



Analysis of District Heating Networks

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Abstract

In the context of district heating networks we consider a model for the distribution of energy through water (for heating or cooling) from a central power station to the consumers. We prove the well posedness of the system, by using the Banach Fixed Point Theorem together with stability estimates for reduced systems. Eventually we consider optimal control problems motivated by applications and we provide the existence of optimal controls in special situations.

Keywords District heating networks · Mixed ODE–PDE system · Control problems

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1 Introduction

District heating (or cooling) networks (briefly DHN), also called teleheating, is a system, which permits to distribute heat, generated in a centralized place called power

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station or combined heat and power plant (briefly CHP), to residential houses, commercial and industrial sites. The heat is transported along a network of pipes, filled with an incompressible fluid, typically water. Despite the initial costs due to the construction of the whole network, this system typically results in a high efficiency, in a smaller production of pollution compared to localized heat generators, and in reduced costs for final users in a long time scale.

Here we present, from a mathematical point of view, a complete teleheating system, modeled by ordinary and partial differential equations on a graph; see also [3, 32]. Such equations describe both the fluid dynamics inside the pipes' network and the energy distribution towards the final consumers. Our main result consists in proving the existence of a unique solution to such a system and the continuous dependence of the solution from the initial and boundary conditions. This result opens the road for the study of optimal ways for heat distribution. In simple cases we also present optimal control problems and we show that optimal controls do exist; see [32] for finding optimal controls in a reduced case and Remark 3.11.

The model considered here is constructed starting from by the classic Euler system [29], which describes the evolution of a fluid through macroscopic quantities: density, linear momentum and energy. For results on Euler systems on networks we refer to [7–9, 12, 23] and references therein. In the present framework, since the fluid's incompressibility, in each pipe the velocity of the fluid is described by an ordinary differential equation. Furthermore at junctions coupling conditions can be given in a fixed form, since no change of characteristics can occur. For compressible flows this is a truly challenging problem.

In the present case, the heat energy satisfies a hyperbolic advection equation, since it is transported along each tube according to the fluid velocity. At junctions suitable coupling conditions provide the mass and energy conservation and distribution rules over the pipes. We use coupling conditions slightly different from those considered in [32] with the aim of avoiding low regularity of the solution. Note that although the energy equations are linear, nonlinear effects can occur due to the coupling. In [31, Sect. 3.2] a minimal example is given, how discontinuous solutions can occur even with smooth input data. In this example two neighboring consumers are directly supplied by the power plant, but also mutually connected by an additional pipe. The flow in this additional pipe is always pointing towards the consumer with the larger demand. Thus, if the demands change, the flow changes its direction and a different temperature can enter the pipe.

Partial differential equations on networks attracted a lot of interest in the last years, mainly due to applications, which range from traffic flow [25] to gas distribution [11, 26], irrigation channels [17, 21], sewer networks [2] and flexible strings [15, 22, 28]. In this setting, coupling conditions at nodes play an important role, since they strongly influence the existence, the well posedness, and the qualitative properties of the solution.

The proof of the main result is obtained through the classical Banach Fixed Point Theorem and through ad-hoc stability estimates. The most delicate part consists in deducing stability and total variation estimates for the energy subsystem. This is due to the fact that the energy evolves according to a transport equation whose characteristic curves move with speed given by the fluid velocity. Even for simple networks the

direction of the flow can change arbitrarily often. This fact produces solutions for the energy functions whose traces at the nodes have low regularity. Such stability estimates e.g. are missing in [33].

The paper is organized as follows. In Sect. 2 we construct in detail the model starting from reduced systems, which describe the hydrodynamics part and the energy part. In Sect. 3 we state and prove the main result of the paper. Moreover we introduce the control problems and we show the existence of optimal controls in various situations. Finally, in the appendix, there are the detailed proofs of the stability of the advection equation w.r.t. the velocity of the flow and several auxiliary results.

2 The Model

In this section, we illustrate in detail the system for the description of the district heating network, starting from the definition of a network used in the paper.

Definition 2.1 The hydraulic network is a couple $\mathcal{G} = (\mathcal{I}, \mathcal{J})$ with the following properties.

1. $\mathcal{I} = \{1, \dots, N_{\mathcal{I}}\}$ is the set of the indices enumerating the edges or pipes. For every $i \in \mathcal{I}$, the corresponding edge I_i is modeled by the one-dimensional interval $[a_i, b_i]$. We denote with A^i the sectional area of the pipe I_i , i.e. $A^i = \pi r_i^2$ where r_i is the constant radius of the i -th tube.
2. $\mathcal{J} = \{J_1, \dots, J_M\}$ is the set of internal nodes. Each node J_j is described by the set

$$J_j = \{j_1, \dots, j_{M_j}\} \subseteq \mathcal{I},$$

- i.e. by the indices of the edges connected to the node J_j . Moreover we denote by $\text{inc}(J_j) \subseteq J_j$ and by $\text{out}(J_j) \subseteq J_j$ respectively the incoming and outgoing edges for the node J_j , according to the edge's parametrization, i.e. we have $i \in \text{inc}(J_j)$ (resp. $i \in \text{out}(J_j)$) with $I_i = [a_i, b_i]$ if the junction J_j is at $x = b_i$ (resp. at $x = a_i$).
3. $\mathcal{I}_P = \{1\} \subseteq \mathcal{I}$, i.e. the first pipe I_1 is the only one connected to the power station. We assume that a_1 corresponds to the power station position.
 4. $\mathcal{I}_H = \{h_1, \dots, h_{N_H}\} \subseteq \mathcal{I}$ consists in all the indices corresponding to the pipes connected to the houses. Therefore, for $h_k \in \mathcal{I}_H$, b_{h_k} denotes the position of the h_k -th house, while a_{h_k} represents the position of the (last) internal node before the consumer's house, thus of some junction J_k .

Remark 2.2 Without loss of generalities, we assume that, for every junction $J \in \mathcal{J}$, the sets of incoming and outgoing edges are not empty, i.e.

$$\text{inc}(J) \neq \emptyset \quad \text{and} \quad \text{out}(J) \neq \emptyset.$$

Note that for every $J \in \mathcal{J}$,

$$J = \text{inc}(J) \cup \text{out}(J) \quad \text{and} \quad \text{inc}(J) \cap \text{out}(J) = \emptyset.$$

Moreover we underline that the incoming and the outgoing edges are not determined by the sign of the velocity of the fluid inside the pipe, but only by the choice of the parametrization.

As stated in the introduction, in each pipe the starting point is the Euler system see [29, Chap. 1]

$$\begin{cases} \partial_t \rho + \partial_x (\rho v) = 0, \\ \partial_t v + v \partial_x v = -\frac{\partial_x p}{\rho}, \\ \partial_t \left(\frac{1}{2} \rho v^2 + \rho e \right) = -\partial_x \left(\rho v \left(\frac{1}{2} v^2 + e + \frac{p}{\rho} \right) \right), \end{cases} \tag{2.1}$$

where t is time, $x \in [a, b]$ the spatial parametrization, ρ denotes the density of the fluid, v is the velocity, p is the pressure, and e is the energy density. In the case of an incompressible fluid, the macroscopic variable ρ is constant, so that the first equation in (2.1) simply becomes $\partial_x v = 0$ and (2.1) reduces to

$$\begin{cases} \partial_x v = 0, \\ \partial_t v + \frac{1}{\rho} \partial_x p = f(v), \\ \partial_t e + v \partial_x e = g(e), \end{cases} \tag{2.2}$$

where the right hand side $f = f(v)$ models the friction as well as gravitational effects, while the source term $g = g(e)$ models the energy loss due to not perfect insulation.

Remark 2.3 Common choices for the source terms f and g in (2.2) are

$$\begin{aligned} f(v) &= -\frac{\lambda}{2d} v|v| - \gamma \partial_x h, \\ g(e) &= -\frac{4\kappa}{d} (T(e) - T_\infty). \end{aligned} \tag{2.3}$$

The term $-\frac{\lambda}{2d} v|v|$ corresponds to the Darcy-Weisbach friction law of the fluid with the pipe’s walls (d is the pipe’s diameter and λ is the friction factor). The source term $-\gamma \partial_x h$ represents static pressure due to different elevations at the ends of the tube, where the constant γ is the gravitational acceleration and $\partial_x h$ is the slope of the pipe. Finally, the term $-\frac{4\kappa}{d} (T(e) - T_\infty)$ models the energy loss due to heat flux over the pipe wall, where κ is the heat transmission coefficient of the pipe, $T(e)$ is the fluid temperature depending on its energy density e and T_∞ is the external temperature.

In a district heating network, the heating power is distributed through water in a system of pipes. This can be described by considering a copy of (2.2) for each pipe $I_i, i \in \mathcal{I}$, of the network via suitable coupling conditions at nodes. The full system then can be divided into three different parts, namely the hydrodynamics part (briefly HYD), the consumer part, and the energy part. We describe them separately.

2.1 The Hydrodynamics Part

The flow in each pipe of the network, except those connected to a house, can be modeled by a copy of the first two equations of the complete incompressible Euler system (2.2), i.e. for every $i \in \mathcal{I} \setminus \mathcal{I}_H$, by the system

$$\begin{cases} \partial_x v^i = 0, \\ \partial_t v^i + \frac{1}{\rho} \partial_x p^i = f(v^i), \end{cases} \quad t > 0, x \in [a_i, b_i]. \tag{2.4}$$

Note that, since the incompressibility assumption holds, namely $\partial_x v^i = 0$, each v^i is a function depending only on time; hence $\partial_x p^i$ depends only on time, so that p^i is affine in x . For every $i \in \mathcal{I} \setminus \mathcal{I}_H$, the initial condition for v^i is given by

$$v^i(0) = v^{i,o} \in \mathbb{R}. \tag{2.5}$$

Moreover, for every junction $J \in \mathcal{J}$ and $t > 0$, we consider the coupling conditions

$$\begin{cases} \sum_{I_j \in J} A^j v^j(t) = 0, \\ p^i(t, b_i) = p^j(t, a_j), \end{cases} \quad i \in \text{inc}(J), j \in \text{out}(J). \tag{2.6}$$

The first equation states the conservation of mass. For constant density the mass flux in pipe j is the product of cross sectional area A^j and fluid velocity v^j . The pipes are assumed to be filled completely for all times. The remaining equations provide the continuity of the pressure at the junction.

Finally, at the CHP we impose, for $t > 0$, the boundary condition

$$p^1(t, a_1) = p^{1,b}(t). \tag{2.7}$$

Note that no initial condition for the pressure on the edges is needed. Indeed, thanks to the particular form of the second equation in (2.4) and to the compatibility condition (2.6), it is sufficient to provide the boundary condition at the CHP (2.7). For a later use, we consider the following assumptions:

- (H.1) $p^{1,b} \in \mathbf{L}_{\text{loc}}^1((0, +\infty); \mathbb{R})$ and $\text{TV}(p^{1,b}) < +\infty$.
- (H.2) $f \in \mathbf{C}^1(\mathbb{R}; \mathbb{R})$ satisfies $vf(v) \leq 0$ for every $v \in \mathbb{R}$.
- (H.3) For every $i \in \mathcal{I}$, $v^{i,o} \in \mathbb{R}$ satisfies $\sum_{I_j \in J} A^j v^{j,o} = 0$.

2.2 The Consumers' Part

For every $k \in \mathcal{I}_H$, denote by J_k the junction at position a_k and let $Q_k(t)$ be the time dependent power demand of the consumer k . For every $k \in \mathcal{I}_H$, the velocity v^k is determined by an ordinary differential equation depending on the power demand Q_k .

More precisely, v^k is governed by the equation

$$\partial_t v^k = \frac{1}{\alpha} \left(Q_k(t) - A^k v^k (e_{J_k}(t) - e^{k,out}) \right), \tag{2.8}$$

where $\alpha > 0$ is a relaxation parameter, $e^{k,out} = e(T_{k,out})$ is the energy related to the fixed temperature level $T_{k,out}$, to which the water inside the pipe is cooled down, and e_{J_k} is the energy at the starting node for consumer k . Note that equation (2.8) pushes v^k to the state $Q_k(t) = A^k v^k (e_{J_k}(t) - e^{k,out})$.

We consider the following assumptions for the consumer subsystem:

- (C.1) There exist two positive constants $\overline{Q}_{max}, \overline{Q}_{min} > 0$ such that, for every $k \in \mathcal{I}_H$, $Q_k \in \mathbf{L}^1_{loc}((0, +\infty); [\overline{Q}_{min}, \overline{Q}_{max}])$.

Finally we give the definition of solution for the system (2.4)–(2.8).

Definition 2.4 Fix $T > 0$ and suppose that the assumptions (H.1), (H.2), (H.3), and (C.1) hold. Assume moreover that for every $k \in \mathcal{I}_H$, the function $t \mapsto e_{J_k}(t)$ is in $\mathbf{L}^1((0, T); \mathbb{R})$.

A couple $(p, v) = ((p^1, \dots, p^{N_{\mathcal{I}}}), (v^1, \dots, v^{N_{\mathcal{I}}}))$ is a solution to (2.4)–(2.8) on the time interval $[0, T]$ if the following conditions are satisfied.

1. For every $i \in \{1, \dots, N_{\mathcal{I}}\}$, the functions p^i , and v^i satisfy the regularity assumptions:
 - (a) $p^i \in \mathbf{L}^1((0, T); \mathbf{C}^1([a_i, b_i]; \mathbb{R}))$ and, for a.e. $t \in [0, T]$, $\partial_x p^i(t)$ does not depend on x .
 - (b) $v^i \in \mathbf{AC}([0, T]; \mathbb{R})$.
2. For every $i \in \{1, \dots, N_{\mathcal{I}}\}$, $v^i(0) = v^{i,o}$ and p^i satisfies, for all $x \in I_i$ and for a.e. $t \in [0, T]$,

$$p^i(t, x) = p^i(t, a_i) + \rho \left(f(v^i(t)) - \dot{v}^i(t) \right) (x - a_i).$$

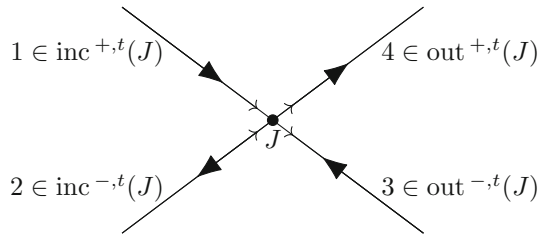
3. v is a solution to (2.8) in the sense of Carathéodory [19], i.e. for every $k \in \mathcal{I}_H$ and for every $t \in [0, T]$,

$$v^k(t) = v^{k,o} + \int_0^t \frac{1}{\alpha} \left(Q^k(s) - v^k(s) (e_{J_k}(s) - e^{k,out}) \right) ds.$$

4. The boundary condition $p^1(t, a_1) = p^{1,b}(t)$ holds for a.e. $t \in [0, T]$.
5. For every junction $J \in \mathcal{J}$

$$\begin{cases} \sum_{I_k \in J} A^k v^k(t) = 0, & \forall t \in [0, T] \\ p^i(t, b_i) = p^j(t, a_j) =: p_J(t), & \text{for a.e. } t \in [0, T], \begin{matrix} i \in \text{inc}(J) \\ j \in \text{out}(J) \end{matrix} \end{cases}$$

Fig. 1 Scheme of the sets $\text{inc}^{\pm,t}(J)$ and $\text{out}^{\pm,t}(J)$ defined in (2.12). The bold arrows indicate the direction of the fluid flow, the narrow arrows indicate the orientation of the edge



2.3 The Energy Part

The energy transport in the network is described by the third equation in (2.2). Thus for every $i \in \mathcal{I}$, we consider the advection equation

$$\partial_t e^i + v^i(t) \partial_x e^i = g(e^i), \tag{2.9}$$

supplemented with the initial condition

$$e^i(0, x) = e^{i,o}(x), \quad x \in I_i, \tag{2.10}$$

and with suitable boundary and coupling conditions. More precisely, the boundary condition applies only at the level of CHP and, when admissible, it reads

$$e^1(t, a_1) = e^{1,b}(t) \tag{2.11}$$

for a.e. $t > 0$. It prescribes the energy produced by the CHP.

The coupling conditions at each internal node $J \in \mathcal{J}$ depend on the solution of a differential equation. Fix an internal node $J \in \mathcal{J}$ and $t > 0$. Define the (possibly time dependent) sets (see Fig. 1)

$$\begin{aligned} \text{inc}^{+,t}(J) &= \{i \in \text{inc}(J) : v^i(t) > 0\}, & \text{inc}^{-,t}(J) &= \{i \in \text{inc}(J) : v^i(t) < 0\}, \\ \text{out}^{+,t}(J) &= \{i \in \text{out}(J) : v^i(t) > 0\}, & \text{out}^{-,t}(J) &= \{i \in \text{out}(J) : v^i(t) < 0\} \end{aligned} \tag{2.12}$$

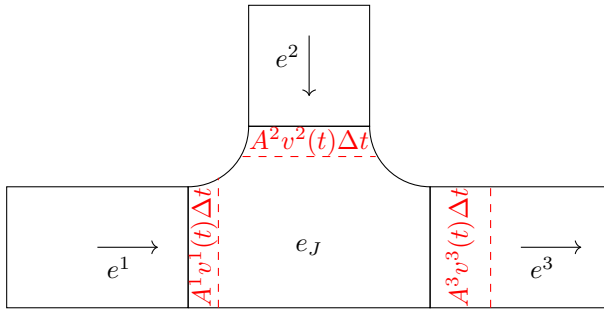


Fig. 2 Schematic model of the node J with two incoming and one outgoing pipes, used in the derivation of the coupling condition (2.14)

and consider the Cauchy problem

$$\left\{ \begin{aligned} \dot{e}_J &= \frac{1}{V_J} \left[\sum_{I_i \in \text{inc}^{+,t}(J)} A^i v^i(t) e^i(t, b_i) - \sum_{I_i \in \text{out}^{-,t}(J)} A^i v^i(t) e^i(t, a_i) \right. \\ &\quad \left. - e_J \sum_{I_i \in \text{inc}^{-,t}(J) \cup \text{out}^{+,t}(J)} A^i |v^i(t)| \right], \\ e_J(0) &= \frac{1}{\sum_{I_i \in \text{inc}^{-,t}(J) \cup \text{out}^{+,t}(J)} A^i |v^i(0)|} \\ &\quad \times \left[\sum_{I_i \in \text{inc}^{+,t}(J)} A^i v^i(0) e^i(0, b_i) - \sum_{I_i \in \text{out}^{-,t}(J)} A^i v^i(0) e^i(0, a_i) \right], \end{aligned} \right. \tag{2.13}$$

where $V_J > 0$ represents the volume inside the junction J . At the outgoing edges according to the fluid velocity, we impose that the energy is equal to e_j : for $t > 0$ and for every $i \in \text{inc}^{-,t}(J)$ and $j \in \text{out}^{+,t}(J)$, we set

$$e^i(t, b_i) = e^j(t, a_j) = e_J(t). \tag{2.14}$$

Remark 2.5 The coupling conditions (2.13) can be justified in the following way.

For $J \in \mathcal{J}$, define $V_J > 0$ the volume inside the junction J . Denote by $e_J(t)$ the temperature inside the junction J at time t , the variation Δe_J of e_J during the time

step Δt is given by

$$\begin{aligned}
 e_J(t) + \Delta e_J &= \frac{1}{V_J} \sum_{I_i \in \text{inc}^{+,t}(J)} A^i v^i(t) e^i(t, b_i) \Delta t \\
 &\quad - \frac{1}{V_J} \sum_{I_i \in \text{out}^{-,t}(J)} A^i v^i(t) e^i(t, a_i) \Delta t \\
 &\quad + \frac{1}{V_J} \left(V_J - \sum_{I_i \in \text{out}^{+,t}(J) \cap \text{inc}^{-,t}(J)} A^i |v^i(t)| \Delta t \right) e_J(t);
 \end{aligned}$$

see Fig. 2. Thus, passing to the limit as $\Delta t \rightarrow 0$, we obtain (2.13).

Remark 2.6 A reasonable alternative to the coupling condition (2.14) could be the following:

$$\begin{cases} \sum_{i \in \text{inc}(J)} A^i v^i(t) e^i(t, b_i) + \sum_{i \in \text{out}(J)} A^i v^i(t) e^i(t, a_i) = 0 \\ e^i(t, b_i) = e^j(t, a_j) \quad i \in \text{inc}^{-,t}(J), j \in \text{out}^{+,t}(J), \end{cases}$$

stating the conservation of energy inside the node and the perfect mixture of energies; see also [3, 32].

The advantage of conditions (2.13)–(2.14) is mainly technical. Indeed (2.13)–(2.14) permit to deduce total variation estimates needed in the proof of the main result.

We consider the following assumptions.

- (E.1) $e^{1,b} \in \mathbf{L}_{\text{loc}}^1((0, +\infty); \mathbb{R})$ and, for every $i \in \mathcal{I}$, $e^{i,o} \in \mathbf{L}^1([a_i, b_i]; \mathbb{R})$ with $\text{TV}(e^{i,o}) < +\infty$.
- (E.2) There exists $G > 0$ such that $|g(e_1) - g(e_2)| \leq G|e_1 - e_2|$ for every $e_1, e_2 \in \mathbb{R}$.

We give now the definition of solution for the energy system.

Definition 2.7 Fix $T > 0$ and assume (E.1) and (E.2). Furthermore, assume that for every $i \in \mathcal{I}$, the function $t \mapsto v^i(t)$ is in $\mathbf{L}^1((0, T); \mathbb{R})$.

A function $e = (e^{\mathcal{I}}, e_{\mathcal{J}}) = ((e^1, \dots, e^{N_{\mathcal{I}}}), (e_{J_1}, \dots, e_{J_M}))$ is a solution to (2.9), (2.10), (2.11), (2.13), and (2.14) on the time interval $[0, T]$ if the following conditions are satisfied.

1. For every $i \in \{1, \dots, N_{\mathcal{I}}\}$, the function $e^i \in \mathbf{C}^0([0, T]; \mathbf{L}^1([a_i, b_i]; \mathbb{R}))$ and, for every $t \in [0, T]$, $e^i(t)$ has finite total variation.
2. For every $J \in \mathcal{J}$, the function $e_J \in \mathbf{L}^1((0, T); \mathbb{R})$ has finite total variation.
3. $e^{\mathcal{I}}$ is a MV-solution to (2.9), i.e. for every $i \in \{1, \dots, N_{\mathcal{I}}\}$ and for every $\varphi \in \mathbf{C}_c^1((-\infty, T[\times \mathbb{R}; \mathbb{R}_+))$ and $k \in \mathbb{R}$,

$$\int_0^T \int_{a_i}^{b_i} (e^i(t, x) - k)^{\pm} (\partial_t \varphi(t, x) + v^i(t) \partial_x \varphi(t, x)) \, dx \, dt$$

$$\begin{aligned}
 &+ \int_0^T \int_{a_i}^{b_i} \operatorname{sgn}^\pm \left(e^i(t, x) - k \right) g(e^i(t, x)) \varphi(t, x) \, dx \, dt \\
 &+ \int_{a_i}^{b_i} \left(e^{i,o}(x) - k \right)^\pm \varphi(0, x) \, dx \\
 &+ \|v_i\|_{L^\infty([0, T]; \mathbb{R})} \int_0^t \left(e_{J, a_i}(t) - k \right)^\pm \varphi(t, a_i) \, dt \\
 &+ \|v_i\|_{L^\infty([0, T]; \mathbb{R})} \int_0^t \left(e_{J, b_i}(t) - k \right)^\pm \varphi(t, b_i) \, dt \geq 0,
 \end{aligned}$$

where $s^+ = \max\{s, 0\}$, $s^- = \max\{-s, 0\}$,

$$\operatorname{sgn}^+(s) = \begin{cases} 1 & \text{if } s > 0, \\ 0 & \text{if } s \leq 0, \end{cases} \quad \operatorname{sgn}^-(s) = \begin{cases} 0 & \text{if } s \geq 0, \\ -1 & \text{if } s < 0, \end{cases}$$

$(J, a_i), (J, b_i) \in \mathcal{J}$ denote the junctions located respectively at $x = a_i$ and $x = b_i$, and e_{J, a_i}, e_{J, b_i} are the Carathéodory solutions to (2.13).

For the definition of solution used in item 3 we refer to [30, Definition 1], see also [34, Definition 2.1] referring to the 1-D case.

2.4 The Complete System

Finally we consider the complete DHN system, composed by hydrodynamic and energy part. It is given by the system of differential equations

$$\begin{cases} \partial_x v^i = 0, & i \in \mathcal{I} \setminus \mathcal{I}_H, \\ \partial_t v^i + \frac{1}{\rho} \partial_x p^i = f(v^i), & i \in \mathcal{I} \setminus \mathcal{I}_H, \\ \partial_t v^k = \frac{1}{\alpha} \left(Q_k(t) - A^k v^k (e_{J_k}(t) - e(T_{out})) \right), & k \in \mathcal{I}_H, \\ \partial_t e^i + v^i(t) \partial_x e^i = g(e^i), & i \in \mathcal{I}, \end{cases} \tag{2.15}$$

for $i \in \mathcal{I}$, with initial conditions

$$\begin{cases} v^i(0) = v^{i,o}, \\ e^i(0, x) = e^{i,o}(x), \end{cases} \tag{2.16}$$

with boundary conditions

$$p^1(t, a_1) = p^{1,b}(t) \quad \text{and} \quad e^1(t, a_1) = e^{1,b}(t) \tag{2.17}$$

and with the coupling conditions (2.6) and (2.14).

We introduce here the concept of solution to the complete system.

Definition 2.8 Given $T > 0$, a triple (e, p, v) is a solution on the time interval $[0, T]$ to (2.15)–(2.16)–(2.17) with the coupling conditions (2.6) and (2.14) if the following conditions are satisfied.

1. $e = (e^{\mathcal{I}}, e_{\mathcal{J}}) = ((e^1, \dots, e^{N_{\mathcal{I}}}), (e_{J_1}, \dots, e_{J_M})),$
 $p = (p^1, \dots, p^{N_{\mathcal{I}}}),$ and $v = (v^1, \dots, v^{N_{\mathcal{I}}}).$
2. For the given node energies $e_{\mathcal{J}}, (p, v)$ is a solution as defined in Definition 2.4.
3. For the given velocity v, e is a solution as defined in Definition 2.7.

3 Well Posedness Result

In this part we deal with the well posedness result for the district heating network, which is stated in Sect. 3.3. In Sect. 3.1 and in Sect. 3.2 we consider well posedness for the hydrodynamics and energy part respectively, which are the basic steps in the proof of the main result.

3.1 Hydrodynamics Part

Here we consider the system (2.4), with the initial conditions (2.5), coupling conditions (2.6), boundary conditions (2.7), and with the conditions (2.8) at consumers' sites.

Theorem 3.1 *Fix $T > 0$ and assume that (H.1), (H.2), (H.3), and (C.1) hold. Then, system (2.4)–(2.8) admits a unique solution, in the sense of Definition 2.4.*

Moreover, the following stability estimate holds. There exists a positive constant $L > 0,$ depending on the time $T,$ such that, for every two sets of initial conditions

$\bar{v}^{i,o}$ and $\tilde{v}^{i,o}$ ($i \in \mathcal{I}$) satisfying (H.3),

two sets of power demands \bar{Q}_k, \tilde{Q}_k ($k \in \mathcal{I}_H$) satisfying (C.1), and two sets of node energy functions

$\bar{e}_{\mathcal{J}}, \tilde{e}_{\mathcal{J}}$ in $\mathbf{L}^1((0, T); \mathbb{R}^M),$ the corresponding solutions (\bar{p}, \bar{v}) and (\tilde{p}, \tilde{v}) satisfy, for a.e. $t \in [0, T],$

$$\sum_{i \in \mathcal{I}} \left\| \bar{v}^i - \tilde{v}^i \right\|_{\mathbf{C}^0([0,t])} \leq L \sum_{J \in \mathcal{J}} \|\bar{e}_J - \tilde{e}_J\|_{\mathbf{L}^1(0,t)} + L \sum_{k \in \mathcal{I}_H} \left\| \bar{Q}_k - \tilde{Q}_k \right\|_{\mathbf{L}^1(0,t)} + L \sum_{i \in \mathcal{I}} \left| \bar{v}^{i,o} - \tilde{v}^{i,o} \right|. \tag{3.1}$$

Proof The proof is very similar to the proof of Theorem 5.3 in [27], but here we use less regularity assumptions on the inputs and the solution. We show that there is a decomposition of the system into pure algebraic and pure ODE parts, where the existence of unique solutions follows directly from standard results about Cauchy problems for solutions in Carathéodory sense [19].

Define the transformation

$$v = \begin{pmatrix} v_0 \\ v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} A_Q^T & A_r^T & A_{PC}^T \end{pmatrix} v = A_Q^T v_0 + A_r^T v_1 + A_{PC}^T v_2, \tag{3.2}$$

where $A_Q^T \in \mathbb{R}^{N_I \times N_H}$ rearranges the velocities of consumer edges $v_0 \in \mathbb{R}^{N_H}$ onto the full velocity vector. The columns of A_Q^T are the unit vectors of the consumer edges in I such that $A_Q A_Q^T = I_{d_{N_H}}$.

The matrix $A_t \in \mathbb{R}^{(M-1) \times N_I}$ maps the edges of a spanning tree of \mathcal{G} onto the full edge set and $A_{PC} \in \mathbb{R}^{N_I - (M-1) \times N_I}$ is the incidence matrix of a set of fundamental cycles of \mathcal{G} corresponding to the spanning tree.

Furthermore we define $A = (A_r, A_r^p)$ as the incidence matrix of the whole graph, where $A_r^p \in \mathbb{R}^{N_I \times 1}$ is the column corresponding to the node of the CHP. A_r is the remaining matrix for the inner nodes and consumers. A formal definition of these matrices can be found in [27]. For those matrices, we use the following properties:

- (i) $R := \begin{pmatrix} A_t \\ A_{PC} \end{pmatrix} \in \mathbb{R}^{N_I \times N_I}$ is nonsingular
- (ii) $A_t A_r \in \mathbb{R}^{M-1 \times M-1}$ is nonsingular
- (iii) $A_{PC} A_r = 0$
- (iv) $A_Q A_t^T = 0$ and $A_Q A_{PC}^T = 0$

A proof for (i)–(iii) can be found in [27]. Property (iv) holds due to the fact that the rows of A_Q only affect consumer edges while those are excluded in the definition of A_t and A_{PC} .

System (2.4)–(2.8) can be compactly written as

$$\begin{aligned}
 \partial_t v + \frac{1}{\rho} A_r p_{\mathcal{J}} &= f(v) - \frac{1}{\rho} A_r^p p^{1,b} \\
 A_r^T v &= 0 \\
 \partial_t (A_Q v) &= \frac{1}{\alpha} \left(Q(t) - A^K (A_Q v) (e_{\mathcal{J}}(t) - e^{out}) \right),
 \end{aligned}
 \tag{3.3}$$

together with the corresponding initial conditions. Here, $p_{\mathcal{J}} \in \mathbf{L}^1((0, T); \mathbb{R}^M)$ is the vector of node pressures. Due to (2.6) this is well-defined. The last equation is the vectorized version of (2.8). We insert the transformation (3.2) and multiply by $\begin{pmatrix} A_t \\ A_{PC} \end{pmatrix}$ from the left. This leads to the equivalent formulation

$$\begin{aligned}
 \partial_t v_0 &= \frac{1}{\alpha} \left(Q(t) - A^K v_0 (e_{\mathcal{J}}(t) - e^{out}) \right) \\
 v_1 &= - \left(A_r^T A_t^T \right)^{-1} \left(A_r A_Q^T \right) v_0 \\
 \partial_t v_2 &= \left(A_{PC} A_{PC}^T \right)^{-1} A_{PC} \left(\frac{1}{\rho} A_r^p p^{1,b} + f(A_t^T v_1 + A_{PC}^T v_2) - A_t^T \partial_t v_1 \right) \\
 p_{\mathcal{J}} &= \rho \left(A_t A_r^T \right)^{-1} A_t \left(\frac{1}{\rho} A_r^p p^{1,b} + f(A_t^T v_1 + A_{PC}^T v_2) - \left(A_t^T \partial_t v_1 + A_{PC}^T \partial_t v_2 \right) \right).
 \end{aligned}
 \tag{3.4}$$

For system (3.4) the requirements for the existence theorem of solutions in Carathéodory sense are met, as the right hand side of the first equation is measurable in t as well as Lipschitz continuous in v_0 . We get a unique solution v_0 and due

to the pure algebraic connection also a unique v_1 . Similarly, the ODE for v_2 has continuously differentiable right hand side and we get existence of a unique v_2 .

Applying Gronwall inequality provides us with bound

$$|v(t)| \leq v_{max}(T) \tag{3.5}$$

after transforming back to the usual coordinates.

The stability estimate (3.1) directly follows from basic ODE theory and Gronwall Lemma, e.g. [6, 24]. □

Remark 3.2 Similar stability estimates can be deduced for the pressure p . We have, for $t > 0$,

$$\begin{aligned} & \| \bar{p} - \tilde{p} \|_{\mathbf{L}^1([0,t]; \mathbf{C}^1([a,b]; \mathbb{R}^{N_{\mathcal{I}}})} \\ & \leq \sum_{i \in \mathcal{I}} \left(\left\| \sup_x \left| \bar{p}^i(x) - \tilde{p}^i(x) \right| \right\|_{\mathbf{L}^1(0,t)} + \left\| \sup_x \left| \partial_x \bar{p}^i(x) - \partial_x \tilde{p}^i(x) \right| \right\|_{\mathbf{L}^1(0,t)} \right). \end{aligned} \tag{3.6}$$

Using the fact that $\partial_x p^i$ does not depend on the space variable, we deduce that, for $t > 0$,

$$\begin{aligned} & \| \bar{p} - \tilde{p} \|_{\mathbf{L}^1([0,t]; \mathbf{C}^1([a,b]; \mathbb{R}^{N_{\mathcal{I}}})} \\ & \leq \sum_{i \in \mathcal{I}} \left(1 + \frac{1}{b_i - a_i} \right) \left(\| \bar{p}(a_i) - \tilde{p}(a_i) \|_{\mathbf{L}^1(0,t)} + \| \bar{p}(b_i) - \tilde{p}(b_i) \|_{\mathbf{L}^1(0,t)} \right) \end{aligned} \tag{3.7}$$

and thus

$$\| \bar{p} - \tilde{p} \|_{\mathbf{L}^1([0,t]; \mathbf{C}^1([a,b]; \mathbb{R}^{N_{\mathcal{I}}})} \leq C \sum_{J \in \mathcal{J}} \| \bar{p}_J - \tilde{p}_J \|_{\mathbf{L}^1([0,t]; \mathbb{R}^M)}, \tag{3.8}$$

where the individual node pressures p_J is taken from $p_{\mathcal{J}}$ in (3.4). The form of the last equation in (3.4) immediately gives stability estimates for $p_{\mathcal{J}}$ due to (H.2) and boundedness of v . Those estimates are similar to the ones for $\partial_t v_2$. For the ease of notations and clarity of the proofs, we omit the estimates for p . Note that only the velocity couples to the energy transport. The triangular structure of (3.4) allows evaluation and stability estimates for v independently of the pressure, thus the results for the coupled system do not depend on p .

Remark 3.3 The constant L in Theorem 3.1 depends exponentially on T , since it is obtained through the Gronwall Lemma. More precisely

$$L = O(1)e^{O(1)T},$$

where the Landau symbol $O(1)$ denotes a suitable constant, which depends on initial data by $\max_{k \in \mathcal{I}} |v^{k,0}|$, but not on T .

3.2 Energy Network

Here we consider system (2.9), supplemented with the initial conditions (2.10), with the coupling conditions (2.14), and with the boundary conditions (2.11).

Theorem 3.4 Fix $T > 0$ and assume (E.1) and (E.2). Then, system (2.9)–(2.14) admits a unique solution, in the sense of Definition 2.7.

Moreover, there exists a positive constant $L > 0$, depending on the time T , such that, for every two sets of initial conditions $\bar{e}^{i,o}, \tilde{e}^{i,o}$, ($i \in \mathcal{I}$) and of boundary data $\bar{e}^{1,b}, \tilde{e}^{1,b}$, both satisfying (E.1), and two sets of velocity functions

$\bar{v}, \tilde{v} \in \mathbf{L}^1((0, T); \mathbb{R}^{N_{\mathcal{I}}})$, the corresponding solutions $\bar{e} = (\bar{e}^{\mathcal{I}}, \bar{e}_{\mathcal{J}})$ and $\tilde{e} = (\tilde{e}^{\mathcal{I}}, \tilde{e}_{\mathcal{J}})$ satisfy, for a.e. $t \in [0, T]$,

$$\begin{aligned} & \sum_{J \in \mathcal{J}} \|\bar{e}_J - \tilde{e}_J\|_{\mathbf{C}^0(0,t)} + \sum_{i \in \mathcal{I}} \|\bar{e}^i(t) - \tilde{e}^i(t)\|_{\mathbf{L}^1(I_i)} \\ & \leq L \sum_{i \in \mathcal{I}} \left(\|\bar{v}^i - \tilde{v}^i\|_{\mathbf{L}^1(0,t)} + \|\bar{e}^{i,o} - \tilde{e}^{i,o}\|_{\mathbf{L}^1(I_i)} + \|\bar{e}^{1,b} - \tilde{e}^{1,b}\|_{\mathbf{L}^1(0,t)} \right). \end{aligned} \tag{3.9}$$

Before we start with the proof, we state a lemma for the stability of the solution on a single edge.

Lemma 3.5 Fix $a, b \in \mathbb{R}$, with $a < b$, $v_1, v_2, e_L^1, e_L^2, e_R^1, e_R^2 \in \mathbf{L}^1(\mathbb{R}^+) \cap \mathbf{L}^\infty(\mathbb{R}^+)$ and $\bar{e}_1, \bar{e}_2 \in \mathbf{L}^\infty(a, b)$, all with bounded total variation. Let $G > 0$ be fixed and let $g: \mathbb{R} \rightarrow \mathbb{R}$ be a Lipschitz continuous function such that $|g(y_1) - g(y_2)| \leq G|y_1 - y_2|$ for every $y_1, y_2 \in \mathbb{R}$. Let e_1 and e_2 be the solutions to the following IBVP problems:

$$P_1 : \begin{cases} \partial_t e_1 + v_1 \partial_x e_1 = g(e_1) \\ e_1(t, a) = e_L^1(t) \\ e_1(t, b) = e_R^1(t) \\ e_1(0, x) = \bar{e}_1(x) \\ x \in (a, b), t > 0 \end{cases} \quad \text{and} \quad P_2 : \begin{cases} \partial_t e_2 + v_2 \partial_x e_2 = g(e_2) \\ e_2(t, a) = e_L^2(t) \\ e_2(t, b) = e_R^2(t) \\ e_2(0, x) = \bar{e}_2(x) \\ x \in (a, b), t > 0. \end{cases}$$

Then, for a.e. $t > 0$, the following stability estimate holds:

$$\begin{aligned} \|e_1(t) - e_2(t)\|_{\mathbf{L}^1(a,b)} & \leq e^{Gt} K \left[\|v_1 - v_2\|_{\mathbf{L}^1(0,t)} + \|\bar{e}_1 - \bar{e}_2\|_{\mathbf{L}^1(a,b)} \right. \\ & \quad \left. + \|e_L^1 - e_L^2\|_{\mathbf{L}^1(0,t)} + \|e_R^1 - e_R^2\|_{\mathbf{L}^1(0,t)} \right], \end{aligned} \tag{3.10}$$

where the constant K depends on the total variation and on the \mathbf{L}^∞ -norm of $\bar{e}_1, \bar{e}_2, e_L, e_R$, and on the \mathbf{L}^∞ -norm of v_1 and v_2 .

Furthermore, if $t \leq \frac{b-a}{v_{max}}$ with $|v_1|, |v_2| \leq v_{max}$, for the fluxes at the boundaries we have

$$\|v_1^- e_1(\cdot, a) - v_2^- e_2(\cdot, a)\|_{\mathbf{L}^1(0,t)} \leq K_{v,a} \|v_1 - v_2\|_{\mathbf{L}^1(0,t)} + K_{I,a} \|\bar{e}_1 - \bar{e}_2\|_{\mathbf{L}^1(a,b)}$$

$$+ K_{L,a} \left\| e_L^1 - e_L^2 \right\|_{\mathbf{L}^1(0,t)}, \tag{3.11}$$

$$\begin{aligned} \left\| v_1^+ e_1(\cdot, b) - v_2^+ e_2(\cdot, b) \right\|_{\mathbf{L}^1(0,t)} &\leq K_{v,b} \|v_1 - v_2\|_{\mathbf{L}^1(0,t)} + K_{I,b} \|\bar{e}_1 - \bar{e}_2\|_{\mathbf{L}^1(a,b)} \\ &+ K_{R,b} \left\| e_R^1 - e_R^2 \right\|_{\mathbf{L}^1(0,t)}, \end{aligned} \tag{3.12}$$

where $v_i^-(t) = -\min\{v_i(t), 0\}$, $v_i^+(t) = \max\{v_i(t), 0\}$. A sequential consideration with (3.10) can remove the bound on t .

The proof of Lemma 3.5 is given in Appendix A.1.

Proof of Theorem 3.4 Existence of solutions to such transport problems on networks have already been proven, e.g. in [33]. We focus on the stability estimate (3.9) that, to our knowledge, has not been shown before.

The proof consists of two steps.

For both steps, we decompose the network problem into shorter time intervals. Those intervals are chosen in such a way that information can not reach one node from another, so that the network problem then decouples into a sequence of localized problems on segments. By defining the intermediate times $0 = t_0 < t_1 < \dots < t_n = t$ with $\Delta t_i = (t_i - t_{i-1})$ such that $\max_{i=1, \dots, n} \Delta t_i \leq \frac{\min_{I_k \in J} (b_k - a_k)}{v_{max}}$, where the constant v_{max} denotes the maximum possible of velocity,

in each interval $[t_{i-1}, t_i]$ the information starting from one node can not reach a neighboring one. Therefore, a local consideration of the nodes with the adjacent edges is sufficient. To ease the notations, we assume here the topological orientation of the edges to point towards the node, such that positive velocities always represent an inflow, whereas negative velocities correspond to outgoing flows (different from the more complicated global definitions from (2.12)).

Step 1: Stability of node values \bar{e}_J and \tilde{e}_J .

From classical results about ODEs, we deduce that, for the junction J , the solutions \bar{e}_J and \tilde{e}_J of (2.13) fulfill

$$|\bar{e}_J(t) - \tilde{e}_J(t)| \leq \hat{K} \left\| \hat{e} - \tilde{e} \right\|_{L^1(0,t)} + K_{J,0} |\bar{e}_J(0) - \tilde{e}_J(0)|, \tag{3.13}$$

where $\hat{e} = \sum_{I_i \in \text{inc}(J)} A^i (v^i)^+ e^i(\cdot, b_i) + \sum_{I_i \in \text{out}(J)} A^i (v^i)^- e^i(\cdot, a_i)$, and \hat{K} and $K_{J,0}$ are suitable constants, not depending on time.

Using (3.11) and (3.12), we have that

$$\left\| \hat{e} - \tilde{e} \right\|_{L^1(0,t)} \leq K_J \|\bar{e}_J - \tilde{e}_J\|_{L^1(0,t)} + K_v \|\bar{v} - \tilde{v}\|_{L^1(0,t)} + K_0 \left\| \bar{e}^0 - \tilde{e}^0 \right\|_{L^1(a,b)} .$$

Inserting it into (3.13) yields

$$\begin{aligned} |\bar{e}_J(t) - \tilde{e}_J(t)| &\leq \alpha(t) + \hat{K} K_J \|\bar{e}_J - \tilde{e}_J\|_{L^1(0,t)} \\ &= \alpha(t) + \int_0^t \hat{K} K_J |\bar{e}_J(\tau) - \tilde{e}_J(\tau)| d\tau , \end{aligned}$$

where

$$\alpha(t) = \hat{K} K_v \|\bar{v} - \tilde{v}\|_{L^1(0,t)} + \hat{K} K_0 \left\| \bar{e}^0 - \tilde{e}^0 \right\|_{L^1(a,b)} + K_{J,0} |\bar{e}_J(0) - \tilde{e}_J(0)|.$$

Applying Gronwall inequality, we obtain

$$|\bar{e}_J(t) - \tilde{e}_J(t)| \leq \alpha(t) e^{\hat{K} K_J t},$$

and, due to the positivity of $\hat{K} K_J$,

$$\begin{aligned} \|\bar{e}_J - \tilde{e}_J\|_{C^0(0,t)} &\leq \alpha(t) e^{\hat{K} K_J t} \\ &\leq K \left(\|\bar{v} - \tilde{v}\|_{L^1(0,t)} + \left\| \bar{e}^0 - \tilde{e}^0 \right\|_{L^1(a,b)} + |\bar{e}_J(0) - \tilde{e}_J(0)| \right), \end{aligned}$$

where $K = e^{\hat{K} K_J T} (\hat{K} K_v + \hat{K} K_0 + K_{J,0})$.

Step 2: Stability estimates on the whole network

Here, for suitable $L_1 > 0$ and $L_2 > 0$, we aim to prove:

$$\begin{aligned} &\sum_{i \in \mathcal{I}} \left\| \bar{e}^i(t, \cdot) - \tilde{e}^i(t, \cdot) \right\|_{L^1(a_i, b_i)} \\ &\leq L_1 \sum_{k \in \mathcal{I}} \left(\left\| \bar{v}^k - \tilde{v}^k \right\|_{L^1(0,t)} + \left\| \bar{e}^{k,0} - \tilde{e}^{k,0} \right\|_{L^1(a_k, b_k)} + \left\| \bar{e}^{1,b} - \tilde{e}^{1,b} \right\|_{L^1(0,t)} \right) \end{aligned} \tag{3.14}$$

$$\begin{aligned} &\sum_{J \in \mathcal{J}} \|\bar{e}_J - \tilde{e}_J\|_{C^0(0,t)} \\ &\leq L_2 \sum_{k \in \mathcal{I}} \left(\left\| \bar{v}^k - \tilde{v}^k \right\|_{L^1(0,t)} + \left\| \bar{e}^{k,0} - \tilde{e}^{k,0} \right\|_{L^1(a_k, b_k)} + \left\| \bar{e}^{1,b} - \tilde{e}^{1,b} \right\|_{L^1(0,t)} \right). \end{aligned} \tag{3.15}$$

For each edge $i \in \mathcal{I}$ with $I_i = [a_i, b_i]$ we can apply Lemma 3.5 and get

$$\begin{aligned} &\sum_{i \in \mathcal{I}} \left\| \bar{e}^i(t, \cdot) - \tilde{e}^i(t, \cdot) \right\|_{L^1(I_i)} \\ &\leq \sum_{k \in \mathcal{I}} \left[K_v^k \left\| \bar{v}^k - \tilde{v}^k \right\|_{L^1(0,t)} + K_I^k \left\| \bar{e}^{k,0} - \tilde{e}^{k,0} \right\|_{L^1(I_k)} \right. \\ &\quad \left. + K_L^k \left\| \bar{e}_L^k - \tilde{e}_L^k \right\|_{L^1(0,t)} + K_R^k \left\| \bar{e}_R^k - \tilde{e}_R^k \right\|_{L^1(0,t)} \right], \end{aligned} \tag{3.16}$$

where e_L, e_R are the left and right boundary values. In case the edge is connected to a node, these are the node energies e_J .

Altogether we have

$$\begin{aligned}
 & \sum_{i \in \mathcal{I}} \left\| \bar{e}^i(t, \cdot) - \tilde{e}^i(t, \cdot) \right\|_{\mathbf{L}^1(I_i)} \\
 & \leq K_v \sum_{k \in \mathcal{I}} \left\| \bar{v}^k - \tilde{v}^k \right\|_{\mathbf{L}^1(0,t)} + K_I \sum_{k \in \mathcal{I}} \left\| \bar{e}^{k,0} - \tilde{e}^{k,0} \right\|_{\mathbf{L}^1(I_k)} \\
 & \quad + K_J \sum_{J \in \mathcal{J}} \left\| \bar{e}_J - \tilde{e}_J \right\|_{\mathbf{L}^1(0,t)} + K_J \left\| \bar{e}^{1,b} - \tilde{e}^{1,b} \right\|_{\mathbf{L}^1(0,t)} \tag{3.17} \\
 & \leq K_v \sum_{k \in \mathcal{I}} \left\| \bar{v}^k - \tilde{v}^k \right\|_{\mathbf{L}^1(0,t)} + K_I \sum_{k \in \mathcal{I}} \left\| \bar{e}^{k,0} - \tilde{e}^{k,0} \right\|_{\mathbf{L}^1(I_k)} \\
 & \quad + K_J t \sum_{J \in \mathcal{J}} \left\| \bar{e}_J - \tilde{e}_J \right\|_{C^0(0,t)} + K_J \left\| \bar{e}^{1,b} - \tilde{e}^{1,b} \right\|_{\mathbf{L}^1(0,t)},
 \end{aligned}$$

with $K_v = \max_{k \in \mathcal{I}} K_v^k$, $K_I = \max_{k \in \mathcal{I}} K_I^k$ and $K_J = \text{deg}(\mathcal{G})(\max_{k \in \mathcal{I}} K_L + \max_{k \in \mathcal{I}} K_R)$, where $\text{deg}(\mathcal{G})$ is the largest node degree in the network.

Consequently, (3.14) follows directly from (3.15).

For a node $J \in \mathcal{J}$ and a single time interval $[t_{i-1}, t_i]$, using the result of step 1, we obtain

$$\begin{aligned}
 & \left\| \bar{e}_J - \tilde{e}_J \right\|_{\mathbf{C}^0([t_{i-1}, t_i])} \\
 & \leq A_J^i \sum_{k \in \mathcal{I}} \left\| \bar{v}^k - \tilde{v}^k \right\|_{\mathbf{L}^1(t_{i-1}, t_i)} + B_J^i \sum_{k \in \mathcal{I}} \left\| \bar{e}^k(t_{i-1}, \cdot) - \tilde{e}^k(t_{i-1}, \cdot) \right\|_{\mathbf{L}^1(I_k)} \\
 & \quad + |\bar{e}_J(t_{i-1}) - \tilde{e}_J(t_{i-1})|, \tag{3.18}
 \end{aligned}$$

for suitable constants A_J^i and B_J^i . Set $A = \max_{i,J} A_J^i$, $B = \max_{i,J} B_J^i$. Then, by iterative insertion, we get

$$\begin{aligned}
 & \sum_{J \in \mathcal{J}} \left\| \bar{e}_J - \tilde{e}_J \right\|_{\mathbf{C}^0([0,t])} = \sum_{J \in \mathcal{J}} \sup_i \left\| \bar{e}_J - \tilde{e}_J \right\|_{\mathbf{C}^0([t_{i-1}, t_i])} \\
 & \leq \sum_{J \in \mathcal{J}} \sum_{i=1}^n \left\| \bar{e}_J - \tilde{e}_J \right\|_{\mathbf{C}^0([t_{i-1}, t_i])} \\
 & \stackrel{(3.18)}{\leq} \sum_{J \in \mathcal{J}} \sum_{i=1}^n \left(A \sum_{k \in \mathcal{I}} \left\| \bar{v}^k - \tilde{v}^k \right\|_{\mathbf{L}^1(t_{i-1}, t_i)} \right. \\
 & \quad \left. + B \sum_{k \in \mathcal{I}} \left\| \left(\bar{e}^k - \tilde{e}^k \right) (t_{i-1}, \cdot) \right\|_{\mathbf{L}^1(I_k)} + |\bar{e}_J - \tilde{e}_J(t_{i-1})| \right) \\
 & \leq \sum_{J \in \mathcal{J}} \left(A \sum_{k \in \mathcal{I}} \left\| \bar{v}^k - \tilde{v}^k \right\|_{\mathbf{L}^1(0,t)} \right)
 \end{aligned}$$

$$\begin{aligned}
 & + B \sum_{i=0}^{n-1} \sum_{k \in \mathcal{I}} \left(\left\| (\tilde{e}^k - \tilde{e}^k)(t_i, \cdot) \right\|_{\mathbf{L}^1(I_k)} + \|e_J - \tilde{e}_J\|_{\mathbf{C}^0([0, t_{i-1}])} \right) \\
 & \stackrel{[(3.17)]}{\leq} \sum_{J \in \mathcal{J}} \left(A \sum_{k \in \mathcal{I}} \left\| \tilde{v}^k - \tilde{v}^k \right\|_{\mathbf{L}^1(0, t)} + B \sum_{i=0}^{n-1} \left(K_v \sum_{k \in \mathcal{I}} \left\| \tilde{v}^k - \tilde{v}^k \right\|_{\mathbf{L}^1(0, t_i)} \right. \right. \\
 & \quad \left. \left. + K_J t \sum_{J \in \mathcal{J}} \| \tilde{e}_J - \tilde{e}_J \|_{\mathbf{C}^0([0, t_i])} + K_I \sum_{k \in \mathcal{I}} \left\| (\tilde{e}^k - \tilde{e}^k)(t_{i-1}, \cdot) \right\|_{\mathbf{L}^1(I_k)} \right) \right. \\
 & \quad \left. + \| \tilde{e}_J - \tilde{e}_J \|_{\mathbf{C}^0([0, t_{i-1}])} \right) \\
 & \stackrel{[(3.10)]}{\leq} \sum_{J \in \mathcal{J}} \left[A \sum_{k \in \mathcal{I}} \left\| \tilde{v}^k - \tilde{v}^k \right\|_{\mathbf{L}^1(0, t)} \right. \\
 & \quad \left. + (n-1) K_I^{(n-2)} B \sum_{i=0}^{n-1} \left(K_v \sum_{k \in \mathcal{I}} \left\| \tilde{v}^k - \tilde{v}^k \right\|_{\mathbf{L}^1(0, t_i)} \right) \right. \\
 & \quad \left. + K_J t \sum_{J \in \mathcal{J}} \| \tilde{e}_J - \tilde{e}_J \|_{\mathbf{C}^0([0, t_i])} + K_I \sum_{k \in \mathcal{I}} \left\| (\tilde{e}^k - \tilde{e}^k)(t_0, \cdot) \right\|_{\mathbf{L}^1(I_k)} \right. \\
 & \quad \left. + \| \tilde{e}_J - \tilde{e}_J \|_{\mathbf{C}^0([0, t_{i-1}])} \right] \\
 & \leq M \left(\left(A + (n-1) B K_I^{(n-2)} K_v \right) \sum_{k \in \mathcal{I}} \left\| \tilde{v}^k - \tilde{v}^k \right\|_{\mathbf{L}^1(0, t)} \right. \\
 & \quad \left. + (n-1)^2 B K_I^{(n-1)} \sum_{k \in \mathcal{I}} \left\| \tilde{e}^{k,0} - \tilde{e}^{k,0} \right\|_{\mathbf{L}^1(I_k)} \right) \tag{3.19}
 \end{aligned}$$

$$\begin{aligned}
 & + (n-1) B K_I^{(n-2)} (K_J t + 1) \sum_{i=1}^{n-1} \sum_{J \in \mathcal{J}} \| \tilde{e}_J - \tilde{e}_J \|_{\mathbf{C}^0([0, t_i])} \Big). \tag{3.20}
 \end{aligned}$$

This is a recursive formula of the form

$$a(i) \leq c_1 + c_2 \sum_{l=1}^{i-1} a(l), \quad a(1) = c_1, \tag{3.21}$$

with

$$\begin{aligned}
 c_1 & = c_1^1 \sum_{k \in \mathcal{I}} \left\| \tilde{v}^k - \tilde{v}^k \right\|_{\mathbf{L}^1(0, t)} + c_1^2 \sum_{k \in \mathcal{I}} \left\| \tilde{e}^{k,0} - \tilde{e}^{k,0} \right\|_{\mathbf{L}^1(I_k)}, \\
 c_1^1 & = M \left(A + (n-1) B K_I^{(n-2)} K_v \right), \quad c_1^2 = M(n-1)^2 B K_I^{(n-1)}, \\
 c_2 & = M(n-1) B K_I^{(n-2)} (K_J t + 1).
 \end{aligned}$$

Hence

$$a(i) \leq c_1 (1 + c_2)^{(i-1)}. \tag{3.22}$$

So the explicit estimate of (3.20) is

$$\begin{aligned} & \sum_{J \in \mathcal{J}} \|\bar{e}_J - \tilde{e}_J\|_{\mathbf{C}^0(0,t)} \\ & \leq (1 + c_2)^{(n-1)} \left(c_1^1 \sum_{k \in \mathcal{I}} \|\bar{v}^k - \tilde{v}^k\|_{\mathbf{L}^1(0,t)} + c_1^2 \sum_{k \in \mathcal{I}} \|\bar{e}^{k,0} - \tilde{e}^{k,0}\|_{\mathbf{L}^1(I_k)} \right) \\ & \leq L_2 \sum_{k \in \mathcal{I}} \left(\|\bar{v}^k - \tilde{v}^k\|_{\mathbf{L}^1(0,t)} + \|\bar{e}^{k,0} - \tilde{e}^{k,0}\|_{\mathbf{L}^1(I_k)} \right). \end{aligned} \tag{3.23}$$

The result follows directly from (3.18). □

Remark 3.6 The constant L in Theorem 3.4 depends exponentially on T . More precisely

$$L = O(1)e^{O(1)T},$$

where the Landau symbol $O(1)$ denotes a suitable constant, which depends on initial data, but not on T . Regarding the initial data, the constants contain only $\max_{i \in \mathcal{I}} \|e^{k,0}\|_\infty$ and the bound on the total variation TV_e .

3.3 The Complete System

In this part, we deal with the well posedness result for the complete system.

Theorem 3.7 *Assume (H.1), (H.3), (E.1), (E.2), and (C.1) hold. Then, for any $T > 0$ the system (2.15)–(2.16)–(2.17)–(2.6)–(2.14) admits a unique solution, in the sense of Definition 2.8.*

Moreover there exists a positive constant $L > 0$ such that for two power demands \bar{Q}_k and $\tilde{Q}_k, k \in \mathcal{I}_H$, two boundary data $\bar{e}^{1,b}, \tilde{e}^{1,b}$ satisfying (E.1) and initial conditions $\bar{v}^{i,o}, \tilde{v}^{i,o}$ and $\bar{e}^{i,o}, \tilde{e}^{i,o}$ ($i \in \mathcal{I}$) satisfying (H.3), denoting by $(\bar{e}, \bar{p}, \bar{v})$ and $(\tilde{e}, \tilde{p}, \tilde{v})$ the corresponding solutions to (2.15)–(2.16)–(2.17)–(2.6)–(2.14), the following stability estimate holds: for $t \in [0, T]$,

$$\begin{aligned} & \sum_{i \in \mathcal{I}} \left[\|\bar{v}^i - \tilde{v}^i\|_{\mathbf{C}^0([0,t])} + \|\bar{e}^i(t) - \tilde{e}^i(t)\|_{\mathbf{L}^1(I_i)} \right] + \sum_{J \in \mathcal{J}} \|\bar{e}_J - \tilde{e}_J\|_{\mathbf{C}^0([0,t])} \\ & \leq L \left(\sum_{k \in \mathcal{I}_H} \|\bar{Q}_k - \tilde{Q}_k\|_{\mathbf{L}^1(0,t)} + \|\bar{e}^{1,b} - \tilde{e}^{1,b}\|_{\mathbf{L}^1(0,t)} \right. \\ & \quad \left. + \sum_{i \in \mathcal{I}} \left| \bar{v}^{i,o} - \tilde{v}^{i,o} \right| + \|\bar{e}^{i,o} - \tilde{e}^{i,o}\|_{\mathbf{L}^1(I_i)} \right). \end{aligned} \tag{3.24}$$

Proof Define the set

$$X_v = \left\{ v \in \mathbf{C}^0([0, T]; \mathbb{R}^{N_{\mathcal{I}}}) : \sum_{j \in J} A^j v^j = 0 \text{ for every } J \in \mathcal{J} \right\}, \tag{3.25}$$

which is a closed subset of the Banach space $\mathbf{C}^0([0, T]; \mathbb{R}^{N_{\mathcal{I}}})$ endowed with the norm

$$\|v\|_{X_v} = \sup_{t \in [0, T]} |v(t)|. \tag{3.26}$$

Consider the operator

$$\begin{aligned} \mathcal{T} : X_v &\longrightarrow X_v \\ v &\longmapsto w \end{aligned} \tag{3.27}$$

where \mathcal{T} is defined by two subsequent steps.

For given initial values e^0 and external inputs e^b we first we define by \mathcal{T}_e the operator producing a solution to the energy subsystem according to Theorem 3.4. The solution $g = \mathcal{T}_e(v)$ is split into $g = (g^{\mathcal{I}}, g_{\mathcal{J}})$, where $g_{\mathcal{J}}$ denotes the node energy functions. Then, with v^0 and Q_K given, the operator \mathcal{T}_v provides the solution to the hydrodynamic part (see Theorem 3.1). Its output we name $w = \mathcal{T}_v(g_{\mathcal{J}})$.

\mathcal{T} is well defined. Fix $v \in X_v$. Then, for every $i \in \{1, \dots, N_{\mathcal{I}}\}$ $v^i \in \mathbf{L}^1([0, T]; \mathbb{R})$.

By Theorem 3.4, there exists a unique $g = (g^{\mathcal{I}}, g_{\mathcal{J}}) = \mathcal{T}_e(v)$, such that, in particular, $g_J \in \mathbf{L}^1([0, T]; \mathbb{R})$ for every $J \in \mathcal{J}$ (see Item 2 in Definition 2.7).

Since the set of internal nodes on pipes connected to the houses, i.e. pipes in the set $\mathcal{I}_{\mathcal{H}}$, are a subset of \mathcal{J} , by Theorem 3.1 there exists a unique $w = \mathcal{T}_v(g_{\mathcal{J}})$ such that, in particular $w \in \mathbf{AC}([0, T]; \mathbb{R}^{N_{\mathcal{I}}})$ and $\sum_{h \in J} A^h w^h = 0$ for every junction $J \in \mathcal{J}$, that is $w \in X_v$.

\mathcal{T} is a contraction. Let $T' \leq T$. Fix two elements $\bar{v}, \tilde{v} \in X_v$ and denote

$$\bar{w} = \mathcal{T}(\bar{v}), \quad \tilde{w} = \mathcal{T}(\tilde{v}).$$

By Theorem 3.4 we deduce that

$$\begin{aligned} \sum_{J \in \mathcal{J}} \|\bar{g}_J - \tilde{g}_J\|_{\mathbf{C}^0([0, T'])} &\leq L_E(T) \sum_{i \in \mathcal{I}} \|\bar{v}^i - \tilde{v}^i\|_{\mathbf{L}^1(0, T')} \\ &\leq L_E(T) T' \sum_{i \in \mathcal{I}} \|\bar{v}^i - \tilde{v}^i\|_{\mathbf{C}^0([0, T'])} \\ &\leq L_E(T) T' N_{\mathcal{I}}^2 \|\bar{v} - \tilde{v}\|_{X_v}. \end{aligned}$$

Therefore, using Theorem 3.1, we deduce that

$$\|\bar{w} - \tilde{w}\|_{X_v} = \sup_{t \in [0, T']} |\bar{w}(t) - \tilde{w}(t)| \leq \sum_{i \in \mathcal{I}} \|\bar{w}^i - \tilde{w}^i\|_{\mathbf{C}^0([0, T'])}$$

$$\begin{aligned} &\leq L_H(T) \sum_{J \in \mathcal{J}} \|\bar{g}_J - \tilde{g}_J\|_{\mathbf{L}^1(0, T')} \leq L_H(T) T' \sum_{J \in \mathcal{J}} \|\bar{g}_J - \tilde{g}_J\|_{\mathbf{C}^0([0, T'])} \\ &\leq L_H(T) L_E(T) T'^2 N_{\mathcal{I}}^2 \|\bar{v} - \tilde{v}\|_{X_v}. \end{aligned}$$

We choose $T' > 0$ such that $L_H(T) L_E(T) T'^2 N_{\mathcal{I}}^2 < 1$, proving that \mathcal{T} is a contraction on $[0, T']$.

Thus the requirements of Banach fixed point theorem [1] are fulfilled, leading to the unique existence of a solution on $[0, T']$.

Since $L_H(T)$ and $L_E(T)$ do not depend on T' , we aim to repeat this procedure to cover the full interval $[0, T]$. According to Remark 3.3 and Remark 3.6 these constants depend on the respective initial conditions only by v_{max} , e_{max} and TV_e .

Due to (3.5), Lemmas A.3 and A.4 these quantities are globally bounded and $L_H(T)$ and $L_E(T)$ can be chosen independently of the individual initial condition. Thus by repeating the contraction finitely many times, we obtain uniquely the solution on $[0, T]$.

Stability estimate. Let \bar{Q}_k and \tilde{Q}_k ($k \in \mathcal{I}_H$) denote two power demands, $\bar{e}^{1,b}$, $\tilde{e}^{1,b}$ two boundary conditions and $\bar{v}^{i,o}$, $\tilde{v}^{i,o}$, $\bar{e}^{i,o}$, $\tilde{e}^{i,o}$ initial conditions for the velocity and energy. Denote respectively with $(\bar{e}, \bar{p}, \bar{v})$ and $(\tilde{e}, \tilde{p}, \tilde{v})$ the corresponding solutions. By Theorem 3.1, we deduce that

$$\begin{aligned} &\sum_{i \in \mathcal{I}} \|\bar{v}^i - \tilde{v}^i\|_{\mathbf{C}^0([0, t])} \\ &\leq L \left(\sum_{J \in \mathcal{J}} \|\bar{e}_J - \tilde{e}_J\|_{\mathbf{L}^1(0, t)} + \sum_{k \in \mathcal{I}_H} \|\bar{Q}_k - \tilde{Q}_k\|_{\mathbf{L}^1(0, t)} + \sum_{i \in \mathcal{I}} |\bar{v}^{i,o} - \tilde{v}^{i,o}| \right) \\ &\leq L \left(t \sum_{J \in \mathcal{J}} \|\bar{e}_J - \tilde{e}_J\|_{\mathbf{C}^0([0, t])} + \sum_{k \in \mathcal{I}_H} \|\bar{Q}_k - \tilde{Q}_k\|_{\mathbf{L}^1(0, t)} + \sum_{i \in \mathcal{I}} |\bar{v}^{i,o} - \tilde{v}^{i,o}| \right). \end{aligned}$$

By Theorem 3.4 we deduce that

$$\begin{aligned} &\sum_{J \in \mathcal{J}} \|\bar{e}_J - \tilde{e}_J\|_{\mathbf{C}^0([0, t])} + \sum_{i \in \mathcal{I}} \|\bar{e}^i(t) - \tilde{e}^i(t)\|_{\mathbf{L}^1(I_i)} \\ &\leq L \sum_{i \in \mathcal{I}} \left(\|\bar{v}^i - \tilde{v}^i\|_{\mathbf{L}^1(0, t)} + \|\bar{e}^{i,o} - \tilde{e}^{i,o}\|_{\mathbf{L}^1(I_i)} \right) + L \|\bar{e}^{1,b} - \tilde{e}^{1,b}\|_{\mathbf{L}^1(0, t)} \\ &\leq L \sum_{i \in \mathcal{I}} \left(t \|\bar{v}^i - \tilde{v}^i\|_{\mathbf{C}^0([0, t])} + \|\bar{e}^{i,o} - \tilde{e}^{i,o}\|_{\mathbf{L}^1(I_i)} \right) + L \|\bar{e}^{1,b} - \tilde{e}^{1,b}\|_{\mathbf{L}^1(0, t)}. \end{aligned}$$

From (3.17), we obtain

$$\begin{aligned}
 & \sum_{i \in \mathcal{I}} \left(\|\tilde{v}^i - \tilde{v}^i\|_{\mathbf{C}^0([0,t])} + \|\tilde{e}^i(t) - \tilde{e}^i(t)\|_{\mathbf{L}^1(I_i)} \right) + \sum_{J \in \mathcal{J}} \|\bar{e}_J - \tilde{e}_J\|_{\mathbf{C}^0([0,t])} \\
 & \leq (K_v t + 1) \sum_{k \in \mathcal{I}} \|\tilde{v}^k - \tilde{v}^k\|_{\mathbf{C}^0([0,t])} + K_I \sum_{k \in \mathcal{I}} \|\tilde{e}^{i,0} - \tilde{e}^{i,0}\|_{\mathbf{L}^1(I_k)} \tag{3.28} \\
 & \quad + (K_J t + 1) \sum_{J \in \mathcal{J}} \|\bar{e}_J - \tilde{e}_J\|_{\mathbf{C}^0([0,t])} + K_J \|\tilde{e}^{1,b} - \tilde{e}^{1,b}\|_{\mathbf{L}^1(0,t)}.
 \end{aligned}$$

In particular, the estimates of Theorems 3.1 and 3.4 hold point wise, such that

$$\begin{aligned}
 & \sum_{i \in \mathcal{I}} \left| \tilde{v}^i(t) - \tilde{v}^i(t) \right| + \sum_{J \in \mathcal{J}} |\bar{e}_J(t) - \tilde{e}_J(t)| \\
 & \leq L \left(\sum_{J \in \mathcal{J}} \|\bar{e}_J - \tilde{e}_J\|_{\mathbf{L}^1(0,t)} + \sum_{i \in \mathcal{I}} \|\tilde{v}^i - \tilde{v}^i\|_{\mathbf{L}^1(0,t)} \right) + L\gamma.
 \end{aligned}$$

where

$$\begin{aligned}
 \gamma = & \sum_{k \in \mathcal{I}_{\mathcal{H}}} \|\bar{Q}_k - \tilde{Q}_k\|_{\mathbf{L}^1(0,t)} + \sum_{i \in \mathcal{I}} \left(\left| \tilde{v}^{i,o} - \tilde{v}^{i,o} \right| + \|\tilde{e}^{i,o} - \tilde{e}^{i,o}\|_{\mathbf{L}^1(I_i)} \right) \\
 & + \|\tilde{e}^{1,b} - \tilde{e}^{1,b}\|_{\mathbf{L}^1(0,t)}.
 \end{aligned}$$

Here we can now apply Gronwall inequality to get

$$\sum_{i \in \mathcal{I}} \left| \tilde{v}^i(t) - \tilde{v}^i(t) \right| + \sum_{J \in \mathcal{J}} |\bar{e}_J(t) - \tilde{e}_J(t)| \leq e^{Lt} \gamma.$$

The monotonicity of the right hand side and (3.28) imply

$$\begin{aligned}
 & \sum_{i \in \mathcal{I}} \left(\|\tilde{v}^i - \tilde{v}^i\|_{\mathbf{C}^0([0,t])} + \|\tilde{e}^i(t) - \tilde{e}^i(t)\|_{\mathbf{L}^1(I_i)} \right) + \sum_{J \in \mathcal{J}} \|\bar{e}_J - \tilde{e}_J\|_{\mathbf{C}^0([0,t])} \\
 & \leq (K t + 1) e^{Lt} \gamma + K_I \sum_{k \in \mathcal{I}} \|\tilde{e}^{i,0} - \tilde{e}^{i,0}\|_{\mathbf{L}^1(I_k)} + K_J \|\tilde{e}^{1,b} - \tilde{e}^{1,b}\|_{\mathbf{L}^1(0,t)},
 \end{aligned}$$

which concludes the proof. □

3.4 Optimal Control Problems

In this part, we consider the complete teleheating system (2.15) as a control system, where the boundary term $e^{1,b}$ at the CHP acts as a control function, and we assume that, given $T > 0$, the solution exists on the time interval $[0, T]$. In this perspective, a

natural problem consists in finding a control function $e^{1,b}$, satisfying assumption **(E.1)**, which minimizes the functional

$$\begin{aligned}
 J(e^{1,b}) &= \sum_{k \in \mathcal{I}_H} \alpha_k \int_0^T \left(Q_k(t) - A^k v^k(t)(e_{J_{k_1}}(t) - e^{k,out}) \right)^2 dt \\
 &+ \sum_{k \in \mathcal{I}_H} \beta_k \left(Q_k(T) - A^k v^k(T)(e_{J_{k_1}}(T) - e^{k,out}) \right)^2 \\
 &+ \gamma_1 \int_0^T |e^{1,b}(t)| dt + \gamma_2 \text{TV} \left(e^{1,b} \right) \\
 &+ \gamma_3 \int_0^T |e^{1,b}(t)v^1(t)| dt + \gamma_4 \text{TV} \left(e^{1,b}v^1 \right),
 \end{aligned}
 \tag{3.29}$$

for suitable coefficients $\alpha_k \geq 0, \beta_k \geq 0$, and $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \geq 0$. The minimization of the first two terms in **(3.29)** aims at producing the required temperature in the houses respectively on the whole time interval and at the final time. The third term in **(3.29)** measures the total energy produced by the CHP. The minimization of the fourth term penalizes too many oscillation in the energy production. The fifth and sixth terms have similar meanings with respect to the total power provided at the CHP. The next result deals with the lower semicontinuity of J .

Proposition 3.8 *Assume that the hypotheses **(H.1)**, **(H.2)**, **(H.3)**, **(C.1)**, **(E.1)**, and **(E.2)** hold. Define the sets*

$$\begin{aligned}
 E_1 &= \mathbf{L}^1((0, T); [0, \bar{e}_{\max}]), \\
 E_2 &= \left\{ e \in \mathbf{L}^1((0, T); [0, \bar{e}_{\max}]) : \text{TV}(e) < +\infty \right\}.
 \end{aligned}
 \tag{3.30}$$

Then the functional $J : E_2 \rightarrow \mathbb{R}$, defined in **(3.29)**, is lower semicontinuous with respect to the \mathbf{L}^1 topology. If $\gamma_2 = 0$ and $\gamma_4 = 0$, then $J : E_1 \rightarrow \mathbb{R}$ is also continuous.

Proof We prove that $J : E_2 \rightarrow \mathbb{R}$ is lower semicontinuous with respect the \mathbf{L}^1 topology. The final statement can be proved similarly.

Fix $\bar{e}^{1,b} \in E_2$ and a sequence $e_n^{1,b} \in E_2$ such that $e_n^{1,b} \rightarrow \bar{e}^{1,b}$ in $\mathbf{L}^1(0, T)$ as $n \rightarrow +\infty$. Denote with $(\bar{e}, \bar{p}, \bar{v})$ and respectively with (e_n, p_n, v_n) the solution to the system **(2.15)** corresponding to the boundary data $\bar{e}^{1,b}$ and respectively to $e_n^{1,b}$, according to Definition 2.8. Note that these solutions exist by Theorem 3.7.

Clearly, by Theorem 3.7, we have that

$$\lim_{n \rightarrow +\infty} \int_0^T |e_n^{1,b}(t)| dt = \int_0^T |\bar{e}^{1,b}(t)| dt$$

and

$$\lim_{n \rightarrow +\infty} \int_0^T |e_n^{1,b}(t)v_n^1(t)| dt = \int_0^T |\bar{e}^{1,b}(t)\bar{v}^1(t)| dt.$$

Moreover, by [18, Theorem 1, Sect. 5.2.1],

$$\liminf_{n \rightarrow +\infty} \text{TV} \left(e_n^{1,b} \right) \geq \text{TV} \left(\bar{e}^{1,b} \right)$$

and

$$\liminf_{n \rightarrow +\infty} \text{TV} \left(e_n^{1,b} v_n^1 \right) \geq \text{TV} \left(\bar{e}^{1,b} \bar{v}^1 \right).$$

By Theorem 3.7, we deduce that

$$\lim_{n \rightarrow +\infty} \|v_n - \bar{v}\|_{\mathbf{C}^0(0,T)} = 0$$

and

$$\lim_{n \rightarrow +\infty} \sup_{t \in [0,T]} \|e_n - \bar{e}\|_{\mathbf{L}^1} = 0.$$

Therefore, for every $k \in \mathcal{I}_H$, we deduce that

$$\begin{aligned} & \lim_{n \rightarrow +\infty} \int_0^T \left(Q_k(t) - A^k v_n^k(t) (e_{J_{k_1}}(t) - e^{k,out}) \right)^2 dt \\ &= \int_0^T \left(Q_k(t) - A^k \bar{v}^k(t) (e_{J_{k_1}}(t) - e^{k,out}) \right)^2 dt \end{aligned}$$

and

$$\begin{aligned} & \lim_{n \rightarrow +\infty} \left(Q_k(T) - A^k v_n^k(T) (e_{J_{k_1}}(T) - e^{k,out}) \right)^2 \\ &= \left(Q_k(T) - A^k \bar{v}^k(T) (e_{J_{k_1}}(T) - e^{k,out}) \right)^2. \end{aligned}$$

The proof is so concluded. □

The next result deals with optimal solutions.

Corollary 3.9 *Assume that (H.1), (H.2), (H.3), (C.1), (E.1), and (E.2) hold. Fix \mathcal{K} , a compact subset of E_1 or E_2 , defined in (3.30), with respect to the \mathbf{L}^1 topology. Then there exists $\bar{e}^{1,b} \in \mathcal{K}$ such that*

$$J \left(\bar{e}^{1,b} \right) = \min_{e^{1,b} \in \mathcal{K}} J \left(e^{1,b} \right).$$

Proof The proof follows the lines of the direct method of calculus of variations; see [14, 16] for more details. Consider a minimizing sequence $e_n^{1,b} \in \mathcal{K}$ for the functional J , i.e.

$$\lim_{n \rightarrow +\infty} J \left(e_n^{1,b} \right) = \inf_{e^{1,b} \in \mathcal{K}} J \left(e^{1,b} \right).$$

Since \mathcal{K} is compact, then there exists $\bar{e}^{1,b} \in \mathcal{K}$ and a subsequence $e_{n_k}^{1,b}$ such that $e_{n_k}^{1,b} \rightarrow \bar{e}^{1,b}$ as $k \rightarrow +\infty$. By using the lower semicontinuity of J (see Proposition 3.8), we deduce that

$$\inf_{e^{1,b} \in \mathcal{K}} J(e^{1,b}) = \lim_{n \rightarrow +\infty} J(e_n^{1,b}) = \liminf_{k \rightarrow +\infty} J(e_{n_k}^{1,b}) \geq J(\bar{e}^{1,b}),$$

concluding the proof. □

Remark 3.10 Note that the previous result holds if \mathcal{K} is a compact subset of E_1 (or of E_2) with respect the L^1 -topology. Therefore Corollary 3.9 can not be applied for general closed and bounded subsets \mathcal{K} of E_1 or E_2 . If instead \mathcal{K} is a closed and bounded subset of a finite dimensional subspace of E_1 or E_2 , then the functional J admits a minimum.

Remark 3.11 The results in the present section deal only with the existence of *optimal* controls which minimize the functional J in (3.29). Conversely, no necessary conditions, that represent the main tool in the search of optimizers, are deduced here. From the analytic point of view, this is a challenging problem since it needs some differentiability properties of the functional J with respect the topology of the control set. Dealing with solutions with low regularity makes the problem hard.

However one can try to find approximate optimal controls using numerical schemes inspired for example by the *gradient descent method* [4] as in [31], even if the present regularity does not fully justifies it. A possible not-optimized algorithm for approximating an optimal control could be the following.

1. Replace the control set \mathcal{K} by a finite-dimensional approximation \mathcal{K}_d .
2. Fix an initial control $e_0 \in \mathcal{K}_d$.
3. Find the numerical gradient $\nabla J(e_0)$ of J at e_0 and construct the control $e_1 = e_0 - \gamma_0 \nabla J(e_0)$, possibly with a projection on the set \mathcal{K}_d , where $\gamma_0 > 0$ is a sufficiently small learning rate.
4. Recursively, given $e_n \in \mathcal{K}_d$, find the numerical gradient $\nabla J(e_n)$ of J at e_n and construct the control $e_{n+1} = e_n - \gamma_n \nabla J(e_n)$, possibly with a projection on the set \mathcal{K}_d , where $\gamma_n > 0$ is a sufficiently small learning rate.

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A Appendix

A.1 Lipschitz continuity of solutions to advection equations

In this section we give, for an advection equation on a bounded interval, a proof of the stability of the solution and the traces of the flux at the boundary w.r.t. the speed. Note that we do require a uniform upper bound on the velocity, but do not restrict the orientation of the flow, i.e. the direction of the flow can change arbitrarily often.

Proof of Lemma 3.5 Observe first that the *broad solution* (i.e. the solution constructed using characteristics curves) to an initial-boundary value problem such as P_1 or P_2 is indeed a MV-solution, as introduced in item 3 of Definition 2.7. The proof of this claim follows the line of [10, Lemma 4.3], see also [13, Formula 4.5] for the definition of solution constructed through characteristics when the boundary datum is not zero. In order to prove the stability estimate (3.10), first we regularize the velocities v_1 and v_2 : there exist sequences $(v_1^v), (v_2^v) \in C^\infty([0, T]; \mathbb{R})$ converging in L^1 respectively to v_1, v_2 . Denote by e_1^v, e_2^v the corresponding solutions to P_1^v and P_2^v , constructed through characteristics. Using [34, Theorem 2.4], we deduce that, for a.e. $t > 0$, $e_i^v(t)$ converges to $e_i(t)$ in $L^1((a, b); \mathbb{R})$, for $i = 1, 2$. The definition of MV-solution in item 3 of Definition 2.7 has the remarkable feature of being stable under convergence in L^1 , therefore we obtain that e_i is the solution to P_i , for $i = 1, 2$.

Using [34, Proposition 2] and adapting the proof of [34, Theorem 2.4] to the present linear setting, we obtain that for all $t \in [0, T]$ the following estimate holds

$$\begin{aligned} \|e_1^v(t) - e_2^v(t)\|_{L^1(a,b)} &\leq e^{Gt} \left((TV_e + 2 e_{\max}) \|v_1^v - v_2^v\|_{L^1(0,t)} \right. \\ &\quad + 2 v_{\max} \|e_L^1 - e_L^2\|_{L^1(0,t)} + 2 v_{\max} \|e_R^1 - e_R^2\|_{L^1(0,t)} \\ &\quad \left. + \|\bar{e}_1 - \bar{e}_2\|_{L^1(a,b)} \right), \end{aligned}$$

where we used the bounds provided by Lemmas A.3 and A.4. Passing to the limit as $v \rightarrow +\infty$, we obtain that (3.10) holds.

In the second part of this proof, we show (3.12), the proof for (3.11) being symmetric.

The case of positive velocity. Here, assuming that v_1 and v_2 are positive velocities, we prove that

$$\begin{aligned} &\|v_1(\cdot)e_1(\cdot, b) - v_2(\cdot)e_2(\cdot, b)\|_{L^1(0,t)} \\ &\leq e^{Gt} \|\bar{e}_1 - \bar{e}_2\|_{L^1(a,b)} + e^{Gt} (\|\bar{e}_2\|_{L^\infty(a,b)} + TV(\bar{e}_2)) \|v_1 - v_2\|_{L^1(0,t)}. \end{aligned} \tag{A.1}$$

The proof is divided into three steps.

Step 1. Here only the initial datum is changed, namely, given $v \geq 0$, e_1 and e_2 are solutions to

$$\begin{cases} \partial_t e_1 + v \partial_x e_1 = g(e_1) \\ e_1(t, a) = e_L^1(t) \\ e_1(t, b) = e_R^1(t) \\ e_1(0, x) = \bar{e}_1(x) \end{cases} \quad \text{and} \quad \begin{cases} \partial_t e_2 + v \partial_x e_2 = g(e_2) \\ e_2(t, a) = e_L^2(t) \\ e_2(t, b) = e_R^2(t) \\ e_2(0, x) = \bar{e}_2(x). \end{cases}$$

Denote with Φ the flux associated to the vector field g , i.e. Φ solves

$$\begin{cases} \frac{d}{dt} \Phi(t; t_0, u_0) = g(\Phi(t; t_0, u_0)) \\ \Phi(t_0; t_0, u_0) = u_0. \end{cases}$$

Note that, for $t \geq t_0$,

$$\frac{d}{du_0} \Phi(t; t_0, u_0) = \exp\left(\int_{t_0}^t g'(\Phi(\tau; t_0, u_0)) d\tau\right) \leq e^{G(t-t_0)}; \quad (A.2)$$

see [5]. Since $v \geq 0$, for $t > 0$ sufficiently small,

$$\begin{aligned} |e_1(t, b) - e_2(t, b)| &= |\Phi(t; 0, \bar{e}_1(X(0; t, b))) - \Phi(t; 0, \bar{e}_2(X(0; t, b)))| \\ &\leq e^{Gt} |\bar{e}_1(X(0; t, b)) - \bar{e}_2(X(0; t, b))|, \end{aligned}$$

where X is the characteristic associated to v . Thus

$$\begin{aligned} &\|v(\cdot)(e_1(\cdot, b) - e_2(\cdot, b))\|_{L^1(0,t)} \\ &\leq e^{Gt} \int_0^t v(\tau) |\bar{e}_1(X(0; \tau, b)) - \bar{e}_2(X(0; \tau, b))| d\tau \\ &\leq e^{Gt} \int_a^b |\bar{e}_1(x) - \bar{e}_2(x)| dx \\ &\leq e^{Gt} \|\bar{e}_1 - \bar{e}_2\|_{L^1(a,b)}. \end{aligned}$$

Step 2. Here we consider the same initial datum, but different velocities $v_1 \geq 0$, $v_2 \geq 0$, namely, given \bar{e} , e_1 and e_2 are solutions to

$$\begin{cases} \partial_t e_1 + v_1 \partial_x e_1 = g(e_1) \\ e_1(t, a) = e_L^1(t) \\ e_1(t, b) = e_R^1(t) \\ e_1(0, x) = \bar{e}(x) \end{cases} \quad \text{and} \quad \begin{cases} \partial_t e_2 + v_2 \partial_x e_2 = g(e_2) \\ e_2(t, a) = e_L^2(t) \\ e_2(t, b) = e_R^2(t) \\ e_2(0, x) = \bar{e}(x). \end{cases}$$

Consider a sequence $\bar{e}^h \in C^\infty([a, b]; \mathbb{R})$ such that

- $\lim_{h \rightarrow +\infty} \|\bar{e}^h - \bar{e}\|_{L^1(a,b)} = 0;$

2. $\|\bar{e}^h\|_{\mathbf{L}^\infty(a,b)} \leq \|\bar{e}\|_{\mathbf{L}^\infty(a,b)}$;
3. $\text{TV}(\bar{e}^h) \leq \text{TV}(\bar{e})$ for every h .

Let $\theta \in [0, 1]$ and define the velocity

$$v_\theta = \theta v_1 + (1 - \theta)v_2.$$

Denote with X_θ and e_θ^h respectively the characteristic associated to the velocity v_θ and the solution to

$$\begin{cases} \partial_t e_\theta^h + v_\theta \partial_x e_\theta^h = g(e_\theta^h) \\ e_\theta^h(t, a) = e_L^2(t) \\ e_\theta^h(t, b) = e_R^2(t) \\ e_\theta^h(0, x) = \bar{e}^h(x). \end{cases}$$

Observe that, for $t \geq t_o$ and $x_o \in (a, b)$

$$\begin{cases} \partial_t \partial_\theta X_\theta(t; t_o, x_o) = v_1(t) - v_2(t) \\ \partial_\theta X_\theta(t_o; t_o, x_o) = 0, \end{cases}$$

so that

$$\partial_\theta X_\theta(t; t_o, x_o) = \int_{t_o}^t (v_1(\tau) - v_2(\tau)) \, d\tau .$$

We have

$$\begin{aligned} & \int_0^t \left| v_1(\tau)e_1^h(\tau, b) - v_2(\tau)e_2^h(\tau, b) \right| \, d\tau \leq \int_0^1 \int_0^t \left| \partial_\theta \left(v_\theta(\tau)e_\theta^h(\tau, b) \right) \right| \, d\tau \, d\theta \\ & \leq \int_0^1 \int_0^t \left| (v_1(\tau) - v_2(\tau)) e_\theta^h(\tau, b) \right| \, d\tau \, d\theta \\ & \quad + \int_0^1 \int_0^t \left| v_\theta(\tau) e^{G\tau} \nabla \bar{e}^h(X_\theta(0; \tau, b)) \int_0^\tau (v_1(s) - v_2(s)) \, ds \right| \, d\tau \, d\theta \\ & \leq \int_0^1 \int_0^t \left| v_1(\tau) - v_2(\tau) \right| e^{G\tau} \left| \bar{e}^h(X_\theta(0; \tau, b)) \right| \, d\tau \, d\theta \\ & \quad + e^{Gt} \int_0^1 \int_0^t v_\theta(\tau) \left| \nabla \bar{e}^h(X_\theta(0; \tau, b)) \right| \int_0^\tau |v_1(s) - v_2(s)| \, ds \, d\tau \, d\theta \\ & \leq e^{Gt} \|\bar{e}\|_{\mathbf{L}^\infty(a,b)} \|v_1 - v_2\|_{\mathbf{L}^1(0,t)} + e^{Gt} \text{TV}(\bar{e}) \|v_1 - v_2\|_{\mathbf{L}^1(0,t)} . \end{aligned}$$

Now, using these calculations and step 1, we deduce that

$$\begin{aligned} \|v_1(\cdot)e_1(\cdot, b) - v_2(\cdot)e_2(\cdot, b)\|_{\mathbf{L}^1(0,t)} & \leq \left\| v_1(\cdot)(e_1(\cdot, b) - e_1^h(\cdot, b)) \right\|_{\mathbf{L}^1(0,t)} \\ & \quad + \left\| v_1(\cdot)e_1^h(\cdot, b) - v_2(\cdot)e_2^h(\cdot, b) \right\|_{\mathbf{L}^1(0,t)} \end{aligned}$$

$$\begin{aligned}
 & + \left\| v_2(\cdot)(e_2^h(\cdot, b) - e_2(\cdot, b)) \right\|_{\mathbf{L}^1(0,t)} \\
 & \leq 2e^{Gt} \left\| \bar{e} - \bar{e}^h \right\|_{\mathbf{L}^1(a,b)} \\
 & \quad + e^{Gt} \left(\|\bar{e}\|_{\mathbf{L}^\infty(a,b)} + \text{TV}(\bar{e}) \right) \|v_1 - v_2\|_{\mathbf{L}^1(0,t)}.
 \end{aligned}$$

Passing to the limit as $h \rightarrow +\infty$, we get

$$\|v_1(\cdot)e_1(\cdot, b) - v_2(\cdot)e_2(\cdot, b)\|_{\mathbf{L}^1(0,t)} \leq e^{Gt} \left(\|\bar{e}\|_{\mathbf{L}^\infty(a,b)} + \text{TV}(\bar{e}) \right) \|v_1 - v_2\|_{\mathbf{L}^1(0,t)}.$$

Step 3. Here we consider different initial data \bar{e}_1, \bar{e}_2 and different velocities $v_1 \geq 0, v_2 \geq 0$, namely, e_1 and e_2 are solutions to

$$\begin{cases} \partial_t e_1 + v_1 \partial_x e_1 = g(e_1) \\ e_1(t, a) = e_L^1(t) \\ e_1(t, b) = e_R^1(t) \\ e_1(0, x) = \bar{e}_1(x) \end{cases} \quad \text{and} \quad \begin{cases} \partial_t e_2 + v_2 \partial_x e_2 = g(e_2) \\ e_2(t, a) = e_L^2(t) \\ e_2(t, b) = e_R^2(t) \\ e_2(0, x) = \bar{e}_2(x). \end{cases}$$

Let e_3 be the solution to the additional problem

$$\begin{cases} \partial_t e_3 + v_1 \partial_x e_3 = g(e_3) \\ e_3(t, a) = e_L^1(t) \\ e_3(t, b) = e_R^1(t) \\ e_3(0, x) = \bar{e}_2(x). \end{cases}$$

Using the results of Step 1 and Step 2, we have that

$$\begin{aligned}
 & \|v_1(\cdot)e_1(\cdot, b) - v_2(\cdot)e_2(\cdot, b)\|_{\mathbf{L}^1(0,t)} \\
 & \leq \|v_1(\cdot)e_1(\cdot, b) - v_1(\cdot)e_3(\cdot, b)\|_{\mathbf{L}^1(0,t)} + \|v_1(\cdot)e_3(\cdot, b) - v_2(\cdot)e_2(\cdot, b)\|_{\mathbf{L}^1(0,t)} \\
 & \leq e^{Gt} \|\bar{e}_1 - \bar{e}_2\|_{\mathbf{L}^1(a,b)} + e^{Gt} \left(\|\bar{e}_2\|_{\mathbf{L}^\infty(a,b)} + \text{TV}(\bar{e}_2) \right) \|v_1 - v_2\|_{\mathbf{L}^1(0,t)},
 \end{aligned}$$

proving (A.1).

The general case.

For $\varepsilon > 0$ we define the sets

$$\begin{aligned}
 A_\varepsilon & = \{t \in [0, t] | v_1 \geq \varepsilon, v_2 \geq \varepsilon\} \\
 B_\varepsilon^1 & = \{t \in [0, t] | v_1 \geq \varepsilon, v_2 \leq -\varepsilon\} \\
 B_\varepsilon^2 & = \{t \in [0, t] | v_1 \leq -\varepsilon, v_2 \geq \varepsilon\} \\
 B_\varepsilon & = B_\varepsilon^1 \cup B_\varepsilon^2 \\
 C_\varepsilon & = [0, t] \setminus (A_\varepsilon \cup B_\varepsilon)
 \end{aligned}$$

and split the estimate for $F_{[0,t]} = \|v_1^+ e_1(\cdot, b) - v_2^+ e_2(\cdot, b)\|_{\mathbf{L}^1([0,t])}$ into the contributions coming from the different subsets of the whole domain

$$F_{[0,t]} = F_{A_\varepsilon} + F_{B_\varepsilon} + F_{C_\varepsilon}$$

For C_ε , all involved velocities have an absolute value smaller than ε , leading to the estimate

$$F_{C_\varepsilon} \leq 2e_{max}t\varepsilon.$$

For the set B_ε , one of the velocities is negative, meaning the corresponding contribution is zero and thus

$$F_{B_\varepsilon} \leq e_{max} \|v_1 - v_2\|_{\mathbf{L}^1(B_\varepsilon)}.$$

The crucial part is the estimate for A_ε , where we can use the results for positive velocities obtained above. The set A_ε is closed: due to the bounded total variation of v , it can be written as a finite union of closed intervals

$$A_\varepsilon = \cup_{i=1}^{n_\varepsilon} I_i^\varepsilon, \quad I_i^\varepsilon = [\underline{t}_i^\varepsilon, \bar{t}_i^\varepsilon] \\ \Delta t_i^\varepsilon = \bar{t}_i^\varepsilon - \underline{t}_i^\varepsilon.$$

The goal is to get an estimate similar to (A.1) for each I_i^ε . Here it is important that the dependencies on the initial datum do not overlap, as that would lead to a boundlessly growing right hand side as ε tends to zero and the number of intervals grows. This is why we use a combination of above estimate and (3.10), where we take special care of the domains of dependency in the spatial integrals.

We start by considering the interval $I_{n_\varepsilon}^\varepsilon$: by (A.1) we get

$$\|v_1 e_1(\cdot, b) - v_2 e_2(\cdot, b)\|_{\mathbf{L}^1(I_{n_\varepsilon}^\varepsilon)} \\ \leq e^{G\Delta t_{n_\varepsilon}^\varepsilon} \left[\|e_1(t_{n_\varepsilon}^\varepsilon, \cdot) - e_2(t_{n_\varepsilon}^\varepsilon, \cdot)\|_{\mathbf{L}^1([a_{n_\varepsilon}, b])} + \|v_1 - v_2\|_{\mathbf{L}^1(I_{n_\varepsilon}^\varepsilon)} (e_{max} + \text{TV}_e) \right] \tag{A.3}$$

with $a_{n_\varepsilon} = b - \int_{\underline{t}_{n_\varepsilon}^\varepsilon}^{\bar{t}_{n_\varepsilon}^\varepsilon} v_1(t) dt$. The bound (A.3) is actually more precise than (A.1): indeed, we take care, in the \mathbf{L}^1 -norm in space, of the true area of dependency induced by the flow velocity.

In order to bound the first term on the right hand side of (A.3), we exploit (3.10):

$$\|e_1(t_{n_\varepsilon}^\varepsilon, \cdot) - e_2(t_{n_\varepsilon}^\varepsilon, \cdot)\|_{\mathbf{L}^1([a_{n_\varepsilon}, b])} \\ \leq e^{G(t_{n_\varepsilon}^\varepsilon - \bar{t}_{n_\varepsilon-1}^\varepsilon)} \left[(\text{TV}_e + 2e_{max}) \|v_1 - v_2\|_{\mathbf{L}^1([\bar{t}_{n_\varepsilon-1}^\varepsilon, t_{n_\varepsilon}^\varepsilon])} \right. \\ \left. + 2 v_{max} \left\| e_R^1 - e_R^2 \right\|_{\mathbf{L}^1([\bar{t}_{n_\varepsilon-1}^\varepsilon, t_{n_\varepsilon}^\varepsilon])} + \|e_1(\bar{t}_{n_\varepsilon-1}^\varepsilon, \cdot) - e_2(\bar{t}_{n_\varepsilon-1}^\varepsilon, \cdot)\|_{\mathbf{L}^1([c_{n_\varepsilon}, b])} \right] \tag{A.4}$$

with $c_{n_\varepsilon} = a_{n_\varepsilon} - v_{max}(t_{n_\varepsilon}^\varepsilon - \bar{t}_{n_\varepsilon-1}^\varepsilon)$. Due to $v_j > 0$ in $I_{n_\varepsilon-1}^\varepsilon$, $j = 1, 2$, the last term in the inequality above can be estimated as follows:

$$\|e_1(\bar{t}_{n_\varepsilon-1}^\varepsilon, \cdot) - e_2(\bar{t}_{n_\varepsilon-1}^\varepsilon, \cdot)\|_{\mathbf{L}^1([c_{n_\varepsilon}, b])}$$

$$\begin{aligned} &\leq e^{G\Delta t_{n_\varepsilon}^\varepsilon} \|e_1(t_{n_\varepsilon-1}^\varepsilon, \cdot) - e_2(t_{n_\varepsilon-1}^\varepsilon)\|_{\mathbf{L}^1([d_{n_\varepsilon-1}, a_{n_\varepsilon-1}])} \\ &\quad + e^{G\Delta t_{n_\varepsilon}^\varepsilon} \text{TV}_e \|v_1 - v_2\|_{\mathbf{L}^1(I_{n_\varepsilon}^\varepsilon)}, \end{aligned} \tag{A.5}$$

where $d_{n_\varepsilon-1} = c_{n_\varepsilon} - \int_{t_{n_\varepsilon-1}^\varepsilon}^{t_{n_\varepsilon}^\varepsilon} v_1(t) dt = a_{n_\varepsilon} - v_{max}(t_{n_\varepsilon}^\varepsilon - t_{n_\varepsilon-1}^\varepsilon) - (b - a_{n_\varepsilon-1})$. Inserting (A.5) into (A.4) and then into (A.3), we get

$$\begin{aligned} &\|v_1 e_1(\cdot, b) - v_2 e_2(\cdot, b)\|_{\mathbf{L}^1(I_{n_\varepsilon}^\varepsilon)} \\ &\leq e^{G(\bar{t}_{n_\varepsilon}^\varepsilon - t_{n_\varepsilon-1}^\varepsilon)} \left[\|v_1 - v_2\|_{\mathbf{L}^1(I_{n_\varepsilon}^\varepsilon)} (e_{max} + \text{TV}_e) \right. \\ &\quad + \|v_1 - v_2\|_{\mathbf{L}^1([\bar{t}_{n_\varepsilon-1}^\varepsilon, t_{n_\varepsilon}^\varepsilon])} (2e_{max} + \text{TV}_e) + \|v_1 - v_2\|_{\mathbf{L}^1(I_{n_\varepsilon-1}^\varepsilon)} (\text{TV}_e) \\ &\quad \left. + 2v_{max} \left\| e_R^1 - e_R^2 \right\|_{\mathbf{L}^1([\bar{t}_{n_\varepsilon-1}^\varepsilon, t_{n_\varepsilon}^\varepsilon])} + \|e_1(t_{n_\varepsilon-1}^\varepsilon, \cdot) - e_2(t_{n_\varepsilon-1}^\varepsilon, \cdot)\|_{\mathbf{L}^1([d_{n_\varepsilon-1}, a_{n_\varepsilon-1}])} \right]. \end{aligned} \tag{A.6}$$

Using (A.6) for the time interval $I_{n_\varepsilon}^\varepsilon$ and (A.3) for $I_{n_\varepsilon-1}^\varepsilon$, leads to

$$\begin{aligned} &\|v_1 e_1(\cdot, b) - v_2 e_2(\cdot, b)\|_{\mathbf{L}^1(I_{n_\varepsilon}^\varepsilon \cup I_{n_\varepsilon-1}^\varepsilon)} \\ &\leq e^{G(\bar{t}_{n_\varepsilon}^\varepsilon - t_{n_\varepsilon-1}^\varepsilon)} \left[\|v_1 - v_2\|_{\mathbf{L}^1([t_{n_\varepsilon-1}^\varepsilon, \bar{t}_{n_\varepsilon}^\varepsilon])} (2e_{max} + 2 \text{TV}_e) \right. \\ &\quad + 2v_{max} \left\| e_R^1 - e_R^2 \right\|_{\mathbf{L}^1([\bar{t}_{n_\varepsilon-1}^\varepsilon, t_{n_\varepsilon}^\varepsilon])} \\ &\quad \left. + \|e_1(t_{n_\varepsilon-1}^\varepsilon, \cdot) - e_2(t_{n_\varepsilon-1}^\varepsilon, \cdot)\|_{\mathbf{L}^1([d_{n_\varepsilon-1}, b])} \right]. \end{aligned} \tag{A.7}$$

We remark that the coefficient of the norm of the difference between the velocities is computed taking care of the time intervals on which the \mathbf{L}^1 -norm is evaluated.

We can proceed by adding the time interval $I_{n_\varepsilon-2}$: in particular, we exploit (A.6) for $I_{n_\varepsilon}^\varepsilon \cup I_{n_\varepsilon-1}^\varepsilon$ and add (A.3) for $I_{n_\varepsilon-2}$, noting that we can bound the last line on the right hand side of (A.6) exploiting (A.4) and (A.5). What we obtain is entirely analogous to (A.7). Proceeding in this way, we get

$$\begin{aligned} F_{A_\varepsilon} &= \sum_{i=1}^{n_\varepsilon} \|v_1 e_1(\cdot, b) - v_2 e_2(\cdot, b)\|_{\mathbf{L}^1(I_i^\varepsilon)} \\ &\leq e^{Gt} \left[\|v_1 - v_2\|_{\mathbf{L}^1([0,t])} (2e_{max} + 2 \text{TV}_e) + 2v_{max} \left\| e_R^1 - e_R^2 \right\|_{\mathbf{L}^1([0,t] \setminus A_\varepsilon)} \right. \\ &\quad \left. + \|\bar{e}_1 - \bar{e}_2\|_{\mathbf{L}^1([a,b])} \right], \end{aligned}$$

Finally, we add the contributions from the three different subsets into which we divided $[0, t]$:

$$\begin{aligned} F_{[0,t]} &= F_{A_\varepsilon} + F_{B_\varepsilon} + F_{C_\varepsilon} \\ &\leq e^{Gt} \left[\|v_1 - v_2\|_{\mathbf{L}^1([0,t])} (2e_{max} + 2 \text{TV}_e) + 2v_{max} \left\| e_R^1 - e_R^2 \right\|_{\mathbf{L}^1([0,t])} \right] \end{aligned}$$

$$\begin{aligned}
 &+ \|\bar{e}_1 - \bar{e}_2\|_{\mathbf{L}^1([a,b])}] \\
 &+ e_{max} \|v_1 - v_2\|_{\mathbf{L}^1(B_\varepsilon[0,t])} + 2e_{max}t\varepsilon.
 \end{aligned}$$

The left hand side does not depend on ε , while the right hand side holds true for all choices of ε , so we can omit the last term, obtaining the desired estimate. \square

Lemma A.1 *Let $s_1, s_2 : [a, b] \rightarrow [a', b']$ be continuous and invertible functions satisfying the following properties:*

1. $s_1(a) = s_2(a)$ and $s_1(b) = s_2(b)$;
2. there exists $K > 0$ such that

$$\begin{aligned}
 |s_1^{-1}(y_1) - s_1^{-1}(y_2)| &\leq K|y_1 - y_2| \\
 |s_2^{-1}(y_1) - s_2^{-1}(y_2)| &\leq K|y_1 - y_2|
 \end{aligned}$$

for every $y_1, y_2 \in [a', b']$.

Fix $f \in \mathbf{L}^1((a', b'); \mathbb{R})$ with finite total variation.

Then

$$\|f \circ s_1 - f \circ s_2\|_{\mathbf{L}^1(a,b)} \leq K \text{TV}(f) \|s_1 - s_2\|_{\mathbf{L}^\infty(a,b)}.$$

Proof We extend f to a function $\tilde{f} : \mathbb{R} \rightarrow \mathbb{R}$ such that \tilde{f} is continuous in a' and b' , $\tilde{f}(x) = 0$ for every $x \in \mathbb{R} \setminus [a' - 1, b' + 1]$, \tilde{f} has finite total variation, and $\tilde{f}(x) = f(x)$ for a.e. $x \in (a', b')$. Then there exists a unique signed Borel measure μ_f such that $\tilde{f}(x) = \mu_f((-\infty, x])$ for every $x \in \mathbb{R}$; see [20, Theorem 3.29].

Define the continuous functions $m(x) = \min\{s_1(x), s_2(x)\}$ and $M(x) = \max\{s_1(x), s_2(x)\}$. We have

$$\begin{aligned}
 \|f \circ s_1 - f \circ s_2\|_{\mathbf{L}^1[a,b]} &= \int_a^b |f(s_1(x)) - f(s_1(x) + \varepsilon(x))| dx \\
 &= \int_a^b |\mu_f((-\infty, s_1(x)]) - \mu_f((-\infty, s_2(x)])| dx \\
 &= \int_a^b |\mu_f((m(x), M(x)])| dx \\
 &\leq \int_a^b \int_{(m(x), M(x))} d|\mu_f| dx \\
 &\leq \int_{a'}^{b'} \int_{[\min\{s_1^{-1}(y), s_2^{-1}(y)\}, \max\{s_1^{-1}(y), s_2^{-1}(y)\}]} dx d|\mu_f| \\
 &\leq \int_{a'}^{b'} |s_1^{-1}(y) - s_2^{-1}(y)| d|\mu_f| \\
 &\leq \int_{a'}^{b'} |s_2^{-1}(s_2(s_1^{-1}(y))) - s_2^{-1}(y)| d|\mu_f|
 \end{aligned}$$

$$\begin{aligned}
 &\leq K \int_{a'}^{b'} \left| s_2(s_1^{-1}(y)) - y \right| \mathbf{d}|\mu_f| \\
 &\leq K \int_{a'}^{b'} \left| s_2(s_1^{-1}(y)) - s_1(s_1^{-1}(y)) \right| \mathbf{d}|\mu_f| \\
 &\leq K \int_{a'}^{b'} \|s_1 - s_2\|_{\mathbf{L}^\infty(a,b)} \mathbf{d}|\mu_f| \\
 &= \text{TV}(f) K \|s_1 - s_2\|_{\mathbf{L}^\infty(a,b)} .
 \end{aligned}$$

concluding the proof. □

Lemma A.2 *Assume e_L, e_R and \bar{e} be functions with bounded total variation. Let $G > 0$ be fixed and let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a globally Lipschitz continuous function with Lipschitz constant G . Denote with e the solution of*

$$\begin{cases} \partial_t e + v \partial_x e = g(e) & x \in (a, b), t > 0 \\ e(t, a) = e_L(t) \\ e(t, b) = e_R(t) \\ e(0, x) = \bar{e}(x) . \end{cases} \tag{A.8}$$

Then, for a.e. $t \in \mathbb{R}^+$, it holds

$$\begin{aligned}
 \text{TV}(e(t)) &\leq \exp(Gt) [\text{TV}(e_L) + |e_L(0+) - \bar{e}(a+)| + \text{TV}(\bar{e}) \\
 &\quad + |\bar{e}(b-) - e_R(0+)| + \text{TV}(e_R)] .
 \end{aligned}$$

Proof The proof follows almost directly using the fact that the solution to (A.8) along the characteristics lines is a solution of an ordinary differential equations with vector field given by g . Thus the amplification, obtained by the Gronwall Lemma, produced by the solution at time t is given by e^{Gt} . Therefore the total variation of $e(t)$ can be control by the total variation of the initial and boundary conditions multiplied by e^{Gt} and by the jumps between the boundary data and the initial condition, again multiplied by the factor e^{Gt} ; see also [34, Theorem 2.3] for a similar situation. □

Lemma A.3 (Energy bound) *Under the assumptions of Theorem 3.4 we have for all $t \leq T$*

$$e(t, x) \leq e_{max}$$

for almost every x .

Proof First consider a single pipe and the corresponding transport equation

$$e_t + v e_x = g(e)$$

with $e(0, x) = e_0(x)$ and $e(t, a) = e_L(t), e(t, b) = e_R(t)$ with $e_0, e_L, e_R \in \mathbf{L}^1$ and essentially bounded. Denoting the essential supremum by

$$e_{max,b} = \max_{f \in \{e_0, e_L, e_R\}} \{\text{ess sup } f\},$$

then we know from basic ODE theory that

$$e(t, x) \leq \Theta(t)e_{max,b} \tag{A.9}$$

for almost every x , where $\Theta(t) = e^{Gt}$. For a node J , the node energy depends on those of all connected pipes. Let

$$\bar{e}_{max}(t) = \max_i e_{max}^i(t)$$

be the maximum essential bound of all edges. Then

$$\dot{e}_J = f(t, e_J) \leq \frac{\text{deg}(\mathcal{G})v_{max}}{V} (e_{max}(t) + e_J)$$

and with Gronwall’s inequality

$$|e_J(t) - e_J(0)| \leq C e^{Ct} \int_0^t e_{max}(\tau) d\tau, \tag{A.10}$$

where $C = \frac{\text{deg}(\mathcal{G})v_{max}}{V}$ is the maximal node degree in the network and V the minimal volume of a junction. For an arbitrary edge in the network, the left and right boundary values come from some nodes in the network. Inserting (A.10) into (A.9) and applying Gronwall one more time gives

$$|e(t, x) - e(0, x)| \leq \exp(e_{max,0}\Theta(t)Cte^{Ct}),$$

where $e_{max,0}$ is the essential supremum of the initial datum of the network and the boundary data at the source. The choice $e_{max} = \exp(e_{max,0}\Theta(T)CTe^{CT})$ concludes the proof. □

Lemma A.4 (Total variation bound) *Under the assumptions of Theorem 3.4 we have for all $t \leq T$*

$$\text{TV}(e(t, \cdot)) \leq \text{TV}_e.$$

Proof Then the total variation of the node values fulfills

$$\text{TV}(e_J) = \int_0^t \left| \frac{1}{V_J} \left[\sum_{I_i \in \text{inc}^{+, \tau}(J)} A^i v^i(\tau) e^i(\tau, b_i) - \sum_{I_i \in \text{out}^{-, \tau}(J)} A^i v^i(\tau) e^i(\tau, a_i) \right] \right| d\tau$$

$$\begin{aligned}
 & -e_J \sum_{I_i \in \text{inc}^{-\tau}(J) \cup \text{out}^{+\tau}(J)} A^i \left| v^i(\tau) \right| \Bigg] d\tau \\
 \leq & \int_0^t \frac{1}{V_J} \text{deg}(\mathcal{G}) A_{\max} v_{\max} e_{\max} d\tau + \int_0^t \frac{1}{V_J} e_J A_{\max} v_{\max} d\tau \\
 \leq & \text{deg}(\mathcal{G}) T \frac{A_{\max}}{V} v_{\max} e_{\max} \cdot \exp\left(T \frac{A_{\max}}{V} v_{\max}\right) =: \text{TV}_{e_J}.
 \end{aligned}$$

Here, $\text{deg}(\mathcal{G})$ is the maximal node degree in the network and V the minimal volume of a junction. Using A.2 we get for the solution along one edge in the network

$$\text{TV} \left(e^i(t, \cdot) \right) \leq \exp(GT) \left[2 \text{TV}_{e_J} + \text{TV}(\bar{e}^i) \right] + 4e_{\max}$$

The full energy solution $e = (e^{\mathcal{I}}, e_{\mathcal{J}})$ then collects the total variation of all edges and nodes (including the boundary).

$$\begin{aligned}
 \text{TV} (e(t, \cdot)) & \leq (M - 1) \text{TV}_{e_J} + \text{TV} (e^{1,b}(t)) \\
 & \quad + N_{\mathcal{I}} \left(\exp(GT) \left[2 \text{TV}_{e_J} + \text{TV}(\bar{e}^i) \right] + 4e_{\max} \right) \\
 & = (M + 2N_{\mathcal{I}} \exp(GT)) \text{TV}_{e_J} + \text{TV} (e^{1,b}(t)) \\
 & \quad + 2N_{\mathcal{I}} \exp(GT) \text{TV}(\bar{e}^i) + 4e_{\max} \\
 & =: \text{TV}_e.
 \end{aligned}$$

□

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