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Quaternionic 1−factorizations and complete sets of rainbow spanning trees

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Abstract

A 1−factorization $\mathcal{F}$ of a complete graph K_{2n} is said to be G−regular, or regular under G, if G is an automorphism group of $\mathcal{F}$ acting sharply transitively on the vertex-set. The problem of determining which groups can realize such a situation dates back to a result by Hartman and Rosa (1985) on cyclic groups and, when n is even, the problem is still open, even though several classes of groups were tested in the recent past. It was recently proved, see Rinaldi (2021) and Mazzuoccolo et al. (2019), that a G−regular 1−factorization together with a complete set of rainbow spanning trees exists whenever n is odd, while the existence for n even was proved when either G is cyclic and n is not a power of 2, or when G is a dihedral group. In this paper we extend this result and prove the existence also for the following classes of groups: Abelian but not cyclic, dicyclic, non cyclic 2−groups with a cyclic subgroup of index 2.

Keywords: Regular 1*-factorizations, complete graph, sharply transitive permutation groups, starter, rainbow spanning trees.*
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1 Introduction

It is well known that the number of non-isomorphic 1−factorizations of K_{2n}, the complete graph on $2n$ vertices, goes to infinity with the positive integer n, [10]. Therefore, attempts to achieve classifications can be done if one imposes additional conditions either on the 1−factorization or on its automorphism group. For example, a precise description of the 1−factorization and of its automorphism group was given when the group is assumed to act multiply transitively on the vertex set, [11].

Few years ago the following question was adressed:

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Question. *Let G be a group of order $2n$. Does there exist a $1-$factorization of K_{2n} admitting G as an automorphism group acting sharply transitively on the vertex-set of K_{2n}?*

A $1-$factorization of K_{2n} satisfying the above condition is said to be G-*regular* or *regular* under G.

This question is a restricted version of problem n.4 in the list of [32], namely the word "sharply" does not appear there, but the two versions are equivalent for abelian groups, since every transitive abelian permutation group is sharply transitive. When n is odd the problem simplifies somewhat: G must be the semi-direct product of Z_2 with its normal complement and G always realizes a $1-$factorization of K_{2n} upon which it acts sharply transitively on vertices, see [3, Remark 1]. When n is even, the complete answer is still unknown.

If G is a cyclic group then Hartman and Rosa proved in [17] that the answer to the above question is negative when n is a power of 2 greater than 2, while it is affirmative for all other values of n. In a most recent past, an affirmative answer was given for several other classes of groups, see for example [7], [3], [4], [28] which respectively consider the class of abelian, dihedral, dicyclic and other nilpotente groups. In [6] and [25] a positive answer was found for the class of $2-$groups with an elementary abelian Frattini subgroup and for some non-solvable groups, respectively. Also, nonexistence results were achieved by assuming the existence of a fixed $1-$factor, [22], [28]. Further results were obtained when the number of fixed $1-$factors is as large as possible, [3], or when the $1-$factors satisfy some additional requests, [5]. Recently, we focused our attention on the existence of G-regular $1-$factorizations of K_{2n} which possess a complete set of rainbow spanning trees, [23],[29].

We recall that a rainbow spanning tree is a spanning tree sharing exactly one edge with each $1-$factor of the given $1-$factorization. In other words, a $1-$factorization of K_{2n} corresponds to a proper edge coloring of K_{2n} with precisely $2n-1$ colors: each color appears exactly n times and corresponds to a $1-$factor. Therefore, a spanning tree is *rainbow* if its edges have distinct colors. It is also usual to say that such a tree is *orthogonal* to the $1-$factorization. We also recall that if T is any subgraph of K_{2n} with exactly $2n-1$ edges, then T is a spanning tree if and only if T is a spanning connected graph, see for instance [33, 6, p.68].

A set of rainbow spanning trees is said to be *a complete set* if the trees form a partition of the edge set of K_{2n}. It is easy to prove that a complete set cannot exist in K_4, so we restrict our discussion to complete sets in K_{2n} with $n \geq 3$. Also, since each rainbow spanning tree has $2n-1$ edges, n is the number of disjoint trees in a complete set.

In [29] it is proved that, regardless of the isomorphism type of G, a $G-$regular $1-$factorization of K_{2n} together with a rainbow spanning tree, whose orbit under a subgroup of G gives rise to a complete set, exists if and only if $n \geq 3$ is an odd number. The problem of determining for which groups G a $G-$regular $1-$factorization, together with a complete set of rainbow spanning trees, exists remains open when the order of G is twice an even number. With some exceptions: in [23] a complete set of rainbow spanning trees was constructed in the family of cyclic regular $1-$factorizations of [17] for each $n \geq 3$, except when $n = 2^s$, $s \geq 2$. In [29] an explicit construction was given for the class of dihedral groups of order twice an even number.

Our main interest fits in the general problem of characterizing $G-$regular $1-$factorizations satisfying additional properties. However, I recall that the problem of determining whether every given $1-$factorization of a complete graph possesses a complete set of rainbow spanning trees dates back to the Brualdi and Hollingsworth conjecture, [8], and to the Constantine conjecture when the trees are asked to be pairwise isomorphic as uncolored trees, [15]. A recent asymptotic result settles both these conjectures for all sufficiently large n, [16]. Nevertheless, the solution for each given n remains nontrivial even if one is allowed to choose the $1-$factorization.

Most of the papers about these conjectures treat the general case by methods of extremal graph theory/probabilistic methods which can be applied for every 1-factorization of K_{2n}. The best known results hold for large n and mainly give lower-bounds on the number of rainbow spanning trees. Together with [16] we recall some other important papers in this direction: [1], [13], [18], [21], [24], [26]. The Brualdi-Hollingsworth conjecture was extended also in [20], by stating that edges of every properly colored K_n (not necessarily colored by a 1-factorization) can be partitioned into rainbow spanning trees. Results are, for example, contained in [2], [12], [24], and for large n, the results of [26] improved the best known bounds for the three conjectures in [8], [15] and [20].

Some examples of $1-$factorizations of K_{2n} satisfying the above conjectures without imposing conditions on n are also available. Constantine himself proved the existence of a suitable $1-$factorization satisfying his conjecture for the case $2n$ a power of 2 or five times a power of two, [15].

Also, a first family of $1-$factorizations for which the conjecture of Brualdi and Hollingsworth can be verified for each $n \geq 3$ was recently shown in [9].

When n is even, the examples of $G-$regular $1-$factorizations together with a complete set of rainbow spanning trees obtained in [23] and [29], involve groups possessing a cyclic subgroup of index 2. In the present paper we feel rather natural to try to extend the analysis in this direction. More precisely, we consider dicyclic groups and abelian groups with a cyclic subgroup of index 2. We obviously exclude the family of cyclic $2-$groups, in fact a regular $1-$factorization does not exist in these cases, [17]. Moreover, we consider all the non-cyclic 2-groups admitting a cyclic subgroup of index 2.

The state of art can be resumed in the following Theorem.

Theorem 1. *Let G be a group of order $2n$, $n > 2$ even. A $G-$regular $1-$factorization together with a complete set of rainbow spanning trees exists whenever G is one of the following: a dihedral group; a dicyclic group; an abelian group admitting a cyclic subgroup of index 2 and different from a cyclic $2-$group; a non-cyclic 2-group admitting a cyclic subgroup of index 2.*

The dihedral case and the cyclic case were considered in [29] and [23], respectively. In the following sections 2, 4, 3, we will show explicit constructions which will prove the existence in all the other cases.

For the sake of completeness, we recall that the finite non-abelian 2-groups (of order ≥ 8) admitting a cyclic subgroup of index 2 are known. Satz 14.9 in [19] divides them into four isomorphism types: (1), (2), (3), (4). Groups of type (1) are dihedral groups, while, each group G of type (2), (3) or (4) is considered in this paper.

In this paper we will make use of the regular $1-$factorizations already obtained in [4] and we refer to [7] for the abelian case. The $1-$factorizations constructed in [4] were referred to as *quaternionic $1-$factorizations*, since a type (2) group is a quaternionic one. This inspired the title of the present paper.

1.1 Preliminaries

We refer to the monograph [33] for the general notions on graphs and $1-$factorizations that will not be explicitely defined here. Let G be a group of even order $2n$. We use for G a multiplicative notation and denote by 1_G its identity, we also use 1 if the group G is clear from the context. Let us denote by V and E the set of vertices and edges of K_{2n}, respectively. We identify the vertices of K_{2n} with the group-elements of G. We shall denote by $[x, y]$ the edge with vertices x and y. Following [7] we always consider G in its right regular permutation representation. In other words, each group-element $g \in G$ is identified with the permutation $V \to V$, $x \mapsto xg$. This action of G on V induces actions on the subsets of V and on sets of such subsets. Hence if $g \in G$ is an arbitrary group-element and S is any subset of V then we write $S \cdot g = \{xg \; : \; x \in S\}$. In particular, if $S = [x, y]$ is an edge, then $[x, y] \cdot g = [xg, yg]$. Furthermore, if U is a collection of subsets of V, then we write $U \cdot g = \{S \cdot g \; : \; S \in U\}$. In particular, if U is a collection of edges of K_{2n} then $U \cdot g = \{[xg, yg] \; : \; [x, y] \in U\}$. The G-orbit of an edge $[x, y]$ has either length $2n$ or n and we speak of a *long* orbit or a *short* orbit, respectively, and we call $[x, y]$ a *long* edge or a *short* edge, respectively. If $[x, y]$ is a short edge, then there is a non-trivial group element g so that $[xg, yg] = [x, y]$. Such a g is unique ($g = x^{-1}y$) and is an involution; we call this g the involution *associated* with the short edge $[x, y]$. Obviously, the element yx^{-1} is an involution as well.

It is easy to show that a 1-factor of K_{2n} which is fixed by G necessarily coincides with a short G-orbit of edges.

If e is an edge, respectively if S is a set of edges, we will denote by $Orb_G(e)$, respectively by $Orb_G(S)$, the orbit of e, respectively of the set S, under the action of G.

If H is a subgroup of G then a system of distinct representatives for the left cosets of H in G will be called a *left transversal* for H in G.

If $[x, y]$ is an edge in K_{2n} we define

$$\partial([x,y]) = \begin{cases} \{xy^{-1}, yx^{-1}\} & \text{if } [x,y] \text{ is long} \\ \\ \{xy^{-1}\} & \text{if } [x,y] \text{ is short} \end{cases}$$

$$\phi([x,y]) = \begin{cases} \{x, y\} & \text{if } [x,y] \text{ is long} \\ \\ \{x\} & \text{if } [x,y] \text{ is short} \end{cases}$$

Roughly speaking, we also say that the edge $[x, y]$ has *difference set* $\partial([x, y])$, or that $\{xy^{-1}, yx^{-1}\}$ are the *differences* of $[x, y]$.

It is clear that all the edges having a same difference set form a unique G-orbit.

If S is a set of edges of K_{2n} we define

$$\partial S = \bigcup_{e \in S} \partial(e) \qquad \phi(S) = \bigcup_{e \in S} \phi(e)$$

where, in either case, the union may contain repeated elements and so, in general, will return a multiset.

In [7, Definition 2.1] a *starter* in a group G of even order is a set $\Sigma = \{S_1, \ldots, S_k\}$ of subsets of E together with associated subgroups $H_1, \ldots, H_k$ which satisfy the following conditions:

(i) $\partial S_1 \cup \cdots \cup \partial S_k = G \setminus \{1_G\}$;

(ii) for $i = 1, \ldots, k$, the set $\phi(S_i)$ is a left transversal for H_i in G;

(iii) for $i = 1, \ldots, k$, H_i must contain the involutions associated with any short edge in S_i.

We note that $G - \{1_G\}$ is a set, so that $\partial S_1 \cup \cdots \cup \partial S_k$ is a list of distinct elements, the edges of $S_1 \cup \cdots \cup S_k$ are all distinct and lie in distinct G-orbits. Hence it also follows S_i can have no edges in common with S_j for $i \neq j$. Moreover, each $\phi(S_i)$ is a set and then the edges of S_i are vertex disjoint.

It is proved in [7], that the existence of a starter in a finite group G of order $2n$ is equivalent to the existence of a $G-$regular $1-$factorization of K_{2n}. Property (i) in previous definition ensures that every edge of K_{2n} will occur in exactly one G-orbit of an edge from $S_1 \cup \ldots \cup S_k$. Properties (ii) and (iii) ensure the union of the H_i-orbits of edges from S_i will form a 1-factor.

Namely, for each index i, we form a $1-$factor $F_i = \cup_{e \in S_i} Orb_{H_i}(e)$, whose stabilizer in G is the subgroup H_i; the G-orbit $Orb_G(F_i) = \{F_i^1, \dots, F_i^{t_i}\}$, which has length $t_i = |G : H_i|$ (the index of H_i in G), is then included in the $1-$factorization.

Observe also that the existence of a $1-$factor, say F_1, which is fixed by G is equivalent to the existence in Σ of a set $S_1 = \{e\}$, where e is a short edge. Moreover, $\phi(S_i)$ and ∂S_i both contain t_i elements and t_i is equal to the number of short edges in S_i plus twice the number of long edges in S_i. It is also true that the unique $1-$factor which contains a chosen edge e with differences in ∂S_i is one of the 1-factors in $\{F_i^1, \dots, F_i^{t_i}\}$.

Suppose $n > 2$ to be even and G to contain a cyclic subgroup H of index 2. Let j be the unique involution in H and let $\{h_1, \dots h_{\frac{n}{2}}\}$ be a set of distinct representatives for the cosets of $\{1, j\}$ in H. Suppose $\Sigma = \{S_1, \dots, S_r\}$ to be a starter in G with associated subgroups $H_1, \dots, H_r$, and such that $S_1 = \{e\}$, with $\partial e = \{j\}$. Let $\mathcal{F}$ be the $G-$regular $1-$factorization equivalent to Σ.

In the following Lemma 1 we describe a subgraph R of K_{2n} which leads to the construction of a complete set of spanning trees orthogonal to $\mathcal{F}$.

Lemma 1. *Let $R = R_2 \cup \cdots \cup R_r$ be a subgraph of K_{2n} such that:*

1. *For each $i \in \{2, \dots, r\}$, the set R_i contains $t_i = [G : H_i]$ edges: one for each $1-$factor of the set $\{F_i^1, \dots, F_i^{t_i}\}$, and the set of distinct elements of ∂R_i coincides with ∂S_i.*
2. *If l is a long edge of R_i, $i \in \{2, \dots, r\}$, then there is exactly one edge $l' \in R_i$ such that $\partial l = \partial l'$ and $l' \notin Orb_H(l)$. While, if l is a short edge of R_i, $i \in \{2, \dots, r\}$, then it is the unique edge of R_i with difference set ∂l.*
3. *There exist two distinct edges e_1 and e_2 of the fixed $1-$factor F_1 such that $Orb_H(e_1) \cap Orb_H(e_2) = \emptyset$ and both $R \cup \{e_1\}$ and $R \cup \{e_2\}$ are spanning connected graphs.*

Let $T_1 = R \cup \{e_1\}$ and $T_2 = Rj \cup \{e_2\}$.
The set $\mathcal{T} = \{T_1 h_1, \dots, T_1 h_{\frac{n}{2}}\} \cup \{T_2 h_1, \dots, T_2 h_{\frac{n}{2}}\}$ is a complete set of rainbow spanning trees.

Proof. Conditions 1 and 3 assures that both $T_1 = R \cup \{e_1\}$ and $R \cup \{e_2\}$ are spanning connected graphs with $2n-1$ edges belonging to distinct $1-$factors, therefore they are spanning rianbow trees. Since $(R \cup \{e_2\})j = Rj \cup \{e_2\} = T_2$, therefore T_2 is a rainbow spanning tree as well. We also conclude that each graph in $\mathcal{T}$ is a rainbow spanning tree. We now prove that $\mathcal{T}$ is a partition of the edge-set of K_{2n}.

Let f be an edge of K_{2n}. We have three possibility: either f is long, or f is short with $\partial f = \{j_1\}$, $j_1 \neq j$, or f is short and $\partial f = \{j\}$. In all these cases we prove that f belongs to a unique spanning tree of $\mathcal{T}$.

Suppose f is a long edge, then there exists a unique $S_i \in \Sigma \setminus \{S_1\}$ such that $\partial f \in \partial S_i$ and f is an edge of $F_i^1 \cup \cdots \cup F_i^{t_i}$. Conditions 1 and 2 assures the existence of $l, l' \in R_i$ such that $\partial f = \partial l = \partial l'$, with $l' \notin Orb_H(l)$, $f \in Orb_G(l) = Orb_G(l')$. Let $g_1, g_2 \in G$ be the unique elements such that $f = lg_1 = l'g_2$. Since $g_1 g_2^{-1} \notin H$, just one of the two elements g_1 or g_2 is in H and then there is a unique graph of the set $\{Rh \mid h \in H\}$ containing f. Therefore f belongs to a unique tree of $\mathcal{T}$.

Now suppose f is short and $\partial f = \{j_1\}$, $j_1 \neq j$. Let l be the unique edge of R with $\partial l = \partial f$. Since $j_1 \notin H$, all the n edges of K_{2n} with difference set $\{j_1\}$ are in $Orb_H(l)$. We conclude that a unique tree of $\mathcal{T}$ contains f.

Finally suppose f to be a short edge with $\partial f = \{j\}$, i.e., $f \in F_1$. Condition 3 implies that F_1 contains the n distinct edges $\{e_1 h_i, e_2 h_i \mid i = 1, \ldots, \frac{n}{2}\}$. Therefore, a unique tree of $\mathcal{T}$ contains f. □

2 Dicyclic groups and complete sets of rainbow spanning trees

In this section we prove the following Proposition 1

Proposition 1. *Let G be a dicyclic group of order $2n \geq 6$. There exists a $G-$regular $1-$factorization of K_{2n} together with a complete set of rainbow spanning trees.*

The dicyclic group G of order $2n = 4s$, $s \geq 2$, can be presented as follows [31, p.189]:

$$G = \langle a, b \,:\; a^{2s} = 1, b^2 = a^s, b^{-1}ab = a^{-1} \rangle.$$

We have $G = \{1, a, \ldots, a^{2s-1}, b, ba, \ldots, ba^{2s-1}\}$ and the relations $a^r b = ba^{-r}$, $ba^r (ba^t)^{-1} = a^{t-r}$, $(ba^r)^{-1} = ba^{r+s}$, $(ba^r)^2 = a^s$ hold for r, $t = 0, 1, \ldots, (2s-1)$. Furthermore a^s is the unique involution in G. In particular, if $s = 2^{m-1}$, then G is a generalized quaternion group of order 2^{m+1}.

We consider the $G-$regular $1-$factorizaion of K_{2n} constructed in [4]. The description is given in terms of starters according to whether s is even or odd.

Starter in the case s even
A starter can be constructed as follows:

$\Sigma = \{S\} \cup \{S_{2i+1} \mid 0 \leq i \leq \frac{s-2}{2}\} \cup \{S_j^* \mid 0 \leq j \leq s-1, j \neq \dfrac{s}{2}\} \cup \{S_s\}$; With:

$S = \{[a^t, a^{-t}],\ t = 1, \ldots \frac{s}{2} - 1\} \cup \{[1, ba^{\frac{s}{2}}]\}$;

$S_{2i+1} = \{[1, a^{2i+1}]\}, \quad 0 \leq i \leq \frac{s-2}{2}$;

$S_j^* = \{[1, ba^j]\}, \quad 0 \leq j \leq s-1, j \neq \dfrac{s}{2}$;

$S_s = \{[1, a^s]\}$.

Take the subgroups:

$< b >= \{1, b, a^s, ba^s\}$ and $< b, a^2 >= \{1, a^2, a^4, \dots, a^{2n-2}, b, ba^2, \dots, ba^{2n-2}\}$.

We have:

$\partial S = \{a^{2t}, a^{-2t} \mid t = 1, \dots, \frac{s}{2} - 1\} \cup \{ba^{\frac{s}{2}}, ba^{-\frac{s}{2}}\}$ and $\phi(S)$ is a left transversal for $< b >$.

$\partial S_{2i+1} = \{a^{2i+1}, a^{-2i-1}\}$ and $\phi(S_{2i+1}) = \{1, a^{2i+1}\}$ is a left transversal for the subgroup $< b, a^2 >$.

$\partial S_j^* = \{ba^j, ba^{j+s}\}$ and $\phi(S_j^*) = \{1, ba^j\}$ is a left transversal for the cyclic subgroup $< a >$.

$\partial S_s = \{a^s\}$ and $\phi(S_s) = \{1\}$.

With the starter above, we construct the following $1-$factors:

$F = Orb_{<b>}(S) = \{[1, ba^{\frac{s}{2}}], [b, a^{\frac{s}{2}}], [a^s, ba^{s+\frac{s}{2}}], [ba^s, a^{s+\frac{s}{2}}], [a^t, a^{-t}], [ba^{-t}, ba^t], [a^{s+t}, a^{s-t}], [ba^{s-t}, ba^{s+t}], t = 1, \dots, \frac{s}{2} - 1\}$.

$F_{2i+1} = Orb_{<b,a^2>}(S_{2i+1}) = \{[a^{2k}, a^{2i+1+2k}], [ba^{2k}, ba^{2k-2i-1}],\ k = 0, \dots, s-1\}$ with $0 \leq i < \frac{s-2}{2}$.

$F_j^* = Orb_{<a>}(S_j^*) = \{[a^k, ba^{j+k}],\ k = 0, \dots, 2s-1\}$ with $0 \leq j \leq s-1$, $j \neq \frac{s}{2}$.

$F_s = Orb_G([1, a^s])$.

These $1-$factors give rise to the $1-$factorization. Namely:
The $1-$factor F is fixed by $< b >$ and its orbit under G yields the $1-$factors:

$$F, Fa, Fa^2, \dots, Fa^{s-1}$$

These $1-$factors cover all long edges with difference set in ∂S.

For each $0 \leq i \leq \frac{s-2}{2}$, the $1-$factor F_{2i+1} is fixed by $< b, a^2 >$ and its orbit under G yields the $1-$factors:

$$F_{2i+1}, F_{2i+1}a$$

These $1-$factors cover all edges with difference set in ∂S_{2i+1}.

For each $0 \leq j \leq s-1,\ j \neq \frac{s}{2}$, the $1-$factor F_j^* is fixed by $< a >$ and its orbit under G yields the $1-$factors:

$$F_j^*, F_j^* b$$

These $1-$factors cover all edges with difference set in ∂S_j^*.

Finally F_s is a fixed $1-$factor which contains all edges with difference set $\{a^s\}$.

Starter in the case s odd
A starter can be constructed as follows:

$\Sigma = \{S\} \cup \{S_i^* \mid 0 \leq i \leq s-1, i \neq \frac{s-1}{2}\} \cup \{S_s\}$. With:

$S = \{[a^t, a^{s-t-1}], [ba^t, ba^{s-t-2}] \mid 0 \leq t \leq \frac{s-3}{2}\} \cup \{[a^{\frac{s-1}{2}}, ba^{s-1}]\}$;

$S_i^* = \{[1, ba^i]\}$, $0 \leq i \leq s-1 \quad i \neq \frac{s-1}{2}$;

$S_s = \{[1, a^s]\}$

We have:

$\partial S = \{a^j,\ 1 \leq j \leq 2s-1, j \neq s\} \cup \{ba^{\frac{s-1}{2}}, ba^{s+\frac{s-1}{2}}\}$ and $\phi(S)$ is a left transversal for $< a^s >$.

$\partial S_i^* = \{ba^i, ba^{i+s}\}$ and $\phi(S_i^*)$ is a left transversal for the subgroup $< a >$.

$\partial S_s = \{a^s\}$ and $\phi(S_s) = \{1\}$.

With the starter above, we construct the following $1-$factors:

$$F = Orb_{<a^s>}(S) = \{[a^t, a^{s-t-1}], [a^{t+s}, a^{2s-t-1}], [ba^t, ba^{s-t-2}], [ba^{t+s}, ba^{2s-t-2}], [a^{\frac{s-1}{2}}, ba^{s-1}], [a^{s+\frac{s-1}{2}}, ba^{2s-1}],\ \mid 0 \leq t \leq \frac{s-3}{2}\}$$

$F_i^* = Orb_{<a>}(S_i^*) = \{[a^r, ba^{i+r}],\ r = 0, \ldots, 2s-1\}$ with $0 \leq i < s-1$, $i \neq \frac{s-1}{2}$.

$F_s = Orb_G([1, a^s])$.

These $1-$factors give rise to the $1-$factorization. Namely:
The $1-$factor F is fixed by $< a^s >$ and its orbit under G yields the $1-$factors:

$$F, Fa, Fa^2, \ldots, Fa^{s-1}, Fb, Fba, Fba^2, \ldots, Fba^{s-1}$$

These $1-$factors cover all long edges with difference set in ∂S.

For each $0 \leq i \leq s-1$, $i \neq \frac{s-1}{2}$, the $1-$factor F_i^* is fixed by $< a >$ and its orbit under G yields the $1-$factors:

$$F_i^*, F_i^* b$$

These $1-$factors cover all edges with difference set in ∂S_i^*.
Finally F_s is a fixed $1-$factor which contains all edges with difference set $\{a^s\}$.

We are now able to construct a complete set of rainbow spanning trees in both of these two cases using the method explained in the previous Lemma 1.

2.1 Case s even

Let $s \equiv 2 \pmod 4$.
Suppose $s \geq 6$. Consider the forest induced by the following set T of edges:

$$T = \{[1, ba^{\frac{s}{2}}], [1, ba^{s+\frac{s}{2}}], [1, a^{2t}], [b, ba^{s+2t}] \mid t = 1, \ldots, \frac{s}{2} - 1\}.$$

We have $[1, ba^{\frac{s}{2}}] \in F$, $[1, ba^{s+\frac{s}{2}}] \in Fa^{\frac{s}{2}}$, we also have: $[1, a^{2t}] \in Fa^t$, $[b, ba^{s+2t}] \in Fa^{\frac{s}{2}+t}$. In fact: $[1, ba^{s+\frac{s}{2}}] = [a^{s+\frac{s}{2}}, ba^s]a^{\frac{s}{2}}$, $[1, a^{2t}] = [a^{-t}, a^t]a^t$, $[b, ba^{s+2t}] = [ba^{s+(\frac{s}{2}-t)}, ba^{s-(\frac{s}{2}-t)}]a^{\frac{s}{2}+t}$. Moreover $\partial[1, a^{2t}] = \partial[b, ba^{s+2t}]$ and these two long edges are in distinct orbits under the action of $< a >$ for each $t = 1, \ldots, \frac{s}{2} - 1$, and also $\partial[1, ba^{\frac{s}{2}}] = \partial[1, ba^{s+\frac{s}{2}}]$ and these two long edges are in distinct orbits under $< a >$.

For each $i = 0, \ldots, \frac{s-2}{2}$, let $T_{2i+1} = \{[1, a^{2i+1}], [b, ba^{2i+1}]\}$. We have $[1, a^{2i+1}] \in F_{2i+1}$ and $[b, ba^{2i+1}] \in F_{2i+1}a$, in fact $[b, ba^{-2i-1}] \in F_{2i+1}$ and then $[ba^{2i+1}, b] \in F_{2i+1}a^{2i+1} = F_{2i+1}a$ since $F_{2i+1}a^2 = F_{2i+1}$. Moreover $\partial[1, a^{2i+1}] = \partial[b, ba^{2i+1}]$ and these two long edges are in distinct orbits under $< a >$.

Set $T' = T \cup (\bigcup_{i=0}^{\frac{s-2}{2}} T_{2i+1})$. The graph T' is a rainbow tree which is given by the union of a star at 1 and a star at b which are connected through the edge $[1, ba^{\frac{s}{2}}]$. Moreover T' covers all the vertices of K_{4s} except for those in the set:

$$\{a^{s+i} \mid 0 \leq i \leq s-1\} \cup \{ba^{s-2t} \mid 0 \leq t \leq \frac{s}{2} - 1\} \cup$$

$$\cup \{ba^{s+2j+1} \mid 0 \leq j \leq \frac{s-2}{2}, j \neq \frac{s-2}{4}\}$$

Consider the star at a^s induced by the set:

$$S_1 = \{[a^s, ba^{s-2t}], [a^s, ba^{s+2j+1}] \mid 0 \leq t \leq \frac{s-2}{2}, 0 \leq j \leq \frac{s-2}{2}, j \neq \frac{s-2}{4}\},$$

together with the star at ba^{s+1} induced by:

$$S_2 = \{[ba^{s+1}, a^{2s-2j}] \mid 1 \leq j \leq \frac{s-2}{2}, j \neq \frac{s-2}{4}\},$$

and the star at ba^{2s-1} induced by the set:

$$S_3 = \{[ba^{2s-1}, a^{s+2t-1}] \mid 1 \leq t \leq \frac{s-2}{2}\}$$

.

Now let:

$$T'' = S_1 \cup S_2 \cup S_3 \cup \{[ba^s, a^{2s-1}], [ba^{\frac{s}{2}+1}, a^{s+\frac{s}{2}+1}]\}.$$

The graph T'' is a tree, T' and T'' are disconnected and all together cover all the vertices of K_{4s}. Moreover, you can partition T'' into the following pairs of edges:

$T_0^* = \{[a^s, ba^s], [ba^{\frac{s}{2}+1}, a^{s+\frac{s}{2}+1}]\}$, $T_1^* = \{[a^s, ba^{s+1}], [ba^s, a^{2s-1}]\}$,

$T_{2j+1}^* = \{[a^s, ba^{s+2j+1}], [ba^{s+1}, a^{2s-2j}]\}$ with $1 \leq j \leq \frac{s-2}{2}, j \neq \frac{s-2}{4}$,

$T_{s-2t}^* = \{[ba^{2s-1}, a^{s+2t-1}], [a^s, ba^{s-2t}]\}$ with $1 \leq t \leq \frac{s-2}{2}$.

Observe that the edges of T_0^* have the same difference set $\{b, ba^s\}$, are in distinct orbits under $< a >$ and they belong to F_0^* and $F_0^* b$, respectively. In fact: $[1, b] \in F_0^*$, F_0^* is fixed by $< a >$ and then: $[a^s, ba^s] \in F_0^*$, moreover $[ba^{\frac{s}{2}+1}, a^{s+\frac{s}{2}+1}] = [ba^{\frac{s}{2}+1}, b^2 a^{\frac{s}{2}+1}] = [a^{-\frac{s}{2}-1} b, ba^{-\frac{s}{2}-1} b] = [a^{-\frac{s}{2}-1}, ba^{-\frac{s}{2}-1}] b \in F_0^* b$.

Observe that the edges of T_1^* have the same difference set $\{ba, ba^{s+1}\}$, are in distinct orbits under $< a >$ and they belong to F_1^* and $F_1^* b$, respectively. In fact: $[1, ba] \in F_1^*$, F_1^* is fixed by $< a >$ and then: $[a^s, ba^{s+1}] \in F_1^*$, moreover: $[a^s b, ba^{s+1} b] \in F_1^* b$ and $[a^s b, ba^{s+1} b] = [ba^s, b^2 a^{-s-1}] = [ba^s, a^{2s-1}]$.

For each $1 \leq j \leq \frac{s-2}{2}, j \neq \frac{s-2}{4}$, the edges of T_{2j+1}^* have the same difference set $\{ba^{2j+1}, ba^{s+2j+1}\}$, are in distinct orbits under $< a >$ and they belong to F_{2j+1}^* and $F_{2j+1}^* b$, respectively. In fact: $[1, ba^{2j+1}] \in F_{2j+1}^*$, F_{2j+1}^* is fixed by $< a >$ and then: $[a^s, ba^{s+2j+1}] \in F_{2j+1}^*$, moreover: $[a^{-s-1}, ba^{-s-1+2j+1}] b \in F_{2j+1}^* b$ and $[a^{-s-1} b, ba^{-s-1+2j+1} b] = [ba^{s+1}, b^2 a^{s-2j}] = [ba^{s+1}, a^{2s-2j}]$.

When $1 \leq t \leq \frac{s}{2} - 1$, the edges of T_{s-2t}^* have the same difference set $\{ba^{s-2t}, ba^{2s-2t}\}$, are in distinct orbits under $< a >$ and they belong to F_{s-2t}^* and $F_{s-2t}^* b$, respectively. In fact: $[1, ba^{s-2t}] \in F_{s-2t}^*$, F_{s-2t}^* is fixed by $< a >$ and then: $[1, ba^{s-2t}] a^{s+2t-1} = [a^{s+2t-1}, ba^{2s-1}] \in F_{s-2t}^*$, moreover: $[1, ba^{s-2t}] b \in F_{s-2t}^* b$ and $[b, ba^{s-2t} b] = [b, b^2 a^{-s+2t}] = [b, a^{2t}]$. Therefore $[b, a^{2t}] \in F_{s-2t}^* b$ and $[ba^{s-2t}, a^s] \in F_{s-2t}^* ba^{s-2t}$ with $F_{s-2t}^* ba^{s-2t} = F_{s-2t}^* a^{-s+2t} b = F_{s-2t}^* b$.

Therefore, the graph $R = T' \cup T''$ satisfies conditions (1) and (2) of Lemma 1. Let now $e_1 = [1, a^s] \in F_s$ and $e_2 = [b, ba^s] \in F_s$, they are in distinct orbits under $< a >$ and both connect T' and T'' in such a way that $R \cup \{e_1\}$ and $R \cup \{e_2\}$ satisfy condition (3) of Lemma 1. We conclude that $\mathcal{T} = \{T_1 a^i \mid 0 \leq i \leq s-1\} \cup \{T_2 a^i \mid 0 \leq i \leq s-1\}$ with $T_1 = R \cup \{e_1\}$ and $T_2 = Ra^s \cup \{e_2\}$ is a complete set of rainbow spanning trees.

If $s = 2$, the dicyclic group is the quaternion group Q_8 and it is easy to observe that $R = T' \cup T$" with $T' = \{[1, ba], [1, ba^3], [1, a], [b, ba], [ba, a^3]\}$ and T" $= \{[a^2, ba^2\}$ is rainbow and satisfies (1) and (2) of Lemma 1 and the above construction can be repeated with $e_1 = [1, a^2]$ and $e_2 = [b, ba^2]$.

For the readers' convenience, in the following Figures 1 we picture $R \cup$

$\{e_1\}$ and $Ra^s \cup \{e_2\}$ when $s = 2$ and we point out e_1 and e_2 with a different color. In the followig Figure 2 we show $R \cup \{e_1\}$ when $s = 6$, in particular we picture the sets T', T'' and the edge e_1 assigning a color to each of them.

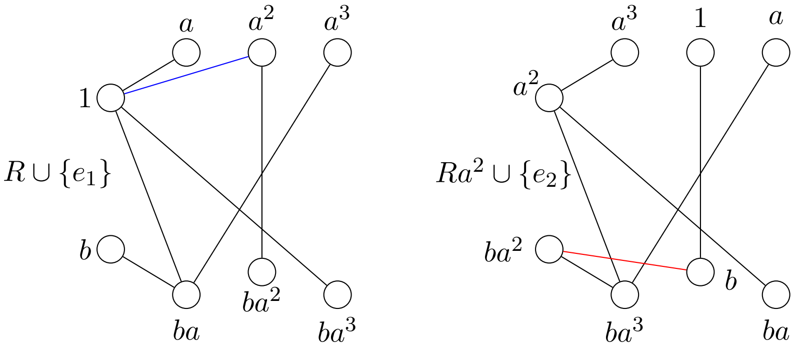

Figure 1: case $s = 2$. Group Q_8.

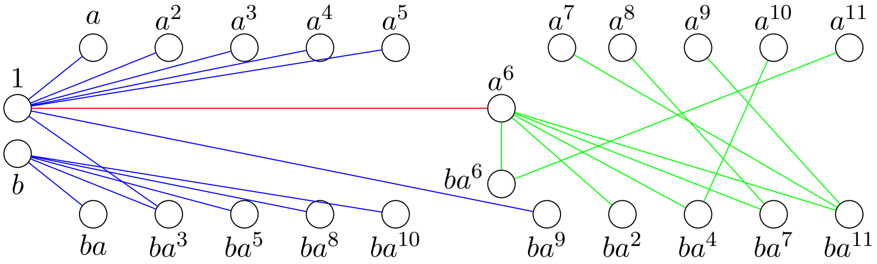

Figure 2: $R \cup \{e_1\}$, case $s = 6$. Dicyclic group of order 24.

Let $s \equiv 0 \pmod 4$.
With a slightly modification of the construction above, we construct a complete set of rainbow spanning trees. Namely, take $T' = T \cup (\bigcup_{i=0}^{\frac{s-2}{2}} T_{2i+1})$ exactly as above and recall that T' is a rainbow tree. It is the union of a star at 1 together with a star at b connected through the edge $[1, ba^{s+\frac{s}{2}}]$. Let

$$S_1 = \{[a^s, ba^{s-2t}], [a^s, ba^{s+2j+1}] \mid 0 \leq t \leq \frac{s-2}{2}, t \neq \frac{s}{4}, 0 \leq j \leq \frac{s-2}{2}\},$$

$$S_2 = \{[ba^{s+1}, a^{2s-2j}] \mid 1 \leq j \leq \frac{s-2}{2}\},$$

$$S_3 = \{[ba^{2s-1}, a^{s+2t-1}] \mid 1 \leq t \leq \frac{s-2}{2} \mid t \neq \frac{s}{4}\},$$

$$T'' = S_1 \cup S_2 \cup S_3 \cup \{[ba^s, a^{2s-1}]\}.$$

It is easy to observe that T'' and $T' \cup \{[ba^{\frac{s}{2}-1}, a^{s+\frac{s}{2}-1}\}$ are trees, they are disconnected and all together cover all the vertices of K_{4s}. Moreover, you can partition $T'' \cup \{[ba^{\frac{s}{2}-1}, a^{s+\frac{s}{2}-1}\}$ into the following pairs of edges:

$T^*_0 = \{[a^s, ba^s], [ba^{\frac{s}{2}-1}, a^{s+\frac{s}{2}-1}]\}$, $T^*_1 = \{[a^s, ba^{s+1}], [ba^s, a^{2s-1}]\}$,

$T^*_{2j+1} = \{[a^s, ba^{s+2j+1}], [ba^{s+1}, a^{2s-2j}]\}$ with $1 \leq j \leq \frac{s-2}{2}$,

$T^*_{s-2t} = \{[ba^{2s-1}, a^{s+2t-1}], [a^s, ba^{s-2t}]\}$ with $1 \leq t \leq \frac{s-2}{2}$, $\mid t \neq \frac{s}{4}$.

Proceeding as above, we can conclude that $R = T' \cup \{[ba^{\frac{s}{2}-1}, a^{s+\frac{s}{2}-1}\} \cup T''$ satisfies conditions (1) and (2) of Lemma 1. Taking $e_1 = [1, a^s] \in F_s$ and $e_2 = [b, ba^s] \in F_s$ the set $\mathcal{T} = \{T_1 a^i \mid 0 \leq i \leq s-1\} \cup \{T_2 a^i \mid 0 \leq i \leq s-1\}$, with $T_1 = R \cup \{e_1\}$ and $T_2 = Ra^s \cup \{e_2\}$, is a complete set of rainbow spanning trees.

In the followig Figure 3 we show $R \cup \{e_1\}$ when $s = 4$, in particular we picture the sets $T' \cup \{[ba^{\frac{s}{2}-1}, a^{s+\frac{s}{2}-1}]\}$, T'' and the edge e_1 assigning a color to each of them.

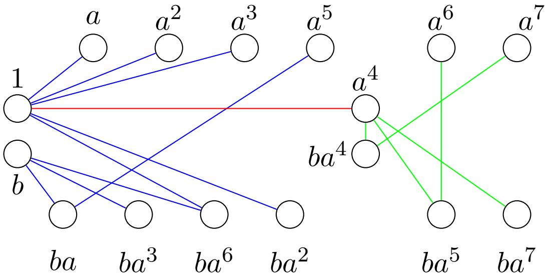

Figure 3: $R \cup \{e_1\}$, case $s = 4$. Dicyclic group of order 16.

2.2 Case s odd

Consider the forest T' induced by the follwing set of edges:

$$\{[1, a^{2t}], [b, ba^{2s-2t}], [1, a^{2t-1}], [b, ba^{2s-2t+1}] \mid 1 \leq t \leq \frac{s-1}{2}\} \cup$$

$$\cup \{[a^{\frac{s+1}{2}}, ba^s], [a^s, ba^{\frac{s-1}{2}}]\}.$$

Observe that T' is rainbow as it contains exactly one edge for each $1-$factor of the set $\{Fa^i, Fba^i \mid 1 \leq i \leq s-1\}$.

More precisely: $[1, a^{2t}] \in Fa^{\frac{s+1}{2}+t}$, $[b, ba^{2s-2t}] \in Fba^{\frac{s-1}{2}-t}$, $[1, a^{2t-1}] \in Fba^{\frac{s-3}{2}+t}$, $[b, ba^{2s-2t+1}] \in Fa^{\frac{s+3}{2}-t}$, $1 \leq t \leq \frac{s-1}{2}$, and also $[a^{\frac{s+1}{2}}, ba^s] \in Fa$, $[a^s, ba^{\frac{s-1}{2}}] \in Fba^{s-1}$.

In fact: $[1, a^{2t}] = [a^{\frac{s-1}{2}-t}, a^{s-\frac{s-1}{2}+t-1}]a^{-\frac{s-1}{2}+t} \in Fa^{-\frac{s-1}{2}+t} = Fa^{\frac{s+1}{2}+t}$,

$[b, ba^{2s-2t}] = [1, a^{2t}]b \in Fa^{\frac{s+1}{2}+t}b = Fa^{s-\frac{s-1}{2}+t}b = Fba^{\frac{s-1}{2}-t}$,
$[1, a^{2t-1}] = [ba^{\frac{s-3}{2}+t}, ba^{\frac{s-1}{2}-t}]ba^{s+\frac{s-3}{2}+t} \in Fba^{s+\frac{s-3}{2}+t} = Fba^{\frac{s-3}{2}+t}$,
$[b, ba^{2s-2t+1}] = [1, a^{2t-1}]b \in Fba^{\frac{s-3}{2}+t}b = Fa^{s-\frac{s-3}{2}-t} = Fa^{\frac{s+3}{2}-t}$,
$[a^{\frac{s+1}{2}}, ba^s] = [a^{\frac{s-1}{2}}, ba^{s-1}]a \in Fa$.
Moreover, you can partition T' into the following pairs of edges:
$\{[1, a^{2t}], [b, ba^{2s-2t}]\}$, $\{[1, a^{2t-1}], [b, ba^{2s-2t+1}]\}$, $1 \leq t \leq \frac{s-1}{2}$ and $\{[a^{\frac{s+1}{2}}, ba^s], [a^s, ba^{\frac{s-1}{2}}]\}$. Two edges in the same pair are in distinct orbits under $< a >$ and have the same difference set. Namely: $\partial[1, a^{2t}] = \partial[b, ba^{2s-2t}] = \{a^{2t}, a^{2s-2t}\}$, $\partial[1, a^{2t-1}] = \partial[b, ba^{2s-2t+1}] = \{a^{2t-1}, a^{2s-2t+1}\}$, $1 \leq t \leq \frac{s-1}{2}$, and $\partial[a^{\frac{s+1}{2}}, ba^s] = \partial[a^s, ba^{\frac{s-1}{2}}] = \{ba^{s+\frac{s-1}{2}}, ba^{\frac{s-1}{2}}\}$.
Consider the forest T'' induced by the following set of edges:

$$\{[1, ba^i], [ba^s, a^{2s-i}] \mid 1 \leq i \leq s-1,\ i \neq \frac{s-1}{2}\} \cup$$

$$\cup \{[a^{s+\frac{s-1}{2}}, ba^{\frac{s-1}{2}}], [a^{s+\frac{s+1}{2}}, ba^{s+\frac{s+1}{2}}]\}$$

.

Observe that: $[a^{s+\frac{s+1}{2}}, ba^{s+\frac{s+1}{2}}] \in F_0^*$ and $[a^{s+\frac{s-1}{2}}, ba^{\frac{s-1}{2}}] \in F_0^*b$. In fact: the first edge is contained in $Orb_{<a>}([1, b])$, while $[a^{s+\frac{s-1}{2}}, ba^{\frac{s-1}{2}}]$ is contained in $Orb_{<a>}([a^s, b])$ with $[a^s, b] = [1, b]b \in F_0^*b$. Moreover, these two edges have the same difference set and are in distinct orbits under $< a >$.
For each i, with $1 \leq i \leq s-1,\ i \neq \frac{s-1}{2}$, we obviously have $[1, ba^i] \in F_i^*$ and $[ba^s, a^{2s-i}] \in F_i^*b$ and both these edges have the same difference set and are in distinct orbits under $< a >$.
The graph $T' \cup T''$ covers all the vertices of K_{4s} and it is formed by two connected components. Namely: a first component is given by a star at 1 connected to a star at ba^s through the edge $[a^{\frac{s+1}{2}}, ba^s]$, plus the two edges $[ba^{\frac{s-1}{2}}, a^s]$, $[ba^{\frac{s-1}{2}}, a^{s+\frac{s-1}{2}}]$. A second component is given by a star at b plus the edge $[ba^{s+\frac{s+1}{2}}, a^{s+\frac{s+1}{2}}]$.
Moreover $R = T' \cup T''$ satisfies conditions (1) and (2) of Lemma 1.
Taking $e_1 = [a^{\frac{s+1}{2}}, a^{s+\frac{s+1}{2}}] \in F_s$ and $e_2 = [ba^{\frac{s+1}{2}}, ba^{s+\frac{s+1}{2}}] \in F_s$ the set $\mathcal{T} = \{T_1 a^i \mid 0 \leq i \leq s-1\} \cup \{T_2 a^i \mid 0 \leq i \leq s-1\}$, with $T_1 = R \cup \{e_1\}$ and $T_2 = Ra^s \cup \{e_2\}$, is a complete set of rainbow spanning trees.

In the followig Figure 4 we show $R \cup \{e_1\}$ when $s = 5$, in particular we picture the sets T', T'' and the edge e_1 assigning a color to each of them.

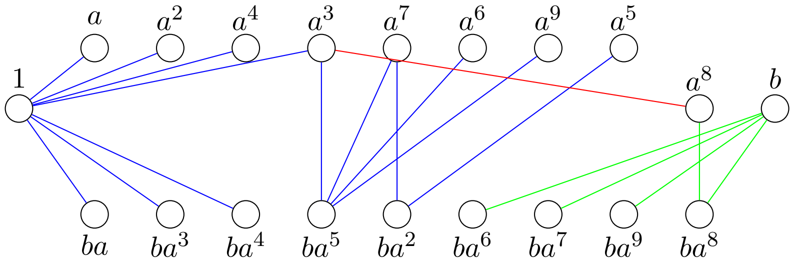

Figure 4: $R \cup \{e_1\}$, case $s = 5$. Dicyclic group of order 20.

3 Abelian groups with a cyclic subgroup of index 2 and complete sets of rainbow spanning trees

Let G be an abelian non-cyclic group G of order $2n$, n even, possesing a cyclic subgroup H of index 2. It is well-known that G is the direct product of the latter subgroup by a cyclic group, say K, of order 2. ?????CITARE???? We have $G = KH$ with $K =< b >$ and $H =< a >$ and G has three involutions: $b, a^{\frac{n}{2}}, ba^{\frac{n}{2}}$.
A starter can be constructed as follows:

If $n > 4$ let $\Sigma = \{S, S'\} \cup \{S_i, 1 \leq i \leq \frac{n}{2} - 1,\ i \neq \frac{n}{4}\} \cup \{S_1^*, S_2^*, S^*\}$.

If $n = 4$ let $\Sigma = \{S, S'\} \cup \{S_1^*, S_2^*, S^*\}$.

With:

$S = \{[a^i, a^{-i+1}],\ 1 \leq i \leq \frac{n}{4}\}$; $\quad S' = \{[a^i, a^{\frac{n}{2}-i}],\ 1 \leq i < \frac{n}{4}\} \cup \{[1, ba^{\frac{n}{4}}\}$;

$S_i = \{[1, ba^i]\}$, $1 \leq i \leq \frac{n}{2} - 1$, $i \neq \frac{n}{4}$; $S_1^* = \{[1, b]\}$; $S_2^* = \{[1, ba^{\frac{n}{2}}]\}$;

$S^* = \{[1, a^{\frac{n}{2}}]\}$.

We have:

$\partial S = \{a^{2i-1}, 1 \leq i \leq \frac{n}{2}\}$; $\partial S' = \{a^{n-2r}, 1 \leq r < \frac{n}{2}\} \cup \{ba^{\frac{n}{4}}, ba^{\frac{3n}{4}}\}$; and both $\phi(S)$ and $\phi(S')$ are left transversal for the subgroup $I = \{1, b, a^{\frac{n}{2}}, ba^{\frac{n}{2}}\}$.

$\partial S_i = \{ba^i, ba^{n-i}\}$ and $\phi(S_i)$ is left transversal for $< a >$.

Finally, we have: $\partial S_1^* = \{b\}$, $\partial S_2^* = \{ba^{\frac{n}{2}}\}$, $\partial S^* = \{a^{\frac{n}{2}}\}$. Moreover, $\phi(S_1^*) = \phi(S_2^*) = \phi(S^*) = \{1\}$.

With the starter above, we construct the following $1-$factors:

$F_S = Orb_I(S)$ whose orbit under G gives the $1-$factors: $F_S, F_S a, \ldots, F_S a^{\frac{n}{2}-1}$.

$F_{S'} = Orb_I(S')$ whose orbit under G gives the $1-$factors: $F_{S'}, F_{S'} a, \ldots, F_{S'} a^{\frac{n}{2}-1}$.

$F_i = Orb_{<a>}(S_i)$ whose orbit under G gives the $1-$factors: F_i, F_ib, for each i with $1 \leq i \leq \frac{n}{2} - 1$, $i \neq \frac{n}{4}$. .

Finally, we have the three fixed $1-$factors $F_1^* = Orb_G([1, b])$, $F_2^* = Orb_G([1, ba^{\frac{n}{2}}])$, $F^* = Orb_G([1, a^{\frac{n}{2}}])$.

Consider the graph R_1 induced by the following set of edges:

$$\{[1, a^{2i-1}], [ba^{\frac{n}{4}}, ba^{\frac{n}{4}+2i-1}], 1 \leq i \leq \frac{n}{4}\}$$

The $\frac{n}{2}$ edges of R_1 belong to the $\frac{n}{2}$ distinct $1-$factors F_S, F_Sa, $\ldots$, $F_Sa^{\frac{n}{2}-1}$. In fact: $[1, a^{2i-1}] = [a^i, a^{-i+1}]a^{i-1} \in F_Sa^{i-1}$ and $[ba^{\frac{n}{4}}, ba^{\frac{n}{4}+2i-1}] = [1, a^{2i-1}]ba^{\frac{n}{4}} \in F_Sa^{\frac{n}{4}+i-1}$. Moreover, for every i, the two edges $[1, a^{2i-1}]$ and $[ba^{\frac{n}{4}}, ba^{\frac{n}{4}+2i-1}]$ have the same difference set and are in distinct orbits under $< a >$.

Consider the graph R_2 induced by the following set of edges:

$$\{[1, a^{\frac{n}{2}-2i}], [ba^{\frac{n}{4}}, ba^{\frac{3n}{4}-2i}], 1 \leq i < \frac{n}{4}\} \cup \{[1, ba^{\frac{n}{4}}], [ba^{\frac{n}{4}}, a^{\frac{n}{2}}]\}$$

The $\frac{n}{2}$ edges of R_2 belong to the $\frac{n}{2}$ distinct $1-$factors $F_{S'}$, $F_{S'}a$, $\ldots$, $F_{S'}a^{\frac{n}{2}-1}$. In fact: $[1, ba^{\frac{n}{4}}] \in F_{S'}$ and $[1, ba^{\frac{n}{4}}]ba^{\frac{n}{4}} = [ba^{\frac{n}{4}}, a^{\frac{n}{2}}] \in F_{S'}a^{\frac{n}{4}}$. Moreover, for every i, the two edges $[1, a^{\frac{n}{2}-2i}]$ and $[ba^{\frac{n}{4}}, ba^{\frac{n}{4}+2i-1}]$ have the same difference set and are in distinct orbits under $< a >$.
$[1, a^{\frac{n}{2}-2i}] = [a^i, a^{\frac{n}{2}-i}]a^{-i} \in F_{S'}a^{\frac{n}{2}-i}$ and $[ba^{\frac{n}{4}}, ba^{\frac{3n}{4}-2i}] = [1, a^{\frac{n}{2}-2i}]ba^{\frac{n}{4}} \in F_{S'}a^{\frac{n}{4}-i}$. Moreover, for every i, these two edges have the same difference set and are in distinct orbits under $< a >$.
Observe that $R_2 = \{[1, ba], [ba, a^2]\}$ whenever $n = 4$.

If $n > 4$, consider the graph R_3 induced by the following set of edges:

$$\{[a^{\frac{n}{4}+2}, ba^{\frac{3n}{4}+1}], [b, a^{\frac{n}{2}-1}]\} \cup \{[1, ba^i], [ba^{\frac{3n}{4}}, a^{\frac{3n}{4}+i}], 1 \leq i \leq \frac{n}{4} - 1\} \cup$$

$$\cup \{[a^{\frac{n}{2}+1}, ba^{\frac{3n}{4}+i+1}], [ba^{\frac{n}{2}-2i}, a^{\frac{3n}{4}-i}], 1 \leq i \leq \frac{n}{4} - 2\}$$

The $n - 4$ edges of R_3 belong to the $n - 4$ distinct $1-$factors F_i, F_ib, $1 \leq i \leq \frac{n}{2} - 1$, $i \neq \frac{n}{4}$. In fact:
$[1, ba^i] \in F_i$ and $[ba^{\frac{3n}{4}}, a^{\frac{3n}{4}+i}] \in F_ib$, $1 \leq i \leq \frac{n}{4} - 1$. Moreover, for each fixed i, these two edges have the same difference set and are in distinct orbits under $< a >$.
$[a^{\frac{n}{2}+1}, ba^{\frac{3n}{4}+i+1}] \in F_{\frac{n}{4}+i}$ and $[ba^{\frac{n}{2}-2i}, a^{\frac{3n}{4}-i}] \in F_{\frac{n}{4}+i}b$, $1 \leq i \leq \frac{n}{4} - 2$. Moreover, for each fixed i, these two edges have the same difference set and are in distinct orbits under $< a >$.
$[a^{\frac{n}{4}+2}, ba^{\frac{3n}{4}+1}] \in F_{\frac{n}{2}-1}$ and $[b, a^{\frac{n}{2}-1}] \in F_{\frac{n}{2}-1}b$. Also these two edges have the same difference set and are in distinct orbits under $< a >$.

Finally, if $n > 4$, let R_4 be the graph induced by the following two short edges: $[a^{\frac{3n}{4}}, ba^{\frac{3n}{4}}] \in F_1^*$, $[ba, a^{\frac{n}{2}+1}] \in F_2^*$, while if $n = 4$ let R_4 be the graph induced by the following two short edges: $[a^3, ba^3] \in F_1^*$, $[b, ba^3] \in F_2^*$.

If $n > 4$, let $R = R_1 \cup R_2 \cup R_3 \cup R_4$, while let $R = R_1 \cup R_2 \cup R_4$ whenever $n = 4$. As above observed, the graph R satisfies conditions (1) and (2) of Lemma 1. Moreover, the graph R has two connected components. In fact, if $n = 4$ the two connected components are clearly indicated in the following figure 5. If $n > 4$, a component is given by the star at $ba^{\frac{3n}{4}}$ containing all the vertices $\{a^{\frac{3n}{4}+i}, 0 \leq i \leq \frac{n}{4} - 1\}$; the other component is obtained as follows: a star at 1 containing all the vertices $\{a^i, 1 \leq i \leq \frac{n}{2} - 1\} \cup \{ba^i, 1 \leq i \leq \frac{n}{4}\}$ plus the edge $[b, a^{\frac{n}{2}-1}]$ with b of degree 1 and the edge $[a^{\frac{n}{4}+2}, ba^{\frac{3n}{4}+1}]$ with $ba^{\frac{3n}{4}+1}$ of degree 1; a star at $a^{\frac{n}{2}+1}$ containing all the vertices $\{ba^{\frac{3n}{4}+i}, 2 \leq i \leq \frac{n}{4} - 1\} \cup \{ba\}$ and which has just the vertex ba in common with the star at 1; a star at $ba^{\frac{n}{4}}$ containing the vertex $a^{\frac{n}{2}}$ together with all the vertices $\{ba^{\frac{n}{4}+i}, 1 \leq i \leq \frac{n}{2} - 1\}$. This star has the unique vertex $ba^{\frac{n}{4}}$ in common with the star at 1 and no vertex in common with the star at $a^{\frac{n}{2}+1}$. Finally, we have the edges $[ba^{\frac{n}{2}-2i}, a^{\frac{3n}{4}-i}]$, $1 \leq i \leq \frac{n}{4} - 2$ which are connected to the star $ba^{\frac{n}{4}}$ and the vertices $a^{\frac{3n}{4}-i}$ have degree 1. Therefore, this component is a tree.

If $n = 4$, let $e_1 = [a, a^3] \in F^*$ and $e_2 = [b, ba^2] \in F^*$, while if $n > 4$, let $e_1 = [a^{\frac{3n}{4}}, a^{\frac{n}{4}}] \in F^*$ and $e_2 = [ba^{\frac{3n}{4}}, ba^{\frac{n}{4}}] \in F^*$. These two edges are in distinct orbits under $< a >$ and the graphs $T_1 = R \cup \{e_1\}$ and $T_2 = R \cup \{e_2\}$ satisfy condition (3) of Lemma 1. Therefore, the set $\mathcal{T} = \{T_1 a^i \mid 0 \leq i \leq s - 1\} \cup \{T_2 a^i \mid 0 \leq i \leq s - 1\}$ is a complete set of rainbow spanning trees. In the following Figures 5 and 6 we show $R \cup \{e_1\}$. In particular, we picture the two connected components of R and the edge e_1 assigning a color to each of them.

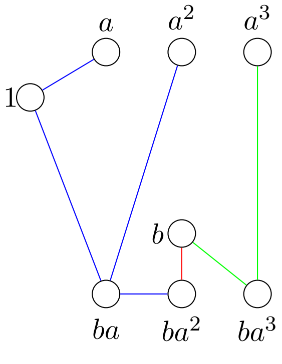

Figure 5: Group $\mathbb{Z}_2 \times \mathbb{Z}_4$.

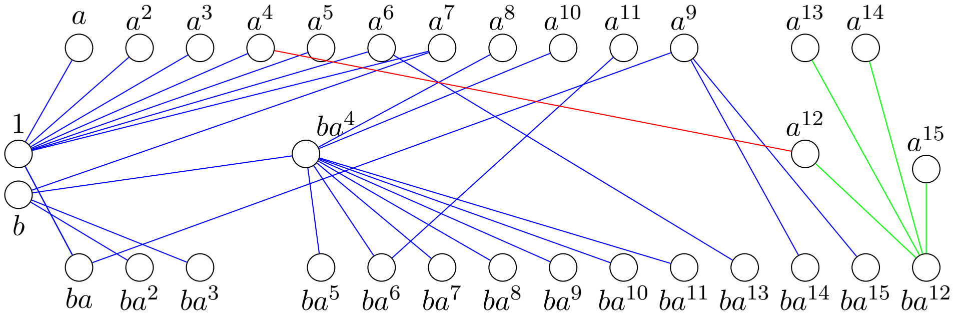

Figure 6: $R \cup \{e_1\}$ Group $\mathbb{Z}_2 \times \mathbb{Z}_{16}$

4 2−groups with a cyclic subgroup of index 2 and complete sets of rainbow spanning trees

Apart from the abelian groups, the dihedral groups and the generalized quaternion groups, for which we refer to [23], [29] and to the previous sections, respectively, there are two more isomorphism types of groups of order $2n = 2^{m+1}$ with a cyclic subgroup of index 2, see Satz 14.9 in [19]. In particular, it is $n \geq 8$ and they can be presented as follows, [19, p.91]:

(i) $\quad G = \langle a, b : a^n = b^2 = 1, bab = a^{\frac{n}{2}-1}\rangle$ (semidihedral group)

(ii) $\quad G = \langle a, b : a^n = b^2 = 1, bab = a^{\frac{n}{2}+1}\rangle$

We consider these two cases separately. For each case, we exhibit a starter, a 1−factorization and a complete set of rainbow spanning trees.

Case (i).

Let $0 \leq r \leq n-1$. Observe that $a^r b = ba^{-r}$ whenever r is even, while $a^r b = ba^{\frac{n}{2}-r}$ whenever r is odd. Moreover, G contains exactly $\frac{n}{2}+1$ involutions: $a^{\frac{n}{2}}$ and ba^r, with r even and $0 \leq r \leq n-2$.

A starter can be constructed as follows:

$$\Sigma = \{S\} \cup \{S_{2t+1}, 0 \leq t \leq \frac{n}{4}-1\} \cup \{S_{2s}, 0 \leq s \leq \frac{n}{2}-1\} \cup$$

$$\cup \{S'_{2r+1}, 0 \leq r \leq \frac{n}{4}-1, r \neq \frac{n}{8}\} \cup \{S^*\}$$

.

With:

$S = \{[a^t, a^{-t}],\ 1 \leq t \leq \frac{n}{4}-1\} \cup \{[1, ba^{\frac{n}{4}+1}]\}$.

$S_{2t+1} = \{[1, a^{2t+1}]\}$, $0 \leq t \leq \frac{n}{4}-1$.

$S_{2s} = \{[1, ba^{2s}]\}$, $0 \leq s \leq \frac{n}{2}-1$.

$S'_{2r+1} = \{[1, ba^{2r+1}]\}$, $0 \leq r \leq \frac{n}{4} - 1$, $r \neq \frac{n}{8}$.

$S^* = \{[1, a^{\frac{n}{2}}]\}$.

We have:

$\partial S = \{a^{2t}, a^{-2t}, 1 \leq t \leq \frac{n}{4} - 1\} \cup \{ba^{\frac{n}{4}+1}, ba^{-\frac{n}{4}+1}\}$ and $\phi(S)$ is a left transversal for the subgroup $< ba >= \{1, ba, a^{\frac{n}{2}}, ba^{\frac{n}{2}+1}\}$.

For each t, $0 \leq t \leq \frac{n}{4} - 1$, we have: $\partial S_{2t+1} = \{a^{2t+1}, a^{-2t-1}\}$ and $\phi(S_{2t+1})$ is a left transversal for the subgroup $< a^2, b >= \{a^{2m}, ba^{2m}, 1 \leq m \leq \frac{n}{2}\}$.

For each s, $0 \leq s \leq \frac{n}{2} - 1$, we have: $\partial S_{2s} = \{ba^{2s}\}$ and $\phi(S_{2s}) = \{1\}$.

For each r, $0 \leq r \leq \frac{n}{4} - 1$, $r \neq \frac{n}{8}$, we have: $\partial S'_{2r+1} = \{ba^{2r+1}, ba^{\frac{n}{2}+2r+1}\}$ and $\phi(S'_{2r+1})$ is a left transversal for the subgroup $< a >$.

Finally, we have $\partial S^* = \{a^{\frac{n}{2}}\}$ and and $\phi(S^*) = \{1\}$.

With the starter above, we construct the following $1-$factors:

$F = Orb_{<ba>}(S)$ whose orbit under G gives the $1-$factors: $F, Fa, \ldots, Fa^{\frac{n}{2}-1}$.

For each t, $0 \leq t \leq \frac{n}{4} - 1$, we obtain the $1-$factors F_{2t+1} and $F_{2t+1}a$, with $F_{2t+1} = Orb_{<a^2,b>}[1, a^{2t+1}]$.

For each s, $0 \leq s \leq \frac{n}{2} - 1$, we have the fixed $1-$factor $F_{2s} = Orb_G([1, ba^{2s}])$.

For each r, $0 \leq r \leq \frac{n}{4} - 1$, $r \neq \frac{n}{8}$, we obtain the $1-$factors F'_{2r+1} and $F'_{2r+1}b$, with $F'_{2r+1} = Orb_{<a>}[1, ba^{2r+1}]$.

We also have the fixed $1-$factor $F^* = Orb_G([1, a^{\frac{n}{2}}])$.

Consider the graph R_1 induced by the following set of edges:

$$\{[1, ba^{\frac{n}{4}+1}], [b, a^{\frac{n}{4}-1}]\} \cup \{[1, a^{2t}], [b, ba^{\frac{n}{2}+2t}], 1 \leq t \leq \frac{n}{4} - 1\}$$

The $\frac{n}{2}$ edges of R_1 belongs to the $\frac{n}{2}$ distinct $1-$factors F, Fa, $\ldots$, $Fa^{\frac{n}{2}-1}$. In fact, observe that $[1, ba^{\frac{n}{4}+1}] \in F$ and $[b, a^{\frac{n}{4}-1}] = [1, ba^{\frac{n}{4}+1}]b \in Fb = Fba^{\frac{n}{2}+1}a^{\frac{n}{2}-1} = Fa^{\frac{n}{2}-1}$. Moreover, these two edges have the same difference set and are in distinct orbits under $< a >$.
Observe also that $[1, a^{2t}] = [a^{-t}, a^t]a^t \in Fa^t$ and $[b, ba^{\frac{n}{2}+2t}] = [1, a^{-2(\frac{n}{4}+t)}]b \in Fa^{-(\frac{n}{4}+t)}b = Fa^{\frac{n}{4}+t-1}$ in fact: if t is even we have $Fa^{-(\frac{n}{4}+t)}b = Fba^{\frac{n}{4}+t} = Fbaa^{\frac{n}{4}+t-1} = Fa^{\frac{n}{4}+t-1}$, while if t is odd we have: $Fa^{-(\frac{n}{4}+t)}b = Fba^{\frac{n}{2}+\frac{n}{4}+t} = Fba^{\frac{n}{2}+1}a^{\frac{n}{4}+t-1} = Fa^{\frac{n}{4}+t-1}$. Moreover, for each t, $1 \leq t \leq \frac{n}{4} - 1$, the two edges $[1, a^{2t}]$ and $[b, ba^{\frac{n}{2}+2t}]$ have the same difference set and they are in distinct orbits under $< a >$.

Let R_2 be the graph induced by the following set of edges:

$$\{[1, a^{2t+1}], [ba, ba^{\frac{n}{2}-2t}], 0 \leq t \leq \frac{n}{4} - 1\}$$

The $\frac{n}{2}$ edges of R_2 belongs to the $\frac{n}{2}$ distinct $1-$factors F_{2t+1}, $F_{2t+1}a$, with $0 \leq t \leq \frac{n}{4} - 1$. In fact, it is $[1, a^{2t+1}] \in F_{2t+1}$ and $[1, a^{2t+1}]ba = [ba, ba^{\frac{n}{2}-2t}] \in F_{2t+1}ba = F_{2t+1}a$. Morevore, it is $\partial[1, a^{2t+1}] = \partial[ba, ba^{\frac{n}{2}-2t}]$ and, for each t with $0 \leq t \leq \frac{n}{4} - 1$, these two edges are in distinct orbits under $< a >$.
Let R_3 be the graph induced by the following set of edges:

$$\{[a^{\frac{n}{2}+\frac{n}{4}-2}, ba^{\frac{n}{4}-2}]\} \cup \{[a^{\frac{n}{2}+1}, ba^{2t+1}], 1 \leq t \leq \frac{n}{2} - 1\}$$

The $\frac{n}{2}$ edges of R_3 are short and they belongs to the $\frac{n}{2}$ distinct fixed $1-$factors F_{2s}, $0 \leq s \leq \frac{n}{2} - 1$. In fact: $\partial[a^{\frac{n}{2}+1}, ba^{2t+1}] = \{ba^{2t+\frac{n}{2}}\}$ with $1 \leq t \leq \frac{n}{2} - 1$. If $1 \leq t \leq \frac{n}{4} - 1$, we have $[a^{\frac{n}{2}+1}, ba^{2t+1}] \in F_{\frac{n}{2}+2t} = F_{2s}$ with $\frac{n}{4} + 1 \leq s \leq \frac{n}{2} - 1$. If $\frac{n}{4} \leq t \leq \frac{n}{2} - 1$, we have $[a^{\frac{n}{2}+1}, ba^{2t+1}] \in F_{\frac{n}{2}+2t} = F_{2s}$ with $0 \leq s \leq \frac{n}{4} - 1$. Finally $\partial[a^{\frac{n}{2}+\frac{n}{4}-2}, ba^{\frac{n}{4}-2}] = ba^{\frac{n}{2}}$ and $[a^{\frac{n}{2}+\frac{n}{4}-2}, ba^{\frac{n}{4}-2}] \in F_{\frac{n}{2}}$.
If $n = 8$ let R_4 be the graph induced by the following set of edges:

$$\{[ba^7, a^6], [a^7, ba^4]\}$$

While, if $n > 8$, let R_4 be the graph induced by the following set of edges:

$$\{[ba^{n-1}, a^{n-2r-2}], [a^{\frac{n}{2}+2r+3}, ba^{4r+4}], 0 \leq r \leq \frac{n}{4} - 2, r \neq \frac{n}{8}\} \cup$$

$$\cup \{[ba^{n-1}, a^{\frac{n}{2}}], [a^{\frac{n}{2}+\frac{n}{4}+3}, ba^{\frac{n}{2}+\frac{n}{4}+2}]\}$$

If $n = 8$, we have: $\partial[ba^7, a^6] = \partial[a^7, ba^4] = \{ba, ba^5\}$, these two edges are in distinct orbits under $< a >$ with $[ba^7, a^6] \in F'_1$ and $[a^7, ba^4] = [ba^7, a^6]a^6b \in F'_1 b$ since F'_1 is fixed by $< a >$.
If $n > 8$, we have:
$\partial[ba^{n-1}, a^{n-2r-2}] = \partial[a^{\frac{n}{2}+2r+3}, ba^{4r+4}] = \{ba^{2r+1}, ba^{\frac{n}{2}+2r+1}\}$ for each fixed r, with $0 \leq r \leq \frac{n}{4} - 2$, $r \neq \frac{n}{8}$. Moreover, these two edges are in distinct orbits under $< a >$ and we have; $[ba^{n-1}, a^{n-2r-2}] \in F'_{2r+1}$ and $[a^{\frac{n}{2}+2r+3}, ba^{4r+4}] = [ba^{n-1}, a^{n-2r-2}]a^{-2r-2}b \in F'_{2r+1}b$ since F'_{2r+1} is fixed by $< a >$. Moreover, we have $\partial[ba^{n-1}, a^{\frac{n}{2}}] = \partial[a^{\frac{n}{2}+\frac{n}{4}+3}, ba^{\frac{n}{2}+\frac{n}{4}+2}] = \{ba^{\frac{n}{2}-1}, ba^{n-1}\}$. These two edges are in distinct orbits under $< a >$ with $[ba^{n-1}, a^{\frac{n}{2}}] \in F'_{\frac{n}{2}-1}$ and $[a^{\frac{n}{2}+\frac{n}{4}+3}, ba^{\frac{n}{2}+\frac{n}{4}+2}] = [ba^{n-1}, a^{\frac{n}{2}}]a^{-\frac{n}{4}-2}b \in F'_{\frac{n}{2}-1}b$ since $F'_{\frac{n}{2}-1}$ is fixed by $< a >$.
The graph $R = R_1 \cup R_2 \cup R_3 \cup R_4$ satisfies conditions (1) and (2) of Lemma 1.
If $n = 8$, the graph R has two connected components: one is given by the three edges: $[ba, ba^2], [ba, ba^4], [ba^4, a^7]$ and the other by the remaining ones.

Let $e_1 = [a^3, a^7] \in F^*$ and $e_2 = [b, ba^4] \in F^*$, these two edges are in distinct orbits under $< a >$ and the graphs $T_1 = R \cup \{e_1\}$ and $T_2 = R \cup \{e_2\}$ satisfy condition (3) of Lemma 1. Therefore, the set $\mathcal{T} = \{T_1 a^i \mid 0 \leq i \leq 3\} \cup \{T_2 a^i \mid 0 \leq i \leq 3\}$ is a complete set of rainbow spanning trees.

If $n > 8$, the graph R has two connected components, both without cycles. More precisely, one component, say R' is given by a star at ba together with the edges of the set $\{[ba^{\frac{n}{4}-2}, a^{\frac{n}{2}+\frac{n}{4}-2}]\} \cup \{[ba^{4t+4}, a^{\frac{n}{2}+2t+3}] \mid t = 0, \ldots, \frac{n}{8}-1\}$. The other component, say R'', is given by four stars: a star at 1, a star at b, a star at $a^{\frac{n}{2}+1}$ and a star at ba^{n-1}. The stars at 1 and at b are connected through the unique common vertex $a^{\frac{n}{4}-1}$. Their union is connected to the star at $a^{\frac{n}{2}+1}$ through the unique common vertex $ba^{\frac{n}{4}+1}$. Finally, the union of these three stars is connected to the star at ba^{n-1} through the unique common edge $[a^{\frac{n}{2}+1}, ba^{n-1}]$. Both these connected components have no cycles. Let $e_1 = [a^{\frac{n}{2}+\frac{n}{4}-2}, a^{\frac{n}{4}-2}] \in F^*$ and and $e_2 = [b, ba^{\frac{n}{2}}] \in F^*$. Observe that $a^{\frac{n}{2}+\frac{n}{4}-2}$ is a vertex of R' while $a^{\frac{n}{4}-2}$ is a vertex of R'', in the same manner b is a vertex of R'' while $ba^{\frac{n}{2}}$ is a vertex of R'. Moreover, e_1 and e_2 are in distinct orbits under $< a >$ and then $T_1 = R \cup \{e_1\}$ and $T_2 = R \cup \{e_2\}$ satisfy condition (3) of Lemma 1. Now, the set $\mathcal{T} = \{T_1 a^i \mid 0 \leq i \leq \frac{n}{2} - 1\} \cup \{T_2 a^i \mid 0 \leq i \leq \frac{n}{2} - 1\}$ is a complete set of rainbow spanning trees.

In the following Figures 7 and 8 we show $R \cup \{e_1\}$ when either $n = 8$ or $n = 16$. In particular we picture the two connected components of R and the edge e_1 assigning a color to each of them.

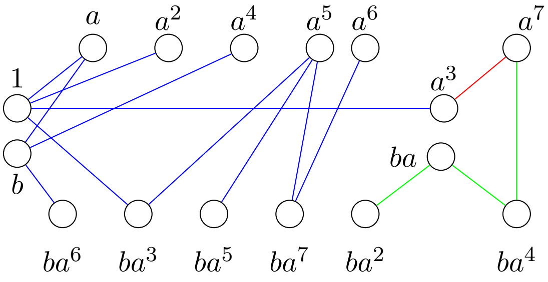

Figure 7: $R \cup \{e_1\}$, case (i) with $n = 8$

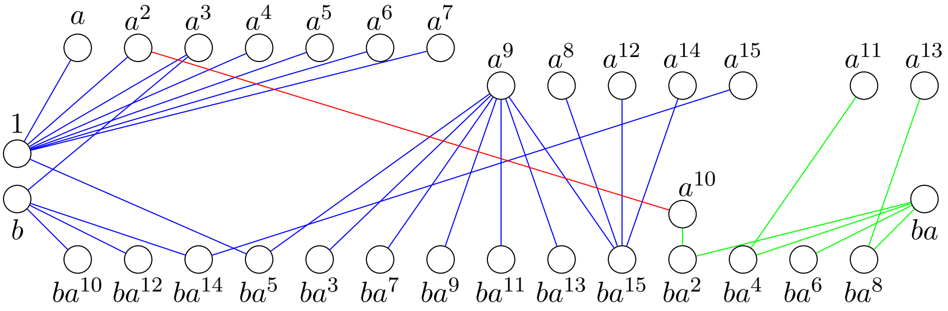

Figure 8: $R \cup \{e_1\}$, case (i) with $n = 16$

Case (ii)
Let $0 \leq r \leq n-1$. Observe that $a^r b = ba^r$ whenever r is even, while $a^r b = ba^{\frac{n}{2}+r}$ whenever r is odd. Moreover, G contains exactly 3 involutions: $a^{\frac{n}{2}}$, b and $ba^{\frac{n}{2}}$.
Let $n > 8$. A starter can be constructed as follows:

$\Sigma = \{S\} \cup \{S_{2t+1}, 0 \leq t \leq \frac{n}{8} - 1 \text{ and } \frac{n}{4} \leq t \leq \frac{n}{4} + \frac{n}{8} - 1\} \cup \{S_{2s}, 1 \leq s \leq \frac{n}{4} - 1, s \neq \frac{n}{8}\} \cup \{S_1^*, S_2^*, S^*\}$

With:

$S = \{[a^t, a^{\frac{n}{2}-t-1}],\ 0 \leq t \leq \frac{n}{4} - 1\} \cup \{[a^{\frac{n}{2}+s}, a^{n-s}],\ 1 \leq s \leq \frac{n}{4} - 1\} \cup \{[a^{\frac{n}{2}+\frac{n}{4}}, ba^{\frac{n}{2}}]\}$.

$S_{2t+1} = \{[1, ba^{2t+1}]\}$, $0 \leq t \leq \frac{n}{8} - 1$ and $\frac{n}{4} \leq t \leq \frac{n}{4} + \frac{n}{8} - 1$.

$S_{2s} = \{[1, ba^{2s}]\}$, $1 \leq s \leq \frac{n}{4} - 1, s \neq \frac{n}{8}$.

$S_1^* = \{[1, b]\}$ $S_2^* = \{[1, ba^{\frac{n}{2}}]\}$ $S^* = \{[1, a^{\frac{n}{2}}]\}$.

We have:

$\partial S = \{a^t, 1 \leq t \leq n-1, t \neq \frac{n}{2}\} \cup \{ba^{\frac{n}{2}+\frac{n}{4}}, ba^{\frac{n}{4}}\}$ and $\phi(S_1) = n$ is a left transversal for the subgroup $< b >= \{1, b\}$.

$\partial S_{2t+1} = \{ba^{2t+1}, ba^{\frac{n}{2}-2t-1}\}$ $\partial S_{2s} = \{ba^{2s}, ba^{n-2s}\}$ and both $\phi(S_{2t+1})$ and $\phi(S_{2s})$ are both left transversal for $< a >$.

Finally, we have: $\partial S_1^* = \{b\}$, $\partial S_2^* = \{ba^{\frac{n}{2}}\}$, $\partial S^* = \{a^{\frac{n}{2}}\}$. Moreover, $\phi(S_1^*) = \phi(S_2^*) = \phi(S^*) = \{1\}$.

With the starter above, we construct the following 1−factors:

$F = Orb_{<b>}(S)$ whose orbit under G gives the 1−factors: $F, Fa, \ldots, Fa^{n-1}$.

$F_{2t+1} = Orb_{<a>}(S_{2t+1})$ whose orbit under G gives the $1-$factors: F_{2t+1}, $F_{2t+1}b$, for each t with $0 \leq t \leq \frac{n}{8} - 1$ and $\frac{n}{4} \leq t \leq \frac{n}{4} + \frac{n}{8} - 1$.

$F_{2s} = Orb_{<a>}(S_{2s})$ whose orbit under G gives the $1-$factors: F_{2s}, $F_{2s}b$, for each s with $1 \leq s \leq \frac{n}{4} - 1, s \neq \frac{n}{8}$.

Finally, we have the three fixed $1-$factors $F_1^* = Orb_G([1,b])$, $F_2^* = Orb_G([1, ba^{\frac{n}{2}}])$, $F^* = Orb_G([1, a^{\frac{n}{2}}])$.

Consider the tree R_1 induced by the following set of edges:

$$\{[b, a^{\frac{n}{4}}], [ba^{\frac{n}{2}}, a^{\frac{n}{4}}]\} \cup \{[1, a^{\frac{n}{2}-2s}], [ba^{\frac{n}{4}}, ba^{\frac{n}{4}+\frac{n}{2}-2s}], 1 \leq s \leq \frac{n}{4} - 1\} \cup$$

$$\cup \{[1, a^{\frac{n}{2}-2t-1}], [ba^{\frac{n}{4}}, ba^{\frac{n}{4}-2t-1}], 0 \leq t \leq \frac{n}{4} - 1\}$$

The n edges of R_1 belongs to the n distinct $1-$factors F, Fa, $\ldots$, Fa^{n-1}. In fact: $[b, a^{\frac{n}{4}}] = [ba^{\frac{n}{2}}, a^{\frac{n}{2}+\frac{n}{4}}]a^{\frac{n}{2}} \in Fa^{\frac{n}{2}}$ and $[ba^{\frac{n}{2}}, a^{\frac{n}{4}}] = [a^{\frac{n}{4}}, b]ba^{\frac{n}{4}} \in Fa^{\frac{n}{2}+\frac{n}{4}}$. Moreover, these two edges have the same difference set and are in distinct orbits under $< a >$.
Observe that $[1, a^{\frac{n}{2}-2s}] = [a^{\frac{n}{2}+s}, a^{n-s}]a^{\frac{n}{2}-s} \in Fa^{\frac{n}{2}-s}$ and $[ba^{\frac{n}{4}}, ba^{\frac{n}{2}+\frac{n}{4}-2s}] \in Fa^{\frac{n}{2}-s}ba^{\frac{n}{4}}$ which is either $Fa^{\frac{n}{2}+\frac{n}{4}-s}$ or $Fa^{\frac{n}{4}-s}$ according to whether s is even or odd. Moreover, for each s, $1 \leq s \leq \frac{n}{4} - 1$, the two edges $[1, a^{\frac{n}{2}-2s}]$ and $[ba^{\frac{n}{4}}, ba^{\frac{n}{2}+\frac{n}{4}-2s}]$ have the same difference set and they are in distinct orbits under $< a >$.
Finally, observe that $[1, a^{\frac{n}{2}-2t-1}] = [a^t, a^{\frac{n}{2}-t-1}] \in Fa^{-t}$ and $[ba^{\frac{n}{4}}, ba^{\frac{n}{4}-2t-1}] = [1, a^{\frac{n}{2}-2t-1}]ba^{\frac{n}{4}} \in Fa^{-t}ba^{\frac{n}{4}}$ which is either $Fa^{\frac{n}{4}-t}$ or $Fa^{\frac{n}{2}+\frac{n}{4}-t}$ according to whether t is even or odd. Moreover, for each t, $0 \leq t \leq \frac{n}{4} - 1$, the two edges $[1, a^{\frac{n}{2}-2t-1}]$ and $[ba^{\frac{n}{4}}, ba^{\frac{n}{4}-2t-1}]$ have the same difference set and they are in distinct orbits under $< a >$.
Let R_2 be the union of the two stars induced by the following set of edges:

$$\{[ba^{\frac{n}{2}}, a^{\frac{n}{2}+\frac{n}{4}+2i}], [ba^{\frac{n}{2}}, a^{\frac{n}{2}+2i}], [a^{\frac{n}{2}+\frac{n}{4}}, ba^{2i}], [a^{\frac{n}{2}+\frac{n}{4}}, ba^{\frac{n}{2}+\frac{n}{4}+2i}], 1 \leq i \leq \frac{n}{8} - 1\}$$

.
The $\frac{n}{2} - 4$ edges of R_2 belongs to the distinct $1-$factors F_{2s}, $F_{2s}b$, $1 \leq s \leq \frac{n}{4} - 1$, $s \neq \frac{n}{8}$. In fact, for each $1 \leq i \leq \frac{n}{8} - 1$, we have: $[ba^{\frac{n}{2}}, a^{\frac{n}{2}+\frac{n}{4}+2i}] \in F_{\frac{n}{4}+2i}b$, $[ba^{\frac{n}{2}}, a^{\frac{n}{2}+2i}] \in F_{2i}b$, $[a^{\frac{n}{2}+\frac{n}{4}}, ba^{2i}] \in F_{\frac{n}{4}+2i}$, $[a^{\frac{n}{2}+\frac{n}{4}}, ba^{\frac{n}{2}+\frac{n}{4}+2i}] \in F_{2i}$. Moreover, for each i we have: $\partial[ba^{\frac{n}{2}}, a^{\frac{n}{2}+2i}] = \partial[a^{\frac{n}{2}+\frac{n}{4}}, ba^{\frac{n}{2}+\frac{n}{4}+2i}]$ and $\partial[ba^{\frac{n}{2}}, a^{\frac{n}{2}+\frac{n}{4}+2i}] = \partial[a^{\frac{n}{2}+\frac{n}{4}}, ba^{2i}]$ and the edges with the same difference set are in distinct orbits under $< a >$.
Let R_3 be the union of the three stars induced by the following set of edges:

$$\{[a^{\frac{n}{2}}, ba^{\frac{n}{2}+2i+1}], [b, a^{\frac{n}{2}+2i+1}], [ba^{\frac{n}{2}}, a^{\frac{n}{2}+\frac{n}{4}+2i+1}], [a^{\frac{n}{2}}, ba^{\frac{n}{4}+2i+1}], 0 \leq i \leq \frac{n}{8} - 1\}$$

. The $\frac{n}{2}$ edges of R_3 belongs to the distinct $1-$factors: F_{2t+1}, $F_{2t+1}b$, with $0 \leq t \leq \frac{n}{8} - 1$ and $\frac{n}{4} \leq t \leq \frac{n}{4} + \frac{n}{8} - 1$. In fact: $[a^{\frac{n}{2}}, ba^{\frac{n}{2}+2i+1}] \in F_{2i+1}$, $[b, a^{\frac{n}{2}+2i+1}] \in F_{2i+1}b$. Also, $[ba^{\frac{n}{2}}, a^{\frac{n}{2}+\frac{n}{4}+2i+1}] \in F_{\frac{n}{2}+\frac{n}{4}-2i-1}$, in fact $[ba^{\frac{n}{2}}, a^{\frac{n}{2}+\frac{n}{4}+2i+1}] = [1, ba^{\frac{n}{2}+\frac{n}{4}-2i-1}]a^{-\frac{n}{4}+2i+1}$, and $[a^{\frac{n}{2}}, ba^{\frac{n}{4}+2i+1}] \in F_{\frac{n}{2}+\frac{n}{4}-2i-1}b$. Moreover we have: $\partial[a^{\frac{n}{2}}, ba^{\frac{n}{2}+2i+1}] = \partial[b, a^{\frac{n}{2}+2i+1}]$ and $\partial[ba^{\frac{n}{2}}, a^{\frac{n}{2}+\frac{n}{4}+2i+1}] = [a^{\frac{n}{2}}, ba^{\frac{n}{4}+2i+1}]$ and edges with the same difference set are in distinct orbits under $< a >$.

Finally, let R_4 be inuduced by the two edges $[a^{\frac{n}{2}+\frac{n}{4}}, ba^{\frac{n}{2}+\frac{n}{4}}] \in F_1^*$ and $[b, a^{\frac{n}{2}}] \in F_2^*$ and which are both short.

The graph $R = R_1 \cup R_2 \cup R_3 \cup R_4$ satisfies conditions (1) and (2) of Lemma 1. It has two connected components. One is given by the union of 5 stars: a star at 1 and a star at $ba^{\frac{n}{4}}$ without common vertices and connected through the unique edge $[ba^{\frac{n}{2}}, a^{\frac{n}{4}}]$; a star at $ba^{\frac{n}{2}}$ with just the two vertices $ba^{\frac{n}{2}}, a^{\frac{n}{4}}$ in common with the previous two stars, a star at b which is connected to the previous three stars through the unique edge $[b, a^{\frac{n}{4}}]$ and a star at $a^{\frac{n}{2}}$ connected to the previous four stars through the unique edge $[b, a^{\frac{n}{2}}]$. The other component of R is a star at $a^{\frac{n}{2}+\frac{n}{4}}$.

Let $e_1 = [a^{\frac{n}{2}+\frac{n}{4}}, a^{\frac{n}{4}}] \in F^*$ and $e_2 = [ba^{\frac{n}{2}+\frac{n}{4}}, ba^{\frac{n}{4}}] \in F^*$, these two edges are in distinct orbits under $< a >$ and the graphs $T_1 = R \cup \{e_1\}$ and $T_2 = R \cup \{e_2\}$ satisfy condition (3) of Lemma 1. Therefore, the set $\mathcal{T} = \{T_1 a^i \mid 0 \leq i \leq \frac{n}{2} - 1\} \cup \{T_2 a^i \mid 0 \leq i \leq \frac{n}{2} - 1\}$ is a complete set of rainbow spanning trees.

If $n = 8$, a starter is given by:

$\Sigma = \{S\} \cup \{S_{2t+1}, 0 \leq t \leq \frac{n}{8} - 1$ and $\frac{n}{4} \leq t \leq \frac{n}{4} + \frac{n}{8} - 1\} \cup \{S_1^*, S_2^*, S^*\}$.

We have the $1-$factors:

$F = Orb_{<b>}(S)$ whose orbit under G gives the $1-$factors: $F, Fa, \ldots, Fa^{n-1}$.

$F_{2t+1} = Orb_{<a>}(S_{2t+1})$ whose orbit under G gives the $1-$factors: F_{2t+1}, $F_{2t+1}b$, for each t with $0 \leq t \leq \frac{n}{8} - 1$ and $\frac{n}{4} \leq t \leq \frac{n}{4} + \frac{n}{8} - 1$.

$F_1^* = Orb_G([1, b])$, $F_2^* = Orb_G([1, ba^{\frac{n}{2}}])$, $F^* = Orb_G([1, a^{\frac{n}{2}}])$.

Then, we repeat the same construction above with the graph $R = R_1 \cup R_3 \cup R_4$.

In the following Figures 9 and 10 we show $R \cup \{e_1\}$. In particular we picture the two connected components of R and the edge e_1 assigning a color to each of them.

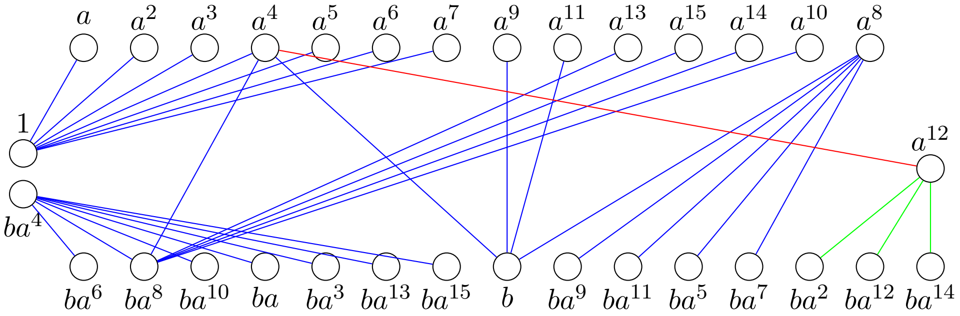

Figure 9: $R \cup \{e_1\}$, case (ii) with $n = 16$

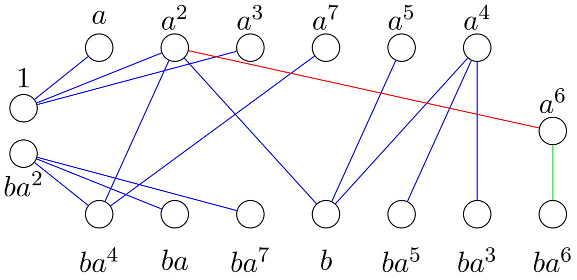

Figure 10: $R \cup \{e_1\}$, case (ii) with $n = 8$

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