

## CONCISENESS OF COPRIME COMMUTATORS IN FINITE GROUPS

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### Abstract

Let  $G$  be a finite group. We show that the order of the subgroup generated by coprime  $\gamma_k$ -commutators (respectively,  $\delta_k$ -commutators) is bounded in terms of the size of the set of coprime  $\gamma_k$ -commutators (respectively,  $\delta_k$ -commutators). This is in parallel with the classical theorem due to Turner-Smith that the words  $\gamma_k$  and  $\delta_k$  are concise.

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### 1. Introduction

Let  $F$  be the free group freely generated by  $x_1, \dots, x_n$ . Any nonidentity element of  $F$  is called a group-word in the variables  $x_1, \dots, x_n$ . Given a group-word, we think of it primarily as a function of  $n$  variables defined on any given group  $G$ . The verbal subgroup  $w(G)$  of  $G$  determined by  $w$  is the subgroup generated by the set  $G_w$  consisting of all values  $w(g_1, \dots, g_n)$ , where  $g_1, \dots, g_n$  are elements of  $G$ . A word  $w$  is said to be concise if, whenever  $G_w$  is finite for a group  $G$ , it always follows that  $w(G)$  is finite. More generally, a word  $w$  is said to be concise in a class of groups  $\mathcal{X}$  if, whenever  $G_w$  is finite for a group  $G \in \mathcal{X}$ , it always follows that  $w(G)$  is finite. In the 1960s P. Hall asked whether every word is concise but later Ivanov proved that this problem has a negative solution in its general form [6] (see also [9, page 439]). On the other hand, many important words are known to be concise. For instance, Turner-Smith [15] showed that the *lower central words*  $\gamma_k$  and the *derived words*  $\delta_k$  are concise; here the words  $\gamma_k$  and  $\delta_k$  are defined by the positions  $\gamma_1 = \delta_0 = x_1$ ,  $\gamma_{k+1} = [\gamma_k, x_{k+1}]$  and  $\delta_{k+1} = [\delta_k, \delta_k]$ . Wilson showed in [16] that the multilinear commutator words (outer commutator words) are concise. It was proved by Merzlyakov [8] that every word is concise in the class of linear groups.

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In [3], a word  $w$  was called boundedly concise in a class of groups  $\mathcal{X}$  if for every integer  $m$  there exists a number  $\nu = \nu(\mathcal{X}, w, m)$  such that whenever  $|G_w| \leq m$  for a group  $G \in \mathcal{X}$  it always follows that  $|w(G)| \leq \nu$ . Fernández-Alcober and Morigi [4] showed that every word which is concise in the class of all groups is boundedly concise. Moreover, they showed that whenever  $w$  is a multilinear commutator word having at most  $m$  values in a group  $G$ , one has  $|w(G)| \leq (m-1)^{(m-1)}$ . Questions on conciseness of words in the class of residually finite groups have been tackled in [1]. It was shown that if  $w$  is a multilinear commutator word and  $q$  a prime-power, then the word  $w^q$  is concise in the class of residually finite groups; and if  $w = \gamma_k$  is the  $k$ th lower central word and  $q$  a prime-power, then the word  $w^q$  is boundedly concise in the class of residually finite groups.

The concept of (bounded) conciseness can be applied in a much wider context. Suppose  $\mathcal{X}$  is a class of groups and  $\phi(G)$  is a subset of  $G$  for every group  $G \in \mathcal{X}$ . One can ask whether the subgroup generated by  $\phi(G)$  is finite whenever  $\phi(G)$  is finite. In the present paper we show bounded conciseness of coprime commutators in finite groups.

The coprime commutators  $\gamma_k^*$  and  $\delta_k^*$  were introduced in [13] as a tool to study properties of finite groups that can be expressed in terms of commutators of elements of coprime orders. Let  $G$  be a finite group. Every element of  $G$  is both a  $\gamma_1^*$ -commutator and a  $\delta_0^*$ -commutator. Now let  $k \geq 2$  and let  $X$  be the set of all elements of  $G$  that are powers of  $\gamma_{k-1}^*$ -commutators. An element  $g$  is a  $\gamma_k^*$ -commutator if there exist  $a \in X$  and  $b \in G$  such that  $g = [a, b]$  and  $(|a|, |b|) = 1$ . For  $k \geq 1$  let  $Y$  be the set of all elements of  $G$  that are powers of  $\delta_{k-1}^*$ -commutators. The element  $g$  is a  $\delta_k^*$ -commutator if there exist  $a, b \in Y$  such that  $g = [a, b]$  and  $(|a|, |b|) = 1$ . The subgroups of  $G$  generated by all  $\gamma_k^*$ -commutators and all  $\delta_k^*$ -commutators will be denoted by  $\gamma_k^*(G)$  and  $\delta_k^*(G)$ , respectively. One can easily see that if  $N$  is a normal subgroup of  $G$  and  $x$  an element whose image in  $G/N$  is a  $\gamma_k^*$ -commutator (respectively, a  $\delta_k^*$ -commutator), then there exists a  $\gamma_k^*$ -commutator  $y \in G$  (respectively, a  $\delta_k^*$ -commutator) such that  $x \in yN$ . It was shown in [13] that  $\gamma_k^*(G) = 1$  if and only if  $G$  is nilpotent and  $\delta_k^*(G) = 1$  if and only if the Fitting height of  $G$  is at most  $k$ . It follows that for every  $k \geq 2$  the subgroup  $\gamma_k^*(G)$  is precisely the last term of the lower central series of  $G$  (which is sometimes denoted by  $\gamma_\infty(G)$ ) while for every  $k \geq 1$  the subgroup  $\delta_k^*(G)$  is precisely the last term of the lower central series of  $\delta_{k-1}^*(G)$ . In the present paper we prove the following results.

**THEOREM 1.1.** *Let  $k \geq 1$  and let  $G$  be a finite group in which the set of  $\gamma_k^*$ -commutators has size  $m$ . Then  $|\gamma_k^*(G)|$  is  $m$ -bounded.*

**THEOREM 1.2.** *Let  $k \geq 0$  and let  $G$  be a finite group in which the set of  $\delta_k^*$ -commutators has size  $m$ . Then  $|\delta_k^*(G)|$  is  $m$ -bounded.*

We remark that the bounds for  $|\gamma_k^*(G)|$  and  $|\delta_k^*(G)|$  in the above results do not depend on  $k$ . Thus, we observe here the phenomenon that in [4] was dubbed ‘uniform conciseness’. We make no attempts to provide explicit bounds for  $|\gamma_k^*(G)|$  and  $|\delta_k^*(G)|$  in Theorems 1.1 and 1.2. Throughout the paper we use the term  $m$ -bounded to mean that the bound is a function of  $m$ .

## 2. Preliminaries

We begin with a well-known result about coprime actions on finite groups. Recall that  $[K, H]$  is the subgroup generated by  $\{[k, h] : k \in K, h \in H\}$ , and  $[K, {}_i H] = [[K, {}_{i-1} H], H]$  for  $i \geq 2$ .

**LEMMA 2.1** [5, Lemma 4.29]. *Let  $A$  act via automorphisms on  $G$ , where  $A$  and  $G$  are finite groups, and suppose that  $(|G|, |A|) = 1$ . Then  $[G, A, A] = [G, A]$ .*

For the following result from [14], recall that a subset  $B$  of a group  $A$  is *normal* if  $B$  is a union of conjugacy classes of  $A$ .

**LEMMA 2.2.** *Let  $A$  be a group of automorphisms of a finite group  $G$  with  $(|A|, |G|) = 1$ . Suppose that  $B$  is a normal subset of  $A$  such that  $A = \langle B \rangle$ . Let  $k \geq 1$  be an integer. Then  $[G, A]$  is generated by the subgroups of the form  $[G, b_1, \dots, b_k]$ , where  $b_1, \dots, b_k \in B$ .*

The following is an elementary property of  $\delta_k^*$ -commutators.

**LEMMA 2.3.** *Let  $G$  be a finite group. For a nonnegative integer  $k$ ,*

$$\delta_k^*(\delta_1^*(G)) = \delta_{k+1}^*(G).$$

**PROOF.** We argue by induction. For  $k = 0$ , the result is obvious by the definition of  $\delta_0^*$ -commutators.

Suppose the result holds for  $k - 1$ . So

$$\delta_{k-1}^*(\delta_1^*(G)) = \delta_k^*(G).$$

It was mentioned in the introduction that  $\delta_{k+1}^*(G) = \gamma_\infty(\delta_k^*(G))$ . By induction,

$$\delta_{k+1}^*(G) = \gamma_\infty(\delta_{k-1}^*(\delta_1^*(G))),$$

and viewing  $\delta_1^*(G)$  as the group under consideration,

$$\gamma_\infty(\delta_{k-1}^*(\delta_1^*(G))) = \delta_k^*(\delta_1^*(G)),$$

as required. □

Here is a helpful observation that we will use in both of our main results. Recall that a Hall subgroup of a finite group is a subgroup whose order is coprime to its index. Also, a finite group  $G$  is metanilpotent if and only if  $\gamma_\infty(G)$  is nilpotent.

**LEMMA 2.4.** *Let  $G$  be a finite metanilpotent group and  $P$  a Sylow  $p$ -subgroup of  $\gamma_\infty(G)$ , and let  $H$  be a Hall  $p'$ -subgroup of  $G$ . Then  $P = [P, H]$ .*

**PROOF.** For simplicity, we write  $K$  for  $\gamma_\infty(G)$ . By passing to the quotient  $G/O_{p'}(G)$ , we may assume that  $P = K$ .

Let  $P_1$  be a Sylow  $p$ -subgroup of  $G$ . So  $G = P_1 H$ . Now  $P_1/P$  is normal in  $G/P$  as  $G/P$  is nilpotent, but also  $P \leq P_1$ ; hence,  $P_1$  is normal in  $G$ . It follows that  $K = [P_1, H]$ , since in a nilpotent group all coprime elements commute. By Lemma 2.1,  $[P_1, H, H] = [P_1, H] = P$ , and so  $P = [P_1, H] = [P, H]$ . □

In the proofs of our main results we often reduce to the following case.

**LEMMA 2.5.** *Let  $i$  and  $m$  be positive integers. Let  $P$  be an abelian  $p$ -group acted on by a  $p'$ -group  $A$  such that*

$$|\{[x, a_1, \dots, a_i] : x \in P, a_1, \dots, a_i \in A\}| = m.$$

*Then  $|[P, A]| = 2^m$ , so is  $m$ -bounded.*

**PROOF.** We enumerate the set  $\{[x, a_1, \dots, a_i] : x \in P, a_1, \dots, a_i \in A\}$  as  $\{c_1, \dots, c_m\}$ . As  $P$  is abelian,

$$[x, a_1, \dots, a_i]^l = [x^l, a_1, \dots, a_i] \quad (\dagger)$$

for all  $x \in P, a_1, \dots, a_i \in A$ , and a positive integer  $l$ .

Consider  $g \in [P, A]$ , which can be expressed as some product  $c_1^{l_1} \cdots c_m^{l_m}$  for nonnegative integers  $l_1, \dots, l_m$ . We claim that  $l_1, \dots, l_m \in \{0, 1\}$ . For, if  $l_j > 1$  with  $j \in \{1, \dots, m\}$ , we know from  $(\dagger)$  that  $c_j^{l_j} \in \{c_1, \dots, c_m\}$ . We replace all such  $c_j^{l_j}$  accordingly, so that  $g$  is now expressed as  $c_1^{k_1} \cdots c_m^{k_m}$  with  $k_1, \dots, k_m \in \{0, 1\}$ . Hence  $|[P, A]| = 2^m$ .  $\square$

The well-known focal subgroup theorem [12, Corollary 10.34, page 255] states that if  $G$  is a finite group and  $P$  a Sylow  $p$ -subgroup of  $G$ , then  $P \cap G'$  is generated by the set of commutators  $\{[g, z] \mid g \in G, z \in P, [g, z] \in P\}$ . In particular, it follows that  $P \cap G'$  can be generated by commutators lying in  $P$ . This observation led to the question on generation of Sylow subgroups of verbal subgroups of finite groups. More specifically, the following problem was addressed in [2].

Given a multilinear commutator word  $w$  and a Sylow  $p$ -subgroup  $P$  of a finite group  $G$ , is it true that  $P \cap w(G)$  can be generated by  $w$ -values lying in  $P$ ?

The answer to this is still unknown. The main result of [2] is that if  $G$  has order  $p^a n$ , where  $n$  is not divisible by  $p$ , then  $P \cap w(G)$  is generated by  $n$ th powers of  $w$ -values. In the present paper we will require a result on generation of Sylow subgroups of  $\delta_k^*(G)$ .

**LEMMA 2.6.** *Let  $k \geq 0$  and let  $G$  be a finite soluble group of order  $p^a n$ , where  $p$  is a prime and  $n$  is not divisible by  $p$ , and let  $P$  be a Sylow  $p$ -subgroup of  $G$ . Then  $P \cap \delta_k^*(G)$  is generated by  $n$ th powers of  $\delta_k^*$ -commutators lying in  $P$ .*

It seems likely that Lemma 2.6 actually holds for all finite groups. In particular, the result in [2] was proved without the assumption that  $G$  is soluble. It seems though that proving Lemma 2.6 for arbitrary groups is a complicated task. Indeed, one of the tools used in [2] is the proof of the Ore conjecture by Liebeck *et al.* [7] that every element of any finite simple group is a commutator. Recently, it was conjectured in [13] that every element of a finite simple group is a commutator of elements of coprime orders. If this is confirmed, then extending Lemma 2.6 to arbitrary groups would be easy. However, the conjecture that every element of a finite simple group is a commutator of elements of coprime orders is proved only for the alternating groups [13] and the

groups  $\text{PSL}(2, q)$  [10]. Thus, we prove Lemma 2.6 only for soluble groups, which is adequate for the purposes of the present paper.

Before we embark on the proof of Lemma 2.6, we note a key result from [2] that we will need.

**LEMMA 2.7.** *Let  $G$  be a finite group, and let  $P$  be a Sylow  $p$ -subgroup of  $G$ . Assume that  $N \leq L$  are two normal subgroups of  $G$ , and use bar notation in the quotient group  $G/N$ . Let  $X$  be a normal subset of  $G$  consisting of  $p$ -elements such that  $\overline{P} \cap \overline{L} = \langle \overline{P} \cap \overline{X} \rangle$ . Then  $P \cap L = \langle P \cap X, P \cap N \rangle$ .*

We are now ready to prove Lemma 2.6.

**PROOF.** Let  $G$  be a counter-example of minimal order. Then  $k \geq 1$ .

By induction on the order of  $G$ , the lemma holds for every proper subgroup and every proper quotient of  $G$ . We observe that  $\delta_1^*(G) < G$  since  $G$  is not perfect, and by Lemma 2.3,  $\delta_{k+1}^*(G) = \delta_k^*(\delta_1^*(G))$ . Since the result holds for  $\delta_1^*(G)$ , it follows that  $P \cap \delta_{k+1}^*(G)$  is generated by  $n$ th powers of  $\delta_k^*$ -commutators in  $G$ . Note that we made use of [2, Remark 3.2].

If  $\delta_{k+1}^*(G) \neq 1$ , by induction the result holds for  $G/\delta_{k+1}^*(G)$ . Combining this with the fact that  $P \cap \delta_{k+1}^*(G)$  can be generated by  $n$ th powers of  $\delta_k^*$ -commutators, we get a contradiction by Lemma 2.7. Hence  $\delta_{k+1}^*(G) = 1$ . Further  $O_{p'}(G) = 1$  since  $G$  is a minimal counter-example. Therefore,  $\delta_k^*(G) \subseteq P$ , so  $P \cap \delta_k^*(G)$  is generated by  $n$ th powers of  $\delta_k^*$ -commutators lying in  $P$ . We have our required contradiction.  $\square$

### 3. Proofs of the main results

We mention here a result of Schur and Wiegold. The much celebrated Schur theorem states that if  $G$  is a group with  $|G/Z(G)|$  finite, then  $|G'|$  is finite. It is implicit in the work of Schur that if  $|G/Z(G)| = m$ , then  $|G'|$  is  $m$ -bounded. However, Wiegold produced a shorter proof of this second statement, which also gives the best possible bound. See Robinson [11, pages 102–103] for details.

For the proof of Theorem 1.2, we require the following result from [13].

**LEMMA 3.1.** *Let  $G$  be a finite group and let  $y_1, \dots, y_k$  be  $\delta_k^*$ -commutators in  $G$ . Suppose  $y_1, \dots, y_k$  normalise a subgroup  $N$  such that  $(|y_i|, |N|) = 1$  for every  $i = 1, \dots, k$ . Then for every  $x \in N$  the element  $[x, y_1, \dots, y_k]$  is a  $\delta_{k+1}^*$ -commutator.*

Now we are ready to begin.

**PROOF OF THEOREM 1.1.** Let  $X$  be the set of all  $\gamma_k^*$ -commutators. We wish to show that if  $|X| = m$ , then  $|\gamma_k^*(G)|$  is  $m$ -bounded. For convenience we write  $K$  for  $\langle X \rangle$ . Of course,  $K = \gamma_\infty(G)$ .

The subgroup  $C_G(X)$  has index at most  $m!$ , so  $|K/Z(K)| \leq m!$  too. By Schur,  $K'$  has  $m$ -bounded order. Therefore, by passing to the quotient, we may assume  $K' = 1$ , and so  $K$  is abelian with  $G$  metanilpotent.

It is enough to bound the order of each Sylow subgroup of  $K$ . We choose a Sylow  $p$ -subgroup  $P$ . By passing to the quotient  $G/O_{p'}(G)$ , we may assume  $K = P$ .

By Lemma 2.4, a Hall  $p'$ -subgroup  $H$  of  $G$  satisfies  $P = [P,_{k-1} H]$ . We know that  $P$  is abelian and  $P$  is normal in  $PH$ .

We denote the set  $\{[x, h_1, \dots, h_{k-1}] : x \in P, h_1, \dots, h_{k-1} \in H\}$  by  $\hat{X}$ .

For  $x \in P$  and  $h_1, \dots, h_{i-1} \in H$ , where  $i \geq 2$ , we note that  $[x, h_1, \dots, h_{i-1}]$  is a  $\gamma_i^*$ -commutator. Therefore,  $\hat{X} \subseteq X$ , and  $|\hat{X}| \leq m$ .

By Lemma 2.5, it follows that  $[[P,_{k-1} H]]$  is  $m$ -bounded. Appealing to Lemma 2.4, we conclude that  $|P|$  is  $m$ -bounded.  $\square$

**PROOF OF THEOREM 1.2.** Let  $X$  be the set of  $\delta_k^*$ -commutators in  $G$ . We wish to show that if  $|X| = m$ , then  $|\delta_k^*(G)|$  is  $m$ -bounded. We recall that  $\delta_k^*(G) = \gamma_\infty(\delta_{k-1}^*(G))$ . For ease of notation we define  $Q := \delta_{k-1}^*(G)$ , and we write  $K$  for  $\delta_k^*(G)$ .

The subgroup  $C_G(X)$  has index at most  $m!$  in  $G$ , so  $|K/Z(K)| \leq m!$  and as in the proof of Theorem 1.1, we may assume  $K' = 1$ . Hence  $K$  is assumed to be abelian with  $Q$  metanilpotent. In what follows, we now restrict to the group  $Q$ .

It is sufficient to show that the order of each Sylow subgroup of  $K$  is  $m$ -bounded. We choose  $P$  a Sylow  $p$ -subgroup of  $K$ . By passing to the quotient  $G/O_{p'}(G)$ , we may assume  $K = P$ .

By Lemma 2.4, a Hall  $p'$ -subgroup  $H$  of  $Q$  satisfies  $P = [P, H]$ . By Lemma 2.6, since  $H$  is generated by its Sylow subgroups, we have that  $H$  is generated by a normal subset  $B$  of powers of  $\delta_{k-1}^*$ -commutators that are of  $p'$  order.

Lemma 2.2 now implies that  $[P, H]$  is generated by subgroups  $[P, b_1, \dots, b_k]$  for  $b_1, \dots, b_k \in B$ . By Lemma 3.1, if  $x \in P$ , then  $[x, b_1, \dots, b_k]$  is a  $\delta_k^*$ -commutator, and we deduce that  $[[P, b_1, \dots, b_k]]$  is  $m$ -bounded.

It follows that the number of generators of  $[P, H]$  is at most  $m$ , and, furthermore, the exponent of  $[P, H]$  is  $m$ -bounded. Hence, the finite abelian  $p$ -group  $P = [P, H]$  has  $m$ -bounded order.  $\square$

## References

- [1] C. Acciarri and P. Shumyatsky, 'On words that are concise in residually finite groups', *J. Pure Appl. Algebra* (2013), to appear, arXiv:1212.0581[math.GR].
- [2] C. Acciarri, G. A. Fernández-Alcober and P. Shumyatsky, 'A focal subgroup theorem for outer commutator words', *J. Group Theory* **15** (2012), 397–405.
- [3] S. Brazil, A. Krasilnikov and P. Shumyatsky, 'Groups with bounded verbal conjugacy classes', *J. Group Theory* **9** (2006), 127–137.
- [4] G. A. Fernández-Alcober and M. Morigi, 'Outer commutator words are uniformly concise', *J. Lond. Math. Soc.* **82** (2010), 581–595.
- [5] I. M. Isaacs, *Finite Group Theory*, Graduate Studies in Mathematics, 92 (American Mathematical Society, Providence, RI, 2008).
- [6] S. V. Ivanov, 'P. Hall's conjecture on the finiteness of verbal subgroups', *Izv. Vyssh. Uchebn. Zaved. Mat.*(6) (1989), 60–70 (in Russian); translation in *Soviet Math. (Iz. VUZ)* **33**(6) (1989), 59–70.
- [7] M. W. Liebeck, E. A. O'Brien, A. Shalev and P. H. Tiep, 'The Ore conjecture', *J. Eur. Math. Soc. (JEMS)* **12**(4) (2010), 939–1008.
- [8] Ju. I. Merzlyakov, 'Verbal and marginal subgroups of linear groups', *Dokl. Akad. Nauk SSSR* **177** (1967), 1008–1011 (in Russian).

- [9] A. Yu. Ol'shanskii, *Geometry of Defining Relations in Groups*, Mathematics and its Applications, 70 (Soviet Series) (Kluwer Academic, Dordrecht, 1991).
- [10] M. A. Pellegrini and P. Shumyatsky, 'Copriime commutators in  $\text{PSL}(2, q)$ ', *Arch. Math.* **99** (2012), 501–507.
- [11] D. J. S. Robinson, *Finiteness Conditions and Generalized Soluble Groups, Part 1*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 62 (Springer, New York-Berlin, 1972).
- [12] J. S. Rose, *A Course on Group Theory* (Dover, New York, NY, 1994).
- [13] P. Shumyatsky, 'Commutators of elements of coprime orders in finite groups', *Forum Math.*, to appear.
- [14] P. Shumyatsky, 'On the exponent of a verbal subgroup in a finite group', *J. Aust. Math. Soc.*, to appear.
- [15] R. F. Turner-Smith, 'Finiteness conditions for verbal subgroups', *J. Lond. Math. Soc.* **41** (1966), 166–176.
- [16] J. Wilson, 'On outer-commutator words', *Canad. J. Math.* **26** (1974), 608–620.

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