \rangle \geq 1\}$, so that the functional description of linear separation is a strightforward consequence of the geometric one, a similar result does not hold, in general, for convex radiant and convex coradiant sets. So the question arises of finding a functional description of such sets which is at the same time simple and general in some way.

We studied in [25] the possibility to represent a convex coradiant separating set G by means of a continuous superlinear function q (we call it a *cogauge*) such that $G = [q \geq 1]$, and we found some interesting results, among which a necessary and sufficient condition for G to admit a continuous superlinear cogauge and a dual characterization of the greatest superlinear cogauge.

For the case of a convex radiant separating set the question of existence does not pose problems: a closed, convex, radiant set C, can be represented in analytical terms through its Minkowski gauge $\mu_C(x) = \inf\{\lambda > 0 : x \in \lambda C\}$, which is a sublinear functional from X to $[0, +\infty]$ with $C = [\mu_C \leq 1]$. Moreover μ_C is continuous on X if and only if $0 \in \operatorname{int} C$.

However the Minkowski gauge suffers from an important drawback. While it is possible to associate to a convex coradiant set G a particular cogauge q in a way that q is linear (rather than superlinear) when G is a halfspace, this does not happen with the Minkowski gauge μ_C: it is easy to note that, since μ_C is always nonnegative on X, it can never turn into a linear function, even if C is a (proper) halfspace.

Thus we study in this paper the possibility to give a different gauge description of a convex radiant set. We say that the positively homogeneous function $p : X \to \overline{\mathbb{R}}$ is a *gauge* of the radiant set $A \subseteq X$ if it holds $A = [p \leq 1]$ but, though this concept is well suited to deal with radiant sets in general (see [21] for details), our attention goes to convex radiant sets $C \subseteq X$, and on sublinear gauges, uniquely. Our results show that the Minkowski gauge is always the greatest gauge of C and we study the conditions under which μ_C is also minimal, that is its unique sublinear gauge. This happens, for instance, if C is bounded, but also for many unbounded sets.

We actually concentrate on the class of sets which admit gauges lower than μ_C. Since $\mu_C(x) > 0$ implies $\mu_C(x) = p(x)$ for any other gauge p of C and the equality $\mu_C(x) = 0$ holds if and only if x belongs to the recession cone of C, the set C admits a gauge lower than μ_C if and only if it admits a gauge which takes negative values (we call it a 'negative gauge' for short), at some points of the recession cone of C.

The necessary and sufficient condition we are looking for is that there exists a point $z \in X$ such that $C \subseteq z + \operatorname{Rec} C$ and this turns out to be equivalent to saying that C is costarshaped (we follow the terminology of Rubinov [22]), that is the complement of a starshaped set. In this case C admits infinitely many sublinear gauges and this family, beside the greatest element μ_C, also admits a least element, denoted m_C, which is closely analogous to the greatest cogauge γ_G of a convex coradiant set $G \subseteq X$, as studied in [25]. In particular we show that their support sets (the subdifferential at the origin for m_C and the superdifferential at the origin for γ_G) are both defined in terms of the same 'ingredients', the outer kernel of the set C (see Definition 2.2) and the polar and reverse polar of C and of its outer kernel. Moreover m_C reduces to a continuous linear functional when C is a closed radiant halfspace.

Further results concerns the relation between hyperbolic and costarshaped convex sets, with particular attention to the case where the origin is an interior point of C, an assumption which is always satisfied in the separation theorem. Indeed a necessary condition for C to admit a negative sublinear gauge is that the barrier cone of C is closed and this is true whenever C is hyperbolic, that is C has a bounded excess over its recession cone. These sets were originally studied in comparison to the so called parabolic sets, for which the distance from the recession cone becomes (in some sense) infinitely large. Hyperbolic and parabolic convex sets are important (and somehow opposite) families in the class of unbounded convex sets. The latter coincides in $\mathbb{R}^n$ with the so called continuous convex sets, i.e. those for which the support function is continuous on $X \backslash \{0\}$ (see e.g. [10, 1, 7, 9]).

The paper is organized as follows: Section 2 presents some preliminary concepts of Starshaped Analysis and some results on kernels which do not concern directly the problem of representing a convex radiant set, but are important in subsequent sections and relevant in themselves. In Section 3 we study sublinear gauge and characterize convex cones as those convex radiant sets which admit improper gauges. Moreover we describe in dual terms the sublinear gauges of C. In Section 4 we concentrate on the conditions under which the Minkowski gauge is not minimal and prove that C admits a gauge lower than μ_C if and only if its outer kernel is nonempty, that is when C is costarshaped. And this happens if and only if C is included in some translation of its recession cone. In Section 5 we prove the existence of the least sublinear gauge for all convex radiant sets and describe its support set by means of polarity relations in a way that makes easier to compute the least gauge in concrete examples. In Section 6 we deal with the relations between hyperbolic and costarshaped convex radiant sets. Among other results we prove that, if the origin belongs to the interior of C, then C is costarshaped if and only if it is hyperbolic and its recession cone has nonempty interior. When X is a reflexive Banach space this is also equivalent to the requirement that the barrier cone to C has a closed, bounded base.

The proof of the above result can be modified to deal with sets whose recession cone has nonempty relative interior. Since this is always true in finite dimensional spaces, we obtain that a necessary and sufficient condition for a set $C \subseteq \mathbb{R}^n$ to be costarshaped is that $C \subset K - K$, where $K = \operatorname{Rec} C$ and $C \subseteq \operatorname{Rec} C + \varepsilon B$. We show with an example that the same conditions fails to holds in infinite dimensional spaces.

We consider a normed real vector space X, in which the closed ball of radius δ centered in x is denoted by $\mathbb{B}_\delta(x) = \mathbb{B}(x, \delta)$, with $\mathbb{B} = \mathbb{B}(0, 1)$, while $\mathbb{U}_\delta$ denotes the open ball, and $\mathbb{S}$ denotes the unit sphere; the closure, interior and boundary of some set $S \subseteq X$ are denoted by $\operatorname{cl} S$, $\operatorname{int} S$ and $\operatorname{bd} S$ respectively; the convex hull and the conic hull of S are denoted, respectively, as $\operatorname{conv} S$ and $\operatorname{cone} S = \{y = \lambda x : x \in S, \lambda > 0\}$. Let X^* be the topological dual space of X endowed with the weak* topology and denote by $\langle x, \ell \rangle$ or equivalently $\ell(c)$ the usual bilinear pairing between $x \in X$ and $\ell \in X^*$. For the closed and the open ball in X^* we will use the notation $\mathbb{B}^*$ and $\mathbb{U}^*$. For a function $f : X \to \overline{\mathbb{R}} = [-\infty, +\infty]$ and $k \in \mathbb{R}$ we denote by $[f \leq k]$ the weak lower level set $\{x \in X : f(x) \leq k\}$ and by $[f \geq k]$ the weak upper level set.

2 Preliminaries on radiant and coradiant sets

We begin this section with some preliminary concepts, as the ones of radiant and coradiant sets, and some known results about polarity relations.

Definition 2.1 *The set $A \subseteq X$ is called* radiant *if $x \in A$, $t \in [0, 1]$ imply that $tx \in A$. The set $B \subseteq X$ is called* coradiant *if its complement $B^C = X \backslash B$ is radiant, that is if either $B = X$ or $0 \notin B$ and $x \in B$, $t \geq 1$ imply that $tx \in B$.*

We deduce that the empty set $\emptyset$ and the set X are both radiant and coradiant. We underline that the terms radiant and coradiant have been used previously with a slightly different meaning. Rubinov [22] uses $t \in (0, 1]$ in the definition of a radiant set, so that the origin can either belong or not belong to a radiant or to a coradiant set. Penot [16] includes convexity in his use of the term radiant.

As the intersection of any family of radiant sets gives a radiant set, we can speak of the radiant hull, rad A, of any set $A \subseteq X$. Similarly we will consider the set

$$\operatorname{shw} A = \{y = \lambda x : x \in A, \lambda \geq 1\},$$

called the *shadow* of A (we should say shadow from the origin, as in [19, p.22], but we will omit to do so, as no confusion will arise), which is the smallest coradiant set containing A whenever $0 \notin A$. If otherwise $0 \in A$, the smallest coradiant set containing A coincides with X.

In the analysis of radiant and coradiant sets the two following concepts have great relevance.

Definition 2.2 *The* kernel *of a set $A \subseteq X$ is the set of points*

$$\mathit{ker}\, A = \{z \in X : z + t(x - z) \in A, \forall x \in A, \forall t \in (0, 1]\}.$$

The outer kernel *of a set $A \subseteq X$, $\mathit{oker}\, A$, is the kernel of its complement A^C, that is the set*

$$\mathit{oker}\, A = \{z \in X : z + t(x - z) \notin A, \forall x \notin A, \forall t \in (0, 1]\}.$$

It is obvious that a set $A \subseteq X$ including the origin is radiant if and only if $0 \in \ker A$ and that a proper set A excluding the origin is coradiant if and only if $0 \in \operatorname{oker} A$. It is easy to see that both the kernel and the outer kernel of a set $A \subseteq X$ are convex sets (if nonempty).

Following Rubinov, we say that a set A is *starshaped* if $\ker \neq \emptyset$, while A is *costarshaped* if $\operatorname{oker} \neq \emptyset$. Thus, roughly speaking, a set is starshaped (resp. costarshaped) if it is the translation of a radiant (resp. coradiant) set. More precisely we say that A is starshaped at the point x_0 if $x_0 \in \ker A$, and that $B \subseteq X$ is costarshaped at x_0 if $x_0 \in \operatorname{oker} B$. Consequently A is starshaped at x_0 if and only if it the set $(A - x_0) \cup \{0\}$ is radiant. Likewise B is costarshaped at x_0 if and only if the set $(B - x_0)\backslash\{0\}$ is coradiant.

We can relate these concepts to the one of penumbra introduced by Rockafellar [19]. Let

$$\operatorname{shw}(y, z) = \{z + t(y - z), t \geq 1\}$$

be the shadow of y from z and

$$\operatorname{shw}(B, z) = \bigcup_{y \in B} \operatorname{shw}(y, z)$$

be the shadow of B from z. The *penumbra* of B from the set S is the set

$$P(B, S) = \bigcup_{z \in S} \bigcup_{y \in B} \operatorname{shw}(y, z) = \bigcup_{z \in S} \operatorname{shw}(B, z).$$

It is easy to see that, if B is costarshaped at z, then $B = \operatorname{shw}(B, z)$ and $\operatorname{oker} B$ is the largest set S such that $P(B, S) = B$.

Of particular importance are those radiant or coradiant sets which are also convex. It is easy to see that a set C is convex if and only if $C \subseteq \ker C$ and that a convex set is radiant if and only if it contains the origin. As in this paper unbounded convex sets have great relevance, it is important to underline the relation between the assumption that the convex set C is radiant and various notions of recession cones to C.

As radial properties are especially meaningful for us, we consider the cone

$$C^{\infty} := \{x \in X : R_x \subseteq C\},$$

with $R_x = \{y = tx, t > 0\}$ for $x \neq 0$ and $R_0 = \{0\}$, which contains all rays issuing from the origin and contained in C. For a nonempty convex set C, it is common to consider the recession cone

$$\operatorname{Rec} C = \{d \in X : x + td \in C, \forall x \in C, \forall t > 0\}.$$

The recession cone of a closed set C is closed and satisfies (see [19])

$$\operatorname{Rec} C = \{d \in X : \exists x \in C, \text{ such that } x + td \in C, \forall t > 0\}. \tag{1}$$

The latter condition immediately shows that $C^{\infty} = \operatorname{Rec} C$ whenever C is convex, closed and radiant. Actually the inclusion $\operatorname{Rec} C \subseteq C$ is a different characterization of those closed convex sets which are radiant (see [15]). On the other hand a nonempty closed convex set C is coradiant if and only if $0 \notin C$ and $C \subseteq \operatorname{Rec} C$.

The four classes of sets that we have introduced (radiant/coradiant, with or without convexity), are very closely related in pairs. Such relations can be cast in the form of separation results as follows.

Proposition 2.3 *[24] The set $A \subseteq X$ is closed and radiant if and only if for every point $x \notin A$ there exists an open convex coradiant set G such that $x \in G$ and $A \cap G = \emptyset$. The set $B \subseteq X$ is closed and coradiant if and only if for every point $x \notin B$ there exists an open convex radiant set C such that $x \in C$ and $A \cap C = \emptyset$.*

To every set $C \subseteq X$ it is possible to associate its *polar set*

$$C^{\circ} = \{\ell \in X^* : \langle c, \ell \rangle \leq 1, \forall c \in C\}.$$

The set C° is closed, convex and radiant in X^*. We can also consider the bipolar $(C^{\circ})^{\circ} = C^{\circ\circ} \subseteq X$, which satisfies $C \subseteq C^{\circ\circ}$ for all sets C, while equality holds if and only if C is closed, convex and radiant. If C is a cone, then $C^{\circ} = C^{-} := \{\ell \in X^* : \langle c, \ell \rangle \leq 0, \forall c \in C\}$, the polar cone of C.

For the description of convex coradiant sets a different notion of polarity is needed (see for instance [13, 14, 25] and the further references contained in the latter). Given a nonempty set $C \subseteq X$ we call *reverse polar* of C the set

$$C^{\oplus} = \{\ell \in X^* : \langle c, \ell \rangle \geq 1, \forall c \in C\}.$$

We adopt the convention that $C^{\oplus} = X^*$ if $C = \emptyset$. It is easy to see that $C^{\oplus}$ is always closed, convex and coradiant in X^*. Moreover C is closed, convex and coradiant if and only if $C = C^{\oplus\oplus}$.

For any set $B \subseteq X^*$, the *support function* of B is the function $\sigma_B : X \to \overline{\mathbb{R}}$ given by

$$\sigma_B(x) = \sup\{\langle x, l \rangle, \, l \in B\}.$$

It is a lower semicontinuous sublinear function, whose support set (the subdifferential at the origin) is given by

$$\partial \sigma_B = \operatorname{cl} \operatorname{conv} B.$$

Likewise we can define the support function $\sigma_D : X^* \to \overline{\mathbb{R}}$ of a set $D \in X$. In this case we call *barrier cone* of D the effective domain of σ_D, that is the set

$$b(D) = \{\ell \in X^* : \sup_{d \in D} \langle d, \ell \rangle < +\infty\} = \operatorname{cone} D^{\circ}.$$

We end this section by proving some results about the outer kernel of closed convex radiant sets which will be useful below.

For the first one convexity is not actually needed.

Proposition 2.4 *Let the proper set $A \subsetneq X$ be radiant. Then $0 \in oker A$ if and only if A is a cone.*

Proof: Let $0 \in \operatorname{oker} A$ and $a \in A$. Then $\lambda a \in A$ for all $\lambda \in [0, 1]$ since A is radiant and $\lambda a \in A$ for all $\alpha \geq 1$ since $0 \in \operatorname{oker} A$. Thus $\lambda a \in A$ for all $\lambda \geq 0$ and A is a cone. If conversely A is a cone, take $z = 0$, $a \in A$ and $\lambda \geq 1$ to obtain $\lambda a \in A$ and $z = 0 \in \operatorname{oker} A$. $\square$

The following result gives an interesting characterization of the outer kernel of any closed convex set.

Proposition 2.5 *Let $C \subseteq X$ be convex and closed. Then*

$$oker\, C = \bigcap_{c \in C} (c - Rec\, C).$$

Proof: We prove first the inclusion $\operatorname{oker} C \subseteq \bigcap_{c\in C}(c - \operatorname{Rec} C)$. If $\operatorname{oker} C = \emptyset$, then the result is trivial. Thus suppose that $y \in \operatorname{oker} C$. Then $0 \in \operatorname{oker}(C - y)$. This means that the set $C - y$ is either a cone (if $0 \in C - y$) or a coradiant set (if $0 \notin C - y$). If the former holds, then $\operatorname{Rec} C = C - y$ and, equivalently, $y + \operatorname{Rec} C = C$, and the inclusion $y \in c - \operatorname{Rec} C$ for all $c \in C$ follows immediately. If $C - y$ is coradiant we can apply Theorem 4.4 in [25] to the set $B = C - y$. It follows that

$$\operatorname{oker} B = \bigcap_{b\in B}(b - \overline{K}),$$

where $\overline{K} = \operatorname{cl}\operatorname{cone} B$. Moreover for a closed coradiant set it holds $\operatorname{cl}\operatorname{cone} B = \operatorname{Rec} B$ and the equality $\operatorname{Rec}(B + y) = \operatorname{Rec} B$ holds for every convex set B and any point $y \in X$. Hence we have the following relations

$$0 \in \operatorname{oker} B = \bigcap_{b\in B}(b - \operatorname{Rec} B) = \bigcap_{b\in B}(b - \operatorname{Rec} C) = \bigcap_{c\in C}(c - y - \operatorname{Rec} C),$$

whence

$$y \in \bigcap_{c\in C}(c - \operatorname{Rec} C). \tag{2}$$

Let conversely $y \in \bigcap_{c\in C}(c - \operatorname{Rec} C)$. Then $c - y \in \operatorname{Rec} C$ for all $c \in C$ and $C - y \subseteq \operatorname{Rec} C$ which implies that $C - y$ is coradiant or a cone. In both cases we have $0 \in \operatorname{oker}(C - y)$ and $y \in \operatorname{oker} C$.

□

Remark 2.6 1. It follows from Proposition 2.5 that the outer kernel of a closed, convex set is closed, besides being convex (if nonempty).

2. Relation (2) can be described in terms of $*$-difference of sets, as described for instance in [8]. Given the subsets $A, B \subseteq X$ let

$$A \overset{*}{-} B = \{x \in X : x + B \subseteq A\}.$$

It is easy to see that $A \overset{*}{-} B$ is convex if both A and B are convex.

Relation (2) can be reformulated as

$$\begin{aligned} z \in \operatorname{oker} C \quad &\iff \quad z \in c - \operatorname{Rec} C,\ \forall c \in C \\ &\iff \quad c - z \in \operatorname{Rec} C,\ \forall c \in C \\ &\iff \quad C - z \subseteq \operatorname{Rec} C \\ &\iff \quad -z \in \operatorname{Rec} C \overset{*}{-} C \end{aligned}$$

so that

$$-\operatorname{oker} C = \operatorname{Rec} C \overset{*}{-} C.$$

3. Notice that the inequality $\langle k, \ell \rangle \leq 0$ holds for all $\ell \in C^\circ$ and all $k \in \operatorname{Rec} C$ and hence $\ell(z) \geq \ell(c)$ holds for all $z \in \operatorname{oker} C$, $c \in C$ and $\ell \in C^\circ$.

The last preliminary result shows that the outer kernel of a convex radiant set C is a coradiant set.

Proposition 2.7 *Let the set $C \subseteq X$ be closed, convex and radiant. Then*

a) oker C is a cone if and only if C is a cone; in this case it holds oker $C = -C$;

b) oker C is coradiant, provided C is not a cone. In this case it holds Rec (oker C) $=$ $-$Rec C.

Proof:

a) As $\operatorname{oker} C$ is a closed set, if it is a cone, then it includes the origin, hence C is a cone by Proposition 2.4. If C is a cone, then $C = \operatorname{Rec} C$ and

$$\operatorname{oker} C = \bigcap_{c \in C} c - \operatorname{Rec} C = 0 - \operatorname{Rec} C = -C$$

by Proposition 2.5.

b) Let $z \in \operatorname{oker} C$ and $\alpha \geq 1$. We have to prove that $\alpha z \in \operatorname{oker} C$. Since C is radiant it holds $\operatorname{Rec} C \subseteq C$. Since $0 \in C$ we have from Proposition 2.5 that $z \in -\operatorname{Rec} C$ and this yields $(\alpha - 1)z \in -\operatorname{Rec} C$. For all $c \in C$ it holds

$$\alpha z = z + (\alpha - 1)z \in c - \operatorname{Rec} C - \operatorname{Rec} C = c - \operatorname{Rec} C,$$

whence

$$\alpha z \in \bigcap_{c \in C} c - \operatorname{Rec} C = \operatorname{oker} C.$$

To show that $\operatorname{Rec}(\operatorname{oker} C) = -\operatorname{Rec} C$ it is enough to note that the relations $z \in c - \operatorname{Rec} C$, which holds for every $c \in C$, and $k \in \operatorname{Rec} C$, imply $z - k \in c - k - \operatorname{Rec} C = c - \operatorname{Rec} C$, whence

$$\operatorname{oker} C - \operatorname{Rec} C \subseteq \bigcap_{c \in C} c - \operatorname{Rec} C = \operatorname{oker} C,$$

which yields $-\operatorname{Rec} C \subseteq \operatorname{Rec}(\operatorname{oker} C)$. Moreover $0 \in C$ implies $\operatorname{oker} C \subseteq -\operatorname{Rec} C$, and then $\operatorname{Rec}(\operatorname{oker} C) \subseteq \operatorname{Rec}(-\operatorname{Rec} C) = -\operatorname{Rec} C$, so that equality holds.

$\square$

3 The gauges of convex radiant sets

To obtain an analytic version of the convex separation for radiant and coradiant sets, given by Proposition 2.3, we need to describe the separating sets by means of some functional forms. We call *gauge* of a closed radiant set A a positively homogeneous function $p : X \to \overline{\mathbb{R}}$ such that $A = [p \leq 1]$. The function $p : X \to \overline{\mathbb{R}}$ is positively homogeneous if $p(\alpha x) = \alpha f(x)$ for all $x \in X$ and all $\alpha > 0$.

The main example of a gauge for a radiant set A is its *Minkowski gauge*, that is the function $\mu_A : X \to \overline{\mathbb{R}}$ given by

$$\mu_A(x) = \inf\{\lambda > 0 :\, x \in \lambda A\}.$$

For a detailed study of the properties of the Minkowski gauge of a radiant set, see [20, 22].

The following proposition gives an elementary result about gauges. Its proof is trivial and hence omitted.

Proposition 3.1 *Given a closed and radiant set $A \subseteq X$, and a gauge $p : X \to \overline{\mathbb{R}}$ of A, we have the following relations:*

i) $p(x) = +\infty$ *if and only if the open ray* $R_x = \{\alpha x : \alpha > 0\}$ *does not intersects* A*;*

ii) $p(x) = \bar{\alpha} \in (0, +\infty)$ *if and only if* $x \in \alpha A$ *for all* $\alpha > \bar{\alpha}$ *and* $x \notin \alpha A$ *for all* $\alpha \in (0, \bar{\alpha})$*;*

iii) $p(x) \leq 0$ *if and only if* $R_x \subseteq A$*, that is* $x \in A^\infty$*.*

Thus for all gauges p of A it holds

$$p(x) = \mu_A(x), \qquad \forall x \notin A^\infty,$$

while

$$p(x) \leq 0 = \mu_A(x) \qquad \forall x \in A^\infty.$$

Thus we see that the Minkowski gauge is the greatest among all possible gauges of A.

For some set A, μ_A is also the least possible gauge, i.e. it is the only gauge. This is true for instance for all bounded sets, and more generally for all sets for which cone A^∞ reduces to the origin. If A^∞ is nonempty there are also gauges with negative values, which are lower than μ_A.

If we consider a convex set $C \subseteq X$ and we look for its sublinear gauges (as we shall restrict to do in the sequel), the question is less trivial, as we are about to see, and the condition $C^\infty \neq \emptyset$ does not imply the existence of a negative gauge, as shown by Example 6.4 below.

To visualize a simple example for which a negative sublinear gauge exists, consider the set $C = (-\infty, 1] \subseteq \mathbb{R}$. All the functions

$$p_\alpha(x) = \begin{cases} x & x > 0 \\ \alpha x & x \leq 0 \end{cases}$$

with $\alpha \geq 0$, are p.h. gauges of C. For $\alpha \in [0, 1]$, p_α is sublinear. The two extreme cases are p_0, which coincides with μ_C, and p_1, which is the least sublinear gauge of C and is linear, rather than sublinear (while μ_C is not).

Thus our first goal is to see to what extent this situation applies to the class of convex radiant sets.

We observe that the condition $0 \in \operatorname{int} C$ is essential in order to obtain a sublinear continuous gauge. On the other hand such condition is always verified by the separating set C in Proposition 2.3.

For the sets which admit a sublinear gauge lower than μ_C (with some abuse of terminology we will call them at times 'negative gauges') a further distinction is found between the cases in which the least gauge exists and those in which it does not exists, or more precisely those which admit a least gauge which is improper. Let's discuss this exceptional situation first, in order to get rid of it.

Let C be a closed, convex cone. In this case every gauge p of C satisfies $p(x) = +\infty$ for all $x \notin C$. For the rest p can take any nonpositive value at all points $x \in C \cap \S$ ($\S$ is the unit sphere), which are uniquely extended to all C by positive homogeneity, to obtain a gauge p which is positively homogeneous and lower than μ_C. The least gauge of C is the function

$$p(x) = \begin{cases} -\infty & x \in C \\ +\infty & x \notin C \end{cases}$$

which is a l.s.c. sublinear gauge of C, as its epigraph is a closed convex cone in $X \times \mathbb{R}$, though not a proper one.

For the improper subsets of X (namely $\emptyset$ and X) we have the following results: if $C = \emptyset$, then $\mu_C = +\infty$ is the only gauge. If $C = X$, then $\mu_C = 0$ and $p(x) = -\infty$ is an improper sublinear gauge.

We can also see that no closed, convex radiant sets C other than cones, admit an improper l.s.c. sublinear gauge.

Proposition 3.2 *Let $C \subseteq X$ be a closed, convex radiant set, which is not a cone, and let $p : X \to \overline{\mathbb{R}}$ be a l.s.c. sublinear gauge of C. Then p is proper.*

Proof: Since C is not a cone, there exists at least one ray R_x such that R_x is neither included in C, nor disjoint from it. Hence p is positive and real valued on R_x. It is well known (see e.g. [26]) that a l.s.c. convex function which is not identically $+\infty$, is either finite valued on its effective domain, or it always takes value $-\infty$. Thus p is proper.

$\square$

Our next result is a dual characterization of the gauges of a convex radiant set.

Theorem 3.3 *Let $C \subseteq X$ be closed, convex and radiant. Then $p : X \to \overline{\mathbb{R}}$ is a sublinear gauge of C if and only if*

$$cl\, rad\, \partial p = C^\circ.$$

Proof: By standard results about support functions we know that, given two closed convex sets D and E and their closed convex hull $F = \mathrm{cl}\,\mathrm{conv}\,\{D \cup E\}$, it holds

$$\sigma_F(x) = \max\{\sigma_D(x), \sigma_E(x)\}.$$

Moreover it is straightforward to verify that, for a convex set $D \subseteq X$ it hold

$$\mathrm{rad}\, D = \mathrm{conv}\,\{D \cup \{0\}\}.$$

Now consider the Minkowski gauge μ_C. For any other gauge p of C, it must hold $p(x) = \mu_C(x)$ for all $x \notin \mathrm{Rec}\, C$ and $p(x) \leq 0$ for all $x \in \mathrm{Rec}\, C$, hence

$$\mu_C(x) = \max\{p(x), 0\}$$

and

$$C^\circ = \partial \mu_C = \mathrm{cl}\,\mathrm{conv}\,(\partial p \cup \{0\}) = \mathrm{cl}\,\mathrm{rad}\,(\partial p).$$

$\square$

The case when C is a closed, convex cone can be commented in the light of Theorem 3.3. Indeed, in this case, we have

$$C^\circ = C^- = \{\ell \in X^* : \ell(x) \leq 0, \forall x \in C\},$$

the negative polar cone of C. Taken any $\bar{c} \in C$ and fixed $M = \{\ell \in C^\circ : \ell(\bar{c}) \leq -1\}$, we have that

$$\mathrm{cone}\, M = \{\ell \in C^- : \ell(\bar{c}) < 0\}$$

and

$$\mathrm{cl}\,\mathrm{cone}\, M = \{\ell \in C^- : \ell(\bar{c}) \leq 0\} = C^-,$$

whence

$$\mathrm{cl}\,\mathrm{rad}\, M = \mathrm{cl}\,\mathrm{cone}\, M = C^- = C^\circ.$$

Thus we see that $p = \sigma_M$ is a gauge of C. Since M can be taken as small as we wish (taking $\alpha\bar{c}$, with $\alpha > 1$), the least gauge does not exists, or at least not one which is proper.

We will see in Section 5 that, for all sets which are not cones, it is possible to find the least sublinear gauge.

4 Convex radiant sets with a negative sublinear gauge

In the light of Theorem 3.3 it is not difficult to give a necessary and sufficient dual condition in order that the set C admits a gauge with negative values.

Theorem 4.1 *Let $C \subseteq X$ be closed, convex and radiant in X. Then C admits a sublinear gauge with negative values if and only if there exists some closed, convex set $H \subset X^*$, with $0 \notin H$ such that*

$$cl\, rad\, (C^\circ \cap H) = C^\circ. \tag{3}$$

Proof: Let $D = C^\circ \cap H$ and $p(x) = \sigma_D(x)$. Then p is a gauge of C, by Theorem 3.3, and p takes negative values because $0 \notin D$. On the other hand, if p is a gauge of C with negative values, then $0 \notin \partial p$ and (3) holds with $H = \partial p$.

□

Remark 4.2 The closure in (3) becomes useless if $0 \in \operatorname{int} C$. Indeed in this case C° is bounded, hence w^*-compact, and it is easy to show that the radiant hull is closed in this case. It becomes essential if $0 \notin \operatorname{int} C$ as shown by the set

$$C = \{(x_1, x_2) \in \mathbb{R}^2 : \, x_1 \geq -1, \, x_2 = 0\},$$

whose polar set is given by

$$C^\circ = \{(y_1, y_2) \in \mathbb{R}^2 : \, -1 \leq y_1 \leq 0\}.$$

It is easy to see that C has a negative gauge but, for any closed, convex set M with $0 \notin M$, the set $\operatorname{rad}(C^\circ \cap M)$ is not closed and properly contained in C°.

Although Theorem 4.1 gives a necessary and sufficient condition for C to admit a negative gauge, the result is not completely satisfactory, as it is not easy to decide whether some specific set C satisfies condition (3) or not. We will see in the next result that a condition which is equivalent to (3), but expressed in primal terms, is that C be costarshaped.

Theorem 4.3 *Let $C \subseteq X$ be closed, convex and radiant in X. Then the following are equivalent:*

a) C admits a negative gauge;

b) $oker\, C \neq \emptyset$;

c) there exists $z \in X$ such that $C \subseteq z + Rec\, C$.

Proof: $(a) \Rightarrow (b)$. Let $p : X \to \overline{\mathbb{R}}$ be a l.s.c. sublinear gauge of C and let $p(x) < 0$ for some $x \in X$. Then $p(x') \leq -1$ for some $x' \in R_x$. We will show that $-x' \in \operatorname{oker} C$. In view of a contradiction, suppose that $-x' \notin \operatorname{oker} C$. Then, by Proposition 2.5, there exists $\bar{c} \in C$ such that $-x' \notin \bar{c} - \operatorname{Rec} C$. Hence $\bar{c} + x' \notin \operatorname{Rec} C$ and $p(\bar{c} + x') > 0$. Using subadditivity of p, we obtain

$$0 < p(\bar{c} + x') \leq p(\bar{c}) + p(x') \leq 1 - 1 = 0,$$

which is not possible.

$(b) \Rightarrow (c)$. Using again Proposition 2.5, if $z \in \operatorname{oker} C$, then $z \in c - \operatorname{Rec} C$ for all $c \in C$. Hence $c \in z + \operatorname{Rec} C$ for all $c \in C$ and $C \subseteq z + \operatorname{Rec} C$.

$(c) \Rightarrow (a)$. We will use Theorem 4.3 with $H = H_z^+ = \{\ell : \langle z, \ell \rangle \geq 1\}$. We can always suppose that $z \neq 0$. Indeed if $z = 0$ in (c), then C is a cone (it follows from the relations $\operatorname{Rec} C \subseteq C \subseteq 0 + \operatorname{Rec} C$) and a negative gauge always exists. Setting $H_z^- = \{\ell : \langle z, \ell \rangle \leq 1\}$ and $H_z^1 = \{\ell : \langle z, \ell \rangle = 1\}$, we have that

$$(\operatorname{Rec} C)^\circ \cap H_z^1 \subseteq (\operatorname{Rec} C)^\circ \cap H_z^- = (z + \operatorname{Rec} C)^\circ \subseteq C^\circ. \tag{4}$$

As H_z^1 is a closed hyperplane in X^*, it is easy to see that

$$(\operatorname{Rec} C)^\circ = \operatorname{cl} \operatorname{cone} \left(H_z^1 \cap (\operatorname{Rec} C)^\circ \right).$$

The inclusion

$$\operatorname{cl} \operatorname{rad} (C^\circ \cap H_z^+) \subseteq C^\circ$$

is immediate, as $C^\circ \cap H_z^+ \subseteq C^\circ$ and C° is closed and radiant.

For the opposite inclusion take $\ell \in C^\circ$ and consider $\delta = \langle z, \ell \rangle$. If $\delta \geq 1$, then $\ell \in H_z^+$ and we are done. If $0 < \delta < 1$, then take $\alpha = 1/\delta > 0$ so that $f = \alpha \ell \in H_z^1 \cap (\operatorname{Rec} C)^\circ$. It follows from (4) that $f \in C^\circ$. Since $f \in H_z^1$ we have that

$$\ell = \delta f \in \operatorname{rad} (C^\circ \cap H_z^1) \subseteq \operatorname{rad} (C^\circ \cap H_z^+).$$

If finally $\langle z, \ell \rangle = 0$, we can take a net $\{\ell_i\}_{i \in I}$ such that $\langle z, \ell_i \rangle > 0$ for all $i \in I$. Hence $\ell_i \in \operatorname{rad} (C^\circ \cap H_z^+)$ as in the previous steps, and $\ell \in \operatorname{cl} \operatorname{rad} (C^\circ \cap H_z^+)$ as desired.

□

5 The least sublinear gauge

We are now ready to prove our main result, namely that the family of all continuous sublinear gauges of C admits a least element. We also characterize its support set in terms of polar sets. For a given closed, convex, radiant set $C \subseteq X$, let $m_C : X \to \overline{\mathbb{R}}$ be defined as

$$m_C(x) = \sup \{\langle x, \ell \rangle : \ell \in C^\circ \cap (\operatorname{oker} C)^\oplus\}.$$

Proposition 5.1 *Let C be a proper, closed, convex and radiant set in X and let*

$$Q = C^\circ \cap (okerC)^\oplus.$$

The following holds:

a) if $okerC = \emptyset$, then $Q = C^\circ$ and $m_C = \mu_C$;

b) if $0 \in okerC$, then $Q = \emptyset$ and m_C is not defined;

c) if $okerC \neq \emptyset$ and $0 \notin okerC$, then $Q \neq \emptyset$, and $0 \notin Q$, so that $m_C \neq \mu_C$.

Proof: The proof is trivial if one recalls that $(\emptyset)^\oplus = X^*$ and $S^\oplus = \emptyset$ when $0 \in S$.

□

Theorem 5.2 *Let $C \subseteq X$ be a proper closed, convex and radiant set, with $0 \notin okerC$. Then*

a) m_C is a l.s.c. sublinear gauge of C;

b) $m_C \leq p$ if $p : X \to \mathbb{R}$ is a sublinear gauge of C;

c) m_C is linear if C is a radiant halfpace, with $0 \in intC$.

Proof: (a) Being the support function of a nonempty, closed, convex set, m_C is lower semicontinuos and sublinear. We have to prove that $[m_C \leq 1] = C$. It holds

$$m_C(x) = \sigma_Q(x) \leq \sigma_{C^\circ}(x) = \mu_C(x)$$

for all $x \in X$ and therefore $x \in C$ implies

$$m_C(x) \leq \mu_C(x) \leq 1$$

and $C \subseteq [m_C \leq 1]$.

To prove the opposite inclusion take $x \notin C$. There exists $\ell \in C^\circ$ such that $\langle x, \ell \rangle > 1$. We have to analyse various instances. Let

$$\delta = \sigma_C(\ell) = \sup\{\langle c, \ell \rangle : \, c \in C\},$$

which satisfies $1 \geq \delta \geq 0$, since $0 \in C$ and $\ell \in C^\circ$. Suppose that $\delta = \sigma_C(\ell) > 0$ and let $\bar{\ell} = \alpha \ell$ with $\alpha = \delta^{-1} \geq 1$.

It holds

$$\sup_{c \in C} \langle c, \bar{\ell} \rangle = 1$$

and hence $\bar{\ell} \in C^{\circ}$. Moreover

$$\langle x, \bar{\ell} \rangle \geq \langle x, \ell \rangle > 1.$$

To obtain a contradiction we need to show that $\bar{\ell} \in Q$, i.e. $\bar{\ell} \in (\operatorname{oker} C)^{\oplus}$, that is

$$\langle y, \bar{\ell} \rangle \geq 1 \qquad \forall y \in \operatorname{oker} C.$$

If there exists $\bar{y} \in \operatorname{oker} C$ such that $\langle \bar{y}, \bar{\ell} \rangle < 1$, then we can choose some positive ε such that $\langle \bar{y}, \bar{\ell} \rangle < 1 - \varepsilon$. Since $\sup_C \langle c, \bar{\ell} \rangle = 1$, there exists $\bar{c} \in C$ such that $\langle \bar{c}, \bar{\ell} \rangle > 1 - \varepsilon$, whence $\langle \bar{c}, \bar{\ell} \rangle > \langle \bar{y}, \bar{\ell} \rangle$. However this is a contradiction to Proposition 2.5, as shown in Remark 2.6 (3).

We obtained the first part of the proof by showing that $\sigma_C(\ell) > 0$ implies that the ray R_ℓ, which is included in $(\operatorname{Rec} C)^-$, satisfies

$$R_\ell \cap C^{\circ} \cap (\operatorname{oker} C)^{\oplus} \neq \emptyset, \tag{5}$$

so that some translate of ℓ stays in Q.

Now suppose that $\sigma_C(\ell) = 0$. We have two more cases to deal with, according to the intersection $\Xi_\ell = R_\ell \cap (\operatorname{oker} C)^{\oplus}$ being empty or not. If $\Xi_\ell \neq \emptyset$, then any functional $\alpha\ell \in \Xi_\ell$ will also satisfy (5), as $\sigma_C(\alpha\ell) = 0$ and $\alpha\ell \in C^{\circ}$ for all $\alpha > 0$.

To conclude the proof, suppose that $\sigma_C(\ell) = 0$ and $\Xi_\ell = \emptyset$. In this case we have that the ray R_ℓ, which is contained in $(\operatorname{Rec} C)^-$, is disjoint from $\operatorname{cone}(\operatorname{oker} C)^{\oplus}$. On the other hand, since $(\operatorname{oker} C)^{\oplus}$ is a closed, convex, coradiant set in X^*, it holds

$$\operatorname{cone}(\operatorname{oker} C)^{\oplus} \subseteq (\operatorname{Rec} C)^- = \operatorname{cl} \operatorname{cone}(\operatorname{oker} C)^{\oplus} \tag{6}$$

and hence ℓ is the limit of a net $\{\ell_\alpha\}$, $\alpha \in I$, with $\ell_\alpha \in \operatorname{cone}(\operatorname{oker} C)^{\oplus}$. That is $\langle x, \ell \rangle = \lim_\alpha \langle x, \ell_\alpha \rangle$ for all $x \in X$.

Fix a weak* neighbourhood N of ℓ such that $\langle x, l \rangle > 1$ for all $l \in N(\ell)$.

Moreover it holds $\sigma_C(\ell) = 0$ so that $\ell \in b(C) = \operatorname{cone} C^{\circ}$. Recalling (6) and the equality $\operatorname{cl} b(C) = (\operatorname{Rec} C)^-$, which holds for every convex set C, we have that

$$(\operatorname{Rec} C)^- = \operatorname{cl}\left(\operatorname{cone} C^{\circ} \cap \operatorname{cone}(\operatorname{oker} C)^{\oplus}\right)$$

so we can actually choose the net ℓ_α to belong to both $\operatorname{cone}(\operatorname{oker} C)^{\oplus}$ and to $\operatorname{cone}(C^{\circ})$.

For any $\alpha \in I$ such that $\ell_\alpha \in N(\ell)$, reasoning as in the first part of the proof, we can find $\bar{\ell} = t\ell$ for some $t \geq 1$ such that $\bar{\ell} \in C^{\circ} \cap (\operatorname{oker} C)^{\oplus}$ with $\langle x, \bar{\ell} \rangle > 1$. This show that $m_C(x) > 1$, and hence, starting from the assumption $x \notin C$, we obtained that $x \notin [m_C \leq 1]$.

(b) Let p be l.s.c. sublinear, with $C = [p \leq 1]$. We need to show that $m_C \leq p$, which is equivalent to

$$C^{\circ} \cap (\operatorname{oker} C)^{\oplus} \subseteq \partial p.$$

Take $q \notin \partial p$. Hence there exists $\bar{x} \in X$ such that

$$\langle \bar{x}, q \rangle > \alpha = p(\bar{x}). \tag{7}$$

Consider three cases depending on the sign of α.

$\alpha > 0$. It holds

$$q(\bar{x}/\alpha) > 1 = p(\bar{x}/\alpha)$$

hence $\bar{x}/\alpha \in C$ (because p is a gauge of C) and $q \notin C^\circ$.

$\alpha = 0$. It holds

$$q(\bar{x}) > 0 = p(\bar{x}).$$

Since $q(\bar{x}) > 0$, then for some $\bar{r} > 0$ we have $q(\bar{r}\bar{x}) > 1$. Since $p(r\bar{x}) = 0$ for all $r > 0$, and $r\bar{x} \in C$ for all $r > 0$, we have $\bar{x} \in \operatorname{Rec} C \subseteq C$ and hence $q \notin C^\circ$.

$\alpha < 0$. It follows that $p(\bar{x}) = \alpha < 0$, whence $p(\bar{x}/-\alpha) = -1$ and $\bar{y} = \bar{x}/\alpha \in \operatorname{oker} C$. Moreover $0 \notin \operatorname{oker} C$ implies that $(\operatorname{oker} C)^\oplus \neq \emptyset$ and $q(\bar{y}) = q(\bar{x}/\alpha) < 1$, which yields $q \notin (\operatorname{oker} C)^\oplus$.

In all cases we have $q \notin C^\circ \cap (\operatorname{oker} C)^\oplus$ and the thesis is proved.

(c) If C is a radiant halfspace, with $0 \in \operatorname{int} C$, one can always find some $\ell \in X^*$, $\ell \neq 0$, such that

$$H = \{x \in X : \langle x, \ell \rangle \leq 1\}$$

and we have the following: $\operatorname{Rec} H = \{x \in X : \langle x, \ell \rangle \leq 0\}$, $\operatorname{oker} H = \{x \in X : \langle x, \ell \rangle \geq 1\}$, with $H^\circ = \{\alpha\ell,\ 0 \leq \alpha \leq 1\}$ and $(\operatorname{oker} H)^\oplus = \{\beta\ell,\ \beta \geq 1\}$, so that

$$H^\circ \cap (\operatorname{oker} H)^\oplus = \{\ell\}$$

and $m_C = \ell$ is linear.

$\square$

Remark 5.3

1. It is an immediate consequence of Theorem 5.2 that μ_C is minimal if and only if $\operatorname{oker} C = \emptyset$.

2. The description of the least sublinear gauge m_C of a convex radiant set is perfectly symmetrical to the one of the greatest superlinear cogauge γ_G of a convex coradiant set, as it was developed in [25]. We showed there that a closed, convex, coradiant set $G \subseteq X$ admits a continuous superlinear cogauge $q : X \to \mathbb{R}$ if and only if it holds $0 \in \operatorname{int} \operatorname{oker} G$. Moreover we found out that the family of all continuous and superlinear cogauges of G admits a greatest element γ_G, where

$$\gamma_G(x) = \inf\{\langle x, \ell \rangle,\ \ell \in G^\oplus \cap (\operatorname{oker} G)^\circ\}.$$

3. In [5] the authors study the existence of a minimal sublinear gauge for a convex radiant set $C \subseteq \mathbb{R}^n$ and show that the minimal gauge is the support function of the set $\Lambda = \{\ell \in C^\circ : \langle c, \ell \rangle = 1, \text{ for some } c \in C\}$, that is the set of support points of the set C°, with supporting functional in C.

   The importance of weak*-support points of the polar in order to characterize the least gauge was already noticed in [25] for convex coradiant sets and in [6] for convex radiant sets. In the latter case the result was described with the language of illuminated points. Observe that the results proved here extend the one in [5] under several aspects. We work in infinite dimensional normed spaces and do not always assume that C includes the origin as an interior point. Moreover we give a clear specifications of the class of sets for which the least gauge is different from the Minkowski gauge and characterize its support set in terms of kernels and polar sets, so that well known calculus rules (see, for instance, [22] for kernels and [14] for polar sets) can be used to find the least gauge in specific cases.

To finish this section, we give one further result which clarifies the relations between the outer kernel of C and the sublevel set $[p \leq -1]$, when p is a gauge of C.

Proposition 5.4 *Let $C \subseteq X$ be closed, convex and radiant. The following holds*

a) If $p : X \to \overline{\mathbb{R}}$ is a gauge of C, then $[p \leq -1] \subseteq -okerC$;

b) $[p \leq -1] = -okerC$ if and only if $p = m_C$.

Proof: The first statement is actually proved in the first few lines of the proof of Theorem 4.3. To prove the second, let $p = m_C$ and take $z \in \operatorname{oker} C$.

It holds

$$m_C(-z) = \sup\{\langle -z, \ell \rangle, \ \ell \in Q\} \leq \sup\{\langle -z, \ell \rangle, \ \ell \in (\operatorname{oker} C)^{\oplus}\} \leq -1,$$

which, together with (a), proves that $[m_C \leq -1] = -\operatorname{oker} C$. To finish the proof, let $p : X \to \overline{\mathbb{R}}$ be a l.s.c. sublinear gauge of C which does not coincide with m_C.

Then there exists $\bar{x}$, necessarily belonging to $\operatorname{Rec} C$, such that

$$m_C(\bar{x}) < p(\bar{x}) \leq 0.$$

Hence, for some $\alpha > 0$ and $\bar{y} = \alpha \bar{x}$, it holds

$$p(\bar{y}) > -1 > m_C(\bar{y}).$$

The inequality on the right shows that $\bar{y} \in -\operatorname{oker} C$, while the one on the left shows that $\bar{y} \notin [p \leq -1]$ and the proof is finished.

$\square$

6 Further results on costarshaped and hyperbolic convex sets

In [6] the authors study the existence of negative gauges for a convex, radiant set C in the framework of conical equivalence of convex sets. Theorem 3.1 in [6] says that if a negative gauge exists, then the barrier cone $b(C)$ is closed. Although Example 6.4 shows that the condition that $b(C)$ is closed is not sufficient for C to admit a negative gauge, still the class of convex sets for which the barrier cone is closed is worth studying. Such sets are called pseudo-hyperbolic in [11] and are a generalization of hyperbolic convex sets (see [2, 3, 11]).

Definition 6.1 *The convex set $C \subseteq X$ is said to be* hyperbolic *if there exists a bounded set D such that*

$$C \subseteq D + Rec\, C.$$

Inside the class of (linearly) unbounded convex sets, they are somehow opposite to continuous (parabolic) convex sets, introduced by Gale and Klee [10], and studied for instance by [1, 7, 9], for which the barrier cone is open or, more precisely, satisfies the equality $b(C) \backslash \{0\} = \operatorname{int} b(C)$.

The statement that a convex set, whose barrier cone is closed, is hyperbolic is given in [2], but Goossens [11] gives a counterexample. We give a characterization of hyperbolicity, also underlining some quantitative aspect, which allows to discuss a further result about subdifferentiability properties of the Minkowski gauge of a hyperbolic convex set at the points of the recession cone.

Theorem 6.2 *For the convex set $C \subseteq X$ and $M > 0$ it holds*

$$C \subseteq M\mathbb{B} + Rec\, C$$

if and only if

$$(Rec\, C)^- = cone\left(C^\circ \setminus \frac{1}{M}\mathbb{U}^*\right). \tag{8}$$

Proof:

To prove necessity, let $\ell \in (\operatorname{Rec} C)^-$, with $\|\ell\| = 1$. Since for all $c \in C$ there exist $k \in \operatorname{Rec} C$ and $b \in \mathbb{B}$ such that $c = k + Mb$, it holds

$$\ell(c) = \ell(k + Mb) \leq M,$$

whence

$$\ell/M \in C^\circ \quad \text{and} \quad \ell/M \notin (1/M)\mathbb{U}^*,$$

which proves that
$$(\operatorname{Rec} C)^- \subseteq \operatorname{cone}\left(C^\circ \setminus \frac{1}{M}\mathbb{U}^*\right).$$
The opposite relation is obvious, as the inclusion
$$\operatorname{cone}\left(C^\circ \setminus \frac{1}{M}\mathbb{U}^*\right) \subseteq \operatorname{cone}(C^\circ) = b(C) \subseteq (\operatorname{Rec} C)^-$$
is true for every convex set C.

To prove sufficiency let $\bar{c} \in C \setminus (M\mathbb{B} + \operatorname{Rec} C)$ and separate $\bar{c}$ from the convex solid set $M\mathbb{B} + \operatorname{Rec} C$. We find $\ell \in X^* \setminus \{0\}$ and $\alpha \in \mathbb{R}$ such that $\|\ell\| = 1$ and
$$\ell(\bar{c}) > \alpha \geq \ell(k + Mb), \qquad \forall k \in \operatorname{Rec} C,\ b \in \mathbb{B};$$
since $\operatorname{Rec} C$ is a cone, it holds $\ell(k) \leq 0$ for all $k \in \operatorname{Rec} C$ and $\ell \in (\operatorname{Rec} C)^-$.

It holds
$$\ell(\bar{c}) > M = \sup\left\{\ell(k + Mb),\ k \in \operatorname{Rec} C,\ b \in \mathbb{B}\right\}.$$

Take M' such that $\ell(\bar{c}) > M' > M$ to show that $\ell(\bar{c})/M' > 1$ and $\ell/M' \notin C^\circ$. Since C° is radiant, no point on the halfline
$$[1, +\infty)\ell/M' = [1/M', +\infty)\ell$$
belongs to C°. Hence, if the open ball $(1/M)\mathbb{U}^*$ is removed from C°, we have
$$\{\alpha\ell,\ \alpha > 0\} \cap \operatorname{cone}\left(C^\circ \setminus \frac{1}{M}\mathbb{U}^*\right) = \emptyset$$
and
$$(\operatorname{Rec} C)^- \not\subseteq \operatorname{cone}\left(C^\circ \setminus \frac{1}{M}\mathbb{U}^*\right).$$
□

Theorem 6.2 shows that hyperbolicity of a convex set is a stronger requirement for a convex set C than asking that the barrier cone $b(C)$ is closed. In the latter case we have that for every point ℓ, with $\|\ell\| = 1$, of the cone $(\operatorname{Rec} C)^-$ there must exists some $\varepsilon > 0$ such that the segment $[0, \varepsilon]\ell$ is contained in C°. For C to be hyperbolic, we also need a further uniformity condition, namely that there exists $\delta > 0$ with $\varepsilon \geq \delta$ for all ℓ. The interested reader finds in [11] an example of some convex set C in $\mathbb{R}^3$ for which the barrier cone is closed, but such uniformity condition fails, and C is not hyperbolic.

Remark 6.3 Proposition 6.2 can be used to formulate a subdifferentiability property of the Minkowski gauge of an hyperbolic convex set C, which is relevant to our purposes. Indeed if we search for gauges of C which minorize μ_C and take negative values, it is important to study the slope of μ_C at points close to the recession cone. Actually for $x \in \operatorname{Rec} C$ we have $\mu_C(x) = 0$ and $0 \in \partial\mu_C(x)$. If there are points on the boundary of $\operatorname{Rec} C$ in which the subdifferential reduces to $\{0\}$, then there can be no gauges of C other than μ_C. We can use Proposition 6.2 to show that this does not happen if C is hyperbolic. To see this, remind that, for any sublinear functional $p : X \to \overline{\mathbb{R}}$, it holds

$$\partial p(x) = \{\ell \in \partial p(0) : \ell(x) = p(x)\}.$$

Thus, for any $x \in \operatorname{Rec} C$, it holds

$$\partial\mu_C(x) = \{\ell \in C^\circ : \ell(x) = 0\}.$$

If we consider the characterization of hyperbolic sets given by (8) it is easy to see that for $x \in \operatorname{Rec} C$ and $\ell \in (\operatorname{Rec} C)^-$, with $\ell(x) = 0$, we have that

$$\operatorname{diam} \partial\mu_C(x) \geq \delta,$$

where the diameter of a set S is $\operatorname{diam} S = \sup\{\|z-s\|,\ z, s \in S\}$, in that the subdifferential $\partial\mu_C(x)$ contains elements whose norm reaches $\delta = 1/M$ on the ray passing through ℓ.

The next example shows that hyperbolic convex sets, though their barrier cone is closed, do not in general admit a gauge lower than μ_C.

Example 6.4 Let $C = \{(x_1, x_2) \in \mathbb{R}^2 : -1 \leq x_1 \leq 1,\ x_2 \geq -\sqrt{1 - x_1^2}\}$, with $\operatorname{Rec} C = \{(0, x_2) : x_2 \geq 0\}$. We have

$$\mu_C(x_1.x_2) = \begin{cases} \sqrt{x_1^2 + x_2^2}, & x_2 \leq 0 \\ \\ |x_1|, & x_2 > 0 \end{cases}$$

As μ_C is continuous on $\mathbb{R}^2$, the same is true for any other gauge p of C. If there existed another gauge p of C, whose sublevel set $[p < 0]$ were nonempty, it would hold $[p < 0] = \operatorname{int} \operatorname{Rec} C$. But $\operatorname{Rec} C$ has an empty interior and hence no gauge of C can take negative values.

$\square$

Though our original purpose, in the analysis of gauges of convex radiant sets, was to give an analytic description of the separation theorem for coradiant sets given by Proposition 2.3, in the study carried out until now we dropped the assumption $0 \in \operatorname{int} C$, which is always satisfied by the separating set C, and developed a more general analysis in which that assumption was not required to hold.

Now we want to show that under this assumption there are further relations between costarshaped and hyperbolic convex sets which are worth being illustrated, including an interesting characterization of costarshaped sets by means of their barrier cone.

Theorem 6.5 *Let the set $C \subseteq X$ be closed and convex, with $0 \in \mathrm{int}\, C$. Then the following statements are equivalent*

a) C is costarshaped;

b) C is hyperbolic, with $\mathrm{int}\, \mathrm{Rec}\, C \neq \emptyset$.

If moreover X is a reflexive Banach space, they are also equivalent to

c) $b(C)$ admits a closed bounded base, i.e. there exists a closed, convex, bounded set B with $0 \notin B$ such that $b(C) = \mathrm{cone}\, B$.

Proof: Suppose first that C is hyperbolic, i.e.

$$C \subseteq \mathrm{Rec}\, C + \mathbb{B}(0, M) \tag{9}$$

for some $M > 0$ and that $\mathbb{B}(\bar{x}, \delta) \subseteq \mathrm{Rec}\, C$ for some $\bar{x} \in X$ and $\delta > 0$.

The latter inclusion is equivalent to $\mathbb{B}(0, \delta) \subseteq (-\bar{x} + \mathrm{Rec}\, C)$, which yields

$$\frac{M}{\delta}\mathbb{B}(0, \delta) = \mathbb{B}(0, M) \subseteq -\frac{M}{\delta}\bar{x} + \mathrm{Rec}\, C.$$

Call $\bar{z} = M\bar{x}/\delta$ and use (9) to obtain

$$C \subseteq \mathrm{Rec}\, C - \bar{z} + \mathrm{Rec}\, C = -\bar{z} + \mathrm{Rec}\, C$$

and $\bar{z} \in \mathrm{oker}\, C$, so that C is costarshaped.

If conversely C is costarshaped and $\bar{z} \in \mathrm{oker}\, C$, then $C - \bar{z}$ is coradiant, or a cone, and $C - \bar{z} \subseteq \mathrm{Rec}\, C$, which implies $C \subseteq \bar{z} + \mathrm{Rec}\, C$, so that C is hyperbolic.

Moreover, using again $\mathrm{oker}\, C \neq \emptyset$, there exists a gauge p of C with negative values, so that $0 \notin \partial p \subseteq C^\circ$. As $0 \in \mathrm{int}\, C$, the polar set C° is bounded, hence weak*-compact and the same holds for ∂p, since $\partial p \subseteq C^\circ$. Hence p is continuous.

This implies that, if $p(\bar{x}) < 0$, then there exists $\delta > 0$ such that $p(y) < 0$ for all $y \in \mathbb{B}(\bar{x}, \delta)$ and $\mathrm{Rec}\, C$ has a nonempty interior.

To prove the last statement of the theorem we recall that, for any convex set C, it holds $(\mathrm{Rec}\, C)^- = \mathrm{cl}\, b(C)$, which becomes $(\mathrm{Rec}\, C)^- = b(C)$ when C is hyperbolic, as the barrier cone is closed. Moreover we refer to [17] in which it is proved that, in a reflexive Banach space X, a cone $K \subseteq X$ has a nonempty interior if and only if its polar cone K^- has a closed bounded base, to conclude the proof.

□

The equivalence between (a) and (b) in Theorem 6.5 can be reformulated to hold in the vector space spanned by C, provided that the interior of the recession cone is nonempty there. As this always holds in finite dimensional spaces, that is the relative interior, $\mathrm{ri}\, C$, of every convex set C is nonempty, we can use the same proof as above to show that the following propostion holds.

Proposition 6.6 *Let C be closed, convex and radiant in $\mathbb{R}^n$. Then the following are equivalent:*

a) C is costarshaped;

b) C is hyperbolic, with $C \subseteq Rec\,C - Rec\,C$.

With reference to Example 6.4, we see that the set C does not admit a negative gauge because C and its recession cone lie in two different linear spaces, that is the span of the recession cone does not coincide with the span of C. The example below, which emerged from a very helpful discussion with E. Ernst, shows that, in a separable Hilbert space, the two conditions of Proposition 6.6 (b) are not sufficient for a set C to be costarshaped.

Example 6.7 Consider the Hilbert space $X = l_2$ and let $e^n \in l_2$, $n \in \mathbb{N}$, denote the sequence whose terms are all equal to 0 except the n-th term, which is equal to 1.

Let $C = \operatorname{cl}\operatorname{conv}\left(\{-e^i,\, i \in \mathbb{N}\} \cup X_+\right)$, where $X_+ = \{x \in l_2 : x_i \geq 0,\, i \in \mathbb{N}\}$ is the nonnegative cone. C is a closed convex radiant set with $\operatorname{Rec} C = X_+$.

Moreover

$$C \subseteq \operatorname{Rec} C + \mathbb{B} \tag{10}$$

so that C is hyperbolic and

$$X = \operatorname{Rec} C - \operatorname{Rec} C, \tag{11}$$

so that both conditions of Proposition 6.6 (b) are satisfied. However any gauge p of C is nonnegative. To see this we will show that

$$(\,[p \leq 1] = C\,) \quad \Longrightarrow \quad (\,p(x) \geq -1,\, \forall x \in X_+\,)\,.$$

Indeed pick $v = (v_1, v_2, ...) \in \operatorname{Rec} C$; as $v_n \to 0$, there is $\bar{n} \in \mathbb{N}$ such that $v_{\bar{n}} = v \cdot e^{\bar{n}} < 1$ and thus

$$-e^{\bar{n}}/2 + v/2 \notin \operatorname{Rec} C$$

as

$$(-e^{\bar{n}}/2 + v/2) \cdot e^{\bar{n}} = -1/2 + v_{\bar{n}}/2 < 0.$$

Thus $p(v/2 - e^{\bar{n}}/2) > 0$, which yields $p(v) + p(-e^{\bar{n}}) > 0$ and $p(v) > -p(-e^{\bar{n}})$.

But $-e^{\bar{n}} \in C$, so that $p(-e^{\bar{n}}) \leq 1$ and $p(v) > -1$ for all $v \in \operatorname{Rec} C$. Since p is positively homogeneous, this implies that p is nonnegative.

□

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