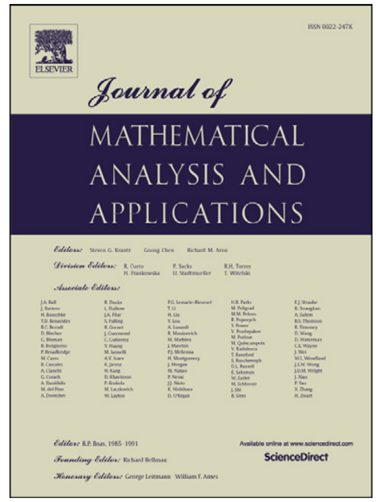

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ON CORON'S PROBLEM FOR THE p-LAPLACIAN

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ABSTRACT. We prove that the critical problem for the p-Laplacian operator admits a nontrivial solution in annular shaped domains with sufficiently small inner hole. This extends Coron's result [4] to a class of quasilinear problems.

1. INTRODUCTION

We want to extend the classical result of Coron [4]. Consider the problem

$$\begin{cases} -\Delta_p u = |u|^{p^*-2}u & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $1 < p < N$, $p^* := Np/(N-p)$ is the critical Sobolev exponent, $\Delta_p u := \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplace operator. Solutions on the whole space will be considered in

$$\mathcal{D}^{1,p}(\mathbb{R}^N) := \{u \in L^{p^*}(\mathbb{R}^N) : \nabla u \in L^p(\mathbb{R}^N; \mathbb{R}^N)\}$$

endowed with the norm

$$\|u\| := \|\nabla u\|_{L^p(\mathbb{R}^N)}.$$

We denote by $W_0^{1,p}(\Omega)$ the closure of $C_c^\infty(\Omega)$ in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ and define on $W_0^{1,p}(\Omega)$ the functional

$$J(u) := \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{1}{p^*}\int_\Omega |u|^{p^*} dx.$$

As it is well-known in tackling problem (1.1) with variational techniques, the main difficulty is due to the fact that the embedding $W_0^{1,p}(\Omega) \subset L^{p^*}(\Omega)$ is not compact. We refer to [14] for a sample of the extensive literature on semi-linear problems involving the critical Sobolev exponent, largely inspired by the pioneering paper of Brezis and Nirenberg [3]. We also define

$$S := \inf\Big\{\int_{\mathbb{R}^N} |\nabla u|^p dx,\, u \in \mathcal{D}^{1,p}(\mathbb{R}^N) \,:\, \int_{\mathbb{R}^N} |u|^{p^*} dx = 1\Big\}$$

the best Sobolev constant, attained by nowhere zero functions in $\mathbb{R}^N$, see e.g. [15]. Equivalently

$$S \;=\; \inf_{\substack{u\in\mathcal{D}^{1,p}(\mathbb{R}^N)\\ u\neq 0}} \frac{\int_{\mathbb{R}^N} |\nabla\, u|^p dx}{\left(\int_{\mathbb{R}^N}\; |u|^{p^*} dx\right)^{\frac{p}{p^*}}}, \tag{1.2}$$

where by a simple scaling argument the infimum remains unchanged if taken on competing functions supported in an arbitrary subdomain of $\mathbb{R}^N$. In light of the Pohozaev identity obtained by Guedda and Veron [9, Corollary 3.1], we know that problem (1.1) does not admit positive solutions on a strictly star-shaped domain.

The main result of the paper is the following

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Theorem 1.1. *Let $2N/(N+2) \leq p \leq 2$, $x_0 \in \mathbb{R}^N$ and radii $R_2 > R_1 > 0$ such that*

$$\{R_1 \leq |x - x_0| \leq R_2\} \subset \Omega, \qquad \{|x - x_0| \leq R_1\} \not\subset \overline{\Omega}. \tag{1.3}$$

Then problem (1.1) admits a positive solution for R_2/R_1 sufficiently large.

Theorem 1.1 is, mainly, a consequence of Lemma 2.3, in which the compactness result [11, Theorem 1.2] and the symmetry result of [5] play a key role. There are several difficulties arising in the present quasilinear setting which are partially highlighted in Lemma 2.3, which make the proof more delicate than for dealing with the semilinear case $p = 2$. One of those is the fact that the classification of all positive solutions of the critical problem in $\mathbb{R}^N$ is not yet available for all $p \in (1, N)$. We observe that an extension of Lemma 2.3 to a broader range of p would immediately yield an extension of Theorem 1.1. We conjecture that the symmetry result of [5] and hence Lemma 2.3 and Theorem 1.1 hold for all values of $p \in (1, N)$. Another open problem, arising in the proof of Lemma 2.3, is the nonexistence of sign-changing solutions of the critical problem in the half-space for $p \neq 2$. Such a limiting problem arises because of the boundary of Ω. We show that in fact only the nonexistence result of the positive solutions of the critical problem in the half-space [11, Theorem 1.1] is needed. The nonexistence of sign-changing solutions to problem (1.1) on strictly star-shaped domains is still an open problem, and this seems to be related to the fact that the unique continuation principle for the p-Laplacian operator is still another major open question. We incidently notice that in [11, p.482] it has been observed that if $\Omega = B(0, 1)$ the unit ball, no nontrivial radial solutions to (1.1) exist if p is in the range of Theorem 1.1. In the case $N = 2$, Theorem 1.1 holds for all $1 < p < 2$, which is the desired range for a p-Laplacian extension of the classical result of Coron. Theorem 1.1 extends [10, Theorem 1.1], where problem (1.1) had been studied assuming that Ω is invariant under the action of a closed subgroup of $O(N)$. In the case Ω is non-symmetric our result on problem (1.1) seems to be the first since Coron's classical paper [4] appeared in 1984. Even though our proof follows the original homotopy argument given in [4] (see also e.g. [14]) for the case $p = 2$, we point out that the present paper provides the first proof of the key fact that the Palais-Smale condition holds at energy levels $c \in (S^{N/p}/N, 2S^{N/p}/N)$ by using the recent results [5, 11]. This allows to carry on with a classical homotopy argument by constructing a pseudo gradient flow, as given e.g. in [14, pp.191-193].
It is an open problem whether (1.1) has nontrivial solutions when a $\mathbb{Z}_2$-homology group of Ω is nontrivial. This is the case for $p = 2$, see the celebrated analysis done in [1]. In several contributions dealing with the semi-linear case $p = 2$, see e.g. [6, 7, 12], it is shown that the existence of a nontrivial solution is possible also in contractible domains, hence conditions on the homology of Ω are not necessary for problem 1.1 to have solutions. A very well-known and challenging problem, even in the case $p = 2$, would be to exploit the combined effect of both the topology and the geometry of Ω in order to characterize the existence of a positive solution to problem (1.1).

2. Proof of Theorem 1.1

In this section we prove Theorem 1.1.

2.1. **Palais-Smale condition.** We define $\mathbb{R}^N_+ := \{x \in \mathbb{R}^N : x_N > 0\}$ and denote by $\mathcal{D}^{1,p}_0(\mathbb{R}^N_+)$ the closure of $C^\infty_c(\mathbb{R}^N_+)$ in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ after extending by zero on $\mathbb{R}^N \setminus \mathbb{R}^N_+$.

Lemma 2.1. *Let $u \in W^{1,p}_0(\Omega)$ be a sign-changing solution to (1.1). Then $J(u) \geq 2S^{N/p}/N$. Moreover, the same conclusion holds for the sign-changing solutions of $-\Delta_p u = |u|^{p^*-2}u$ in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ or in $\mathcal{D}^{1,p}_0(\mathbb{R}^N_+)$.*

Proof. If $u \in W_0^{1,p}(\Omega)$ is a sign-changing solution to (1.1), then $u^\pm \in W_0^{1,p}(\Omega)\setminus\{0\}$ and by testing (1.1) with $u^\pm$ yields

$$\int_\Omega |\nabla u^+|^p dx = \int_\Omega |u^+|^{p^*} dx, \qquad \int_\Omega |\nabla u^-|^p dx = \int_\Omega |u^-|^{p^*} dx.$$

In turn, using the definition of (1.2), we obtain

$$J(u) = J(u^+) + J(u^-) = \frac{1}{N}\|u^+\|_{p^*}^{p^*} + \frac{1}{N}\|u^-\|_{p^*}^{p^*} \geq 2S^{N/p}/N,$$

concluding the proof. The same argument works for the problem on $\mathbb{R}^N$ and on $\mathbb{R}^N_+$. □

The following lemma is a consequence of the recent result [5].

Lemma 2.2. *Let $2N/(N+2) \leq p \leq 2$ and $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ be a positive solution of $-\Delta_p u = |u|^{p^*-2}u$. Then up to translation, and for a suitable $a > 0$,*

$$u(x) = \Big(Na\Big(\frac{N-p}{p-1}\Big)^{p-1}\Big)^{(N-p)/p^2}(a + |x|^{p/(p-1)})^{(p-N)/p}, \quad a.e. \ \text{on } \mathbb{R}^N.$$

Proof. By [5] for some strictly decreasing function $v : [0, +\infty) \to (0, +\infty)$ and for some $x_0 \in \mathbb{R}^N$ there holds $u(x) = v(|x - x_0|)$. The assertion then follows by [8, Theorem 2.1(ii)] (see also [2]). □

Lemma 2.3. *Assume that $2N/(N+2) \leq p \leq 2$. Then J satisfies the Palais-Smale condition for all $c \in (S^{N/p}/N, 2S^{N/p}/N)$.*

Proof. Assume that for some $c \in (S^{N/p}/N, 2S^{N/p}/N)$, $(u_n) \in W_0^{1,p}(\Omega)$ is such that $J(u_n) \to c$, and $J'(u_n) \to 0$ in $W^{-1,p'}(\Omega)$. We define on $\mathcal{D}^{1,p}(\mathbb{R}^N)$

$$J_\infty(u) := \int_{\mathbb{R}^N} \frac{|\nabla u|^p}{p} dx - \int_{\mathbb{R}^N} \frac{|u|^{p^*}}{p^*} dx.$$

On $\mathcal{D}_0^{1,p}(\mathbb{R}^N_+)$ we define the same functional J_∞ extending by zero on $\mathbb{R}^N \setminus \mathbb{R}^N_+$.
By applying [11, proof of Theorem 1.2], which extends [13], passing if necessary to a subsequence, we can infer that there exists a (possibly trivial) solution $v_0 \in W_0^{1,p}(\Omega)$ of

$$-\Delta_p u = |u|^{p^*-2}u \quad \text{in} \quad \Omega,$$

$k \in \mathbb{N} \cup \{0\}$, nontrivial solutions $\{v_1, ..., v_k\}$ of

$$-\Delta_p u = |u|^{p^*-2}u \quad \text{in} \quad H_i, \quad i \in \{0, 1, ...k\},$$

where H_i is either $\mathbb{R}^N$ or (up to rotation and translation) $\mathbb{R}^N_+$, with either $v_i \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ or (respectively) $v_i \in \mathcal{D}_0^{1,p}(\mathbb{R}^N_+)$, and there exist k sequences $\{y_n^i\}_n \subset \bar{\Omega}$ and $\{\lambda_n^i\}_n \subset \mathbb{R}_+$, satisfying

$$\frac{1}{\lambda_n^i}\,\mathrm{dist}\,(y_n^i, \partial\Omega) \to \infty, \quad n \to \infty,$$

if $H_i \equiv \mathbb{R}^N$ or

$$\frac{1}{\lambda_n^i}\,\mathrm{dist}\,(y_n^i, \partial\Omega) < \infty, \quad n \to \infty,$$

if (up to rotation and translation) $H_i \equiv \mathbb{R}^N_+$, and

$$\|u_n - v_0 - \sum_{i=1}^k (\lambda_n^i)^{(p-N)/p} v_i((\cdot - y_n^i)/\lambda_n^i)\| \to 0, \quad n \to \infty,$$

$$\|u_n\|^p \to \sum_{i=0}^k \|v_i\|^p, \quad n \to \infty,$$

$$J(v_0) + \sum_{i=1}^{k} J_\infty(v_i) = c. \tag{2.1}$$

The restriction on the levels c and Lemma 2.1 immediately yield the bound $k \leq 1$. If $k = 0$ compactness holds and we are done. If instead $k = 1$, we have two cases, namely $v_0 \equiv 0$ or $v_0 \not\equiv 0$. If $v_0 \not\equiv 0$, since

$$J(v_0) \geq S^{N/p}/N, \qquad J_\infty(v_1) \geq S^{N/p}/N,$$

(actually $J(v_0) > S^{N/p}/N$, as the Sobolev constant is never achieved on bounded domains) we obtain a contradiction by combining (2.1) with the assumption $c < 2S^{N/p}/N$. If, instead, $v_0 \equiv 0$, then formula (2.1) reduces to $J(v_1) = c$. Again by Lemma 2.1 v_1 does not change sign and by the nonexistence result [11, Theorem 1.1] $H_1 \equiv \mathbb{R}^N$, namely $v_1 \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ solves

$$\begin{aligned} -\Delta_p u = u^{p^*-1} \quad &\text{in} \quad \mathbb{R}^N, \\ u > 0 \quad &\text{in} \quad \mathbb{R}^N. \end{aligned} \tag{2.2}$$

Now, by Lemma 2.2, after translation in the origin, for a suitable value of $a > 0$ v_1 is a Talenti function

$$v_1(x) = \Big(Na\Big(\frac{N-p}{p-1}\Big)^{p-1}\Big)^{(N-p)/p^2}(a + |x|^{p/(p-1)})^{(p-N)/p},$$

whose associated energy is $c = J_\infty(v_1) = S^{N/p}/N$ [15], since v_1 achieves the best Sobolev constant S. This is a contradiction again, since $c > S^{N/p}/N$. This concludes the proof. □

Remark 2.1. The above compactness property holds for a more general class of functionals. Let Ω be a smooth bounded domain of $\mathbb{R}^N$ and, as in [11], define on $W_0^{1,p}(\Omega)$

$$\phi(u) := \int_\Omega \frac{|\nabla u|^p}{p} + a(x)\frac{|u|^p}{p} - \frac{|u|^{p^*}}{p^*} dx,$$

and consider the following hypotheses on a:

- H1) $a \in L^{N/p}(\Omega)$.
- H2) The Palais-Smale sequences are bounded. This occurs e.g. assuming
$$\inf_{\|\nabla u\|_{L^p}=1} \int_\Omega |\nabla u|^p + a(x)|u|^p dx > 0.$$
- H3) For every nontrivial critical point u of ϕ, there holds
$$\phi(u) \geq S^{N/p}/N$$
(this is the case e.g. if a is a nonnegative function).

With the same proof of Lemma 2.3 we can achieve that if $2N/(N+2) \leq p \leq 2$, then ϕ satisfies the Palais-Smale condition for all $c \in (S^{N/p}/N, 2S^{N/p}/N)$.

2.2. **Proof of Theorem 1.1 concluded.** Let R_1, R_2 be the radii of the annulus as in the statement of Theorem 1.1. As observed in [4, 14], without loss of generality, we may assume that $x_0 = 0$, $R_1 = 1/(4R)$ and $R_2 = 4R$ where $R > 0$ will be chosen sufficiently large. Let us set $\Sigma := \{x \in \mathbb{R}^N : |x| = 1\}$ and consider the family of functions

$$u_t^\sigma(x) := \left[\frac{1-t}{(1-t)^p + |x - t\sigma|^{\frac{p}{p-1}}}\right]^{\frac{N-p}{p}} \in \mathcal{D}^{1,p}(\mathbb{R}^N), \qquad \text{for } \sigma \in \Sigma \text{ and } t \in [0,1).$$

Moreover, let us now consider a function $\varphi \in C_c^\infty(\Omega)$ be such that $0 \le \varphi \le 1$ on Ω, $\varphi = 1$ on $\{1/2 < |x| < 2\}$ and $\varphi = 0$ outside $\{1/4 < |x| < 4\}$, then define

$$\varphi_R(x) := \begin{cases} \varphi(Rx) & \text{on } 0 \le |x| < \frac{1}{R}, \\ 1 & \text{on } \frac{1}{R} \le |x| < R, \\ \varphi(x/R) & \text{on } |x| \ge R. \end{cases}$$

Finally, let us set

$$w_t^\sigma(x) := u_t^\sigma(x)\varphi_R(x) \in W_0^{1,p}(\Omega), \quad w_0(x) := u_0(x)\varphi_R(x), \quad u_0(x) := \Big[\frac{1}{1+|x|^{\frac{p}{p-1}}}\Big]^{\frac{N-p}{p}}.$$

Then, we have the following

Lemma 2.4. *For $\sigma \in \Sigma$ and $t \in [0,1)$, $\|u_t^\sigma\| = \|u_0\|$, $\|u_t^\sigma\|_{p^*} = \|u_0\|_{p^*}$ and $\|u_t^\sigma\|^p = S\|u_t^\sigma\|_{p^*}^p$. Furthermore, there holds*

$$\lim_{R\to\infty} \sup_{\sigma\in\Sigma, t\in[0,1)} \|w_t^\sigma - u_t^\sigma\| = 0.$$

Proof. The first properties of u_t^σ follow by [15]. In the following C will denote a generic positive constant, independent of $\sigma \in \Sigma$ and $t \in [0,1)$, which may vary from line to line. We have the inequality

$$\int_{\mathbb{R}^N} |\nabla(w_t^\sigma - u_t^\sigma)|^p dx \le C\sum_{i=1}^4 \mathbb{I}_i,$$

where we have set

$$\begin{aligned} \mathbb{I}_1 &:= \int_{\mathbb{R}^N\setminus B_{2R}} |\nabla u_t^\sigma|^p dx, \\ \mathbb{I}_2 &:= \int_{B_{(2R)^{-1}}} |\nabla u_t^\sigma|^p dx, \\ \mathbb{I}_3 &:= \frac{1}{R^p}\int_{B_{4R}\setminus B_{2R}} |u_t^\sigma|^p dx, \\ \mathbb{I}_4 &:= R^p\int_{B_{(2R)^{-1}}} |u_t^\sigma|^p dx. \end{aligned}$$

Taking into account that

$$|\nabla u_t^\sigma(x)| \le \frac{C}{((1-t)^p + |x-t\sigma|^{\frac{p}{p-1}})^{\frac{N}{p}}} \le C \quad |x| \le \frac{1}{2}, \qquad |\nabla u_t^\sigma(x)| \le \frac{C}{|x|^{\frac{N-1}{p-1}}} \quad |x| \ge 2,$$

we obtain

$$\begin{aligned} \mathbb{I}_1 &= \int_{\mathbb{R}^N\setminus B_{2R}} |\nabla u_t^\sigma|^p dx \le C\int_{\mathbb{R}^N\setminus B_{2R}} \frac{1}{|x|^{\frac{p(N-1)}{p-1}}} dx \le \frac{C}{R^{\frac{N-p}{p-1}}}, \\ \mathbb{I}_2 &= \int_{B_{(2R)^{-1}}} |\nabla u_t^\sigma|^p dx \le C\int_{B_{(2R)^{-1}}} dx \le \frac{C}{R^N}. \end{aligned}$$

Moreover, we have

$$\begin{aligned} \mathbb{I}_3 &= \frac{1}{R^p}\int_{B_{4R}\setminus B_{2R}} \Big[\frac{1-t}{(1-t)^p + |x-t\sigma|^{\frac{p}{p-1}}}\Big]^{N-p} dx \le \frac{C}{R^p}\int_{B_{4R}\setminus B_{2R}} \frac{1}{|x|^{\frac{p(N-p)}{p-1}}} dx \le \frac{C}{R^{\frac{N-p}{p-1}}}, \\ \mathbb{I}_4 &= R^p\int_{B_{(2R)^{-1}}} \Big[\frac{1-t}{(1-t)^p + |x-t\sigma|^{\frac{p}{p-1}}}\Big]^{N-p} dx \le R^pC\int_{B_{(2R)^{-1}}} dx \le \frac{C}{R^{N-p}}. \end{aligned}$$

This concludes the proof. □

Let us now define

$$S(u) := \frac{\|\nabla u\|^p}{\|u\|^p_{L^{p^*}(\mathbb{R}^N)}}, \qquad u \in \mathcal{D}^{1,p}(\mathbb{R}^N) \setminus \{0\}, \tag{2.3}$$

with the understanding that

$$S(u;\Omega) = \frac{\|\nabla u\|^p_{L^p(\Omega)}}{\|u\|^p_{L^{p^*}(\Omega)}}, \qquad u \in W^{1,p}_0(\Omega) \setminus \{0\}, \tag{2.4}$$

after extending by zero outside Ω.

As a consequence of Lemma 2.4, we have the following

Lemma 2.5. *If $v_t^\sigma(x) := \|w_t^\sigma\|^{-1}_{L^{p^*}(\mathbb{R}^N)} w_t^\sigma(x)$ and $v_0(x) = \|w_0\|^{-1}_{L^{p^*}(\mathbb{R}^N)} w_0(x)$, then*

$$\lim_{R\to\infty} S(v_t^\sigma;\Omega) = S(u_t^\sigma) = S,$$

uniformly with respect to $\sigma \in \Sigma$ and $t \in [0,1)$.

We observe that J satisfies the Palais-Smale condition between the levels $S^{N/p}/N$ and $2S^{N/p}/N$. Therefore, as it can be readily verified, the functional $S(\cdot;\Omega)$, constrained to

$$\mathcal{M} = \{u \in W^{1,p}_0(\Omega) : \|u\|^{p^*}_{p^*} = 1\},$$

satisfies the Palais-Smale condition between S and ϖS, for some $\varpi > 1$ depending upon p and N. Then, taking Lemma 2.5 into account, and assuming by contradiction that the problem does not admit any positive solution, by arguing exactly as in [14, pp.191-193] one proves Theorem 1.1 by performing a well-established deformation argument on $S(\cdot;\Omega)$ as restricted to $\mathcal{M}$, yielding a contradiction with the geometrical properties (1.3) of Ω. We point out that under our assumption $2N/(N+2) \le p$, it follows $p^* \ge 2$ so that $\mathcal{M}$ is a $C^{1,1}$ smooth manifold. □

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