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Regularity Results for Bounded Solutions to Obstacle Problems with Non-standard Growth Conditions

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Abstract. In this paper, we consider a class of obstacle problems of the type

$$\min\left\{\int_\Omega f(x,Dv)\,\mathrm{d}x\,:\,v\in\mathcal{K}_\psi(\Omega)\right\}$$

where ψ is the obstacle, $\mathcal{K}_\psi(\Omega)=\{v\in u_0+W_0^{1,p}(\Omega,\mathbb{R}):v\geq\psi$ a.e. in $\Omega\}$, with $u_0\in W^{1,p}(\Omega)$ a fixed boundary datum, the class of the admissible functions and the integrand $f(x,Dv)$ satisfies non standard (p,q)-growth conditions. We prove higher differentiability results for bounded solutions of the obstacle problem under dimension-free conditions on the gap between the growth and the ellipticity exponents. Moreover, also the Sobolev assumption on the partial map $x\mapsto A(x,\xi)$ is independent of the dimension n and this, in some cases, allows us to manage coefficients in a Sobolev class below the critical one $W^{1,n}$.

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1. Introduction

We prove higher differentiability results for solutions to variational obstacle problems of the form

$$\min\left\{\int_\Omega f(x,Dv)\,\mathrm{d}x\,:\,v\in\mathcal{K}_\psi(\Omega)\right\},\tag{1.1}$$

where Ω is a bounded open set of $\mathbb{R}^n$, $n>2$, $\psi:\Omega\mapsto[-\infty,+\infty)$ belonging to the Sobolev class $W^{1,p}_{\mathrm{loc}}(\Omega)$ is the *obstacle* and

$$\mathcal{K}_\psi(\Omega)=\{v\in u_0+W_0^{1,p}(\Omega,\mathbb{R}):v\geq\psi\text{ a.e. in }\Omega\}$$

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is the class of the admissible functions, with $u_0 \in W^{1,p}(\Omega)$ a fixed boundary datum.

We shall consider integrands f such that $\xi \mapsto f(x,\xi)$ is $\mathcal{C}^2$ and there exists $\tilde{f} : \Omega \times [0,\infty) \to [0,\infty)$ such that

$$f(x,\xi) = \tilde{f}(x,|\xi|).$$

Moreover, we assume that there exist positive constants $\tilde{\nu}, \tilde{L}$, exponents p, q with $2 \le p < q < p+1 < +\infty$ and a parameter $0 \le \mu \le 1$ such that the following assumptions are satisfied

$$\langle D_{\xi\xi} f(x,\xi)\lambda, \lambda\rangle \ge \tilde{\nu}(\mu^2 + |\xi|^2)^{\frac{p-2}{2}}|\lambda|^2 \tag{F1}$$

$$|D_{\xi\xi} f(x,\xi)| \le \tilde{L}\left[(\mu^2 + |\xi|^2)^{\frac{p-2}{2}} + (\mu^2 + |\xi|^2)^{\frac{q-2}{2}}\right] \tag{F2}$$

for almost every $x \in \Omega$ and every $\xi, \lambda \in \mathbb{R}^n$.

Note that, following [11], the assumptions (F1) and (F2) and the dependence on the modulus imply that there exists a positive constant $\tilde{\ell}$ such that

$$\frac{1}{\tilde{\ell}}(|\xi|^2 - \mu^2)^{\frac{p}{2}} \le f(x,\xi) \le \tilde{\ell}\left[(\mu^2 + |\xi|^2)^{\frac{p}{2}} + (\mu^2 + |\xi|^2)^{\frac{q}{2}}\right] \tag{F3}$$

for almost every $x \in \Omega$ and every $\xi \in \mathbb{R}^n$, i.e. the functional f has non-standard growth conditions of (p,q)-type as defined and introduced by Marcellini [43,44] and then widely investigated (see for example [5,22,23]) and more recent [45,46].

Concerning the dependence on the $x-$variable, we assume that there exists a non-negative function $k(x) \in L^{\frac{p+2}{p-q+1}}$ such that

$$|D_{x\xi} f(x,\xi)| \le k(x)\left[(\mu^2 + |\xi|^2)^{\frac{p-1}{2}} + (\mu^2 + |\xi|^2)^{\frac{q-1}{2}}\right] \tag{F4}$$

for almost every $x \in \Omega$ and every $\xi \in \mathbb{R}^n$.

Let us observe that, in case of standard growth conditions, $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a solution to the obstacle problem (1.1) in $\mathcal{K}_\psi(\Omega)$ if and only if $u \in \mathcal{K}_\psi(\Omega)$ and u is a solution to the variational inequality

$$\int_\Omega \langle A(x,Du(x)), D(\varphi(x) - u(x))\rangle \mathrm{d}x \ge 0 \qquad \forall \varphi \in W^{1,\infty}_{\mathrm{loc}}(\Omega) \text{ and } \varphi \ge \psi, \tag{1.2}$$

where the operator $A(x,\xi) : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is defined as follows

$$A(x,\xi) = D_\xi f(x,\xi).$$

It is clear that, in case of standard growth, a density argument shows the validity of (1.2) for every $\varphi \in \mathcal{K}_\psi(\Omega)$. Here, dealing with non-standard growth, it is worth observing that (1.2) holds true also for solutions to (1.1). More precisely, due to our assumptions $q - p < 1$ on the gap between the ellipticity exponent p and the growth exponent q, the validity of (1.2) can be easily checked as done at the beginning of the proof of the Theorem 1.1 below.

We want to stress that this is not obvious in case of non-standard growth conditions: already for unconstrained problems, the relation between minima and extremals, i.e. solutions of the corresponding Euler Lagrange system, is an issue that requires a careful investigation (see for example [6] and for constrained problems see the very recent paper [19]).

From assumptions (F1)–(F4), we deduce the existence of positive constants ν, L, ℓ such that the following p-ellipticity and q-growth conditions are satisfied by the map A:

$$\langle A(x,\xi)-A(x,\eta),\xi-\eta\rangle \geq \nu|\xi-\eta|^2\left(\mu^2+|\xi|^2+|\eta|^2\right)^{\frac{p-2}{2}} \tag{A1}$$

$$|A(x,\xi)-A(x,\eta)| \leq L|\xi-\eta|\left[\left(\mu^2+|\xi|^2+|\eta|^2\right)^{\frac{p-2}{2}}+\left(\mu^2+|\xi|^2+|\eta|^2\right)^{\frac{q-2}{2}}\right] \tag{A2}$$

$$|A(x,\xi)| \leq \ell\left[\left(\mu^2+|\xi|^2\right)^{\frac{p-1}{2}}+\left(\mu^2+|\xi|^2\right)^{\frac{q-1}{2}}\right], \tag{A3}$$

$$|D_xA(x,\xi)| \leqslant k(x)\left[(\mu^2+|\xi|^2)^{\frac{p-1}{2}}+(\mu^2+|\xi|^2)^{\frac{q-1}{2}}\right] \tag{$\tilde{\text{A}}$4}$$

for almost every $x\in\Omega$ and for every $\xi,\eta\in\mathbb{R}^n$.

Thanks to a characterization of the Sobolev spaces due to Hajlasz [37], we deduce from ($\tilde{\text{A}}$4) that there exists a non-negative function $\kappa \in L^{\frac{p+2}{p-q+1}}_{\mathrm{loc}}(\Omega)$ such that

$$|A(x,\xi)-A(y,\xi)| \leq (\kappa(x)+\kappa(y))\,|x-y|\left[\left(\mu^2+|\xi|^2\right)^{\frac{p-1}{2}}+\left(\mu^2+|\xi|^2\right)^{\frac{q-1}{2}}\right] \tag{A4}$$

for almost every $x,y\in\Omega$ and for all $\xi\in\mathbb{R}^n$. As far as we know, regularity results concerning local minimizers of integral functionals of the Calculus of Variations under an assumption on the dependence on the x-variable of this type, have been obtained, for the first time, in [38,39].

The study of the regularity properties of solutions to obstacle problems has been the object of intense interest in the last years and it has been usually observed that the regularity of the obstacle influences the regularity of the solutions to the problem: for linear problems the solutions are as regular as the obstacle; this is no longer the case in the nonlinear setting for general integrands without any specific structure. Hence along the years, there has been an intense research activity in which extra regularity has been imposed on the obstacle to balance the nonlinearity (see [2,3,20,21]).

Here, as we already said, we are interested in higher differentiability results since in case of non-standard growth, many questions are still open. In [4,8,9,18,24–26,29,30,35,36,42,47,51] the authors analyzed how an extra differentiability of integer or fractional order of the gradient of the obstacle provides an extra differentiability to the gradient of the solutions, also in case of standard growth. However, since no extra differentiability properties for the solutions can be expected even if the obstacle ψ is smooth, unless some assumption is given on the x-dependence of the operator A, the higher differentiability results for the solutions of systems or for the minimizers of functionals in the case of unconstrained problems (see [1,10,12,17,27,28,31–33,49,50]) have been useful and source of inspiration also for the constrained case. Differentiability results for solutions defined by duality when the coefficients are in $W^{1,n}$ can be found in [41].

For higher regularity results for solutions to non-autonomous elliptic problems, we also refer to [40], and to very recent paper [15], where both unconstrained and constrained problems are treated and the optimal assumption for the obstacle is given to get the Lipschitz regularity for the solutions.

It is well known that, for unconstrained problems with (p,q)-growth, the boundedness of the minimizers can play a crucial role to get regularity for the gradient, under weaker assumptions on the gap between p and q and on the data of the problem (see [5]). Here, we will prove that the same phenomenon happens for the bounded solutions to obstacle problems with (p,q)-growth.

More precisely, we prove the following

Theorem 1.1. *Let $u \in \mathcal{K}_\psi(\Omega)$ be a solution to the obstacle problem (1.1) and let $A(x,\xi)$ satisfy the assumptions (A1)–(A4) with $2 \le p < q < \min\{p+1, p^* = \frac{np}{n-p}\}$. Then, if $\psi \in L^\infty_{\mathrm{loc}}(\Omega)$ the following implication holds*

$$D\psi \in W^{1,\frac{p+2}{p+2-q}}_{\mathrm{loc}}(\Omega) \implies \left(\mu^2 + |Du|^2\right)^{\frac{p-2}{4}} Du \in W^{1,2}_{\mathrm{loc}}(\Omega),$$

with the following estimate

$$\int_{B_{\frac{R}{4}}} |DV_p(Du(x))|^2 \,\mathrm{d}x \le \frac{c(\|\psi\|^2_{L^\infty} + \|u\|^2_{L^{p^*}(B_R)})}{R^{\frac{p+2}{2}}} \cdot \int_{B_R} \left[1 + \left|D^2\psi(x)\right|^{\frac{p+2}{p+2-q}} + |D\psi(x)|^{\frac{p+2}{p+2-q}} + \kappa^{\frac{p+2}{p-q+1}} + |Du(x)|^p\right] \mathrm{d}x. \tag{1.3}$$

We first observe that the assumption of boundedness of the obstacle ψ is needed to get the boundedness of the solutions (see Theorem 2.4). Therefore, if we want to remove the hypothesis $\psi \in L^\infty$, it is sufficient to deal with a priori bounded minimizers. In this case, we can remove also the hypothesis $q < p^*$.

Let us compare, now, our result with the previous ones. All previous higher regularity results for solutions to obstacle problem in case of non-standard growth have been obtained under a Sobolev assumption $W^{1,r}(\Omega)$ with $r \ge n$ on the dependence on x of the operator A, some of them reveal also crucial to prove local Lipschitz results for the obstacle problem, see for instance in [7,14]. Dealing with bounded solutions, we are able to prove our result assuming that the partial map $x \mapsto A(x;\xi)$ belongs to a Sobolev class that is not related to the dimension n but to the ellipticity and the growth exponents p and q of the functional and this assumption in case $\frac{p+2}{p-q+1} < n$ (i.e. $p < n-2$ and $q < \frac{n-1}{n}p + \frac{n-2}{n}$) improves the higher differentiability result obtained in [26]. Moreover, our result is obtained under a weaker assumption also on the gradient of the obstacle, indeed previous result assumed $\psi \in W^{1,2q-p}$ (see [26]) while our hypothesis is $\psi \in W^{1,\frac{p+2}{p+2-q}}$, and under our assumption on the gap, i.e. $q - p < 1$, it results $W^{1,2q-p} \hookrightarrow W^{1,\frac{p+2}{p+2-q}}$.

Note that for $p = q$ we recover exactly our previous result [8] concerning the obstacle problem with standard growth.

On the other hand, our result extends to the solutions of constrained problems the higher differentiability result obtained in [12] for the solutions to unconstrained problems in case of the integrand f is uniformly convex only at infinity.

To prove Theorem 1.1, we first verify the validity of the variational inequality also in the case of non standard growth and then we combine an a priori estimate for the second derivatives of the local solutions, obtained using the difference quotient method, with a suitable approximation argument. The local boundedness of the obstacle, and then of the solutions, allows us to use two interpolation inequalities that give the higher local integrability $L^{\frac{2(p+2)}{p+2-q}}$ for the gradient of the obstacle and the higher local integrability L^{p+2} for the gradient of the solutions. Such higher integrability is the key tool to weaken the assumption on κ that is the function that control the dependence on x-variable of the operator $\mathcal{A}$.

We conclude observing that, if the minimizer u is assumed a priori in a Lebesgue space L^r with $r > \frac{np}{n-p-2}$ instead of assuming $u \in L^\infty$ the interpolation inequality of Lemma 2.1 still gives a higher integrability result for Du, i.e. $Du \in L^{\frac{r}{r+2}(p+2)}$. Such higher integrability allows us to obtain the same higher differentiability result of Theorem 1.1 assuming $\kappa \in L^{\frac{r}{(r-p)}\frac{(p+2)}{p-q+1}}$. We'd like to point out that for $p < n-2$ and $q < \frac{1}{n}(n - \frac{r}{r-p})p + \frac{1}{n}(n - 2\frac{r}{r-p})$ we get $\frac{r}{(r-p)}\frac{(p+2)}{p-q+1} < n$ that means that we obtain the regularity result again under a Sobolev assumption on the dependence on the x-variable below the critical one $W^{1,n}$.

2. Notations and Preliminary Results

In this paper we shall denote by C or c a general constant that may vary on different occasions, even within the same line of estimates. Relevant dependencies on parameters and special constants will be suitably emphasized using parentheses or subscripts. With the symbol $B(x,r) = B_r(x) = \{y \in \mathbb{R}^n : |y-x| < r\}$ we will denote the ball centered at x of radius r. We shall omit the dependence on the center when no confusion arises.

Here we recall some results that will be useful in the following.

The main tools in the proof of Theorem 1.1 are the following Gagliardo-Nirenberg-type inequalities that we state as lemmas. The proofs of inequalities (2.1) and (2.2) can be found in [5, Appendix A]. For the proof of (2.3) see for example [48].

Lemma 2.1. *For any $\phi \in C_0^1(\Omega)$ with $\phi \geq 0$, and any C^2 map $v : \Omega \to \mathbb{R}^N$, we have*

$$\int_\Omega \phi^{\frac{m}{m+1}(p+2)}(x)|Dv(x)|^{\frac{m}{m+1}(p+2)}\mathrm{d}x$$
$$\leq (p+2)^2 \left(\int_\Omega \phi^{\frac{m}{m+1}(p+2)}(x)|v(x)|^{2m}\mathrm{d}x\right)^{\frac{1}{m+1}}$$

$$\cdot \left[\left(\int_\Omega \phi^{\frac{m}{m+1}(p+2)}(x) \left| D\phi(x) \right|^2 \left| Dv(x) \right|^p \mathrm{d}x \right)^{\frac{m}{m+1}} \right.$$
$$\left. + n \left(\int_\Omega \phi^{\frac{m}{m+1}(p+2)}(x) \left| Dv(x) \right|^{p-2} \left| D^2 v(x) \right|^2 \mathrm{d}x \right)^{\frac{m}{m+1}} \right], \tag{2.1}$$

for any $p \in (1, \infty)$ *and* $m > 1$. *Moreover, for any* $\mu \in [0, 1]$

$$\begin{aligned} &\int_\Omega \phi^2(x) \left(\mu^2 + |Dv(x)|^2 \right)^{\frac{p}{2}} |Dv(x)|^2 \mathrm{d}x \\ &\leq c \|v\|^2_{L^\infty(\mathrm{supp}(\phi))} \int_\Omega \phi^2(x) \left(\mu^2 + |Dv(x)|^2 \right)^{\frac{p-2}{2}} \left| D^2 v(x) \right|^2 \mathrm{d}x \\ &\quad + c \|v\|^2_{L^\infty(\mathrm{supp}(\phi))} \int_\Omega \left(\phi^2(x) + |D\phi(x)|^2 \right) \left(\mu^2 + |Dv(x)|^2 \right)^{\frac{p}{2}} \mathrm{d}x, \end{aligned} \tag{2.2}$$

for a constant $c = c(p)$.

Lemma 2.2. *Let* $u \in L^p(\Omega) \cap W^{2,r}(\Omega)$ *with* $1 \leq p \leq \infty$ *and* $1 \leq r \leq \infty$. *Then* $u \in W^{1,q}(\Omega)$ *where* q *is such that* $\frac{1}{q} = \frac{1}{2} \left(\frac{1}{p} + \frac{1}{r} \right)$ *and*

$$\|Du\|_{L^q} \leq C \|u\|^{\frac{1}{2}}_{W^{2,r}} \|u\|^{\frac{1}{2}}_{L^p} \tag{2.3}$$

The following is an higher differentiability result to the solutions to (1.1) when the energy density function f satisfies standard growth conditions. The proof can be found in [8].

Theorem 2.3. *Let* $A(x, \xi)$ *satisfy the conditions* (A1)–(A4) *with* $p = q \geq 2$ *and let* $u \in \mathcal{K}_\psi(\Omega)$ *be a solution to the obstacle problem* (1.2). *Then, if* $\psi \in L^\infty_{\mathrm{loc}}(\Omega)$ *the following implication*

$$D\psi \in W^{1, \frac{p+2}{2}}_{\mathrm{loc}}(\Omega) \implies \left(\mu^2 + |Du|^2 \right)^{\frac{p-2}{4}} Du \in W^{1,2}_{\mathrm{loc}}(\Omega),$$

holds true.

Next result has been proved in [7, Theorem 1.1]

Theorem 2.4. *Let* u *in* $K_\psi(\Omega)$ *be a solution of* (1.1) *under the assumptions* (A1) and (A2) with $2 \leq p \leq q$ such that

$$\begin{array}{ll} p \leq q < p^* = \frac{np}{n-p} & \text{if } p < n \\ p \leq q < \infty & \text{if } p \geq n \end{array}$$

If the obstacle $\psi \in L^\infty_{\mathrm{loc}}(\Omega)$, then $u \in L^\infty_{\mathrm{loc}}(\Omega)$ and the following estimate

$$\sup_{B_{R/2}} |u| \leq \left[\sup_{B_R} |\psi| + \left(\int_{B_R} |u(x)|^{p^*} \mathrm{d}x \right) \right]^\gamma \tag{2.4}$$

holds for every ball $B_R \Subset \Omega$, for $\gamma(n, p, q) > 0$ and $c = c(\ell, \nu, p, q, n)$. We'd like to remark that in a very recent paper [16] the same result has been proved under sharp assumptions on the gap between p and q.

We will use the auxiliary function $V_p : \mathbb{R}^n \to \mathbb{R}^n$, defined as

$$V_p(\xi) := \left(\mu^2 + |\xi|^2\right)^{\frac{p-2}{4}} \xi, \tag{2.5}$$

for which the following estimates hold (see [34]).

Lemma 2.5. *Let $1 < p < \infty$. There is a constant $c = c(n,p) > 0$ such that*

$$c^{-1}\left(\mu^2 + |\xi|^2 + |\eta|^2\right)^{\frac{p-2}{2}} \le \frac{|V_p(\xi) - V_p(\eta)|^2}{|\xi - \eta|^2} \le c\left(\mu^2 + |\xi|^2 + |\eta|^2\right)^{\frac{p-2}{2}}, \tag{2.6}$$

for any $\xi, \eta \in \mathbb{R}^n$ and $\xi \neq \eta$. Moreover, for a C^2 function g, there is a constant $C(p)$ such that

$$C^{-1}\left|D^2 g\right|^2 \left(\mu^2 + |Dg|^2\right)^{\frac{p-2}{2}} \le |D\left(V_p(Dg)\right)|^2 \le C\left|D^2 g\right|^2 \left(\mu^2 + |Dg|^2\right)^{\frac{p-2}{2}}. \tag{2.7}$$

Now we state a well-known iteration lemma (the proof can be found for example in [34, Lemma 6.1]).

Lemma 2.6. (Iteration Lemma) *Let $h : [\rho, R] \to \mathbb{R}$ be a nonnegative bounded function, $0 < \theta < 1$, $A, B \ge 0$ and $\gamma > 0$. Assume that*

$$h(r) \le \theta h(s) + \frac{A}{(s-r)^\gamma} + B$$

for all $\rho \le r < s \le R_0 < R$. Then

$$h(\rho) \le \frac{cA}{(R_0 - \rho)^\gamma} + cB,$$

where $c = c(\theta, \gamma) > 0$.

2.1. Difference Quotient

To get the regularity of the solutions of the problem (1.1), we shall use the difference quotient method. We recall here the definition and basic results.

Definition 2.7. Given $h \in \mathbb{R}^n$, for every function $F : \mathbb{R}^n \to \mathbb{R}$ the finite difference operator is defined by

$$\tau_h F(x) = F(x+h) - F(x).$$

We recall some properties of the finite difference operator that will be needed in the sequel. We start with the description of some elementary properties that can be found, for example, in [34].

Proposition 2.8. *Let F and G be two functions such that $F, G \in W^{1,p}(\Omega)$, with $p \ge 1$, and let us consider the set*

$$\Omega_{|h|} := \left\{x \in \Omega : dist(x, \partial\Omega) > |h|\right\}.$$

Then

(d1) $\tau_h F \in W^{1,p}(\Omega_{|h|})$ *and*

$$D_i(\tau_h F) = \tau_h(D_i F).$$

(d2) *If at least one of the functions F or G has support contained in $\Omega_{|h|}$ then*

$$\int_\Omega F(x)\,\tau_h G(x)\,\mathrm{d}x = \int_\Omega G(x)\,\tau_{-h} F(x)\,\mathrm{d}x.$$

(d3) *We have*

$$\tau_h(FG)(x) = F(x+h)\tau_h G(x) + G(x)\tau_h F(x).$$

The next result about finite difference operator is a kind of integral version of Lagrange Theorem.

Lemma 2.9. *If $0 < \rho < R$, $|h| < \frac{R-\rho}{2}$, $1 < p < +\infty$, and $F, DF \in L^p(B_R)$ then*

$$\int_{B_\rho} |\tau_h F(x)|^p \,\mathrm{d}x \le c(n,p)|h|^p \int_{B_R} |DF(x)|^p \,\mathrm{d}x.$$

Moreover

$$\int_{B_\rho} |F(x+h)|^p \,\mathrm{d}x \le \int_{B_R} |F(x)|^p \,\mathrm{d}x.$$

We conclude this section recalling this result that is proved in [34].

Lemma 2.10. *Let $F : \mathbb{R}^n \to \mathbb{R}^N$, $F \in L^p(B_R)$ with $1 < p < +\infty$. Suppose that there exist $\rho \in (0, R)$ and $M > 0$ such that*

$$\sum_{s=1}^{n} \int_{B_\rho} |\tau_{s,h} F(x)|^p \mathrm{d}x \le M^p |h|^p$$

for every $h < \frac{R-\rho}{2}$. Then $F \in W^{1,p}(B_R, \mathbb{R}^N)$. Moreover

$$\|DF\|_{L^p(B_\rho)} \le M.$$

2.2. Approximation Lemma

We report a Lemma which will be the main tool in the second part of the proof of our main result. For the proof of this Lemma we refer to [13].

Lemma 2.11. *Let $f : \Omega \times \mathbb{R}^n \to [0, \infty)$ be a Carathéodory function such that $\xi \mapsto f(x,\xi)$ is $\mathcal{C}^2$ and there exists $\tilde{f} : \Omega \times [0,\infty) \to [0,\infty)$ such that $f(x,\xi) = \tilde{f}(x,|\xi|)$. Moreover, let us assume that f satisfies assumptions* (F1)–(F4). *Then there exists a sequence $(f_\varepsilon)_\varepsilon$ of Carathéodory functions $f_\varepsilon : \Omega \times \mathbb{R}^n \to [0,\infty)$, monotonically convergent to f, such that*

(i) *for a.e. $x \in \Omega$, for every $\xi \in \mathbb{R}^n$ and for every $\varepsilon_1 < \varepsilon_2$, we have*

$$f_{\varepsilon_2}(x,\xi) \le f_{\varepsilon_1}(x,\xi) \le f(x,\xi)$$

(ii) *there exists $\bar{\nu} > 0$ depending only on p and $\tilde{\nu}$ such that*

$$\langle D_{\xi\xi} f_\varepsilon(x,\xi)\lambda, \lambda\rangle \geqslant \bar{\nu}(\mu^2 + |\xi|^2)^{\frac{p-2}{2}} |\lambda|^2$$

for a.e. $x \in \Omega$, for every $\xi \in \mathbb{R}^n$,

(iii) *there exist* K_0, K_1 *independent of* ε *and* $\bar{K}_1$ *depending on* ε *such that*

$$K_0(|\xi|^p - \mu^2) \leq f_\varepsilon(x,\xi) \leq K_1\left[(\mu^2 + |\xi|^2)^{\frac{p}{2}} + (\mu^2 + |\xi|^2)^{\frac{q}{2}}\right],$$

$$f_\varepsilon(x,\xi) \leq \bar{K}_1(\varepsilon)(\mu^2 + |\xi|^2)^{\frac{p}{2}},$$

for a.e. $x \in \Omega$*, for every* $\xi \in \mathbb{R}^n$*,*

(iv) *there exists a constant* $C(\varepsilon) > 0$ *such that*

$$|D_{x\xi} f_\varepsilon(x,\xi)| \leq k(x)\left[(\mu^2 + |\xi|^2)^{\frac{p-1}{2}} + (\mu^2 + |\xi|^2)^{\frac{q-1}{2}}\right]$$

$$|D_{x\xi} f(x,\xi)| \leq C(\varepsilon) k(x)(\mu^2 + |\xi|^2)^{\frac{p-1}{2}}$$

for a.e. $x \in \Omega$*, for every* $\xi \in \mathbb{R}^n$*.*

3. Proof of the Theorem 1.1

The proof of the theorem is obtained in two steps: first we establish the a priori estimate and then we conclude through an approximation argument.

Proof. Step 1: The a priori estimate.

To get the a priori estimate we first need to prove the validity of the variational inequality (1.2) also in the case of non-standard growth conditions.

Suppose that u is a local solution to the obstacle problem in $\mathcal{K}_\psi(\Omega)$ such that

$$Du \in W^{1,2}_{\mathrm{loc}}(\Omega) \qquad \text{and} \qquad \left(\mu^2 + |Du|^2\right)^{\frac{p-2}{4}} Du \in W^{1,2}_{\mathrm{loc}}(\Omega). \tag{3.1}$$

Thanks to our assumptions on the exponents p and q we can deduce from Theorem 2.4 that the solution u to (1.1) is bounded. Such boundedness, with the a priori assumption (3.1) on the second derivatives of u, allows us to apply Lemma 2.1 to get the higher integrability $Du \in L^{p+2}_{\mathrm{loc}}(\Omega)$.

Concerning the obstacle ψ, by the assumptions $\psi \in L^\infty(\Omega)$ and $D^2\psi \in L^{\frac{p+2}{p+2-q}}(\Omega)$, applying Lemma 2.2, we have $D\psi \in L^{\frac{2(p+2)}{p+2-q}}(\Omega) \hookrightarrow L^{p+2}(\Omega)$.

Note that $Du \in L^{p+2}_{\mathrm{loc}}(\Omega)$ (and then, obviously, $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$) implies that the variational inequality (1.2), by a simple density argument, holds true for every $\varphi \in W^{1,q}_{\mathrm{loc}}(\Omega)$.

Indeed, since $u \in \mathcal{K}_\psi(\Omega)$, for every $v \geq 0$ and every $\varepsilon > 0$ it results $u + \varepsilon v \geq \psi$, therefore if $v \in W^{1,q}_{\mathrm{loc}}(\Omega)$ by minimality of u

$$\int_\Omega f(x, Du(x))\, \mathrm{d}x \leq \int_\Omega f(x, Du + \varepsilon Dv(x))\, \mathrm{d}x$$

or equivalently

$$\int_\Omega \Big[f(x, Du(x) + \varepsilon Dv(x)) - f(x, Du(x))\Big]\, \mathrm{d}x \geq 0.$$

Hence, we have

$$\varepsilon \int_\Omega \int_0^1 \langle D_\xi f(x, Du(x) + \theta\varepsilon Dv(x)), Dv(x)\rangle\, \mathrm{d}\theta\, \mathrm{d}x \geq 0$$

and also

$$\int_\Omega \int_0^1 \langle D_\xi f(x, Du(x) + \theta\varepsilon Dv(x)), Dv(x)\rangle \, \mathrm{d}\theta \, \mathrm{d}x \geq 0$$

where we divided both side of previous inequality by ε. We observe that

$$\begin{aligned} 0 &\leq \int_\Omega \int_0^1 \langle D_\xi f(x, Du(x) + \theta\varepsilon Dv), Dv\rangle \, \mathrm{d}\theta \, \mathrm{d}x \\ &\leq \int_\Omega \int_0^1 |D_\xi f(x, Du + \theta\varepsilon Dv(x))||Dv(x)| \, \mathrm{d}\theta \, \mathrm{d}x \\ &\leq \int_\Omega \int_0^1 (\mu^2 + |Du + \theta\varepsilon Dv(x)|^2)^{\frac{q-1}{2}} |Dv(x)| \, \mathrm{d}\theta \, \mathrm{d}x \\ &\leq c \int_\Omega (\mu^2 + |Du(x)|^2 + \varepsilon^2 |Dv(x)|^2)^{\frac{q-1}{2}} |Dv(x)| \, \mathrm{d}x, \end{aligned} \tag{3.2}$$

where in the last inequality we used Lemma 8.3 in [34].

Therefore, since $v \in W^{1,q}_{\mathrm{loc}}(\Omega)$, by the growth assumption (A3), assuming without loss of generality $\varepsilon < 1$, we get

$$\int_0^1 \langle D_\xi f(x, Du(x) + \theta\varepsilon Dv(x)), Dv(x)\rangle \, \mathrm{d}\theta \leq \mu^q + |Du|^q + |Dv|^q \in L^1(\Omega).$$

Then, applying dominated convergence theorem in (3.2), we have

$$\begin{aligned} &\lim_{\varepsilon \to 0} \int_\Omega \int_0^1 \langle D_\xi f(x, Du(x) + \theta\varepsilon Dv(x)), Dv(x)\rangle \, d\theta \, \mathrm{d}x \\ &\quad = \int_\Omega \langle D_\xi f(x, Du(x)), Dv(x)\rangle \, \mathrm{d}x \geq 0 \end{aligned}$$

for every $v \in W^{1,q}_0(\Omega)$, $v \geq 0$. At this point it is standard to verify the inequality (1.2)

$$\int_\Omega \langle D_\xi f(x, Du(x)), D\varphi(x) - Du(x)\rangle \, \mathrm{d}x \geq 0.$$

Now we have to choose suitable test functions φ in (1.2) that involve the different quotient of the solution and at the same time satisfy the conditions $\varphi \in W^{1,q}_{\mathrm{loc}}(\Omega)$ and $\varphi \geq \psi$ in Ω. To do this, we proceed similarly to what has been done in [8,18].

Let us fix a ball $B_R \Subset \Omega$ and arbitrary radii $\frac{R}{2} < r < s < t < \lambda r < R$, with $1 < \lambda < 2$. Let us consider a cut off function $\eta \in C^\infty_0(B_t)$ such that $\eta \equiv 1$ on B_s and $|D\eta| \leq \frac{c}{t-s}$. From now on, with no loss of generality, we suppose $R < 1$.

Let $v \in W^{1,q}_0(\Omega)$ be such that

$$u - \psi + \tau v \geq 0 \qquad \forall \tau \in [0,1], \tag{3.3}$$

and observe that $\varphi = u + \tau v \geq \psi$ for all $\tau \in [0,1]$. For $|h| < \frac{R}{4}$, we consider

$$v_1(x) = \eta^2(x) \left[(u - \psi)(x + h) - (u - \psi)(x)\right],$$

so we have $v_1 \in W_0^{1,p+2}(\Omega)$, and, for any $\tau \in [0,1]$, v_1 satisfies (3.3). Indeed, for a.e. $x \in \Omega$ and for any $\tau \in [0,1]$

$$\begin{aligned}u(x)-\psi(x)+\tau v_1(x) &= u(x)-\psi(x)+\tau\eta^2(x)\left[(u-\psi)(x+h)-(u-\psi)(x)\right]\\ &= \tau\eta^2(x)(u-\psi)(x+h)+(1-\tau\eta^2(x))(u-\psi)(x)\geq 0,\end{aligned}$$

since $u \in \mathcal{K}_\psi(\Omega)$ and $0 \leq \eta \leq 1$. Therefore, from $q-p<1$ we have $L^{p+2}(\Omega) \hookrightarrow L^q(\Omega)$ and so we can use $\varphi = u+\tau v_1$ as a test function in inequality (1.2), thus getting

$$0 \leq \int_\Omega \langle A(x,Du(x)), D\left[\eta^2(x)\left[(u-\psi)(x+h)-(u-\psi)(x)\right]\right]\rangle \mathrm{d}x. \quad (3.4)$$

Similarly, we define

$$v_2(x) = \eta^2(x-h)\left[(u-\psi)(x-h)-(u-\psi)(x)\right],$$

and we have $v_2 \in W_0^{1,p+2}(\Omega)$, the inequality (3.3) still is satisfied for any $\tau \in [0,1]$, and we can use $\varphi = u+\tau v_2$ as test function in (1.2), obtaining

$$0 \leq \int_\Omega \langle A(x,Du(x)), D\left[\eta^2(x-h)\left[(u-\psi)(x-h)-(u-\psi)(x)\right]\right]\rangle \mathrm{d}x,$$

and by means of a change of variable, we have

$$0 \leq \int_\Omega \langle A(x+h,Du(x+h)), D\left[\eta^2(x)\left[(u-\psi)(x)-(u-\psi)(x+h)\right]\right]\rangle \mathrm{d}x. \quad (3.5)$$

Now we can add (3.4) and (3.5), thus getting

$$\begin{aligned}0 \leq &\int_\Omega \langle A(x,Du(x)), D\left[\eta^2(x)\left[(u-\psi)(x+h)-(u-\psi)(x)\right]\right]\rangle \mathrm{d}x\\ &+\int_\Omega \langle A(x+h,Du(x+h)), D\left[\eta^2(x)\left[(u-\psi)(x)-(u-\psi)(x+h)\right]\right]\rangle \mathrm{d}x,\end{aligned}$$

that is

$$\begin{aligned}0 \leq \int_\Omega &\langle A(x,Du(x))-A(x+h,Du(x+h)),\\ &D\left[\eta^2(x)\left[(u-\psi)(x+h)-(u-\psi)(x)\right]\right]\rangle \mathrm{d}x,\end{aligned}$$

which implies

$$\begin{aligned}0 \geq &\int_\Omega \langle A(x+h,Du(x+h))-A(x,Du(x)), \eta^2(x)D\left[(u-\psi)(x+h)-(u-\psi)(x)\right]\rangle \mathrm{d}x\\ &+\int_\Omega \langle A(x+h,Du(x+h))-A(x,Du(x)), 2\eta(x)D\eta(x)\\ &\left[(u-\psi)(x+h)-(u-\psi)(x)\right]\rangle \mathrm{d}x.\end{aligned}$$

Previous inequality can be rewritten as follows

$$\begin{aligned}0 \geq &\int_\Omega \langle A(x+h,Du(x+h))-A(x+h,Du(x)), \eta^2(x)(Du(x+h)-Du(x))\rangle \mathrm{d}x\\ &-\int_\Omega \langle A(x+h,Du(x+h))-A(x+h,Du(x)), \eta^2(x)(D\psi(x+h)-D\psi(x))\rangle \mathrm{d}x\\ &+\int_\Omega \langle A(x+h,Du(x+h))-A(x+h,Du(x)), 2\eta(x)D\eta(x)\tau_h\left(u-\psi\right)(x)\rangle \mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&+\int_\Omega \langle A(x+h,Du(x))-A(x,Du(x)),\eta^2(x)(Du(x+h)-Du(x))\rangle \mathrm{d}x\\
&-\int_\Omega \langle A(x+h,Du(x))-A(x,Du(x)),\eta^2(x)(D\psi(x+h)-D\psi(x))\rangle \mathrm{d}x\\
&+\int_\Omega \langle A(x+h,Du(x))-A(x,Du(x)),2\eta(x)D\eta(x)\tau_h\left(u-\psi\right)(x)\rangle \mathrm{d}x\\
=:&\ I+II+III+IV+V+VI,
\end{aligned}
\tag{3.6}
$$

so we have

$$
I\le |II|+|III|+|IV|+|V|+|VI|. \tag{3.7}
$$

The ellipticity assumption (A1) implies

$$
I\ge \nu\int_\Omega \eta^2(x)|\tau_h Du(x)|^2\left(\mu^2+|Du(x+h)|^2+|Du(x)|^2\right)^{\frac{p-2}{2}}\mathrm{d}x. \tag{3.8}
$$

By virtue of assumption (A2), we have

$$
\begin{aligned}
|II|\le &\ L\int_\Omega \eta^2(x)|\tau_h Du(x)|\Big[(\mu^2+|Du(x)|^2+|Du(x+h)|^2)^{\frac{p-2}{2}}\\
&+(\mu^2+|Du(x)|^2+|Du(x+h)|^2)^{\frac{q-2}{2}}\Big]|\tau_h D\psi(x)|\mathrm{d}x\\
=&\ L\int_\Omega \eta^2(x)|\tau_h Du(x)|(\mu^2+|Du(x)|^2+|Du(x+h)|^2)^{\frac{p-2}{2}}|\tau_h D\psi(x)|\mathrm{d}x\\
&+L\int_\Omega \eta^2(x)|\tau_h Du(x)|(\mu^2+|Du(x)|^2+|Du(x+h)|^2)^{\frac{q-2}{2}}|\tau_h D\psi(x)|\mathrm{d}x\\
=:&\ II_1+II_2.
\end{aligned}
\tag{3.9}
$$

Let us consider the term II_1. If we apply Young's inequality with exponents $(2,2)$ and Hölder's inequality with exponents $\left(\frac{p+2}{4},\frac{p+2}{p-2}\right)$, we get

$$
\begin{aligned}
II_1\le &\ \varepsilon\int_\Omega \eta^2(x)|\tau_h Du(x)|^2(\mu^2+|Du(x)|^2+|Du(x+h)|^2)^{\frac{p-2}{2}}\mathrm{d}x\\
&+c_\varepsilon\int_\Omega \eta^2(x)|\tau_h D\psi(x)|^2(\mu^2+|Du(x)|^2+|Du(x+h)|^2)^{\frac{p-2}{2}}\mathrm{d}x\\
\le &\ \varepsilon\int_\Omega \eta^2(x)|\tau_h Du(x)|^2(\mu^2+|Du(x)|^2+|Du(x+h)|^2)^{\frac{p-2}{2}}\mathrm{d}x\\
&+c_\varepsilon\left(\int_{B_t}|\tau_h D\psi(x)|^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{4}{p+2}}\\
&\cdot\left(\int_{B_t}(\mu^2+|Du(x)|^2+|Du(x+h)|^2)^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{p-2}{p+2}},
\end{aligned}
$$

where we also used the properties of η. Since $D\psi\in W^{1,\frac{p+2}{p+2-q}}_{\mathrm{loc}}(\Omega)$ and $2\le p<q<p+1$, we also have $D\psi\in W^{1,\frac{p+2}{2}}_{\mathrm{loc}}(\Omega)$, and using both estimates of Lemma 2.9, we get

$$
II_1\le \varepsilon\int_\Omega \eta^2(x)|\tau_h Du(x)|^2(\mu^2+|Du(x)|^2+|Du(x+h)|^2)^{\frac{p-2}{2}}\mathrm{d}x
$$

$$+c_\varepsilon|h|^2\left(\int_{B_{\lambda r}}|D^2\psi(x)|^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{4}{p+2}}\cdot\left(\int_{B_{\lambda r}}(\mu^2+|Du(x)|^2)^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{p-2}{p+2}}. \tag{3.10}$$

To estimate the term II_2, applying Young's inequality with exponents $(2,2)$ and Hölder's inequality with exponents $\left(\frac{p+2}{2(p+2-q)},\frac{p+2}{2q-p-2}\right)$, we get

$$\begin{aligned}II_2 &\le \varepsilon\int_\Omega \eta^2(x)|\tau_h Du(x)|^2(\mu^2+|Du(x)|^2+|Du(x+h)|^2)^{\frac{p-2}{2}}\mathrm{d}x\\&\quad+c_\varepsilon\int_\Omega \eta^2(x)|\tau_h D\psi(x)|^2(\mu^2+|Du(x)|^2+|Du(x+h)|^2)^{\frac{2q-p-2}{2}}\mathrm{d}x\\&\le \varepsilon\int_\Omega \eta^2(x)|\tau_h Du(x)|^2(\mu^2+|Du(x)|^2+|Du(x+h)|^2)^{\frac{p-2}{2}}\mathrm{d}x\\&\quad+c_\varepsilon\left(\int_{B_t}|\tau_h D\psi(x)|^{\frac{p+2}{p+2-q}}\mathrm{d}x\right)^{\frac{2(p+2)-2q}{p+2}}\\&\quad\cdot\left(\int_{B_t}(\mu^2+|Du(x)|^2+|Du(x+h)|^2)^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{2q-p-2}{p+2}}\end{aligned}$$

where we used also the properties of η. Since $D\psi\in W^{1,\frac{p+2}{p+2-q}}_{\mathrm{loc}}(\Omega)$, we may use the first and the second estimate of Lemma 2.9, thus obtaining

$$\begin{aligned}II_2 &\le \varepsilon\int_\Omega \eta^2(x)|\tau_h Du(x)|^2(\mu^2+|Du(x)|^2+|Du(x+h)|^2)^{\frac{p-2}{2}}\mathrm{d}x\\&\quad+c_\varepsilon|h|^2\left(\int_{B_{\lambda r}}|D^2\psi(x)|^{\frac{p+2}{p+2-q}}\mathrm{d}x\right)^{\frac{2(p+2-q)}{p+2}}\\&\quad\cdot\left(\int_{B_{\lambda r}}(\mu^2+|Du(x)|^2)^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{2q-p-2}{p+2}}.\end{aligned}\tag{3.11}$$

Plugging (3.10) and (3.11) into (3.9), we get

$$\begin{aligned}|II| &\le 2\varepsilon\int_\Omega \eta^2(x)|\tau_h Du(x)|^2(\mu^2+|Du(x)|^2+|Du(x+h)|^2)^{\frac{p-2}{2}}\mathrm{d}x\\&\quad+c_\varepsilon|h|^2\left(\int_{B_{\lambda r}}|D^2\psi(x)|^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{4}{p+2}}\cdot\left(\int_{B_{\lambda r}}(\mu^2+|Du(x)|^2)^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{p-2}{p+2}}\\&\quad+c_\varepsilon|h|^2\left(\int_{B_{\lambda r}}|D^2\psi(x)|^{\frac{p+2}{p+2-q}}\mathrm{d}x\right)^{\frac{2(p+2-q)}{p+2}}\\&\quad\cdot\left(\int_{B_{\lambda r}}(\mu^2+|Du(x)|^2)^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{2q-p-2}{p+2}}.\end{aligned}\tag{3.12}$$

Arguing analogously, by virtue of assumption (A2) we have

$$\begin{aligned}|III| \le 2L\int_\Omega \eta(x)|D\eta(x)||\tau_h Du(x)|&\left[\left(\mu^2+|Du(x+h)|^2+|Du(x)|^2\right)^{\frac{p-2}{2}}\right.\\&\left.+\left(\mu^2+|Du(x+h)|^2+|Du(x)|^2\right)^{\frac{q-2}{2}}\right]|\tau_h\left(u-\psi\right)(x)|\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&= c\int_\Omega \eta(x)|D\eta(x)||\tau_h Du(x)|\left(\mu^2+|Du(x+h)|^2+|Du(x)|^2\right)^{\frac{p-2}{2}}\\
&\qquad |\tau_h\left(u-\psi\right)(x)|\mathrm{d}x\\
&\quad+\int_\Omega \eta(x)|D\eta(x)||\tau_h Du(x)|\left(\mu^2+|Du(x+h)|^2+|Du(x)|^2\right)^{\frac{q-2}{2}}\\
&\qquad |\tau_h\left(u-\psi\right)(x)|\mathrm{d}x\\
&:= III_1+III_2. \qquad (3.13)
\end{aligned}
$$

Using Young's inequality with exponents $(2,2)$, Hölder's inequality with exponents $\left(\frac{p+2}{4},\frac{p+2}{p-2}\right)$, and the properties of η, we have

$$
\begin{aligned}
III_1 &\le \varepsilon\int_\Omega \eta^2(x)|\tau_h Du(x)|^2(\mu^2+|Du(x)|^2+|Du(x+h)|^2)^{\frac{p-2}{2}}\mathrm{d}x\\
&\quad+c_\varepsilon\int_\Omega |\tau_h(u-\psi)(x)|^2|D\eta(x)|^2(\mu^2+|Du(x)|^2+|Du(x+h)|^2)^{\frac{p-2}{2}}\mathrm{d}x\\
&\le \varepsilon\int_\Omega \eta^2(x)|\tau_h Du(x)|^2(\mu^2+|Du(x)|^2+|Du(x+h)|^2)^{\frac{p-2}{2}}\mathrm{d}x\\
&\quad+\frac{c_\varepsilon}{(t-s)^2}\left(\int_{B_t}|\tau_h(u-\psi)(x)|^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{4}{p+2}}\\
&\quad\cdot\left(\int_{B_t}(\mu^2+|Du(x)|^2+|Du(x+h)|^2)^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{p-2}{p+2}},
\end{aligned}
$$

and Lemma 2.9 implies

$$
\begin{aligned}
III_1 &\le \varepsilon\int_\Omega \eta^2(x)|\tau_h Du(x)|^2(\mu^2+|Du(x)|^2+|Du(x+h)|^2)^{\frac{p-2}{2}}\mathrm{d}x\\
&\quad+\frac{c_\varepsilon|h|^2}{(t-s)^2}\left(\int_{B_{\lambda r}}|D(u-\psi)(x)|^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{4}{p+2}}\\
&\quad\cdot\left(\int_{B_{\lambda r}}(\mu^2+|Du(x)|^2)^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{p-2}{p+2}}. \qquad (3.14)
\end{aligned}
$$

Similarly, using Young's inequality with exponents $(2,2)$, Hölder's inequality with exponents $\left(\frac{p+2}{2(p+2-q)},\frac{p+2}{2q-p-2}\right)$, the properties of η and Lemma 2.9, we get

$$
\begin{aligned}
III_2 &\le \varepsilon\int_\Omega \eta^2(x)|\tau_h Du(x)|^2(\mu^2+|Du(x)|^2+|Du(x+h)|^2)^{\frac{p-2}{2}}\mathrm{d}x\\
&\quad+\frac{c_\varepsilon|h|^2}{(t-s)^2}\left(\int_{B_{\lambda r}}|D(u-\psi)|^{\frac{p+2}{p+2-q}}\mathrm{d}x\right)^{\frac{2(p+2)-2q}{p+2}}\\
&\quad\cdot\left(\int_{B_{\lambda r}}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)\mathrm{d}x\right)^{\frac{2q-p-2}{p+2}}. \qquad (3.15)
\end{aligned}
$$

Plugging (3.14) and (3.15) into (3.13), we get

$$
|III|\le 2\varepsilon\int_\Omega \eta^2(x)|\tau_h Du(x)|^2(\mu^2+|Du(x)|^2+|Du(x+h)|^2)^{\frac{p-2}{2}}\mathrm{d}x
$$

$$
\begin{aligned}
&+\frac{c_\varepsilon|h|^2}{(t-s)^2}\left(\int_{B_{\lambda r}}|D(u-\psi)(x)|^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{4}{p+2}}\\
&\cdot\left(\int_{B_{\lambda r}}(\mu^2+|Du(x)|^2)^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{p-2}{p+2}}\\
&+\frac{c_\varepsilon|h|^2}{(t-s)^2}\left(\int_{B_{\lambda r}}|D(u-\psi)|^{\frac{p+2}{p+2-q}}\mathrm{d}x\right)^{\frac{2(p+2)-2q}{p+2}}\\
&\cdot\left(\int_{B_{\lambda r}}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)\mathrm{d}x\right)^{\frac{2q-p-2}{p+2}}. \qquad (3.16)
\end{aligned}
$$

For what concerns the term IV, assumption (A4) implies

$$
\begin{aligned}
|IV| \leq\ & |h|\int_\Omega \eta^2(x)\left(\kappa(x+h)+\kappa(x)\right)\\
&\left[\left(\mu^2+|Du(x)|^2\right)^{\frac{p-1}{2}}+\left(\mu^2+|Du(x)|^2\right)^{\frac{q-1}{2}}\right]|\tau_h Du(x)|\mathrm{d}x\\
=\ & |h|\int_\Omega \eta^2(x)\left(\kappa(x+h)+\kappa(x)\right)\left(\mu^2+|Du(x)|^2\right)^{\frac{p-1}{2}}|\tau_h Du(x)|\mathrm{d}x\\
&+|h|\int_\Omega \eta^2(x)\left(\kappa(x+h)+\kappa(x)\right)\left(\mu^2+|Du(x)|^2\right)^{\frac{q-1}{2}}|\tau_h Du(x)|\mathrm{d}x\\
=:\ & IV_1+IV_2. \qquad (3.17)
\end{aligned}
$$

If we use Young's inequality with exponents $(2,2)$ and the properties of η, we obtain

$$
\begin{aligned}
IV_2 &\leq |h|\int_\Omega \eta^2(x)\left(\kappa(x+h)+\kappa(x)\right)\left(\mu^2+|Du(x)|^2\right)^{\frac{q-1}{2}}|\tau_h Du(x)|\mathrm{d}x\\
&\leq \varepsilon\int_\Omega \eta^2(x)\,|\tau_h Du(x)|^2\left(\mu^2+|Du(x+h)|^2+|Du(x)|^2\right)^{\frac{p-2}{2}}\mathrm{d}x\\
&\quad + c_\varepsilon|h|^2\int_{B_t}\left(\kappa(x+h)+\kappa(x)\right)^2\left(\mu^2+|Du(x)|^2\right)^{\frac{2q-p}{2}}\mathrm{d}x.
\end{aligned}
$$

Using Hölder's inequality with exponents $\left(\frac{p+2}{2(p-q+1)},\frac{p+2}{2q-p}\right)$ and Lemma 2.9 we have

$$
\begin{aligned}
IV_2 &\leq \varepsilon\int_\Omega \eta^2(x)\,|\tau_h Du(x)|^2\left(\mu^2+|Du(x+h)|^2+|Du(x)|^2\right)^{\frac{p-2}{2}}\mathrm{d}x\\
&\quad +c_\varepsilon|h|^2\left(\int_{B_{\lambda r}}\kappa^{\frac{p+2}{p-q+1}}(x)\mathrm{d}x\right)^{\frac{2p-2q+2}{p+2}}\cdot\left(\int_{B_t}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)\mathrm{d}x\right)^{\frac{2q-p}{p+2}}. \qquad (3.18)
\end{aligned}
$$

Analogously, since $\kappa\in L^{\frac{p+2}{p-q+1}}_{\mathrm{loc}}(\Omega)\hookrightarrow L^{p+2}_{\mathrm{loc}}(\Omega)$, using Young's inequality with exponents $(2,2)$ Hölder's inequality with exponents $\left(\frac{p+2}{2},\frac{p+2}{p}\right)$, the properties of η and Lemma 2.9, we get

$$
IV_1 \leq \varepsilon\int_\Omega \eta^2(x)\,|\tau_h Du(x)|^2\left(\mu^2+|Du(x+h)|^2+|Du(x)|^2\right)^{\frac{p-2}{2}}\mathrm{d}x
$$

$$+c_\varepsilon|h|^2\left(\int_{B_{\lambda r}}\kappa^{p+2}(x)\mathrm{d}x\right)^{\frac{2}{p+2}}\cdot\left(\int_{B_t}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)\mathrm{d}x\right)^{\frac{p}{p+2}}. \tag{3.19}$$

Plugging (3.18) and (3.19) into (3.17), we obtain

$$\begin{aligned}|IV|\le{}& 2\varepsilon\int_\Omega\eta^2(x)\left|\tau_hDu(x)\right|^2\left(\mu^2+|Du(x+h)|^2+|Du(x)|^2\right)^{\frac{p-2}{2}}\mathrm{d}x\\&+c_\varepsilon|h|^2\left(\int_{B_{\lambda r}}\kappa^{\frac{p+2}{p-q+1}}(x)\mathrm{d}x\right)^{\frac{2p-2q+2}{p+2}}\cdot\left(\int_{B_t}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)\mathrm{d}x\right)^{\frac{2q-p}{p+2}}\\&+c_\varepsilon|h|^2\left(\int_{B_{\lambda r}}\kappa^{p+2}(x)\mathrm{d}x\right)^{\frac{2}{p+2}}\cdot\left(\int_{B_t}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)\mathrm{d}x\right)^{\frac{p}{p+2}}.\end{aligned} \tag{3.20}$$

The condition (A4) also entails

$$\begin{aligned}|V|\le{}&|h|\int_\Omega\eta^2(x)\left(\kappa(x+h)+\kappa(x)\right)\left(\mu^2+|Du(x)|^2\right)^{\frac{p-1}{2}}\left|\tau_hD\psi(x)\right|\mathrm{d}x\\&+|h|\int_\Omega\eta^2(x)\left(\kappa(x+h)+\kappa(x)\right)\left(\mu^2+|Du(x)|^2\right)^{\frac{q-1}{2}}\left|\tau_hD\psi(x)\right|\mathrm{d}x\\\le{}&|h|\left(\int_{B_t}\left(\kappa(x+h)+\kappa(x)\right)^{p+2}\mathrm{d}x\right)^{\frac{1}{p+2}}\cdot\left(\int_{B_t}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)dx\right)^{\frac{p-1}{p+2}}\\&\cdot\left(\int_{B_t}\left|\tau_hD\psi(x)\right|^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{2}{p+2}}\\&+|h|\left(\int_{B_t}\left(\kappa(x+h)+\kappa(x)\right)^{\frac{p+2}{p-q+1}}\mathrm{d}x\right)^{\frac{p-q+1}{p+2}}\cdot\left(\int_{B_t}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)dx\right)^{\frac{q-1}{p+2}}\\&\cdot\left(\int_{B_t}\left|\tau_hD\psi(x)\right|^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{2}{p+2}}\\\le{}&c|h|^2\left(\int_{B_{\lambda r}}\kappa^{p+2}(x)\mathrm{d}x\right)^{\frac{1}{p+2}}\cdot\left(\int_{B_t}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)\mathrm{d}x\right)^{\frac{p-1}{p+2}}\\&\cdot\left(\int_{B_{\lambda r}}\left|D^2\psi(x)\right|^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{2}{p+2}}\\&+c|h|^2\left(\int_{B_{\lambda r}}\kappa^{\frac{p+2}{p-q+1}}(x)\mathrm{d}x\right)^{\frac{p-q+1}{p+2}}\cdot\left(\int_{B_t}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)\mathrm{d}x\right)^{\frac{q-1}{p+2}}\\&\cdot\left(\int_{B_{\lambda r}}\left|D^2\psi(x)\right|^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{2}{p+2}}\end{aligned} \tag{3.21}$$

where we used Hölder's inequality with exponents $\left(p+2,\frac{p+2}{p-1},\frac{p+2}{2}\right)$ and $\left(\frac{p+2}{p-q+1},\frac{p+2}{q-1},\frac{p+2}{2}\right)$, the properties of η and Lemma 2.9.

Finally, using again assumption (A4), the properties of η, Hölder's inequality and Lemma 2.9, we have

$$\begin{aligned}|VI|\le{}&2|h|\int_\Omega\eta(x)\left|D\eta(x)\right|\left(\kappa(x+h)+\kappa(x)\right)\left(\mu^2+|Du(x)|^2\right)^{\frac{p-1}{2}}\left|\tau_h\left(u-\psi\right)(x)\right|\mathrm{d}x\\&+2|h|\int_\Omega\eta(x)\left|D\eta(x)\right|\left(\kappa(x+h)+\kappa(x)\right)\left(\mu^2+|Du(x)|^2\right)^{\frac{q-1}{2}}\left|\tau_h\left(u-\psi\right)(x)\right|\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{c|h|}{t-s}\left(\int_{B_t}(\kappa(x+h)+\kappa(x))^{p+2}\,\mathrm{d}x\right)^{\frac{1}{p+2}}\\
&\quad\cdot\left(\int_{B_t}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)\mathrm{d}x\right)^{\frac{p-1}{p+2}}\\
&\quad\cdot\left(\int_{B_t}\left|\tau_h\left(u-\psi\right)(x)\right|^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{2}{p+2}}\\
&\quad+\frac{c|h|}{t-s}\left(\int_{B_t}(\kappa(x+h)+\kappa(x))^{\frac{p+2}{p-q+1}}\,\mathrm{d}x\right)^{\frac{p-q+1}{p+2}}\\
&\quad\cdot\left(\int_{B_t}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)\mathrm{d}x\right)^{\frac{q-1}{p+2}}\\
&\quad\cdot\left(\int_{B_t}\left|\tau_h\left(u-\psi\right)(x)\right|^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{2}{p+2}}\\
&\leq \frac{c|h|^2}{t-s}\left(\int_{B_{\lambda r}}\kappa(x)^{p+2}\mathrm{d}x\right)^{\frac{1}{p+2}}\cdot\left(\int_{B_t}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)dx\right)^{\frac{p-1}{p+2}}\\
&\quad\cdot\left(\int_{B_{\lambda r}}\left|D\left(u-\psi\right)(x)\right|^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{2}{p+2}}\\
&\quad+\frac{c|h|^2}{t-s}\left(\int_{B_{\lambda r}}\kappa(x)^{\frac{p+2}{p-q+1}}\mathrm{d}x\right)^{\frac{p-q+1}{p+2}}\cdot\left(\int_{B_t}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)dx\right)^{\frac{q-1}{p+2}}\\
&\quad\cdot\left(\int_{B_{\lambda r}}\left|D\left(u-\psi\right)(x)\right|^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{2}{p+2}}.
\end{aligned}
\tag{3.22}
$$

Inserting (3.8), (3.12), (3.16), (3.20), (3.21) and (3.22) into (3.7) we infer

$$
\begin{aligned}
&\nu\int_\Omega\eta^2(x)|\tau_hDu(x)|^2\left(\mu^2+|Du(x+h)|^2+|Du(x)|^2\right)^{\frac{p-2}{2}}\mathrm{d}x\\
&\leq 6\varepsilon\int_\Omega\eta^2(x)|\tau_hDu(x)|^2(\mu^2+|Du(x)|^2+|Du(x+h)|^2)^{\frac{p-2}{2}}\mathrm{d}x\\
&\quad+c_\varepsilon|h|^2\left(\int_{B_{\lambda r}}|D^2\psi(x)|^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{4}{p+2}}\cdot\left(\int_{B_{\lambda r}}\left(\mu^2+|Du(x)|^2\right)^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{p-2}{p+2}}\\
&\quad+c_\varepsilon|h|^2\left(\int_{B_{\lambda r}}|D^2\psi(x)|^{\frac{p+2}{p+2-q}}\mathrm{d}x\right)^{\frac{2(p+2-q)}{p+2}}\cdot\left(\int_{B_{\lambda r}}\left(\mu^2+|Du(x)|^2\right)^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{2q-p-2}{p+2}}\\
&\quad+\frac{c_\varepsilon|h|^2}{(t-s)^2}\left(\int_{B_{\lambda r}}|D(u-\psi)(x)|^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{4}{p+2}}\cdot\left(\int_{B_{\lambda r}}\left(\mu^2+|Du(x)|^2\right)^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{p-2}{p+2}}\\
&\quad+\frac{c_\varepsilon|h|^2}{(t-s)^2}\left(\int_{B_{\lambda r}}|D(u-\psi)|^{\frac{p+2}{p+2-q}}\mathrm{d}x\right)^{\frac{2(p+2)-2q}{p+2}}\\
&\quad\cdot\left(\int_{B_{\lambda r}}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)\mathrm{d}x\right)^{\frac{2q-p-2}{p+2}}\\
&\quad+c_\varepsilon|h|^2\left(\int_{B_{\lambda r}}\kappa^{\frac{p+2}{p-q+1}}(x)\mathrm{d}x\right)^{\frac{2p-2q+2}{p+2}}\cdot\left(\int_{B_t}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)\mathrm{d}x\right)^{\frac{2q-p}{p+2}}\\
&\quad+c_\varepsilon|h|^2\left(\int_{B_{\lambda r}}\kappa^{p+2}(x)\mathrm{d}x\right)^{\frac{2}{p+2}}\cdot\left(\int_{B_t}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)\mathrm{d}x\right)^{\frac{p}{p+2}}\\
&\quad+c|h|^2\left(\int_{B_{\lambda r}}\kappa^{p+2}(x)\mathrm{d}x\right)^{\frac{1}{p+2}}\cdot\left(\int_{B_t}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)\mathrm{d}x\right)^{\frac{p-1}{p+2}}
\end{aligned}
$$

$$
\begin{aligned}
&\cdot\left(\int_{B_{\lambda r}}\left|D^2\psi(x)\right|^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{2}{p+2}}\\
&+c|h|^2\left(\int_{B_{\lambda r}}\kappa^{\frac{p+2}{p-q+1}}(x)\mathrm{d}x\right)^{\frac{p-q+1}{p+2}}\cdot\left(\int_{B_t}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)\mathrm{d}x\right)^{\frac{q-1}{p+2}}\\
&\cdot\left(\int_{B_{\lambda r}}\left|D^2\psi(x)\right|^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{2}{p+2}}\\
&+\frac{c|h|^2}{t-s}\left(\int_{B_{\lambda r}}\kappa(x)^{p+2}\mathrm{d}x\right)^{\frac{1}{p+2}}\cdot\left(\int_{B_t}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)dx\right)^{\frac{p-1}{p+2}}\\
&\cdot\left(\int_{B_{\lambda r}}\left|D\left(u-\psi\right)(x)\right|^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{2}{p+2}}\\
&+\frac{c|h|^2}{t-s}\left(\int_{B_{\lambda r}}\kappa(x)^{\frac{p+2}{p-q+1}}\mathrm{d}x\right)^{\frac{p-q+1}{p+2}}\cdot\left(\int_{B_t}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)dx\right)^{\frac{q-1}{p+2}}\\
&\cdot\left(\int_{B_{\lambda r}}\left|D\left(u-\psi\right)(x)\right|^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{2}{p+2}}
\end{aligned}
$$

Choosing $\varepsilon=\frac{\nu}{12}$, we can reabsorb the first term from the right-hand side to the left-hand one, thus getting

$$
\begin{aligned}
&\nu\int_{\Omega}\eta^2(x)|\tau_h Du(x)|^2\left(\mu^2+|Du(x+h)|^2+|Du(x)|^2\right)^{\frac{p-2}{2}}\mathrm{d}x\\
&\le c|h|^2\left(\int_{B_{\lambda r}}|D^2\psi(x)|^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{4}{p+2}}\cdot\left(\int_{B_{\lambda r}}\left(\mu^2+|Du(x)|^2\right)^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{p-2}{p+2}}\\
&\quad+c|h|^2\left(\int_{B_{\lambda r}}|D^2\psi(x)|^{\frac{p+2}{p+2-q}}\mathrm{d}x\right)^{\frac{2(p+2-q)}{p+2}}\cdot\left(\int_{B_{\lambda r}}\left(\mu^2+|Du(x)|^2\right)^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{2q-p-2}{p+2}}\\
&\quad+\frac{c|h|^2}{(t-s)^2}\left(\int_{B_{\lambda r}}|D(u-\psi)(x)|^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{4}{p+2}}\cdot\left(\int_{B_{\lambda r}}\left(\mu^2+|Du(x)|^2\right)^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{p-2}{p+2}}\\
&\quad+\frac{c|h|^2}{(t-s)^2}\left(\int_{B_{\lambda r}}|D(u-\psi)|^{\frac{p+2}{p+2-q}}\mathrm{d}x\right)^{\frac{2(p+2)-2q}{p+2}}\\
&\quad\cdot\left(\int_{B_{\lambda r}}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)\mathrm{d}x\right)^{\frac{2q-p-2}{p+2}}\\
&\quad+c|h|^2\left(\int_{B_{\lambda r}}\kappa^{\frac{p+2}{p-q+1}}(x)\mathrm{d}x\right)^{\frac{2p-2q+2}{p+2}}\cdot\left(\int_{B_t}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)\mathrm{d}x\right)^{\frac{2q-p}{p+2}}\\
&\quad+c|h|^2\left(\int_{B_{\lambda r}}\kappa^{p+2}(x)\mathrm{d}x\right)^{\frac{2}{p+2}}\cdot\left(\int_{B_t}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)\mathrm{d}x\right)^{\frac{p}{p+2}}\\
&\quad+c|h|^2\left(\int_{B_{\lambda r}}\kappa^{p+2}(x)\mathrm{d}x\right)^{\frac{1}{p+2}}\cdot\left(\int_{B_t}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)\mathrm{d}x\right)^{\frac{p-1}{p+2}}\\
&\quad\cdot\left(\int_{B_{\lambda r}}\left|D^2\psi(x)\right|^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{2}{p+2}}\\
&\quad+c|h|^2\left(\int_{B_{\lambda r}}\kappa^{\frac{p+2}{p-q+1}}(x)\mathrm{d}x\right)^{\frac{p-q+1}{p+2}}\cdot\left(\int_{B_t}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)\mathrm{d}x\right)^{\frac{q-1}{p+2}}\\
&\quad\cdot\left(\int_{B_{\lambda r}}\left|D^2\psi(x)\right|^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{2}{p+2}}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{c|h|^2}{t-s}\left(\int_{B_{\lambda r}}\kappa(x)^{p+2}\mathrm{d}x\right)^{\frac{1}{p+2}}\cdot\left(\int_{B_t}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)dx\right)^{\frac{p-1}{p+2}}\\
&\cdot\left(\int_{B_{\lambda r}}|D\left(u-\psi\right)(x)|^{\frac{p+2}{2}}\,\mathrm{d}x\right)^{\frac{2}{p+2}}\\
&+\frac{c|h|^2}{t-s}\left(\int_{B_{\lambda r}}\kappa(x)^{\frac{p+2}{p-q+1}}\mathrm{d}x\right)^{\frac{p-q+1}{p+2}}\cdot\left(\int_{B_t}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)dx\right)^{\frac{q-1}{p+2}}\\
&\cdot\left(\int_{B_{\lambda r}}|D\left(u-\psi\right)(x)|^{\frac{p+2}{2}}\,\mathrm{d}x\right)^{\frac{2}{p+2}}.
\end{aligned}
$$

Now we apply Young's inequalities and since $u\in\mathcal{K}_\psi(\Omega)$, we have

$$
\begin{aligned}
&\nu\int_\Omega\eta^2(x)|\tau_hDu(x)|^2\left(\mu^2+|Du(x+h)|^2+|Du(x)|^2\right)^{\frac{p-2}{2}}\mathrm{d}x\\
&\le 10\varepsilon|h|^2\int_{B_{\lambda r}}\left(\mu^2+|Du(x)|^2\right)^{\frac{p+2}{2}}\mathrm{d}x\\
&\quad+c_\varepsilon|h|^2\int_{B_{\lambda r}}|D^2\psi(x)|^{\frac{p+2}{2}}\mathrm{d}x+c_\varepsilon|h|^2\int_{B_{\lambda r}}|D^2\psi(x)|^{\frac{p+2}{p+2-q}}\mathrm{d}x\\
&\quad+\frac{c_\varepsilon|h|^2}{(t-s)^{\frac{p+2}{2}}}\int_{B_{\lambda r}}|D(u-\psi)(x)|^{\frac{p+2}{2}}\mathrm{d}x\\
&\quad+\frac{c_\varepsilon|h|^2}{(t-s)^{\frac{p+2}{p+2-q}}}\int_{B_{\lambda r}}|D(u-\psi)|^{\frac{p+2}{p+2-q}}\mathrm{d}x\\
&\quad+c_\varepsilon|h|^2\int_{B_{\lambda r}}\kappa^{\frac{p+2}{p-q+1}}(x)\mathrm{d}x+c_\varepsilon|h|^2\int_{B_{\lambda r}}\kappa^{p+2}(x)\mathrm{d}x. \qquad (3.23)
\end{aligned}
$$

By Young's inequalities of exponents $\left(p+2-q,\frac{p+2-q}{p+1-q}\right)$ we can estimate the thirdlast integral appearing in the right hand side of the previous inequality as

$$
\begin{aligned}
&\frac{c_\varepsilon|h|^2}{(t-s)^{\frac{p+2}{p-q+2}}}\int_{B_{\lambda r}}|D(u-\psi)|^{\frac{p+2}{p+2-q}}\,\mathrm{d}x\\
&\quad\le\frac{c_\varepsilon|h|^2}{(t-s)^{\frac{p+2}{p-q+2}}}\int_{B_{\lambda r}}|Du(x)|^{\frac{p+2}{p+2-q}}\mathrm{d}x+\frac{c_\varepsilon|h|^2}{(t-s)^{\frac{p+2}{p-q+2}}}\int_{B_{\lambda r}}|D\psi(x)|^{\frac{p+2}{p+2-q}}\,\mathrm{d}x\\
&\quad\le c_\varepsilon|h|^2\int_{B_{\lambda r}}|Du(x)|^{p+2}\,\mathrm{d}x+\frac{c_\varepsilon|h|^2}{(t-s)^{\frac{p+2}{p-q+1}}}|B_R|\\
&\quad+\frac{c_\varepsilon|h|^2}{(t-s)^{\frac{p+2}{p-q+2}}}\int_{B_{\lambda r}}|D\psi(x)|^{\frac{p+2}{p+2-q}}\,\mathrm{d}x,\\
&\quad\le\frac{c_\varepsilon|h|^2}{(t-s)^{\frac{p+2}{p-q+1}}}|B_R|+\varepsilon|h|^2\int_{B_{\lambda r}}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)\mathrm{d}x+\frac{c_\varepsilon|h|^2}{(t-s)^{\frac{p+2}{p-q+2}}}\\
&\qquad\int_{B_{\lambda r}}|D\psi(x)|^{\frac{p+2}{p+2-q}}\,\mathrm{d}x,
\end{aligned}
$$

and similarly, using Young's inequality with exponents $(2,2)$, we get

$$\frac{c_\varepsilon |h|^2}{(t-s)^{\frac{p+2}{2}}}\int_{B_{\lambda r}} |D(u-\psi)|^{\frac{p+2}{2}}\,\mathrm{d}x$$
$$\le \frac{c_\varepsilon |h|^2}{(t-s)^{p+2}}|B_R| + \varepsilon|h|^2\int_{B_{\lambda r}}(\mu^{p+2}+|Du(x)|^{p+2})\,\mathrm{d}x + \frac{c_\varepsilon |h|^2}{(t-s)^{\frac{p+2}{2}}}$$
$$\int_{B_{\lambda r}}|D\psi(x)|^{\frac{p+2}{2}}\,\mathrm{d}x.$$

So, from (3.23), we get

$$\nu\int_\Omega \eta^2(x)|\tau_h Du(x)|^2\left(\mu^2+|Du(x+h)|^2+|Du(x)|^2\right)^{\frac{p-2}{2}}\mathrm{d}x$$
$$\le 12\varepsilon|h|^2\int_{B_{\lambda r}}(\mu^2+|Du(x)|^2)^{\frac{p+2}{2}}\mathrm{d}x$$
$$+\frac{c_\varepsilon|h|^2}{(t-s)^{\frac{p+2}{p-q+2}}}\int_{B_{\lambda r}}|D\psi(x)|^{\frac{p+2}{p+2-q}}\mathrm{d}x + \frac{c_\varepsilon|h|^2}{(t-s)^{\frac{p+2}{2}}}\int_{B_{\lambda r}}|D\psi(x)|^{\frac{p+2}{2}}\mathrm{d}x$$
$$+c_\varepsilon|h|^2\int_{B_{\lambda r}}|D^2\psi(x)|^{\frac{p+2}{2}}\mathrm{d}x + c_\varepsilon|h|^2\int_{B_{\lambda r}}|D^2\psi(x)|^{\frac{p+2}{p+2-q}}\mathrm{d}x$$
$$+c_\varepsilon|h|^2\int_{B_{\lambda r}}\kappa^{\frac{p+2}{p-q+1}}(x)\mathrm{d}x + c_\varepsilon|h|^2\int_{B_{\lambda r}}\kappa^{p+2}(x)\mathrm{d}x$$
$$+\frac{c_\varepsilon|h|^2}{(t-s)^{\frac{p+2}{p-q+1}}}|B_R| + \frac{c_\varepsilon|h|^2}{(t-s)^{p+2}}|B_R|. \tag{3.24}$$

Using, in the left hand side of the previous estimate, the right-hand side of the inequality (2.6) in Lemma 2.5 , we get

$$\nu\int_\Omega \eta^2(x)\left|\tau_h V_p\left(Du(x)\right)\right|^2\mathrm{d}x$$
$$\le 12\varepsilon|h|^2\int_{B_{\lambda r}}(\mu^2+|Du(x)|^2)^{\frac{p+2}{2}}\mathrm{d}x$$
$$+\frac{c_\varepsilon|h|^2}{(t-s)^{\frac{p+2}{p-q+2}}}\int_{B_{\lambda r}}|D\psi(x)|^{\frac{p+2}{p+2-q}}\mathrm{d}x + \frac{c_\varepsilon|h|^2}{(t-s)^{\frac{p+2}{2}}}\int_{B_{\lambda r}}|D\psi(x)|^{\frac{p+2}{2}}\mathrm{d}x$$
$$+c_\varepsilon|h|^2\int_{B_{\lambda r}}|D^2\psi(x)|^{\frac{p+2}{2}}\mathrm{d}x + c_\varepsilon|h|^2\int_{B_{\lambda r}}|D^2\psi(x)|^{\frac{p+2}{p+2-q}}\mathrm{d}x$$
$$+c_\varepsilon|h|^2\int_{B_{\lambda r}}\kappa^{\frac{p+2}{p-q+1}}(x)\mathrm{d}x + c_\varepsilon|h|^2\int_{B_{\lambda r}}\kappa^{p+2}(x)\mathrm{d}x$$
$$+\frac{c_\varepsilon|h|^2}{(t-s)^{\frac{p+2}{p-q+1}}}|B_R| + \frac{c_\varepsilon|h|^2}{(t-s)^{p+2}}|B_R|.$$

Dividing both sides by $|h|^2$ and using Lemma 2.10 and the properties of η, we have

$$\nu\int_{B_s}\left|DV_p\left(Du(x)\right)\right|^2\mathrm{d}x$$

$$
\begin{aligned}
&\leq 12\varepsilon \int_{B_{\lambda r}} (\mu^2+|Du(x)|^2)^{\frac{p+2}{2}} \mathrm{d}x \\
&\quad + \frac{c_\varepsilon}{(t-s)^{\frac{p+2}{p-q+2}}} \int_{B_{\lambda r}} |D\psi(x)|^{\frac{p+2}{p+2-q}} \mathrm{d}x + \frac{c_\varepsilon}{(t-s)^{\frac{p+2}{2}}} \int_{B_{\lambda r}} |D\psi(x)|^{\frac{p+2}{2}} \mathrm{d}x \\
&\quad + c_\varepsilon \int_{B_{\lambda r}} |D^2\psi(x)|^{\frac{p+2}{2}} \mathrm{d}x + c_\varepsilon \int_{B_{\lambda r}} |D^2\psi(x)|^{\frac{p+2}{p+2-q}} \mathrm{d}x \\
&\quad + c_\varepsilon \int_{B_{\lambda r}} \kappa^{\frac{p+2}{p-q+1}}(x) \mathrm{d}x + c_\varepsilon \int_{B_{\lambda r}} \kappa^{p+2}(x) \mathrm{d}x \\
&\quad + \frac{c_\varepsilon}{(t-s)^{\frac{p+2}{p-q+1}}} |B_R| + \frac{c_\varepsilon}{(t-s)^{p+2}} |B_R|. \qquad (3.25)
\end{aligned}
$$

Now, by virtue of left-hand side of inequality (2.7) of Lemma 2.5

$$
\begin{aligned}
&\int_{B_s} \left(\mu^2+|Du(x)|^2\right)^{\frac{p-2}{2}} \left|D^2u(x)\right|^2 \mathrm{d}x \leq \int_{B_s} |DV_p(Du(x))|^2 \mathrm{d}x \\
&\leq 12\varepsilon \int_{B_{\lambda r}} (\mu^2+|Du(x)|^2)^{\frac{p+2}{2}} \mathrm{d}x \\
&\quad + \frac{c_\varepsilon}{(t-s)^{\frac{p+2}{p-q+2}}} \int_{B_{\lambda r}} |D\psi(x)|^{\frac{p+2}{p+2-q}} \mathrm{d}x + \frac{c_\varepsilon}{(t-s)^{\frac{p+2}{2}}} \int_{B_{\lambda r}} |D\psi(x)|^{\frac{p+2}{2}} \mathrm{d}x \\
&\quad + c_\varepsilon \int_{B_{\lambda r}} |D^2\psi(x)|^{\frac{p+2}{2}} \mathrm{d}x + c_\varepsilon \int_{B_{\lambda r}} |D^2\psi(x)|^{\frac{p+2}{p+2-q}} \mathrm{d}x \\
&\quad + c_\varepsilon \int_{B_{\lambda r}} \kappa^{\frac{p+2}{p-q+1}}(x) \mathrm{d}x + c_\varepsilon \int_{B_{\lambda r}} \kappa^{p+2}(x) \mathrm{d}x \\
&\quad + \frac{c_\varepsilon}{(t-s)^{\frac{p+2}{p-q+1}}} |B_R| + \frac{c_\varepsilon}{(t-s)^{p+2}} |B_R|. \qquad (3.26)
\end{aligned}
$$

By virtue of the local boundedness of u, the second interpolation inequality of Lemma 2.1 yields

$$
\begin{aligned}
&\int_\Omega \eta^2(x) \left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}} |Du(x)|^2 \mathrm{d}x \\
&\quad \leq c\|u\|^2_{L^\infty(\mathrm{supp}(\eta))} \int_\Omega \eta^2(x) \left(\mu^2+|Du(x)|^2\right)^{\frac{p-2}{2}} \left|D^2u(x)\right|^2 \mathrm{d}x \\
&\qquad + c\|u\|^2_{L^\infty(\mathrm{supp}(\eta))} \int_\Omega \left(|\eta(x)|^2+|D\eta(x)|^2\right) \left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}} \mathrm{d}x.
\end{aligned}
$$

and so, combining this last estimate with (3.26), and using the properties of η, we get

$$
\begin{aligned}
&\int_{B_r} \left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}} |Du(x)|^2 \mathrm{d}x \\
&\leq 12\varepsilon c_\varepsilon \|u\|^2_{L^\infty(B_{\lambda r})} \int_{B_{\lambda r}} (\mu^2+|Du(x)|^2)^{\frac{p+2}{2}} \mathrm{d}x \\
&\quad + \frac{c_\varepsilon \|u\|^2_{L^\infty(B_{\lambda r})}}{(t-s)^{\frac{p+2}{p-q+2}}} \int_{B_{\lambda r}} |D\psi(x)|^{\frac{p+2}{p+2-q}} \mathrm{d}x + \frac{c_\varepsilon \|u\|^2_{L^\infty(B_{\lambda r})}}{(t-s)^{\frac{p+2}{2}}} \int_{B_{\lambda r}} |D\psi(x)|^{\frac{p+2}{2}} \mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&+c_\varepsilon\|u\|^2_{L^\infty(B_{\lambda r})}\int_{B_{\lambda r}}|D^2\psi(x)|^{\frac{p+2}{2}}\mathrm{d}x+c_\varepsilon\|u\|^2_{L^\infty(B_{\lambda r})}\int_{B_{\lambda r}}|D^2\psi(x)|^{\frac{p+2}{p+2-q}}\mathrm{d}x\\
&+c_\varepsilon\|u\|^2_{L^\infty(B_{\lambda r})}\int_{B_{\lambda r}}\kappa^{\frac{p+2}{p-q+1}}(x)\mathrm{d}x+c_\varepsilon\|u\|^2_{L^\infty(B_{\lambda r})}\int_{B_{\lambda r}}\kappa^{p+2}(x)\mathrm{d}x\\
&+\frac{c_\varepsilon\|u\|^2_{L^\infty(B_{\lambda r})}}{(t-s)^{\frac{p+2}{p-q+1}}}|B_R|+\frac{c_\varepsilon\|u\|^2_{L^\infty(B_{\lambda r})}}{(t-s)^{p+2}}|B_R|\\
&+\frac{c_\varepsilon\|u\|^2_{L^\infty(B_{\lambda r})}}{(t-s)^2}\int_{B_{\lambda r}}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}\mathrm{d}x. \qquad (3.27)
\end{aligned}
$$

Now let us notice that, since $2\le p<q<p+1$, we have $\frac{p+2}{2}<\frac{p+2}{p+2-q}$ and $p+2<\frac{p+2}{p-q+1}$.

So using Young's inequality with exponents $\left(\frac{2}{q-p},\frac{2}{p+2-q}\right)$, we get

$$
\begin{aligned}
\int_{B_{\lambda r}}|D\psi(x)|^{\frac{p+2}{2}}\mathrm{d}x&\le c|B_R|^{\frac{q-p}{2}}\left(\int_{B_{\lambda r}}|D\psi(x)|^{\frac{p+2}{p+2-q}}\mathrm{d}x\right)^{\frac{p+2-q}{2}}\\
&\le c|B_R|+c\int_{B_{\lambda r}}|D\psi(x)|^{\frac{p+2}{p+2-q}}\mathrm{d}x,
\end{aligned}
$$

and similarly

$$
\int_{B_{\lambda r}}|D^2\psi(x)|^{\frac{p+2}{2}}\mathrm{d}x\le c|B_R|+c\int_{B_{\lambda r}}|D^2\psi(x)|^{\frac{p+2}{p+2-q}}\mathrm{d}x.
$$

Moreover, since $p+2<\frac{p+2}{p-q+1}$, by Young's inequality with exponents $\left(\frac{1}{q-p},\frac{1}{p-q+1}\right)$, we have

$$
\begin{aligned}
\int_{B_{\lambda r}}\kappa^{p+2}(x)\mathrm{d}x&\le c|B_R|^{q-p}\left(\int_{B_{\lambda r}}\kappa^{\frac{p+2}{p-q+1}}(x)\mathrm{d}x\right)^{p-q+1}\\
&\le c|B_R|+c\int_{B_{\lambda r}}\kappa^{\frac{p+2}{p-q+1}}(x)\mathrm{d}x.
\end{aligned}
$$

So, since $t-s<1$, (3.27)becomes

$$
\begin{aligned}
&\int_{B_r}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}|Du(x)|^2\mathrm{d}x\le 12\varepsilon c_\varepsilon\|u\|^2_{L^\infty(B_{\lambda r})}\int_{B_{\lambda r}}(\mu^2+|Du(x)|^2)^{\frac{p+2}{2}}\mathrm{d}x\\
&+c_\varepsilon\|u\|^2_{L^\infty(B_{\lambda r})}\left[\int_{B_{\lambda r}}\kappa^{\frac{p+2}{p-q+1}}(x)\mathrm{d}x+\int_{B_{\lambda r}}|D^2\psi(x)|^{\frac{p+2}{p+2-q}}\mathrm{d}x\right]\\
&+\frac{c_\varepsilon\|u\|^2_{L^\infty(B_{\lambda r})}}{(t-s)^{\frac{p+2}{p-q+1}}}\left[\int_{B_{\lambda r}}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}\mathrm{d}x+\int_{B_{\lambda r}}|D\psi(x)|^{\frac{p+2}{p+2-q}}\mathrm{d}x+|B_R|\right],
\end{aligned}
$$

and since $0\le\mu\le 1$, we get

$$
\begin{aligned}
&\int_{B_r}|Du(x)|^{p+2}\mathrm{d}x\le\int_{B_r}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}|Du(x)|^2\mathrm{d}x\\
&\le 12\varepsilon c_\varepsilon\|u\|^2_{L^\infty(B_{\lambda r})}\int_{B_{\lambda r}}(\mu^2+|Du(x)|^2)^{\frac{p+2}{2}}\mathrm{d}x\\
&\quad+c_\varepsilon\|u\|^2_{L^\infty(B_{\lambda r})}\left[\int_{B_{\lambda r}}\kappa^{\frac{p+2}{p-q+1}}(x)\mathrm{d}x+\int_{B_{\lambda r}}|D^2\psi(x)|^{\frac{p+2}{p+2-q}}\mathrm{d}x\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{c_\varepsilon\|u\|^2_{L^\infty(B_{\lambda r})}}{(t-s)^{\frac{p+2}{p-q+1}}}\left[\int_{B_{\lambda r}}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}\mathrm{d}x+\int_{B_{\lambda r}}|D\psi(x)|^{\frac{p+2}{p+2-q}}\mathrm{d}x+|B_R|\right]\\
&\le 12\varepsilon c_\varepsilon\|u\|^2_{L^\infty(B_{\lambda r})}\int_{B_{\lambda r}}|Du(x)|^{p+2}\mathrm{d}x\\
&+c_\varepsilon\|u\|^2_{L^\infty(B_{\lambda r})}\left[\int_{B_{\lambda r}}\kappa^{\frac{p+2}{p-q+1}}(x)\mathrm{d}x+\int_{B_{\lambda r}}|D^2\psi(x)|^{\frac{p+2}{p+2-q}}\mathrm{d}x\right]\\
&+\frac{c_\varepsilon\|u\|^2_{L^\infty(B_{\lambda r})}}{(t-s)^{\frac{p+2}{p-q+1}}}\left[\int_{B_{\lambda r}}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}\mathrm{d}x+\int_{B_{\lambda r}}|D\psi(x)|^{\frac{p+2}{p+2-q}}\mathrm{d}x+|B_R|\right],
\end{aligned}
$$

Choosing ε such that $12\varepsilon\ \|u\|^2_{L^\infty(B_R)}\le\frac{1}{2}$, previous estimate becomes

$$
\begin{aligned}
&\int_{B_r}|Du(x)|^{p+2}\,\mathrm{d}x\le\frac{1}{2}\int_{B_{\lambda r}}|Du(x)|^{p+2}\mathrm{d}x\\
&+c\|u\|^2_{L^\infty(B_{\lambda r})}\left[\int_{B_{\lambda r}}\kappa^{\frac{p+2}{p-q+1}}(x)\mathrm{d}x+\int_{B_{\lambda r}}|D^2\psi(x)|^{\frac{p+2}{p+2-q}}\mathrm{d}x\right]\\
&+\frac{c\|u\|^2_{L^\infty(B_{\lambda r})}}{(t-s)^{\frac{p+2}{p-q+1}}}\left[\int_{B_{\lambda r}}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}\mathrm{d}x+\int_{B_{\lambda r}}|D\psi(x)|^{\frac{p+2}{p+2-q}}\mathrm{d}x+|B_R|\right],
\end{aligned}
\tag{3.28}
$$

where $c=c(p,q,L,\nu,\mu)$ is independent of t and s.

Since (3.28) is valid for any $\frac{R}{2}<r<s<t<\lambda r<R<1$, taking the limit as $s\to r$ and $t\to\lambda r$, we get

$$
\begin{aligned}
&\int_{B_r}|Du(x)|^{p+2}\,\mathrm{d}x\le\frac{1}{2}\int_{B_{\lambda r}}|Du(x)|^{p+2}\,\mathrm{d}x\\
&+c\|u\|^2_{L^\infty(B_R)}\left[\int_{B_R}|D^2\psi(x)|^{\frac{p+2}{p+2-q}}\mathrm{d}x+\int_{B_R}\kappa^{\frac{p+2}{p-q+1}}(x)\mathrm{d}x\right]\\
&+\frac{c\|u\|^2_{L^\infty(B_R)}}{r^{\frac{p+2}{p-q+1}}(\lambda-1)^{\frac{p+2}{p-q+1}}}\left[|B_R|+\int_{B_R}|D\psi(x)|^{\frac{p+2}{p+2-q}}\,\mathrm{d}x+\int_{B_R}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}\,\mathrm{d}x\right]
\end{aligned}
\tag{3.29}
$$

Now, setting

$$
h(r)=\int_{B_r}|Du(x)|^{p+2}\,\mathrm{d}x,
$$

$$
A=c\|u\|^2_{L^\infty(B_R)}\left[|B_R|+\int_{B_R}|D\psi(x)|^{\frac{p+2}{p+2-q}}\,\mathrm{d}x+\int_{B_R}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}\,\mathrm{d}x\right],
$$

and

$$
B=c\|u\|^2_{L^\infty(B_R)}\left[\int_{B_R}|D^2\psi(x)|^{\frac{p+2}{p+2-q}}\mathrm{d}x+\int_{B_R}\kappa^{\frac{p+2}{p-q+1}}(x)\mathrm{d}x\right],
$$

we obtain

$$
h(r)\le\frac{1}{2}h(\lambda r)+\frac{A}{r^{\frac{p+2}{p-q+1}}(\lambda-1)^{\frac{p+2}{p-q+1}}}+B
$$

Thus, we can apply Lemma 2.6, with

$$
\theta=\frac{1}{2}\qquad\text{and}\qquad\gamma=\frac{p+2}{p-q+1},
$$

obtaining

$$\int_{B_r} |Du(x)|^{p+2}\, \mathrm{d}x$$
$$\le c\|u\|^2_{L^\infty(B_R)} \left[\int_{B_R} |D^2\psi(x)|^{\frac{p+2}{p+2-q}} \mathrm{d}x + \int_{B_R} \kappa^{\frac{p+2}{p-q+1}}(x)\mathrm{d}x + \right]$$
$$+ \frac{c\|u\|^2_{L^\infty(B_R)}}{R^{\frac{p+2}{p-q+1}}} \left[|B_R| + \int_{B_R} |D\psi(x)|^{\frac{p+2}{p+2-q}}\, \mathrm{d}x + \int_{B_R} \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}}\, \mathrm{d}x\right]$$

Since $R < 1$, the previous estimate can be written as follows

$$\int_{B_{\frac{R}{2}}} |Du(x)|^{p+2}\, \mathrm{d}x \le \frac{c\|u\|^2_{L^\infty(B_R)}}{R^{\frac{p+2}{p+2-q}}} \int_{B_R} \left[1 + \left|D^2\psi(x)\right|^{\frac{p+2}{p+2-q}} + |D\psi(x)|^{\frac{p+2}{p+2-q}} + \kappa^{\frac{p+2}{p-q+1}}(x) + |Du(x)|^p\right] \mathrm{d}x. \tag{3.30}$$

Plugging the last inequality in (3.25) and choosing $\eta \in C_0^\infty(B_{\frac{R}{2}})$ such that $\eta \equiv 1$ on $B_{\frac{R}{4}}$ we get

$$\int_{B_{\frac{R}{4}}} |DV_p(Du(x))|^2\, \mathrm{d}x \le \frac{c\|u\|^2_{L^\infty(B_R)}}{R^{\frac{p+2}{p+2-q}}} \int_{B_R} \left[1 + \left|D^2\psi(x)\right|^{\frac{p+2}{p+2-q}} + |D\psi(x)|^{\frac{p+2}{p+2-q}} + \kappa^{\frac{p+2}{p-q+1}}(x) + |Du(x)|^p\right] \mathrm{d}x.$$

that by virtue of estimate (2.4), gives us the a priori estimate with

$$\int_{B_{\frac{R}{4}}} |DV_p(Du(x))|^2\, \mathrm{d}x \le \frac{c(\|\psi\|^2_{L^\infty} + \|u\|^2_{L^{p^*}(B_R)})}{R^{\frac{p+2}{p+2-q}}} \cdot \int_{B_R} \left[1 + \left|D^2\psi(x)\right|^{\frac{p+2}{p+2-q}} + |D\psi(x)|^{\frac{p+2}{p+2-q}} + \kappa^{\frac{p+2}{p-q+1}}(x) + |Du(x)|^p\right] \mathrm{d}x. \tag{3.31}$$

with $c = c(p, q, L, \nu, \mu)$.

Step 2: The Approximation. Now we conclude the proof by passing to the limit in the approximating problem. The limit procedure is standard see, e.g., [12].

Let $u \in \mathcal{K}_\psi(\Omega)$ be a solution to (1.1) and let f_ε be the sequence obtained applying Lemma 2.11 to the integrand f. Let us fix a ball $B_R \Subset \Omega$ and let $u_\varepsilon \in u + W_0^{1,p}(B_R)$ be the solution to the minimization problem

$$\min\left\{\int_{B_R} f_\varepsilon(x, Dv(x))\mathrm{d}x :\ v \in \mathcal{K}_\psi(B_R)\right\}.$$

By Theorem 2.3, the minimizers u_ε satisfy the a priori assumptions at (3.1), i.e. $\left(\mu^2 + |Du_\varepsilon|^2\right)^{\frac{p-2}{4}} Du_\varepsilon \in W^{1,2}_{\mathrm{loc}}(\Omega)$, and therefore we are legitimated to use estimate (3.31) thus obtaining

$$\int_{B_{\frac{R}{4}}} |DV_p(Du_\varepsilon(x))|^2 \, \mathrm{d}x \le \frac{c(\|\psi\|^2_{L^\infty} + \|u_\varepsilon\|^2_{L^{p^*}(B_R)})}{R^{\frac{p+2}{p+2-q}}} \cdot \int_{B_R} \left[1 + \left|D^2\psi(x)\right|^{\frac{p+2}{p+2-q}} + |D\psi(x)|^{\frac{p+2}{p+2-q}} + \kappa^{\frac{p+2}{p-q+1}} + |Du_\varepsilon(x)|^p\right] \mathrm{d}x. \tag{3.32}$$

By the first inequality of growth conditions at (iii) of Lemma 2.11 and the minimality of u_ε we get

$$\begin{aligned}\int_{B_R} |Du_\varepsilon(x)|^p \, \mathrm{d}x &\le C(K_0) \int_{B_R} f_\varepsilon(x, Du_\varepsilon(x)) \, \mathrm{d}x \\ &\le C(K_0) \int_{B_R} f_\varepsilon(x, Du(x)) \, \mathrm{d}x \\ &\le C(K_0) \int_{B_R} f(x, Du(x)) \, \mathrm{d}x \,,\end{aligned}$$

where in the last estimate we used the second inequality at (i) of Lemma 2.11.

Since $f(x, Du) \in L^1_{\mathrm{loc}}(\Omega)$ by assumption, we deduce, up to subsequences, that there exists $\bar{u} \in W^{1,p}_0(B_R) + u$ such that

$$u_\varepsilon \rightharpoonup \bar{u} \qquad \text{weakly in } W^{1,p}_0(B_R) + u \,.$$

Note that, since $u_\varepsilon \in \mathcal{K}_\psi$ for every ε and $\mathcal{K}_\psi$ is a closed set, we have $\bar{u} \in \mathcal{K}_\psi$. Our next aim is to show that $\bar{u}$ is a solution to our obstacle problem over the ball B_R.

To this aim, fix $\varepsilon_0 > 0$ and observe that the lower semicontinuity of the functional $w \mapsto \int_{B_R} f_{\varepsilon_0}(x, Dw) \, \mathrm{d}x$, the minimality of u_ε and the monotonicity of the sequence of f_ε yield

$$\begin{aligned}\int_{B_R} f_{\varepsilon_0}(x, D\bar{u}(x)) \, \mathrm{d}x &\le \lim_{\varepsilon \to 0} \int_{B_R} f_{\varepsilon_0}(x, Du_\varepsilon(x)) \, \mathrm{d}x \\ &\le \int_{B_R} f_{\varepsilon_0}(x, Du(x)) \, \mathrm{d}x \le \int_{B_R} f(x, Du(x)) \, \mathrm{d}x\end{aligned}$$

We now use monotone convergence Theorem in the left hand side of previous estimate to deduce that

$$\int_{B_R} f(x, D\bar{u}(x)) \, \mathrm{d}x = \lim_{\varepsilon_0 \to 0} \int_{B_R} f_{\varepsilon_0}(x, D\bar{u}(x)) \, \mathrm{d}x \le \int_{B_R} f(x, Du(x)) \, \mathrm{d}x$$

Therefore, we have proved that the limit function $\bar{u} \in W^{1,p}(B_R) + u$ is a solution to the minimization problem

$$\min \left\{ \int_\Omega f(x, Dw(x)) \mathrm{d}x : w \in W^{1,p}_0(B_R) + u, \, w \in \mathcal{K}_\psi \right\}.$$

Since by the strict convexity of the functional the solution is unique, we conclude that $u = \bar{u}$. It is quite routine to show that the convergence of u_ε to u is strong in $W^{1,p}_{\mathrm{loc}}(B_R)$.

The strong convergence of u_ε to u in $W^{1,p}(B_R)$ implies also that u_ε converges strongly to u in $L^{p^*}(B_R)$ and hence the conclusion follows passing to the limit as $\varepsilon \to 0$ in estimate (3.32). □

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