ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 26 (2026) #P2.03
https://doi.org/10.26493/1855-3974.3409.c13
(Also available at http://amc-journal.eu)

Cycle separating cuts in possible counterexamples to the cycle double cover and the Berge-Fulkerson conjectures

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Received 12 June 2024, accepted 6 March 2025, published online 21 January 2026

Abstract

It is known that smallest counterexamples to the Cycle Double Cover Conjecture and Berge-Fulkerson Conjecture (if they exist) are cyclically 4- and 5-edge-connected, respectively. We further analyse small cycle separating cuts in possible counterexamples. We prove that if a smallest counterexample G to the CDC Conjecture contains a cycle separating 4-cut S, then the behaviour of the admissible CDC coverings along the dangling edges of the two 4-poles induced by S is uniquely determined among more than 2^{19} a priori possibilities. Similarly, for the Berge-Fulkerson Conjecture, we prove that among more than 2^{111} a priori possibilities, there are only 13 pairs of admissible sets that could occur along the dangling edges of a 5-cut in a smallest counterexample.

Keywords: Snark, cyclic connectivity, cycle double cover, Berge-Fulkerson conjecture.

Math. Subj. Class. (2020): 05C70, 05C15

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[*] The first author was partially supported by VEGA 1/0173/25 and APVV-23-0076.

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1 Introduction

The Cycle Double Cover Conjecture (see [8, 9]) and the Berge-Fulkerson Conjecture (see [4, 7]) share a compelling similarity rooted in their common focus on the concept of edge coverings. The former asserts that every bridgeless graph can be covered by cycles in such a way that each edge belongs to precisely two of them. The latter posits the existence of six perfect matchings, each covering every edge exactly twice in a bridgeless cubic graph. Although the Cycle Double Cover Conjecture is stated for general bridgeless graphs, a well-known reduction, via the splitting lemma [3], narrows it to the family of bridgeless cubic graphs. The existence of the required covering is trivially guaranteed for both conjectures if the cubic graph is 3-edge-colorable, hence the relevant family of graphs to be studied is that of non-3-edge-colorable ones.

Throughout the years, further restrictions have been imposed on the structure of a potential minimum counterexample to the Cycle Double Cover Conjecture. In [5], it is proven that a minimum counterexample should have a girth (i.e. the length of a shortest cycle) of at least 12. More recently, the first two authors considered the effect of cyclic edge-connectivity on potential minimum counterexamples for the Berge-Fulkerson Conjecture. Precisely, they showed in [6] that a minimum counterexample to the Berge-Fulkerson Conjecture must be cyclically 5-edge-connected. On the other hand, in 1980, Jaeger and Swart [2] conjectured that non-3-edge-colorable cubic graphs with cyclic connectivity grater than 6 do not exist.

In this paper, we adapt the techniques employed in [6]. Our primary focus is twofold. First, we explore the structure of a possible minimum cyclically 4-edge-connected counterexample for the Cycle Double Cover Conjecture. Second, we redirect our attention to the Berge-Fulkerson Conjecture, investigating the cyclically 5-edge-connected case. Despite the inability to exclude the existence of a minimum counterexample with the given cyclic edge-connectivity, we provide, in both cases, new strong restrictions on its structure.

2 Notation

We introduce notation and auxiliary results that we will use in the following sections to prove our main results.

A *multipole* is a pair (V, E) consisting of a set of vertices V and a set of edges E. Each edge possesses two ends, each of which may be incident with a vertex. If an edge has both ends incident with a vertex, it is called a *proper edge*. If exactly one of the ends of an edge is incident with a vertex, the edge is called a *dangling edge*. Finally, if none of them is incident with a vertex, the edge is called an *isolated edge*. An end of an edge which is not incident with a vertex is called a *semiedge*.

A *k-pole* is a multipole with precisely k semiedges. An *ordered k-pole* is a k-pole with a linear ordering of its semiedges. In the following, when colorings of the edges of a k-pole are considered, it is always implicitly assumed that if we assign a color to a given edge, then the same color is also assigned to all (possible) semiedges of that edge.

Conversely, if we claim that a semiedge has a certain color then the same holds for its edge as well. Then, from now on, we indifferently say that a color is assigned either to an edge or to a semiedge.

We extend several concepts traditionally defined for graphs to multipoles in the most natural way. Specifically, a subgraph of a multipole is defined as a multipole (V', E'), where $V' \subseteq V$ and $E' \subseteq E$, such that every end of an edge in E' is either a semiedge or is incident to a vertex in V'. Using this framework, we can generalize the notion of a k-factor of a graph to a k-factor of a multipole. A k-factor of a multipole is a subgraph where every vertex of the multipole is incident to exactly k edges.

A *cycle* is a graph with at least one edge and every vertex of even degree. The main parameter considered along this paper is the cyclic-edge-connectivity. A graph G is *cyclically k-edge-connected* if it does not contain an edge-cut S such that $|S| < k$ and $G - S$ contains at least two components containing cycles. The *cyclic connectivity* of a graph G is the greatest k such that G is cyclically k-edge-connected. In the rest of the paper, we often say *minimum counterexample*, instead of *possible minimum counterexample* to one of the considered conjectures, for brevity. However, we do not intend to suggest whether such counterexamples should actually exist or not. Moreover, we always consider counterexamples with the minimum number of vertices, though not necessarily with the fewest edges among them.

For the sake of completeness, we state here Cycle Double Cover and Berge Fulkerson Conjectures.

Conjecture 2.1 (Cycle Double Cover Conjecture, 1973, [8, 9]). *Every bridgeless graph admits a collection of cycles such that every edge of G belongs to exactly two of them.*

Conjecture 2.2 (Berge-Fulkerson Conjecture, 1971, [4, 7]). *Every bridgeless cubic graph G admits six perfect matchings such that every edge of G belongs to exactly two of them.*

3 On 4-edge-cuts in a minimum counterexample to the CDC-Conjecture

In a minimum counterexample to the Cycle Double Cover Conjecture there cannot be 2-valent vertices and 2- or 3-edge-cuts, as follows by a standard argument which uses minimality. Hence, we focus on 4-edge-cuts in a minimum counterexample to the Cycle Double Cover Conjecture. To do so, we apply a similar technique to the one used by the first two authors in [6]. Note that here we consider graphs that are not necessarily cubic, but we can repeat all arguments also in the cubic case as we will remark at the end of this section.

Definition 3.1. Let $H = (V, E)$ be a multipole. A *CDC-coloring* of H is a function φ which assigns to every element in E a 2-subset of the set of colors $\{1, 2, \dots, t\}$ for some integer t, in such a way that any color occurs an even number of times along the edges incident to a vertex.

It is straightforward that a CDC-coloring of a graph G is equivalent to the existence of a collection of cycles of G covering each edge of G twice. Moreover, the edge-induced subgraph of G by a color in $\{1, 2, ..., t\}$ is a cycle in G.

Remark 3.2. A stronger version of the Cycle Double Cover Conjecture suggests that it is possible to assume t at most 5 (see [1] and also [10, 11] for a comprehensive survey on cycle covers). Clearly, a counterexample for the stronger version could be not a counterexample for the general version. Moreover, a potential minimum counterexample for the general version could be not minimum for the stronger one. However, our arguments never require more than five colors, so the structural restrictions of a minimum counterexample for one conjecture extend to the other as well.

Now, we consider the behavior of a CDC-coloring on the four semiedges of a 4-pole in more detail. Let H be an ordered 4-pole. Each color of a CDC-coloring occurs an even number of times on the semiedges of H. Now we show that in every CDC-coloring of H, the pairs of colors in the semiedges can be expressed as an overlap of two edge-colorings of the semiedges. Each of these two colorings uses at most two colors and each color appears in at most one of these two colorings. An edge-coloring of the semiedges of an ordered 4-pole where ALL semiedges receive the same color is said to be of *type A*. For $i = 2, 3, 4$, an edge-coloring of the semiedges of an ordered 4-pole with two colors, such that the first semiedge has the same color as the i-th one is said to be of *type* T_i.

Let φ be a CDC-coloring of an ordered 4-pole H. If all the four semiedges have the same pair of colors in φ, H is said to be of *type AA*. Otherwise, if exactly one color appears in all semiedges, then φ is of type AT_i for some $i \in \{2, 3, 4\}$. Moreover, every color occurs on an even number of edges. Hence, the only remaining cases to consider are the ones when four colors are present in φ on the four semiedges and each of them appears on exactly two semiedges. In this case, we can partition the four colors in two subsets such that the two colors in the first subset give a coloring of type T_i and the other two colors a coloring of type T_j. Such CDC-coloring will be denoted as of *type* T_iT_j, for some $i, j \in \{2, 3, 4\}$.

Proposition 3.3. *Each CDC-coloring of the semiedges of an ordered* 4*-pole is of type* XY *where* $X, Y \in \{A, T_2, T_3, T_4\}$.

From now on, we do not distinguish a CDC-coloring from another by the specific set of colors used for the semiedges, but only by the types of colorings. Moreover a CDC-coloring of type XY and a CDC-coloring of type YX for $X, Y \in \{A, T_2, T_3, T_4\}$ are always considered of the same type. Hence, we have exactly 10 types of CDC-colorings of an ordered 4-pole, namely AA, AT_i for $i \in \{2, 3, 4\}$, and T_iT_j for $i, j \in \{2, 3, 4\}$, $i \leq j$.

We denote by $\mathcal{C}$ the set of these 10 types of CDC-colorings, and we denote by $\mathcal{C}(H)$ the set of admissible types of CDC-colorings for a given ordered 4-pole H. In this context, we say that a type of CDC-coloring is admissible for H, if H admits at least one CDC-coloring of such a type. A priori $\mathcal{C}(H)$ is one of the 2^{10} elements of the power set of $\mathcal{C}$. Instead of directly working with subsets of $\mathcal{C}$, we prefer to construct an auxiliary graph M and to identify $\mathcal{C}(H)$ with a suitable subgraph of M.

The graph M has four vertices, denoted by A, T_2, T_3, and T_4, and every vertex is connected to every other vertex and to itself by a loop[1]. The vertices of M correspond to the four possible types of edge-colorings of the semiedges of a 4-pole as introduced before. Each of the ten edges (here and later we always refer to loops as edges with two semiedges incident to the same vertex) corresponds to a different type of CDC-coloring. More precisely, the one obtained by the composition of the two edge-colorings of its semiedges. Six copies of the graph M are depicted in Figure 1. In the leftmost copy, vertices are labeled, and the same arrangement of vertices is implicitely assumed in the subsequent copies. We associate a subgraph of M, denoted by H^*, to every ordered 4-pole H in the following way. Consider the set $\mathcal{C}(H)$ and recall that each of its elements corresponds to an edge of M. Then, we define H^* as the subgraph of M induced by all edges which correspond to an element of $\mathcal{C}(H)$.

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[1] In this paper, M as well as the auxiliary graph N introduced in Section 4 and their subgraphs are the only graphs that include loops. For simplicity, we continue to refer to them as graphs, as this terminology does not cause any confusion. Note that every vertex of M (and N) has exactly one loop, which contributes two to its degree.

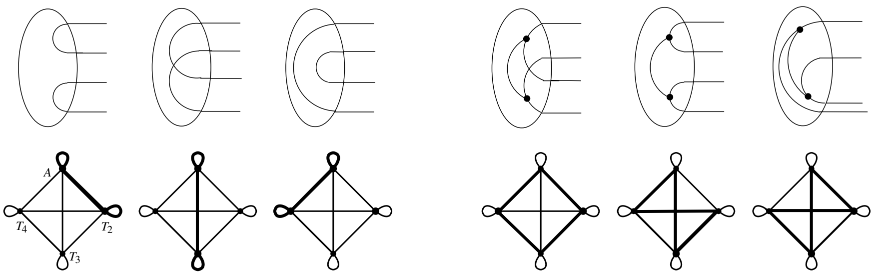

Figure 1: The subgraphs of M associated to some ordered acyclic 4-poles.

We show in Figure 1 six possible ordered acyclic 4-poles (indeed all the possible cubic 4-poles), which will be useful in our main proof, and their associated subgraphs of M. A dumbbell subgraph consists of two vertices, each with a loop, and an edge connecting these two vertices. In particular, note that acyclic poles with no vertex are associated to a dumbbell subgraph of M, while the three acyclic poles with two vertices are associated to a 4-cycle. It is worth noting that this point constitutes one of the main obstructions compared to what was obtained in [6] to rule out the existence of 4-edge-cuts in a minimal counterexample to the Berge-Fulkerson Conjecture. In that case, each of the six dumbbell subgraphs of M was associated to an acyclic 4-pole. Here instead we have in three cases different types of admissible subgraphs which impose less restrictive conditions in what follows.

3.1 Bichromatic chains

Let φ be a CDC-coloring of a 4-pole H. Let s be a semiedge of H and denote by c_1 one of the two colors in $\varphi(s)$ and by c_2 a color not in $\varphi(s)$ (here we also admit that c_2 is a color in $\{1, 2, \dots, t\}$ unused in φ). Consider the subgraph H' of H induced by all the edges e such that $\varphi(e)$ contains exactly one of c_1 and c_2. Let K be the connected component of H' which contains the semiedge s. Clearly, K contains at least two semiedges, let s' denote one different from s. Every path BC in K containing s and s' will be called a c_1-c_2-*bichromatic chain*. Note that it may happen that only one of the colors c_1 and c_2 appears in a c_1-c_2-bichromatic chain. Starting from φ, we can obtain a new CDC-coloring of H by performing a *color switch along the* c_1-c_2-*bichromatic chain* BC. This consists in an interchange of the two colors c_1 and c_2 for all edges (or semiedges) of BC. Note also that in the case of a cubic 4-pole, K itself is always a path beginning and ending with dangling edges and then BC is uniquely determined.

Now, we prove some necessary conditions for a subgraph of M associated to an ordered 4-pole.

Lemma 3.4. *Let H be an ordered 4-pole. Then, the subgraph H^* of M has no vertex of degree 1 and no vertex whose only incident edge is a loop*[2]*.*

Proof. Let X, Y be two arbitrary elements, possibly the same, of the set $\{A, T_2, T_3, T_4\}$. Consider the vertex of M corresponding to the element X and assume, by contradiction,

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[2] The first two authors would like to point out that the proof of the corresponding Lemma 2.4 in [6] is not entirely accurate. In that paper, we neglected to differentiate between the two distinct cases: $Y = A$ or $Y = T_i$. Nevertheless, this issue can be easily addressed by applying a slight modification of the argument presented here.

that XY is the unique edge of M in H^* incident to X (note that if $X = Y$ then XY is a loop). Consider a CDC-coloring φ of H of type XY.

If $Y = A$, without loss of generality, we can assume that φ assigns color 1 to all the four semiedges of the ordered 4-pole H. Choose a color which does not appear in the four semiedges, say 2. Whereas, if $Y = T_i$ for $i \in \{2, 3, 4\}$, assume, without loss of generality, that the two colors which defines the edge-coloring Y are 1 and 2. In both cases, consider a 1-2-bichromatic chain BC in H starting from the first semiedge and ending in another semiedge. By a color switch along BC we obtain a CDC-coloring of type XZ, where, in all cases, Z is different from Y. Hence, XY and XZ are two distinct edges of H^* incident to X, that is X is not a vertex of degree 1 in H^* and it is not incident uniquely to the loop XX. □

Lemma 3.5. *Let H be an ordered 4-pole. If the subgraph H^* of M contains a loop XX with $X \in \{A, T_2, T_3, T_4\}$, then at least one of the following holds:*

- *the two edges XY and YY belong to H^* for some $Y \neq X$;*
- *the edges XY, XZ and YZ belong to H^* for two distinct Y, Z different from X.*

Proof. Let X be an element of the set $\{A, T_2, T_3, T_4\}$ and assume XX is an edge of H^*. By Lemma 3.4, XX cannot be the unique edge of H^* incident to X, then H^* also contains a further edge XY incident to X, where $Y \neq X$. Consider a CDC-coloring φ of H of type XX.

If $X = A$, without loss of generality, we can assume that φ assigns color 1 and 2 to all the four semiedges of the ordered 4-pole H. Consider two colors, say 3 and 4, which do not appear in the four semiedges.

Consider a 1-3-bichromatic chain BC_1 and a 2-4-bichromatic chain BC_2 in H both starting from the first semiedge. By a color switch along BC_1 we obtain a CDC-coloring of type XY ($Y \neq X$) while by a color switch along BC_2 we obtain a CDC-coloring of type XZ ($Z \neq X$). Since the pairs of colors involved in the two bichromatic chains are different, then we can perform the color switches along BC_1 and BC_2 at the same time, thus obtaining a CDC-coloring of type YZ. If $Y = Z$, then XY and YY belong to H^*, otherwise XY, XZ and YZ belong to H^* and the assertion follows in this case.

Now we consider the case $X = T_i$ for some $i \in \{2, 3, 4\}$. Without loss of generality, we can assume that φ assigns colors $1, 2, 3$ and 4 to the four semiedges of the ordered 4-pole H and the first and the i^{th} semiedges receive colors 1 and 3. Consider a 1-2-bichromatic chain BC_1 and a 3-4-bichromatic chain BC_2 in H starting from the first semiedge and ending in another semiedge. By a color switch along BC_1 we obtain a CDC-coloring of type XY ($Y \neq X$) while by a color switch along BC_2 we obtain a CDC-coloring of type XZ ($Z \neq X$). As before, since the pairs of colors involved in the two bichromatic chains are different, then we can perform the color switches along BC_1 and BC_2 at the same time, thus obtaining the assertion. □

We consider a minimum counterexample G for the CDC-conjecture. As already observed, G must be cyclically 4-edge-connected. Here, we show that if G admits a 4-edge-cut S separating two cycles of G, then we can uniquely characterize the two subgraphs of M associated to the two 4-poles separated by S. Note that this case is the unique of $\binom{2^{10}+1}{2} = 2^{19} + 2^9$ a priori possibilities (such a number is obtained by the combinations with repetition by choosing 2 items from 2^{10} objects).

Theorem 3.6. *Let G be a possible minimum counterexample to the Cycle Double Cover Conjecture and let S be a 4-edge-cut separating two cycles of G. Denote by G_1 and G_2 the two 4-poles separated by S. Then, there exist i, j, k such that $\{i, j, k\} = \{2, 3, 4\}$ and the edge-sets of G_1^* and G_2^* are equal to*

$$\{AA, AT_k, AT_j, T_kT_j\} \text{ and } \{T_iT_i, T_jT_j, T_kT_k, T_iT_k, T_iT_j\}.$$

Proof. Assume there exists a counterexample to the Cycle Double Cover conjecture, and let G be one of minimum order. It is well-known that we can assume G of minimum degree 3 and without cycle-separating 2- and 3-edge-cuts.

Consider the two subgraphs G_1^* and G_2^* of M. First observe that for $a \in \{1, 2\}$, G_a^* has at least one edge. Indeed, if G_a^* contains no edge, this implies that G_a does not admit a CDC-coloring. Every 4-pole with a cycle has more vertices than every acyclic 4-pole. Hence, by glueing together G_a and an arbitrary acyclic 4-pole, we obtain a bridgeless graph smaller than G which does not admit a CDC-coloring. This is a contradiction by minimality of G.

Furthermore, since G is a counterexample to the CDC conjecture, then G_1^* and G_2^* must be edge-disjoint. Otherwise, G_1 and G_2 admit a CDC-coloring of the same type and, up to permutation of colors, we can glue such CDC-colorings together to obtain a CDC-coloring of G.

Claim 1. G_a^* contains at least one edge of each of the subgraphs associated to the six acyclic ordered 4-poles depicted in Figure 1.

Proof of Claim 1. If this is not the case, let H be the acyclic ordered 4-pole such that H^* and G_a^* are edge-disjoint. The graph obtained by glueing together H and G_a is a counterexample and it is smaller than G, it yields a contradiction. □

Claim 2. G_a^* cannot contain all edges of a subgraph associated to one of the acyclic ordered 4-poles depicted in Figure 1.

Proof of Claim 2. Suppose there exists an acyclic ordered 4-pole H whose edge-set is a subset of the edge-set of G_a^*. The graph obtained by glueing together H and G_b, $b \neq a$, is a counterexample smaller than G, that yields to a contradiction. □

Now we show that AA belongs either to G_1^* or G_2^*. Suppose not, that is AA does not belong to G_a^* for $a = 1, 2$. For every $j \in \{2, 3, 4\}$ exactly one of the two edges AT_j and T_jT_j should belong to G_1^* and the other one to G_2^*. For otherwise, the dumbbell subgraph with vertices in A and T_j and G_a^* are edge-disjoint for at least one a, contradiction by Claim 1. Moreover, both G_1^* and G_2^* have degree of A different from 1 by Lemma 3.4. Then, the three edges AT_j belong to the same subgraph, without loss of generality say G_1^*, and it follows that all three loops T_jT_j belong to G_2^*. At least one of G_1^* and G_2^* contains at most one of the three edges T_jT_k for $j \neq k$. In all cases, there is either a degree 1 vertex or an isolated loop in one of the two subgraphs, a contradiction by Lemma 3.4. We conclude that AA belongs to G_a^* for some $a \in \{1, 2\}$, say G_1^*.

Since AA is a loop, one of the two cases in Lemma 3.5 occurs, but by Claim 2 the first one is not possible. Then, there exist $j, k \in \{2, 3, 4\}$, with $j \neq k$, such that AT_j, AT_k, T_jT_k are also edges of G_1^*. Let t be the unique element of $\{2, 3, 4\}$ different from j and k. The edge AT_t is not in G_2^* since G_2^* cannot have a degree 1 vertex in A. Hence, all the three loops T_jT_j, T_tT_t and T_kT_k belong to G_2^* by Claim 1.

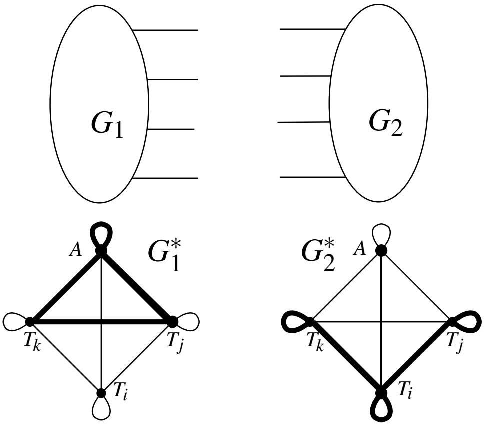

Figure 2: The case described in the statement of Theorem 3.6.

Finally, both T_tT_j and T_tT_k belong to G_2^* otherwise G_2^* has an isolated loop. The edge AT_t is the unique edge of M which does not belong to one of the two subgraphs so far. We have already proved that it cannot belong to G_2^* and, similarly, it cannot belong to G_1^* otherwise G_1^* has a degree 1 vertex in T_t, contradiction by Lemma 3.4. The edge-sets of G_1^* and G_2^* are as stated by the theorem. □

In order to prove that a minimum counterexample cannot admit a 4-edge-cut, it is thus sufficient to establish that a 4-pole of either of the two types described in Theorem 3.6 cannot exist. We suspect that neither can exist, and we leave it as a conjecture.

Conjecture 3.7. *No 4-pole has*

$$\{AA, AT_k, AT_j, T_kT_j\} \text{ or } \{T_iT_i, T_jT_j, T_kT_k, T_iT_k, T_iT_j\}$$

as a set of admissible CDC-colorings, for any choice of distinct values of i, j, k in $\{2, 3, 4\}$.

Since all the acyclic 4-poles considered in the proof are cubic, our argument can be repeated if we consider a minimum cubic counterexample. In this case, we can prove that if such multipoles would exist, then they cannot be 3-edge-colorable.

Next lemma analyzes some general necessary properties of the graph H^* when H is a 3-edge-colorable cubic 4-pole.

Lemma 3.8. *Let H be a 3-edge-colorable cubic 4-pole. Then the following occurs:*

- *H^* has an edge of type AX for some $X \in \{A, T_2, T_3, T_4\}$*
- *if H is connected, the minimum degree of H^* is at least 1.*

Proof. Consider a 3-edge-coloring α of the edges of H with colors $1, 2, 3$. Clearly the three 2-factors C_{ij} each of which induced by the edges of H colored i and j, $i \neq j$, form a CDC-coloring φ of H. Observe that the colors of α never appear all together on the semiedges of H, as ensured by standard parity arguments. Hence there exists at least one 2-factor C_{ij} that encounters all the semiedges of H, so that φ must be of type AX for some $X \in \{A, T_2, T_3, T_4\}$.

If H is connected, then for every $t \in \{2,3,4\}$ consider a path Q_t starting from the first semiedge of H and ending in the t-th semiedge. Then the family of cycles given by $\{C_{12} \Delta Q_t,\ C_{13} \Delta Q_t,\ C_{23} \Delta Q_t,\ Q_t\}$, where $A \Delta B$ indicates the symmetric difference of the sets A and B, form a CDC-coloring of H of type T_tX for some $X \in \{A, T_2, T_3, T_4\}$. □

The following statement is a corollary of Lemma 3.8 and Theorem 3.6.

Corollary 3.9. *If a cubic minimum counterexample to the CDC-conjecture is not cyclically* 5*-edge-connected, then both the components* G_1 *and* G_2 *separated by a cyclic* 4*-edge-cut must be non* 3*-edge-colorable* 4*-poles containing a cycle.*

Finally, also considering Remark 3.2, we can summarize all previous considerations in the following statement.

Theorem 3.10. *Let* G *be a graph and let* S *be a* 4*-edge-cut separating two cycles of* G*. Denote by* G_1 *and* G_2 *the two* 4*-poles separated by* S*. Assume* G *satisfies (at least) one of the following.*

(1) G *is a minimum counterexample to the CDC-conjecture;*

(2) G *is a minimum cubic counterexample to the CDC-conjecture;*

(3) G *is a minimum counterexample to the* 5*-CDC-conjecture;*

(4) G *is a minimum cubic counterexample to the* 5*-CDC-conjecture.*

Then, there exist i, j, k *such that* $\{i,j,k\} = \{2,3,4\}$ *and the edge-sets of* G_1^* *and* G_2^* *are equal to*

$$\{AA, AT_k, AT_j, T_kT_j\} \text{ and } \{T_iT_i, T_jT_j, T_kT_k, T_iT_k, T_iT_j\}.$$

4 On 5-edge-cuts in a possible minimum counterexample to BF-Conjecture

In this section we employ the same technique as in the previous one to study cyclic 5-edge-cuts in a possible minimum counterexample to the Berge-Fulkerson Conjecture. As the Berge-Fulkerson conjecture, in its basic form, is stated for cubic graphs, multipoles studied here will be cubic.

We will use the following definition.

Definition 4.1. Let $H = (V, E)$ be a cubic multipole. A *Berge-Fulkerson coloring*, BF-coloring for short, is a function φ which assigns to every element in E a 2-subset of the set of colors $\{1,2,3,4,5,6\}$, in such a way that the subsets assigned to any two adjacent edges are disjoint.

Clearly, a BF-coloring of a cubic graph G is equivalent to the existence of six perfect matchings of G covering each edge of G twice. Moreover, each of the six color classes induces a perfect matching of G.

From now on, we will refer to a 5-pole, implicitly understanding it to be both ordered and cubic.

Let H be a 5-pole. Then, since every perfect matching intersects an odd number of times every 5-edge-cut, each of the six colors of a BF-coloring of H appears an odd number of times on the semiedges of H. Hence, a color can occur $1, 3$, or 5 times. If there is one color appearing 5 times, any other color must appear only once: we denote this type of BF-coloring by (12345).

In all other cases, two colors appear three times and each of the remaining four colors appears once. Let $x < y$ and $x' < y'$ belong to $\{1, 2, 3, 4, 5\}$. We will say that a BF-coloring is of type $(xy)(x'y')$ if one of the two colors which appear three times on the semiedges is *not* present on the semiedges in positions x and y, and the other one is *not* present in positions x' and y'. Along the paper, we do not distinguish between the types $(xy)(x'y')$ and $(x'y')(xy)$. Moreover, if $(x, y) = (x', y')$, we will denote the coloring of type of $(xy)(xy)$ simply by $[xy]$.

Hence, there are 56 distinct types of BF-colorings: 45 of type $(xy)(x'y')$, with either $x \neq x'$ or $y \neq y'$, 10 of type $[xy]$ and, in addition, the type (12345).

In all types of BF-colorings, we will call a color of φ which appears exactly once along the semiedges of the 5-pole as *lonely*. With a slight abuse of terminology, we will also use the term lonely position/semiedge, when we refer to a semiedge which receives (at least) one lonely color. For a given ordered 5-pole, we say a type of BF-coloring admissible if the 5-pole admits at least one BF-coloring of such a type. The set of admissible types of BF-colorings can be represented as a subgraph of an auxiliary graph N (see Figure 3). The set of vertices of N is $\{(xy) | x, y \in \{1, 2, 3, 4, 5\}, x < y\} \cup \{(12345)\}$. The type $(xy)(x'y')$ of a BF-coloring will be represented by an edge between the two vertices (xy) and $(x'y')$ of N. The type (12345) will be represented by a loop over the vertex (12345) and, similarly, a type $[xy]$ is represented by a loop over the vertex (xy).

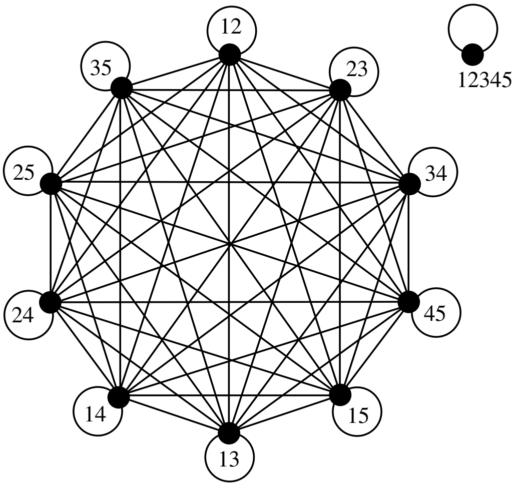

Figure 3: The auxiliary graph N. We omit the parentheses to make the figure more simple.

Given a 5-pole H, denote by $\mathcal{C}(H)$ the set of all admissible types of BF-colorings for H. It follows from the above description that $\mathcal{C}(H)$ can be represented as a subgraph of N. From now on we denote this subgraph by H^*. A priori H^* could be any subgraph of N: hence, for a given H, H^* is one of the 2^{56} possible subgraphs of N. This number can actually be significantly reduced by using Kempe switches, similarly as in [6].

4.1 Kempe switches

Let φ be a BF-coloring of an ordered cubic 5-pole H. Let s be a semiedge of H and denote by c_1 one of the two colors in $\varphi(s)$ and by c_2 one of the four colors not in $\varphi(s)$. Consider the subgraph of H induced by all edges e such that $\varphi(e) \cap \{c_1, c_2\}$ is not empty. Let K be the connected component of such a subgraph which contains the semiedge s. Clearly, K is a path and contains exactly two semiedges. Let s' denote the one different from s. Moreover, every edge of K in φ receives exactly one of the two colors c_1 and c_2 and these colors alternate along the edges of the path K. The component K will be called a c_1-c_2-Kempe chain. Starting from φ, we can obtain a different BF-coloring of H by performing a Kempe switch, that is an interchange of the two colors c_1 and c_2 along all edges of K.

Notice also that, given a BF-coloring φ on a 5-pole H, it could be possible to perform more than one Kempe switch. For example, consider a BF-coloring of type $(xy)(x'y')$. Denote by 1 and 4 the colors appearing three times on the semiedges, by 2 and 3 the colors in positions x and y, respectively, and by 5 and 6 the colors in positions x' and y', respectively. It is then possible to consider two independent Kempe chains with colors 1 and c with $c \in \{2, 3\}$ and 4 and c' with $c' \in \{5, 6\}$. Performing both the Kempe switches along these chains results in another BF-coloring of H.

The exact size of the set of all types of BF-colorings that can be obtained by performing Kempe switches starting from φ in a pole H cannot be determined a priori. Indeed it depends on the specific BF-coloring φ and on the structure of the ordered 5-pole. However, previous observations ensure that it always contains at least one element different from φ.

Now we can formalize the possibility to perform Kempe switches in terms of types of BF-colorings, i.e. in terms of edges of the auxiliary graph N. Let φ be a BF-coloring of a 5-pole H, let c_1 and c_2 be any two distinct colors and consider a c_1-c_2-Kempe chain beginning and ending on the dangling edges. Observe that it exists if and only if the symmetric difference of the set of dangling edges colored c_1 and the set of dangling edges colored c_2 is non-empty (this is equivalent to the fact that it has two or four elements in the case of a 5-pole). Let us call such a pair of colors *switchable*. Note that, for two colors, being switchable or not does not depend on φ but only on the type of φ.

Let H be a 5-pole, let α be an edge of H^* and let $\{1, 2, \ldots, 6\}$ be the set of six colors representing α. Further, let $i, j, k, m \in \{1, 2, \ldots, 6\}$ be any four pairwise distinct colors such that the colors i and j are switchable and k and m are switchable. Consider the corresponding i-j- and the k-m-Kempe chains in H. Then the edge-set of H^* contains at least:

(i) an edge β corresponding to a type of BF-coloring resulting from a Kempe switch on a i-j-Kempe chain K_1 in H, and

(ii) an edge γ corresponding to a type of BF-coloring resulting from a Kempe switch on a k-m-Kempe chain K_2 in H, and

(iii) the edge δ corresponding to the type of BF-coloring resulting by performing both Kempe switches on K_1 and K_2.

We will refer to this property of the edge α of H^* by saying that, from α, we can *perform one (cases* (i) *and* (ii)*) or two (case* (iii)*) Kempe switches*. Notice that, if we know $\alpha \in E(H^*)$, we cannot determine the specific type of β, γ and δ without the exact knowledge of the particular 5-pole and of the specific BF-coloring φ corresponding to α.

However, we can assure that at least a β and a γ and the corresponding δ belong to $E(H^*)$, with α, β, γ and δ not necessarily pairwise distinct.

Definition 4.2. Let M be a subgraph of the auxiliary graph N. We say that M is *closed under one or two Kempe switches* if for every $\alpha \in E(M)$ there exist β, γ and δ in $E(M)$ obtainable via one or two Kempe switches as described above.

4.2 BF-colorings of some small 5-poles

In this subsection we enlight the types of admissible BF-colorings of some particular poles that we need later for our purpose. Notice that, there exist only two acyclic 5-poles which are the two poles P and Q depicted in Figure 4. In the figure, they are represented with their associated subgraphs of N, P^* and Q^* respectively.

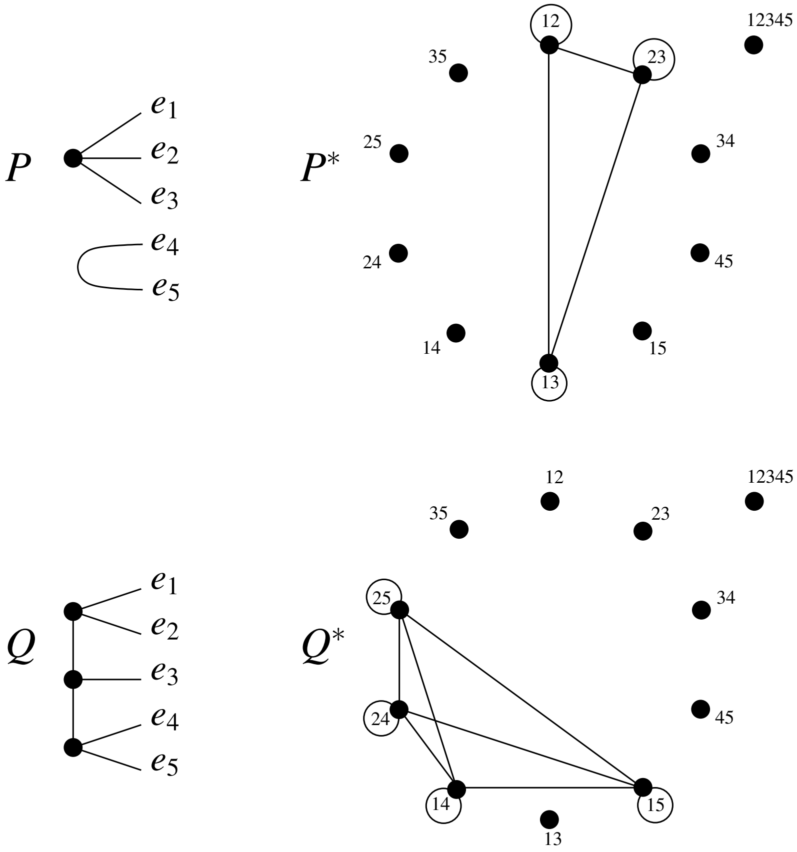

Figure 4: The acyclic 5-poles P and Q and their associated subgraphs P^* and Q^* of N.

In Figure 5 the 5-pole C, whose vertex-induced subgraph is a 5-cycle, is depicted, together with its associated subgraph C^* of N. From now on, with a slight abuse of terminology, we will denote such a pole simply as a *5-cycle*. Observe that reordering the semiedges of these 5-poles gives rise to isomorphic subgraphs P^*, Q^* and C^* of N, but with different vertex sets.

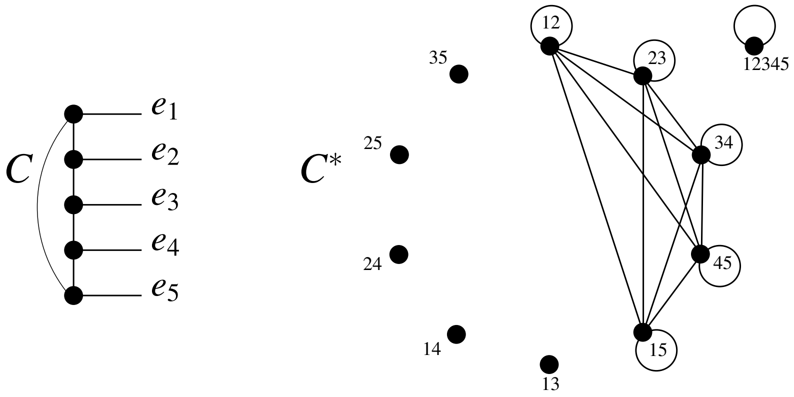

Figure 5: An ordered 5-pole C and its associated subgraph C^*.

4.3 A possible counterexample to BF-Conjecture

Now we consider a minimum possible counterexample G to the BF-conjecture. In [6] it is proved that G should be a cyclically 5-edge-connected non 3-edge-colorable cubic graph. Here, we show that if G admits a cyclic 5-edge-cut S, then the coloring types of the two 5-poles correspond to one of 13 possibilities out of more than 2^{111}. The number of possibilities is given by the combinations with repetition by choosing 2 items from 2^{56} objects: this number is equal to $\binom{2^{56}+1}{2} = 2^{111} + 2^{55}$.

Let R and L be the two 5-poles arising from the edge-cut S, and let R^* and L^* be their associated subgraphs of N, respectively. Clearly, R^* and L^* must be edge-disjoint, for otherwise R and L would admit BF-colorings φ_1 and φ_2 of the same type. Thus, by considering a suitable choice of colors of both poles, we could glue them together, obtaining a BF-coloring on G. Moreover, to be admissible, R^* and L^* must satisfy the following necessary conditions.

- R^* and L^* are closed under one or two Kempe switches (see Definition 4.2).
- Both R^* and L^* do not have as a subgraph any of the associated subgraphs A^* to an acyclic 5-pole A. Indeed, assume that R^* does admit such a subgraph. Replacing R with A would lead to a smaller counterexample to the BF-Conjecture, since A^* would remain edge-disjoint with L^*.
- Both R^* and L^* are not edge-disjoint from all the associated subgraphs A^* to an acyclic 5-pole A. Indeed, assume the complement of R^* admits such a subgraph. Then, replacing L with A would lead to a smaller counterexample to the BF-conjecture, since A^* would remain edge-disjoint with R^*.

In what follows we provide a brief description of the program we implemented to identify pairs of types of 5-poles which can occur in a smallest counterexample to Berge-Fulkerson conjecture.

First, we identified all Kempe closed subsets of K. But as $|K| = 56$, the number of its subsets is 2^{56}, which is too large to consider all of them. Many of these subsets differ from other ones only by permuting the dangling edges. To manage this number of subsets, we eliminated some number of cases that can be obtained by a permutation of dangling edges and are not minimal in some sense explained further. Since the elimination is also time consuming, we had to find the “right amount” of elimination. Therefore we proceeded as follows. We decomposed K into four disjoint subsets:

- $A = \{(12345)\}$
- $B = \{(ij)(km); |\{i, j, k, m\}| = 4\}$
- $C = \{(ij)(im); |\{i, j, m\}| = 3\}$
- $D = \{(ij)(ij); i \neq j\}$

It can be easily seen that $|A| = 1$, $|B| = 15$, $|C| = 30$, $|D| = 10$.

We fixed an order of elements of C. Then each subset C' of C corresponds to a binary vector from $\mathbb{Z}_2^{56}$, and this in turn corresponds to an integer from $\{0, 1, \ldots, 2^{56} - 1\}$. We consider only those subsets C' of C where this integer is smaller or equal than all $5!$ integers corresponding to permutations of dangling edges. For all subsets C' of C, we checked if C' is minimal with respect to permutation of five dangling edges. Then, for all subsets D' of D we checked if, starting from a coloring in $C' \cup D'$ and performing one or two Kempe switches, whether each of the resulting colorings is either outside $C \cup D$ or in $C' \cup D'$. Here we consider all subsets D' of D and not only minimal ones, since it showed up to be more convenient. If yes, we verified that it does not contain a subset corresponding to an acyclic 5-pole P^* or its complement, conf. Figure 4. Further we checked if $C' \cup D'$ is minimal under permutation of five dangling edges. If the answer was yes, we proceeded to identify suitable subsets of $A \cup B$.

For all subsets A' of A and $B'$of B we inspected whether for every coloring of $A' \cup B'$, the BF-colorings obtained by performing one or two Kempe switches are in $A' \cup B'$ or outside $A \cup B$. The same for the sets B and C.

Finally, for each A', B', C', D' that passed the above checks we verified whether all subsets $A' \cup B' \cup C' \cup D'$ are closed under one or two Kempe switches and does not contain Q^* or its complement.

We alltogether obtained 883 sets of types of colorings that fulfil all these requirements. Among them, after performing all permutations of dangling edges, we checked that there are exactly 13 disjoint pairs. These pairs are listed in the Appendix and depicted in Figures 6 and 7.

Observe that, in cases from I to XII, R^* (grey edges) is a copy of K_5 with loops on vertices plus the (12345) coloring. This is the graph induced by the admissible types of BF-colorings of a 5-cycles. Hence, by minimality of G, in these cases, we can assume that the pole R is a 5-cycle.

In the remaining part of the section we study 3-edge-colorability of the 5-poles R and L. In the following lemma we state some general necessary conditions for the associated graph H^*, when H is a 3-edge-colorable 5-pole.

In the next lemma, we extend the notion of lonely colors to a 3-edge-coloring of a 5-pole in the natural way. We say that a color is lonely if it appears exactly one time along the semiedges in the considered 3-edge-coloring.

Lemma 4.3. *A 3-edge-colorable 5-pole H which has a 3-edge-coloring with lonely colors in positions x and y admits a BF-coloring of each of the following types: $[xy]$, $[x_1y]$, $[xy_1]$, $(xy)(x_1y)$, $(xy)(xy_1)$ and $(x_1y)(xy_1)$, for some $x_1 \neq x, y$ and $y_1 \neq y, x$.*

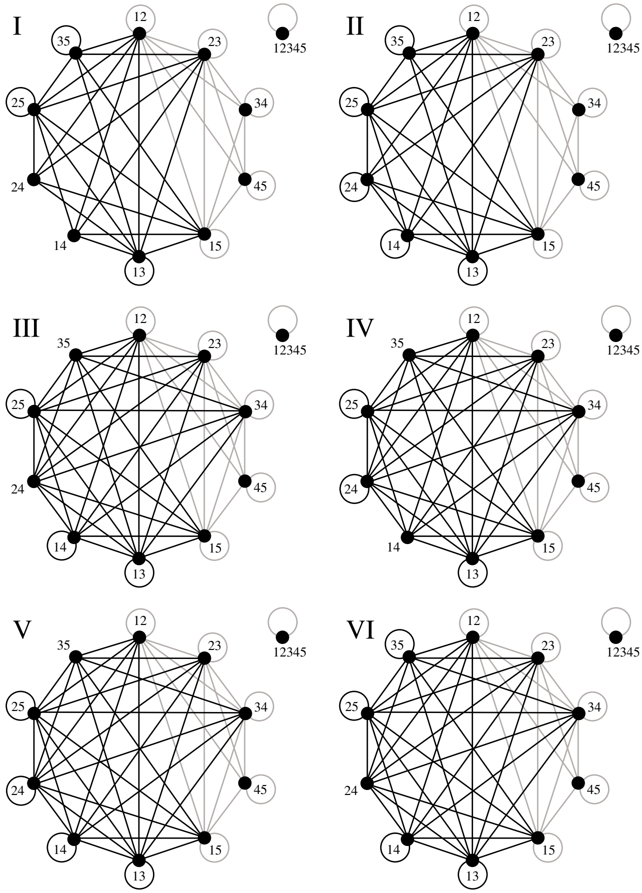

Figure 6: Possible pairs of subgraphs R^* and L^*, cases I to VI

Proof. Observe that if the pole H admits two different 3-edge-colorings c and c' with lonely colors in positions x,y and x',y', respectively, then it admits BF-colorings of types $[xy]$, $[x'y']$ and $(xy)(x'y')$. Indeed, the first two ones are given by an overlap of two copies of c and c', respectively, and the third one is given by the overlap of a copy of c and a copy of c'.

Consider a 3-edge-coloring c of H with lonely colors in positions x and y. Denote by x and y respectively also their colors and denote by z the remaining color of c. Consider a Kempe chain with colors x and z and exchange the colors along it. Since the chain ends in a semiedge in position x_1, $x_1 \neq x, y$, a 3-edge-coloring of H with lonely colors in positions x_1 and y is obtained. The same procedure with a Kempe chain of colors y and z proves that H admits a 3-edge-coloring with lonely colors in positions x and y_1, where $y_1 \neq x, y$. Hence H admits BF-colorings of all types $[xy]$, $[x_1y]$, $[xy_1]$, $(xy)(x_1y)$, $(xy)(xy_1)$ and $(x_1y)(xy_1)$. □

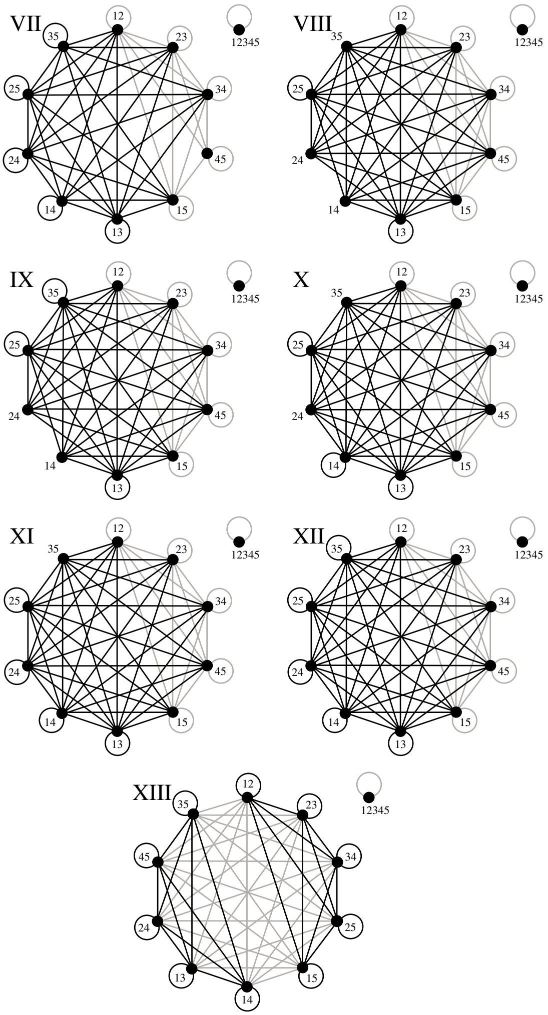

Figure 7: Possible pairs of subgraphs R^* and L^*, cases VII to XIII. We rearrange the labels of the vertices in case XIII for the sake of readability.

As a consequence of Lemma 4.3, we obtain that, for a 3-edge-colorable 5-pole H, in the associated subgraph H^* there are at least 3 distinct loops. Moreover, if a position i, $i \in \{1, ..., 5\}$, appears in one of such loops arising from a 3-edge-coloring, then it must appear at least another time in a different loop. Hence, assume the subgraphs L^* (black edges) in cases from I to XIII (see Figures 6 and 7) are associated to a 3-edge-colorable 5-pole L. Then, all cases are ruled out by previous observations on loops but II, VII, XII and XIII. Cases II and VII do not satisfy the necessary condition in Lemma 4.3, since there is at least a pair of loops $[xy]$ and $[x_1y]$ for which $(xy)(x_1y)$ is not an edge. Hence, we conclude that, if the pole L is 3-edge-colorable, the only possibilities for L^* are cases XII and XIII. Finally, also the subgraph R^* of case XIII cannot be associated to a 3-edge-colorable pole R, since it does not contain any loop.

We summarize what have been discussed in the following proposition.

Proposition 4.4. *If G is a cyclically 5-edge-connected minimum counterexample to the Berge-Fulkerson Conjecture and it is not cyclically 6-edge-connected, then every cycle separating 5-edge-cut separates two non-acyclic 5-poles L and R such that, up to interchanging the roles of L and R,*

- *both L and R admit a BF-coloring and the BF-colorings for L and R are exactly the ones of the subgraphs L^* and R^* of cases from I to XIII depicted in* Figures 6 *and* 7*;*
- *in cases from I to XII the pole R is a 5-cycle, so that G has girth exactly 5;*
- *in cases from I to XI, L is a non 3-edge-colorable pole;*
- *in case XIII, R is a non 3-edge-colorable pole.*

5 Conclusion

In this paper, we have significantly narrowed down the potential structure of a minimal counterexample for the Berge-Fulkerson Conjecture and the Cycle Double Cover Conjecture. However, we have not been able to completely rule out the possibility that a minimal counterexample could have cyclic connectivity 5 or 4, respectively, in either case. Some alternative and new ideas seem necessary to exclude these remaining cases in the future.

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Appendix:

In this appendix we list the edge-set of the subgraphs L^* and R^* represented in Figures 6 and 7.

I to XII
R^*:
(12)(15) (12)(23) (12)(34) (12)(45) [12]
(15)(23) (23)(34) (23)(45) [23]
(15)(34) (34)(45) [34]
(15)(45) [45]
[15]
(12345)

I
L^*:
(12)(35) (13)(35) (15)(35) (23)(35) (25)(35) [35]
(12)(25) (15)(25) (13)(25) (14)(25) (23)(25) (24)(25) [25]
(12)(24) (13)(24) (15)(24) (23)(24)
(12)(14) (14)(23) (14)(15) (13)(14)
(12)(13) (13)(15) (13)(23) [13]

II
L^*:
(12)(35) (13)(35) (15)(35) (23)(35) (25)(35) [35]
(12)(25) (13)(25) (14)(25) (15)(25) (23)(25) (24)(25) [25]
(12)(24) (13)(24) (14)(24) (15)(24) (23)(24) [24]
(12)(14) (13)(14) (14)(15) (14)(23) [14]
(12)(13) (13)(15) (13)(23) [13]

III
L^*:
(12)(35) (13)(35) (15)(35) (23)(35) (24)(35) (25)(35) (34)(35)
(12)(25) (13)(25) (14)(25) (15)(25) (23)(25) (24)(25) (25)(34) [25]
(12)(24) (13)(24) (15)(24) (14)(24) (23)(24) (24)(34)
(12)(14) (13)(14) (14)(15) (14)(23) (14)(34) [14]
(12)(13) (13)(15) (13)(23) (13)(34) [13]

IV
L^*:
(12)(35) (13)(35) (15)(35) (23)(35) (24)(35) (25)(35) (34)(35)
(12)(25) (13)(25) (14)(25) (15)(25) (23)(25) (24)(25) (25)(34) [25]
(12)(24) (13)(24) (14)(24) (15)(24) (23)(24) (24)(34) [24]
(12)(14) (13)(14) (14)(15) (14)(23) (14)(34)
(12)(13) (13)(15) (13)(23) (13)(34) [13]

V

L^*:
(12)(35) (13)(35) (15)(35) (23)(35) (24)(35) (25)(35) (34)(35)
(12)(25) (13)(25) (14)(25) (15)(25) (23)(25) (24)(25) (25)(34) [25]
(12)(24) (13)(24) (14)(24) (15)(24) (23)(24) (24)(34) [24]
(12)(14) (13)(14) (14)(15) (14)(23) (14)(34) [14]
(12)(13) (13)(15) (13)(23) (13)(34) [13]

VI
L^*:
(12)(35) (13)(35) (15)(35) (23)(35) (24)(35) (25)(35) (34)(35) [35]
(12)(25) (13)(25) (14)(25) (15)(25) (23)(25) (24)(25) (25)(34) [25]
(12)(24) (13)(24) (14)(24) (15)(24) (23)(24) (24)(34)
(12)(14) (13)(14) (14)(15) (14)(23) (14)(34) [14]
(12)(13) (13)(15) (13)(23) (13)(34) [13]

VII
L^*:
(12)(35) (13)(35) (15)(35) (23)(35) (24)(35) (25)(35) (34)(35) [35]
(12)(25) (13)(25) (14)(25) (15)(25) (23)(25) (24)(25) (25)(34) [25]
(12)(24) (13)(24) (14)(24) (15)(24) (23)(24) (24)(34) [24]
(12)(14) (13)(14) (14)(15) (14)(23) (14)(34) [14]
(12)(13) (13)(15) (13)(23) (13)(34) [13]

VIII
L^*:
(12)(35) (13)(35) (25)(35) (14)(35) (15)(35) (23)(35) (24)(35) (34)(35) (35)(45)
(12)(25) (13)(25) (14)(25) (15)(25) (23)(25) (24)(25) (25)(34) (25)(45) [25]
(12)(24) (13)(24) (15)(24) (23)(24) (24)(34) (24)(45)
(12)(14) (13)(14) (14)(15) (14)(23) (14)(34) (14)(45)
(12)(13) (13)(15) (13)(23) (13)(34) (13)(45) [13]

IX
L^*:
(12)(35) (13)(35) (25)(35) (14)(35) (15)(35) (23)(35) (24)(35) (34)(35) (35)(45) [35]
(12)(25) (13)(25) (14)(25) (15)(25) (23)(25) (24)(25) (25)(34) (25)(45) [25]
(12)(24) (13)(24) (15)(24) (23)(24) (24)(34) (24)(45)
(12)(14) (13)(14) (14)(15) (14)(23) (14)(34) (14)(45)
(12)(13) (13)(15) (13)(23) (13)(34) (13)(45) [13]

X
L^*:
(12)(35) (13)(35) (25)(35) (14)(35) (15)(35) (23)(35) (24)(35) (34)(35) (35)(45)
(12)(25) (13)(25) (14)(25) (15)(25) (23)(25) (24)(25) (25)(34) (25)(45) [25]
(12)(24) (13)(24) (14)(24) (15)(24) (23)(24) (24)(34) (24)(45)
(12)(14) (13)(14) (14)(15) (14)(23) (14)(34) (14)(45) [14]
(12)(13) (13)(15) (13)(23) (13)(34) (13)(45) [13]

XI

L^*:

(12)(35) (13)(35) (25)(35) (14)(35) (15)(35) (23)(35) (24)(35) (34)(35) (35)(45)
(12)(25) (13)(25) (14)(25) (15)(25) (23)(25) (24)(25) (25)(34) (25)(45) [25]
(12)(24) (13)(24) (14)(24) (15)(24) (23)(24) (24)(34) (24)(45) [24]
(12)(14) (13)(14) (14)(15) (14)(23) (14)(34) (14)(45) [14]
(12)(13) (13)(15) (13)(23) (13)(34) (13)(45) [13]

XII

L^*:

(12)(35) (13)(35) (25)(35) (14)(35) (15)(35) (23)(35) (24)(35) (34)(35) (35)(45) [35]
(12)(25) (13)(25) (14)(25) (15)(25) (23)(25) (24)(25) (25)(34) (25)(45) [25]
(12)(24) (13)(24) (14)(24) (15)(24) (23)(24) (24)(34) (24)(45) [24]
(12)(14) (13)(14) (14)(15) (14)(23) (14)(34) (14)(45) [14]
(12)(13) (13)(15) (13)(23) (13)(34) (13)(45) [13]

XIII
R^*:
(12)(35) (15)(35) (23)(35) (25)(35) (34)(35)
(12)(45) (15)(45) (23)(45) (25)(45) (34)(45)
(12)(24) (15)(24) (23)(24) (24)(25) (24)(34)
(12)(13) (13)(15) (13)(23) (13)(25) (13)(34)
(12)(14) (14)(15) (14)(23) (14)(25) (14)(34)
(12345)

L^*:
(13)(35) (14)(35) (24)(35) (35)(45) [35]
(13)(45) (14)(45) (24)(45) [45]
(13)(24) (14)(24) [24]
(13)(14) [14]
$[13]$
(12)(15) (12)(23) (12)(34) (12)(45) [12]
(15)(23) (23)(34) (23)(45) [23]
(15)(34) (34)(45) [34]
(15)(45) [45]
$[15]$