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Perturbation theory for nonlinear Schrödinger equations

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Abstract

Treating the nonlinear term of the Gross–Pitaevskii nonlinear Schrödinger equation as a perturbation of an isolated discrete eigenvalue of the linear problem one obtains a Rayleigh–Schrödinger power series. This power series is proved to be convergent when the parameter representing the intensity of the nonlinear term is less in absolute value than a threshold value, and it gives a stationary solution to the nonlinear Schrödinger equation.

Keywords: nonlinear Schrödinger equation, perturbation theory, Rayleigh–Schrödinger power series

Mathematics Subject Classification numbers: 35Q55, 81Q15

1. Introduction

Nonlinear Schrödinger equation (hereafter NLS) is a research topic with a large variety of applications [24]: from problems in nonlinear optics to the analysis of quantum dynamics of Bose–Einstein condensates. In particular, the study of its stationary solutions has attracted increasing attention, and, apart from the few cases in which the solution exists in explicit form, the analysis has mainly focused variational methods or on approximation methods based on both semiclassical and perturbative techniques.

Variational methods are widely used in order to construct bound states for NLS with a linear potential, typically by solving a minimisation problem; for instance, this is done by [18] where they proved the existence of a small amplitude stationary solution that bifurcates from the zero solution (see also [21, 22] where nonlinear scattering is considered for NLS with, respectively, one and two nonlinear bound states and the references there in). Similarly, in [5] variational methods have been applied to prove that for the minimiser of the nonlinear Hartree energy functional a symmetry breaking effect occurs.

For what concerns semiclassical methods, they have been successfully used in this framework where several authors have been able to demonstrate the existence, in the semiclassical limit using variational techniques, of stationary solution concentrated around the critical points of the potential [1, 13, 16, 27]. Also the occurrence of bifurcation phenomena has been discussed in the semiclassical limit [19].

On the other side, the perturbative approach takes up the underlying idea of Rayleigh-Schrödinger series expansion, where the solution is written as a formal series of powers whose coefficients are determined recursively and where the convergence of the series is under investigation. Typically in these cases the perturbation is represented by the nonlinear term, and the unperturbed Schrödinger equation, where the nonlinear term is absent, admits isolated eigenvalues. Several applications of this idea have been developed over the years [2, 3, 8, 12, 25, 26] limited, in general, to a formal analysis of the series without proving its convergence. In fact, we should emphasise that the problem of convergence of the power series has been solved for some kind of nonlinear Schrödinger equations; more precisely, the spinless real Hartree–Fock model and the Thomas–Fermi–Von-Weizsäcker model has been considered by [8] proving, in particular, that in the first model the Rayleigh–Schrödinger perturbation series has a positive convergence radius.

Finally, it should be mentioned that numerical methods based on discrete Galerkin approximations or spectral splitting methods are widely and effectively used for the study of time-dependent NLS (see [4, 6, 7, 20, 23] and references therein).

In this paper we aim to give a rigorous basis to the perturbative approach for computing the stationary solution of the NLS by going so far as to demonstrate, under fairly general assumptions, the convergence of the Rayleigh–Schrödinger series when the perturbative parameter, which measures the intensity of the nonlinear perturbation, is less in absolute value than a given threshold. In this way it is shown that the steady states associated with isolated and nondegenerate eigenvalues of the linear operator transform into stationary solutions of the NLS when nonlinearity is switched on, and the latter can be computed very efficiently through the convergent perturbative series. Finally, it is also possible to give a lower estimate of the radius of convergence of the power series.

The paper is organised as follows. In section 2 we describe the model, we write the formal power series of the stationary solutions and we state the convergence result in theorem 1. In section 3 we state and prove some technical preliminary results. In section 4 we obtain the convergence of the perturbative series proving thus theorem 1. In sections 5 and 6 we discuss a couple of one-dimensional examples: namely in section 5 we consider the case of an infinite well potential, in this case we are also able to compare the perturbative results with the exact ones; in section 6 we compute the perturbative series in the case where the potential is the harmonic one. The discussion of these two models is, in some sense, 'pedagogical'; indeed, by means of numerical experiments it is possible to see that the coefficients of the power series expansion rapidly decreases and then one can guess the convergence radius of the power series. Finally, in section 7 we draw some closing comments. A small technical appendix closes the paper.

2. Main results

2.1. Assumptions

We consider the time-independent nonlinear Schrödinger equation

$$H\psi + \nu|\psi|^2\psi = E\psi \, , \; \psi \in L^2\left(\mathbb{R}^d\right) \, , \tag{1}$$

where $H = -\Delta + V$ is a linear operator formally defined on $L^2(\mathbb{R}^d)$. The nonlinear term plays the role of perturbation and its strength $\nu \in \mathbb{C}$ is a small perturbative parameter.

Hypothesis 1. The potential V is assumed to be a real-valued piecewise continuous function bounded from below:

$$V(x) \geqslant \Gamma \, , \forall x \in \mathbb{R}^d \, , \tag{2}$$

for some $\Gamma \in \mathbb{R}$.

Remark 1. We assume that the potential $V(x)$ is a piecewise continuous function bounded from below for the sake of simplicity. In fact, we must remark that one could extend our treatment to the case where some milder conditions on $V(x)$ are assumed; however, we do not dwell on those details here. On the other side, it might be interesting to consider the case in which $V(x)$ is given by means of an attractive Dirac's δ [11]; this case does not fall under the hypothesis 1.

Hence, H admits a self-adjoint extension, still denoted by H, on a self-adjointness domain $\mathcal{D}(H) \subset L^2(\mathbb{R}^d)$.

Hypothesis 2. The discrete spectrum of H is not empty: $\sigma_d(H) \neq \emptyset$, and admits a non degenerate eigenvalue $e_0 \in \sigma_d(H)$ with associated eigenvector $\phi_0 \in \mathcal{D}(H)$:

$$H\phi_0 = e_0\phi_0, \phi_0 \in L^2\left(\mathbb{R}^d\right) \, . \tag{3}$$

Hereafter we can assume, for simplicity's sake and without loss in generality, that the unperturbed eigenvector ϕ_0 is normalised to one, i.e.:

$$\|\phi_0\|_{L^2} = 1 \, .$$

In the following we denote

$$\Lambda = \mathrm{dist}\left[\sigma\left(H\right) \setminus \left\{e_0\right\}, e_0\right] > 0 \, .$$

Remark 2. Since $\phi_0 \in \mathcal{D}(H)$ and the potential V is bounded from below then it follows that $\phi_0 \in H^1$ because

$$\|\nabla\phi_0\|_{L^2}^2 = \langle -\Delta\phi_0, \phi_0\rangle_{L^2} = e_0\|\phi_0\|_{L^2}^2 - \langle V\phi_0, \phi_0\rangle \leqslant (e_0 - \Gamma)\,\|\phi_0\|_{L^2}^2 \, .$$

Thus

$$\phi_0 \in \mathcal{D}\left(H\right) \cap L^6\left(\mathbb{R}^d\right)$$

follows from this fact and from the Gagliardo–Nirenberg inequality [9]

$$\|f\|_{L^p} \leqslant C_{p,d}\|\nabla f\|^\rho \|f\|^{1-\rho} \, , \rho = \frac{d}{2} - \frac{d}{p} \tag{4}$$

for some positive constant $C_{p,d}$ and where

$$p \in \left\{ \begin{array}{l} [2, +\infty] \text{ if } d = 1; \\ [2, +\infty) \text{ if } d = 2; \\ [2, 2d/(d-2)] \text{ if } d > 2. \end{array} \right. \, . \tag{5}$$

2.2. Formal solutions

We look for a *formal* stationary solution to (1) close to the solution to the linear problem (3) by means of a formal power series

$$E := E(\nu) = \lim_{N\to+\infty} E_N(\nu) \text{ and } \psi := \psi(x,\nu) = \lim_{N\to+\infty} \psi_N(x,\nu) , \tag{6}$$

where

$$E_N(\nu) = \sum_{n=0}^{N} \nu^n e_n \text{ and } \psi_N(x,\nu) = \sum_{n=0}^{N} \nu^n \phi_n(x) \tag{7}$$

and where e_n and ϕ_n are defined by induction as follows. In fact, E and ψ depend on the perturbative parameter ν; sometimes, for simplicity, we will omit this dependence when this fact does not cause misunderstanding.

Remark 3. We should underline that the following formulas make sense provided that the vectors u_n and v_n below belongs to $L^2(\mathbb{R}^d)$ and $\phi_n \in \mathcal{D}(H) \cap L^6(\mathbb{R}^d)$; we will discuss this point in section 3.

Let e_ℓ and ϕ_ℓ be defined for any $\ell = 0,1,\ldots,n-1$, where $\langle \phi_0, \phi_\ell \rangle_{L^2} = 0$ for any $\ell = 1,2,\ldots,n-1$, and let

$$v_{n-1} = \sum_{m=0}^{n-1} \sum_{\ell=0}^{n-1-m} \phi_m \bar{\phi}_\ell \phi_{n-1-m-\ell} \, , u_n = \sum_{m=1}^{n-1} e_m \phi_{n-m} \, .$$

We define

$$e_n = \langle \phi_0, v_{n-1} \rangle_{L^2} \tag{8}$$

and

$$\varphi_n = e_n \phi_0 + u_n - v_{n-1} \, .$$

By construction it follows that

$$\langle u_n, \phi_0 \rangle_{L^2} = 0 \text{ and } \langle \varphi_n, \phi_0 \rangle_{L^2} = 0 ,$$

that is $\varphi_n \in \Pi^\perp L^2$, where $\Pi^\perp = 1 - \Pi$ and Π is the projection operator on the space spanned by ϕ_0. Hence, the resolvent operator $[H - e_0]^{-1}$ is bounded on $\Pi^\perp L^2$ and we can define

$$\phi_n = [H - e_0]^{-1} \varphi_n = [H - e_0]^{-1} \Pi^\perp \varphi_n = \Pi^\perp [H - e_0]^{-1} \varphi_n \in \Pi^\perp L^2 \, . \tag{9}$$

Lemma 1. *Let e_n and $\phi_n \in \Pi^\perp L^2$ be defined by induction for any $n \geqslant 1$ as in (8) and (9). Let E_N and ψ_N be defined as in (7). Let*

$$r_N := H\psi_N + \nu |\psi_N|^2 \psi_N - E_N \psi_N , \tag{10}$$

Then r_N is a power series in ν with finitely many terms where all the coefficients of the powers ν^n, with $n \leqslant N$, are exactly zero.

Remark 4. Since e_0 is a simple and isolated eigenvalue of the selfadjoint operator H and since $\varphi_n \perp \Pi L^2$ for any $n \geqslant 1$ then:

$$\|\phi_n\|_{L^2} \leqslant \frac{1}{\Lambda} \|\varphi_n\|_{L^2} \, . \tag{11}$$

Proof. By formally substituting (7) and (6) in (1) we then have to check that

$$\sum_{n=0}^{\infty}\nu^{n}H\phi_{n}+\nu\sum_{n=0}^{\infty}\nu^{n}\phi_{n}\sum_{m=0}^{\infty}\nu^{m}\phi_{m}\sum_{\ell=0}^{\infty}\nu^{\ell}\bar{\phi}_{\ell}=\sum_{m=0}^{\infty}\nu^{m}e_{m}\sum_{n=0}^{\infty}\nu^{n}\phi_{n}. \tag{12}$$

This equation can be written as

$$\sum_{n=0}^{\infty}\nu^{n}H\phi_{n}+\sum_{n=0}^{\infty}\nu^{n+1}v_{n}=\sum_{n=0}^{\infty}\nu^{n}\left[e_{0}\phi_{n}+u_{n}+e_{n}\phi_{0}\right]$$

where u_n and v_n are defined above. By equating the term with the same power of the perturbative parameter ν we have that

$$H\phi_{0}=e_{0}\phi_{0}\,,\ \text{for}\ \ n=0\,, \tag{13}$$

which is satisfied by assumption, and

$$H\phi_{n}+v_{n-1}=e_{n}\phi_{0}+e_{0}\phi_{n}+u_{n}\,,\ \text{for}\ \ n\geqslant 1\,.$$

If we multiply both side by ϕ_0 then

$$\langle\phi_{0},H\phi_{n}\rangle_{L^{2}}+\langle\phi_{0},v_{n-1}\rangle_{L^{2}}=e_{n}\|\phi_{0}\|_{L^{2}}^{2}+e_{0}\langle\phi_{0},\phi_{n}\rangle_{L^{2}}+\langle\phi_{0},u_{n}\rangle_{L^{2}}$$

from which it follows that

$$e_{n}=\frac{\langle\phi_{0},v_{n-1}\rangle_{L^{2}}-\langle\phi_{0},u_{n}\rangle_{L^{2}}}{\|\phi_{0}\|_{L^{2}}^{2}}=\frac{\langle\phi_{0},v_{n-1}\rangle_{L^{2}}}{\|\phi_{0}\|_{L^{2}}^{2}}=\langle\phi_{0},v_{n-1}\rangle_{L^{2}}\,,$$

since $\phi_0 \perp u_n$ and $\|\phi_0\|_{L^2} = 1$. If we denote now

$$\varphi_{n}=e_{n}\phi_{0}+u_{n}-v_{n-1}$$

then $\varphi_n \perp \phi_0$ and thus we get

$$\phi_{n}=\left[H-e_{0}\right]^{-1}\varphi_{n}\,.$$

□

2.3. Main result

Here we state our main result.

Theorem 1. *Let $d = 1,2,3$ and let Hypotheses 1 and 2 be satisfied. Then, there exists $\nu^\star > 0$ such that for any ν such that $|\nu| < \nu^\star$ the nonlinear Schrödinger equation (1) admits a stationary solution $\psi(x,\nu) \in \mathcal{D}(H) \cap L^6(\mathbb{R}^d)$, associated to an energy $E(\nu)$, given by means of the strong-convergent power series*

$$\psi\left(x,\nu\right)=\sum_{n=0}^{\infty}\nu^{n}\phi_{n}\left(x\right)\ \textit{and}\ \ E\left(\nu\right)=\sum_{n=0}^{\infty}\nu^{n}e_{n}\,, \tag{14}$$

where $\phi_n(x)$ and e_n are given in lemma 1.

Remark 5. It is worth noting that the stationary solution ψ given by (14) is not normalised to one, that is, to the value of the norm of the unperturbed eigenvector ϕ_0, which is assumed, for convenience of argument, to be equal to 1. In fact, a simple calculation gives that

$$\|\psi\|_{L^2}^2 = \sum_{n=0}^{\infty} \nu^n r_n \text{ where } r_n = \sum_{m=0}^{n} \langle \phi_{n-m}, \phi_m \rangle .$$

In particular

$$r_0 = \|\phi_0\|_{L^2}^2 = 1$$

$$r_1 = \langle \phi_0, \phi_1 \rangle + \langle \phi_1, \phi_0 \rangle = 0, \text{since } \phi_1 \in \Pi^\perp L^2 ,$$

and

$$r_2 = \langle \phi_0, \phi_2 \rangle + \langle \phi_1, \phi_1 \rangle + \langle \phi_2, \phi_0 \rangle = \|\phi_1\|_{L^2}^2 > 0, \text{since } \phi_2 \in \Pi^\perp L^2 .$$

Thus

$$\|\psi\|_{L^2}^2 = 1 + \nu^2 g(\nu)$$

where $g(\nu)$ is the analytic function obtained by means of the perturbative procedure for ν in a neighbourhood of $\nu = 0$ and such that $g(0) > 0$. If one looks for a normalised solution may act as follows. Let

$$\tilde{\psi} = \frac{\psi}{\|\psi\|_{L^2}}$$

be the normalised stationary solution to the equation

$$H\tilde{\psi} + \tilde{\nu}|\tilde{\psi}|^2\tilde{\psi} = E\tilde{\psi}$$

where E is still given by (14) and where

$$\tilde{\nu} := \tilde{\nu}(\nu) = \nu\|\psi\|_{L^2}^2 = \nu\left[1 + \nu^2 g(\nu)\right] . \tag{15}$$

Such a relation is invertible with inverse function

$$\nu := \nu(\tilde{\nu}) .$$

In conclusion, if one look for the normalised solution to the equation

$$H\psi + \nu|\psi|^2\psi = E\psi \tag{16}$$

for a given value of the parameter ν let $\nu^\star$ be such that $\tilde{\nu}(\nu^\star) = \nu$, let ψ and E be the perturbative solutions given by (14) corresponding to such a value of $\nu^\star$; then $\psi/\|\psi\|_{L^2}$ and E are the normalised solution to (16).

In addition, by means of the scaling $\psi = \nu^{-2}\omega$ then (1) takes the form of the ν-normalised equation

$$H\omega + |\omega|^2\omega = E\omega \tag{17}$$

where we have just seen that

$$\|\omega\|_{L^2}^2 = \nu\|\psi\|_{L^2}^2 = \nu\left[1 + \nu g(\nu)\right] .$$

Thus, for ν in a neighbourhood of 0, we can find a continuous curve $(E(\nu), \|\omega\|_{L^2})$, near the point $(e_0, 0)$, for solution to (17). Recall that the analysis of the slope of this curve is important in the stability analysis of the stationary state (see, e.g. the 'slope condition' in [14]).

3. L^p estimates

As anticipated in remark 3 it turns out that formulas (8) and (9) make sense provided that v_n and φ_n belongs to L^2. Hence, we have to prove that ϕ_n belongs to $L^2 \cap L^6$ for any n. In order to obtain a L^p-norm estimate of the vectors ϕ_n we make use of the Gagliardo–Nirenberg inequality (4).

Lemma 2. *Let $V(x)$ be a potential bounded from below (2); let p and $C_{p,d}$ as given in (4) and (5). Concerning the H^1 and L^p norms of ϕ_n we have that*

$$\|\phi_n\|_{H^1} \leqslant \mu_1 \|\varphi_n\|_{L^2}$$

and

$$\|\phi_n\|_{L^p} \leqslant \mu_2(p,d)\,\|\varphi_n\|_{L^2}$$

for some constants

$$\mu_1 = \left[\frac{1}{\Lambda^2} + \frac{1}{\Lambda} + \frac{e_0 - \Gamma}{\Lambda^2}\right]^{1/2} \text{ and } \mu_2(p,d) := C_{p,d}\frac{1}{\Lambda^{1-\rho}}\left[\frac{1}{\Lambda} + \frac{e_0 - \Gamma}{\Lambda^2}\right]^{\rho/2} \tag{18}$$

independent of n.

Proof. From remark 4 we have (11). Then, we have to estimate

$$\|\nabla\phi_n\|_{L^2} = \|\nabla[H - e_0]^{-1}\varphi_n\|_{L^2}.$$

Since $V(x)$ is bounded from below, $V \geqslant \Gamma$, then

$$\begin{aligned}\|\nabla[H - e_0]^{-1}\varphi_n\|_{L^2}^2 &= \langle [H - e_0]^{-1}\varphi_n, -\Delta[H - e_0]^{-1}\varphi_n\rangle_{L^2} \\ &= \langle [H - e_0]^{-1}\varphi_n, \varphi_n\rangle_{L^2} - \langle [H - e_0]^{-1}\varphi_n, (V - e_0)[H - e_0]^{-1}\varphi_n\rangle_{L^2} \\ &\leqslant \langle [H - e_0]^{-1}\varphi_n, \varphi_n\rangle_{L^2} + (e_0 - \Gamma)\,\|[H - e_0]^{-1}\varphi_n\|_{L^2}^2\end{aligned}$$

since $-\Delta = H - e_0 - (V - e_0)$. Hence

$$\|\nabla[H - e_0]^{-1}\varphi_n\|_{L^2} \leqslant \left[\frac{1}{\Lambda} + \frac{e_0 - \Gamma}{\Lambda^2}\right]^{1/2} \|\varphi_n\|_{L^2}.$$

Therefore, we can conclude that

$$\begin{aligned}\|\phi_n\|_{L^p} &\leqslant C_{p,d}\|\nabla\phi_n\|_{L^2}^{\rho}\|\phi_n\|_{L^2}^{1-\rho} \\ &\leqslant C_{p,d}\frac{1}{\Lambda^{1-\rho}}\|\varphi_n\|_{L^2}^{1-\rho}\left[\frac{1}{\Lambda} + \frac{e_0 - \Gamma}{\Lambda^2}\right]^{\rho/2}\|\varphi_n\|_{L^2}^{\rho} \\ &\leqslant \mu_2\|\varphi_n\|_{L^2}\end{aligned}$$

where $\mu_2(p,d)$ is the constant (18) dependent on p and d but independent of n. □

Lemma 3. *Let $V(x)$ be a potential bounded from below: $V \geqslant \Gamma$. Let $d = 1, 2, 3$ and let $\phi_j \in \mathcal{D}(H) \cap L^6$ for $j = 0, \ldots, n-1$; then $\phi_n \in \mathcal{D}(H) \cap L^6$.*

Proof. Indeed, $u_j \in L^2$, $j = 0, \ldots, n$, by construction. Concerning v_j we have that it belong to L^2 for any $j = 0, \ldots, n-1$ from the Hölder inequality. Hence E_n is well defined and φ_j belongs to L^2 for any $j = 1, \ldots, n$. From this fact and since $\varphi_j \perp \phi_0$, $j = 1, \ldots, n$, then $\phi_n = [H - e_0]^{-1}\varphi_n \in \mathcal{D}(H)$. Finally, by lemma 2 then $\phi_n \in L^6$ where we apply the Gagliardo-Nirenberg inequality (4) with $p = 6$. □

Remark 6. In fact, $\phi_j \in L^p$ for any $p \in [1,+\infty]$ if $d=1$, $p\in[1,+\infty)$ if $d=2$ and $p \leqslant 2d/(d-2)$ if $d>2$.

Remark 7. From lemma 3 and from remark 2 then we have proved that $\phi_n \in \mathcal{D}(H)\cap L^6$ for any $n=0,1,2,\ldots$, and u_n, $v_{n-1}\in L^2$ for any $n=1,2,\ldots$.

4. Are the formal series (7) convergent as *N* goes to infinity? Proof of theorem 1

In order to prove the convergence of the perturbation series we give the following results.

Lemma 4. *Let*

$$c_n := \|\phi_n\|_{L^2}\,, d_n := \|\phi_n\|_{L^6} \text{ and } b_n := |e_n|\,,\ n=0,1,2,\ldots,$$

then

$$\begin{aligned} b_n &\leqslant \sum_{m=0}^{n-1} d_m \sum_{\ell=0}^{n-1-m} d_\ell d_{n-1-\ell-m} \\ c_n &\leqslant \frac{1}{\Lambda}\left[2\sum_{m=0}^{n-1} d_m \sum_{\ell=0}^{n-1-m} d_\ell d_{n-1-\ell-m} + \sum_{m=1}^{n-1} b_m c_{n-m}\right] \\ d_n &\leqslant \mu_2(6,d)\left[2\sum_{m=0}^{n-1} d_m \sum_{\ell=0}^{n-1-m} d_\ell d_{n-1-\ell-m} + \sum_{m=1}^{n-1} b_m c_{n-m}\right] \end{aligned}$$

Proof. In order to prove the result above we remark that

$$b_n = |e_n| \leqslant \|v_{n-1}\|_{L^2}\,, c_n = \|\phi_n\|_{L^2} \leqslant \frac{1}{\Lambda}\|\varphi_n\|_{L^2}$$

and

$$d_n = \|\phi_n\|_{L^6} \leqslant \mu_2(6,d)\,\|\varphi_n\|_{L^2}\,,$$

from lemma 2, where

$$\|\varphi_n\|_{L^2} \leqslant |e_n| + \|v_{n-1}\|_{L^2} + \|u_n\|_{L^2} \leqslant 2\|v_{n-1}\|_{L^2} + \|u_n\|_{L^2}\,. \tag{19}$$

Hence, the above result follows since

$$\begin{aligned} \|v_{n-1}\|_{L^2} &\leqslant \sum_{m=0}^{n-1}\sum_{\ell=0}^{n-1-m} \|\phi_m\|_{L^6}\|\phi_\ell\|_{L^6}\|\phi_{n-1-\ell-m}\|_{L^6} \\ &= \sum_{m=0}^{n-1}\sum_{\ell=0}^{n-1-m} d_\ell d_m d_{n-1-\ell-m} \end{aligned} \tag{20}$$

and

$$\|u_n\|_{L^2} \leqslant \sum_{m=1}^{n-1} b_m c_{n-m}\,. \tag{21}$$

□

Lemma 5. *Let us assume that*

$$d_j \leqslant \delta e^{\alpha j} \frac{1}{(j+1)^2}, ,j = 0, \ldots, n-1, \tag{22}$$

for some $\alpha > 0$ *and where*

$$\delta = d_0 = \|\phi_0\|_{L^2}.$$

Then

$$\sum_{m=0}^{n-1} d_m \sum_{\ell=0}^{n-1-m} d_\ell d_{n-1-\ell-m} \leqslant C_1 \delta^3 e^{\alpha(n-1)} \frac{1}{(n+1)^2} \tag{23}$$

for any $n \geqslant 1$ *and some* $C_1 \leqslant 4 \cdot 4.7^2$.

Remark 8. By construction, (22) holds true for $j = 0$.

Proof. Indeed, from (22) it turns out that

$$\sum_{m=0}^{n-1} d_m \sum_{\ell=0}^{n-1-m} d_\ell d_{n-1-\ell-m} \leqslant \delta^3 e^{\alpha(n-1)} I$$

where we set

$$I := \sum_{m=0}^{n-1} \frac{1}{(m+1)^2} \sum_{\ell=0}^{n-1-m} \frac{1}{(\ell+1)^2 (n-m-\ell)^2}. \tag{24}$$

A simple estimate proves that

$$\sum_{\ell=0}^{n-1-m} \frac{1}{(\ell+1)^2 (n-m-\ell)^2} = \frac{2}{(n-m)^2} + J(n-1-m) \leqslant \frac{4.7}{(n-m)^2}$$

where $J(n)$ has been defined and estimated in appendix. Therefore,

$$\begin{aligned} I &\leqslant \sum_{m=0}^{n-1} \frac{4.7}{(n-m)^2 (m+1)^2} \\ &\leqslant \frac{2 \cdot 4.7}{n^2} + 4.7 J(n-1) \leqslant \frac{4.7^2}{n^2} \leqslant \frac{4.7^2}{(n+1)^2} \frac{(n+1)^2}{n^2} \leqslant \frac{4 \cdot 4.7^2}{(n+1)^2} \end{aligned}$$

from which the statement follows. □

Remark 9. From lemma 5 and from (20) it follows that

$$\|v_{n-1}\|_{L^2} \leqslant C_1 \delta^3 e^{\alpha(n-1)} \frac{1}{(n+1)^2}.$$

In fact, $\delta = 1$ because we have chosen the normalisation condition $\|\phi_0\|_{L^2} = 1$.

Lemma 6. *Let us assume that*

$$b_j \leqslant \beta e^{\alpha(j-1)} \frac{1}{(j+1)^2}, \, j = 1, \ldots, n-1, \tag{25}$$

and

$$c_j \leqslant \gamma e^{\alpha j} \frac{1}{(j+1)^2}, \, j = 1, \ldots, n-1, \tag{26}$$

for some $\beta \geqslant 4b_1 = 4|e_1|$ *and where*

$$\gamma = \max\left[4c_1, 1\right], \; c_1 = 4\|\phi_1\|_{L^2}$$

and where $\alpha > 0$ *has been introduced in lemma 5. Then*

$$\sum_{m=1}^{n-1} b_m c_{n-m} \leqslant C_2 \beta \gamma e^{\alpha(n-1)} \frac{1}{(n+1)^2}, \tag{27}$$

for some $C_2 \leqslant 2.7$.

Remark 10. By construction, (25) and (26) hold true for $j = 1$.

Proof. The proof immediately follows since

$$\sum_{m=1}^{n-1} b_m c_{n-m} \leqslant \beta \gamma e^{\alpha(n-1)} J(n)$$

where $J(n) \leqslant \frac{2.7}{(n+1)^2}$ (see appendix). □

Remark 11. In fact, the estimate of the constants C_1 and C_2 are far to be optimal. Numerical analysis suggests that a sharp estimate for the term I defined in (24) has the form

$$I = \frac{g(n)}{(n+1)^2} \text{ where } g(n) \leqslant g(10) = 10.44589874,$$

that is

$$C_1 \leqslant 10.45.$$

Concerning C_2 from appendix numerical analysis proves that

$$C_2 \leqslant 1.52.$$

Remark 12. From lemma 6 and from (21) it follows that

$$\|u_n\|_{L^2} \leqslant C_2 \beta \gamma e^{\alpha(n-1)} \frac{1}{(n+1)^2}.$$

Remark 13. From remarks 9 and 12 and from (19) it follows that

$$\|\varphi_n\|_{L^2} \leqslant C_3 e^{\alpha(n-1)} \frac{1}{(n+1)^2},$$

where

$$C_3 = \left[2C_1\delta^3 + C_2\beta\gamma\right].$$

Collecting lemmas 4–6, we have that

$$b_n \leqslant C_1 \delta^3 e^{\alpha(n-1)} \frac{1}{(n+1)^2} \tag{28}$$

$$c_n \leqslant \gamma \frac{e^{\alpha n}}{(n+1)^2} \frac{e^{-\alpha}}{\gamma \Lambda} C_3 \tag{29}$$

$$d_n \leqslant \delta \frac{e^{\alpha n}}{(n+1)^2} \frac{\mu_2(6,d)\, e^{-\alpha} C_3}{\delta} \tag{30}$$

if (22), (25) and (26) hold true.

In particular, if we choose

$$\beta = \max\left[4b_1, C_1\delta^3\right]$$

and $\alpha > 0$ large enough such that

$$\frac{e^{-\alpha}}{\gamma\Lambda} C_3 \leqslant 1 \text{ and } \frac{\mu_2(6,d)\, e^{-\alpha} C_3}{\delta} \leqslant 1 \tag{31}$$

then we have that (22), (25) and (26) hold true for $j = n$, too.

In conclusion, we have proved that

Lemma 7. *There exists four positive constants $\alpha > 0$ large enough, $\beta > 0$, $\gamma > 0$ and $\delta > 0$ independent of n such that the following estimates*

$$b_n \leqslant \beta e^{\alpha(n-1)} \frac{1}{(n+1)^2},$$
$$c_n \leqslant \gamma e^{\alpha n} \frac{1}{(n+1)^2},$$
$$d_n \leqslant \delta e^{\alpha n} \frac{1}{(n+1)^2},$$

hold true for any $n = 1, 2, \ldots$.

Remark 14. From remark 13 and from lemma 2 it follows that ψ_N is norm convergent in H^1.

Finally:

Theorem 2. *Let $d = 1,2,3$ and let ν be such that $|\nu| < e^{-\alpha}$ where $\alpha > 0$ is large enough as given in lemma 7. Then the power series E_N is absolutely convergent, and the power series ψ_N is norm convergent in L^2 and L^6, and the power series $H\psi_N = \sum_{n=0}^N \nu^n H\phi_n$ is norm convergent in L^2 .*

Proof. Convergence of E_N and ψ_N directly comes from lemma 7. Concerning the convergence of $\sum_{n=0}^N \nu^n H\phi_n$ we simply remark that

$$\|H\phi_n\|_{L^2} = \|H[H-e_0]^{-1}\varphi_n\|_{L^2} \leqslant \|\varphi_n\|_{L^2} + |e_0|\, \|[H-e_0]^{-1}\varphi_n\|_{L^2}$$
$$\leqslant \left(1 + |e_0|\Lambda^{-1}\right)\|\varphi_n\|_{L^2} \leqslant C_4 e^{\alpha n}\frac{1}{(n+1)^2}$$

for some $C_4 > 0$, and thus the formal power series

$$\sum_n \nu^n H\phi_n$$

is norm convergent in the space L^2 if $|\nu| < e^{-\alpha}$. □

So far we have proved that there exists vectors u, w, $\varphi \in L^2$, $v \in L^6$ and $z \in H^1$ such that

$$\psi_N \to u \text{ in } L^2$$
$$\psi_N \to v \text{ in } L^6$$
$$H\psi_N \to w \text{ in } L^2$$
$$\psi_N \to z \text{ in } H^1$$
$$\sum_{n=1}^N \nu^n \varphi_n \to \varphi \text{ in } L^2$$

as N goes to ∞. First of all we remark that

$$u = \sum_{n=0}^{\infty} \nu^n \phi_n = \phi_0 + \sum_{n=1}^{\infty} \nu^n \phi_n = \phi_0 + \sum_{n=1}^{\infty} \nu^n [H - e_0]^{-1} \varphi_n$$
$$= \phi_0 + [H - e_0]^{-1} \varphi$$

since $[H - e_0]^{-1}$ is a bounded operator on the eigenspace orthogonal to ϕ_0, and where the convergence of the infinite sum has to be intended in the space L^2. Hence

$$u \in \mathcal{D}(H) .$$

Furthermore, we immediately have that

$$\|u - z\|_{L^2} \leqslant \|u - \psi_N\|_{L^2} + \|z - \psi_N\|_{L^2}$$
$$\leqslant \|u - \psi_N\|_{L^2} + \|z - \psi_N\|_{H^1} \to 0 ,$$

hence $u = z$. Similarly, from (4) for some $\rho \in [0, 1]$ we have that

$$\|v - z\|_{L^6} \leqslant \|v - \psi_N\|_{L^6} + \|z - \psi_N\|_{L^6}$$
$$\leqslant \|v - \psi_N\|_{L^6} + C_{6,d} \|z - \psi_N\|_{H^1}^{\rho} \|z - \psi_N\|_{L^2}^{(1-\rho)}$$
$$\leqslant \|v - \psi_N\|_{L^6} + \|z - \psi_N\|_{H^1}^{\rho} \left[\|z\|_{L^2} + \|\psi_N\|_{L^2}\right]^{(1-\rho)} \to 0 ,$$

hence $v = z$. In conclusion, there exists a vector $\psi \in \mathcal{D}(H)$ such that

$$\psi_N \to \psi \text{ in } L^2 , L^6 \text{ and } H^1 ,$$

and

$$H\psi = [H - e_0]\psi + e_0\psi = [H - e_0]\left(\phi_0 + [H - e_0]^{-1}\varphi\right) + e_0\psi$$
$$= e_0\psi + \varphi$$
$$H\psi_N = H\sum_{n=0}^{N} \nu^n \phi_n = H\phi_0 + \sum_{n=1}^{N} \nu^n H\phi_n = H\phi_0 + \sum_{n=1}^{N} \nu^n H [H - e_0]^{-1} \varphi_n$$
$$= e_0 \sum_{n=0}^{N} \nu^n \phi_n + \sum_{n=1}^{N} \nu^n \varphi_n \to e_0\psi + \varphi = H\psi$$

as $N \to \infty$.

Thus we have proved the following result.

Lemma 8. *$\psi_N \to \psi \in \mathcal{D}(H)$ in L^2, L^6 and H^1, and $H\psi_N \to H\psi$ in L^2.*

Finally, it is not hard to see that ψ is a stationary solution associated to the energy E to (1). Indeed:

Lemma 9. *Let*

$$r_N := H\psi_N + \nu|\psi_N|^2\psi_N - E_N\psi_N$$

then

$$\lim_{N\to\infty} \|r_N\|_{L^2} = 0 . \tag{32}$$

Proof. A simple straightforward calculation gives that

$$\begin{aligned} r_N &= \sum_{n=0}^{N} \nu^n H\phi_n + \sum_{n,m,\ell=0}^{N} \nu^{n+m+\ell+1} \phi_n \phi_m \bar{\phi}_\ell - \sum_{n,m=0}^{N} \nu^{n+m} e_m \phi_n \\ &= \sum_{n=0}^{N} \nu^n H\phi_n + \sum_{n=1}^{3N+1} \nu^n \left[\sum_{m=0}^{n-1} \sum_{\ell=0}^{n-1-m} \phi_m \bar{\phi}_\ell \phi_{n-m-\ell-1} \right] - \sum_{n=0}^{2N} \nu^n \left[\sum_{m=0}^{n} e_m \phi_{n-m} \right] \\ &= \sum_{n=N+1}^{3N+1} \nu^n v_{n-1} - \sum_{n=N+1}^{2N} \nu^n u_n \end{aligned}$$

where the two power series $\sum_{n=0}^{\infty} \nu^n v_{n-1}$ and $\sum_{n=1}^{\infty} \nu^n u_n$ are norm-L^2 convergent for ν small enough. Then (32) follows. □

Now, we are ready to complete the proof of theorem 1.

Indeed, let

$$r := H\psi + \nu|\psi|^2\psi - E\psi = a_N + b_N + c_N + r_N$$

where

$$\begin{aligned} a_N &= H\psi - H\psi_N \to 0 \text{ in } L^2 \\ b_N &= \nu \left[|\psi|^2\psi - |\psi_N|^2\psi_N \right] \to 0 \text{ in } L^2 \\ c_N &= -\left[E\psi - E_N\psi_N \right] \to 0 \text{ in } L^2. \end{aligned}$$

From the above results immediately follows that

$$\|a_N\|_{L^2} \to 0 \text{ and } \|c_N\|_{L^2} \to 0$$

as N goes to infinity. Concerning b_N one notes that

$$\begin{aligned} \|b_N\|_{L^2} &\leqslant \| (\psi - \psi_N) |\psi|^2 \|_{L^2} + \| (\psi - \psi_N) \bar{\psi} \psi_N \|_{L^2} + \| \left(\bar{\psi} - \bar{\psi}_N \right) |\psi_N|^2 \|_{L^2} \\ &\leqslant \| |\psi|^2 \|_{L^3} \|\psi - \psi_N\|_{L^6} + \|\psi - \psi_N\|_{L^6} \|\bar{\psi}\psi_N\|_{L^3} + \| |\psi_N|^2 \|_{L^3} \|\psi - \psi_N\|_{L^6} \\ &\leqslant \left[\| |\psi|^2 \|_{L^3} + \|\bar{\psi}\psi_N\|_{L^3} + \| |\psi_N|^2 \|_{L^3} \right] \|\psi - \psi_N\|_{L^6} \to 0 \end{aligned}$$

as N goes to infinity. From these facts and since (32) then theorem 1 is proved.

Remark 15. Since the constants β, γ, δ, Λ, C_1, C_2, C_3, C_4 and $\mu_2(6,d)$ can be estimated then one can obtain the value of the parameter α solution to (31). Hence, the estimate $\nu^\star < e^{-\alpha}$ of the radius of convergence follows.

5. A Toy model—infinite well potential

Let us consider, in dimension one, the infinite well potential of the form:

$$V(x) = \begin{cases} 0 & \text{if } |x| < \pi \\ +\infty & \text{if } |x| \geqslant \pi \end{cases} .$$

5.1. Linear time-independent Schrödinger equation

The linear operator H is formally defined as follows:

$$H\psi = -\psi'' , \; x \in (-\pi, +\pi) , \; \psi \in L^2((-\pi, +\pi))$$

with Dirichlet boundary conditions

$$\psi(-\pi) = \psi(+\pi) = 0. \tag{33}$$

By means of a straightforward calculation it follows that the spectrum of H is purely discrete and it is given by means of simple eigenvalues

$$\lambda_j = \frac{1}{4} j^2 \, , \; j = 1, 2, \dots \, ,$$

with associated normalised eigenvectors

$$q_j(x) = \frac{1}{\sqrt{\pi}} \begin{cases} \cos(jx/2) \, , \text{ odd } j \\ \sin(jx/2) \, , \text{ even } j \end{cases} .$$

The resolvent operator is given by

$$\left([H - z]^{-1} \psi \right)(x) = \sum_{j=1}^{\infty} \frac{1}{\lambda_j - z} q_j(x) \langle q_j, \psi \rangle_{L_2} . \tag{34}$$

5.2. Perturbation theory

By making use of the perturbation formula we compute now the coefficients of the formal power series (7) where

$$e_0 = \lambda_1 = \frac{1}{4}$$

is the first unperturbed eigenvalue with associated unperturbed eigenvector

$$\phi_0(x) = q_1(x) = \frac{1}{\sqrt{\pi}} \cos(x/2) \, .$$

Remark 16. Here, we have considered, for argument's sake, the formal power series (7) associated to the first eigenvalue λ_1. Similarly, the same method may be applied to the unperturbed eigenvalues λ_j for any $j > 1$.

The perturbation theory exploited in lemma 1 gives that

$$v_0 = \phi_0^3 \, , \; u_1 = 0 \, , \; e_1 = \frac{\|\phi_0^2\|^2}{\|\phi_0\|^2} = \frac{3}{4\pi}$$

and

$$\varphi_1 = e_1 \phi_0 - \phi_0^3 .$$

Finally

$$\begin{aligned} \phi_1 = \left([H - e_0]^{-1} \varphi_1 \right)(x) &= \sum_{j=2}^{\infty} \frac{1}{\lambda_j - e_0} q_j(x) \langle q_j, \varphi_1 \rangle_{L_2} \\ &= -\frac{1}{\lambda_3 - e_0} q_3(x) \langle q_3, \phi_0^3 \rangle_{L_2} = -\frac{1}{8[\pi]^{3/2}} \cos\left(\frac{3}{2} x \right) . \end{aligned}$$

By means of a straightforward calculation the other terms follow; for instance

$$\begin{aligned} e_2 &= -\frac{3}{32\pi^2} \, , \; e_3 = \frac{15}{256\pi^3} \, , \; e_4 = -\frac{69}{2048\pi^4} \\ e_5 &= \frac{75}{4096\pi^5} \, , \; e_6 = -\frac{1257}{131072\pi^6} \, , \; e_7 = \dots \end{aligned}$$

Table 1. Infinite well potential—table of values corresponding to the case of defocusing nonlinearities when $\nu = 0.1$ and $\nu = +1$.

	$\nu = 0.1$			$\nu = 1$		
N	E_N	$\|\psi_N\|_{L^2}$	$\|r_N\|_{L^2}$	E_N	$\|\psi_N\|_{L^2}$	$\|r_N\|_{L^2}$
0	0.25	1	$0.25 \cdot 10^{-1}$	0.25	1	$0.25 \cdot 10^{0}$
1	0.273 873	1.000 007 916	$0.16 \cdot 10^{-3}$	0.488 732	1.000 791 259	$0.16 \cdot 10^{-1}$
2	0.273 778	1.000 007 728	$0.30 \cdot 10^{-5}$	0.479 234	1.000 614 941	$0.28 \cdot 10^{-2}$
3	0.273 780	1.000 007 730	$0.52 \cdot 10^{-7}$	0.481 123	1.000 634 507	$0.48 \cdot 10^{-3}$
4	0.273 780	1.000 007 730	$0.88 \cdot 10^{-9}$	0.480 777	1.000 632 165	$0.81 \cdot 10^{-4}$
5	0.273 780	1.000 007 730	$0.15 \cdot 10^{-10}$	0.480 837	1.000 632 442	$0.13 \cdot 10^{-4}$
6	0.273 780	1.000 007 730	$0.24 \cdot 10^{-12}$	0.480 827	1.000 632 410	$0.22 \cdot 10^{-5}$

Table 2. Infinite well potential—table of values corresponding to the case of focusing nonlinearities when $\nu = -0.1$ and $\nu = -1$.

	$\nu = -0.1$			$\nu = -1$		
N	E_N	$\|\psi_N\|_{L^2}$	$\|r_N\|_{L^2}$	E_N	$\|\psi_N\|_{L^2}$	$\|r_N\|_{L^2}$
0	0.25	1	$0.25 \cdot 10^{-1}$	0.25	1	$0.25 \cdot 10^{0}$
1	0.226 168	1.000 007 916	$0.17 \cdot 10^{-3}$	0.011 268	1.000 791 259	$0.17 \cdot 10^{-1}$
2	0.226 032	1.000 008 160	$0.30 \cdot 10^{-5}$	0.001 769	1.000 992 585	$0.32 \cdot 10^{-2}$
3	0.226 030	1.000 008 108	$0.53 \cdot 10^{-7}$	−0.000 121	1.001 018 592	$0.58 \cdot 10^{-3}$
4	0.226 030	1.000 008 108	$0.90 \cdot 10^{-9}$	−0.000 467	1.001 021 668	$1.00 \cdot 10^{-4}$
5	0.226 030	1.000 008 108	$0.15 \cdot 10^{-10}$	−0.000 527	1.001 022 048	$0.17 \cdot 10^{-4}$
6	0.226 030	1.000 008 108	$0.25 \cdot 10^{-12}$	−0.000 537	1.001 022 093	$0.28 \cdot 10^{-5}$

and

$$
\begin{aligned}
\phi_2(x) &= \frac{1}{64\pi^{5/2}}\left[3\cos\left(\frac{3x}{2}\right)+\cos\left(\frac{5x}{2}\right)\right] \\
\phi_3(x) &= -\frac{1}{512\pi^{7/2}}\left[9\cos\left(\frac{3x}{2}\right)+5\cos\left(\frac{5x}{2}\right)+\cos\left(\frac{7x}{2}\right)\right] \\
\phi_4(x) &= \frac{1}{4906\pi^{9/2}}\left[27\cos\left(\frac{3x}{2}\right)+20\cos\left(\frac{5x}{2}\right)+7\cos\left(\frac{7x}{2}\right)+\cos\left(\frac{9x}{2}\right)\right] \\
\phi_5(x) &= -\frac{1}{32768\pi^{11/2}}\left[81\cos\left(\frac{3x}{2}\right)+75\cos\left(\frac{5x}{2}\right)+35\cos\left(\frac{7x}{2}\right)+\right. \\
&\qquad \left.+9\cos\left(\frac{9x}{2}\right)+\cos\left(\frac{11x}{2}\right)\right] \\
\phi_6(x) &= \frac{1}{262144\pi^{13/2}}\left[243\cos\left(\frac{3x}{2}\right)+275\cos\left(\frac{5x}{2}\right)+154\cos\left(\frac{7x}{2}\right)+\right. \\
&\qquad \left.+54\cos\left(\frac{9x}{2}\right)+11\cos\left(\frac{11x}{2}\right)+\cos\left(\frac{13x}{2}\right)\right] \\
\phi_7(x) &= \dots .
\end{aligned}
$$

In tables 1 and 2 we compute the values of E_N, $\|\psi_N\|_{L^2}$ and $\|r_N\|_{L^2}$, where r_N is the remainder term defined by (10), for different values of N and for $\nu = \pm 0.1$ and $\nu = \pm 1$. It turns out that the formal power series (7) rapidly converges and that the norm of the remainder term r_N rapidly decreases when N increases.

5.3. Nonlinear time-independent Schrödinger equation

The nonlinear time-independent equation (1) becomes

$$-\psi'' + \nu|\psi|^2\psi = E\psi \, , \; x \in (-\pi, +\pi) \, , \tag{35}$$

with Dirichlet boundary conditions (33). If we restrict our attention to the case of $\nu \in \mathbb{R}$ and $E \in \mathbb{R}$ then we known that the stationary solution is, up to a constant phase factor, a real-valued function. The proof of this result is quite similar to the one of lemma 3.7 given by [17]. Indeed, if we multiply both sides of (35) by $\bar{\psi}$, we obtain that

$$-\psi''\bar{\psi} + \nu|\psi|^4 = E|\psi|^2$$

and similarly

$$-\bar{\psi}''\psi + \nu|\psi|^4 = E|\psi|^2 .$$

Hence

$$0 = \psi''\bar{\psi} - \bar{\psi}''\psi = \left(\psi'\bar{\psi} - \bar{\psi}'\psi\right)'$$

from which follows that

$$\psi'\bar{\psi} - \bar{\psi}'\psi = C, \forall x \in (-\pi, +\pi) \, ,$$

for some constant C. Recalling that $\psi(\pm\pi) = 0$ then $C = 0$ and thus $\theta = \arg(\psi)$ is a constant term.

Therefore, stationary solutions ψ to (35) may be assumed to be real-valued and they satisfy to the equation

$$-\psi'' + \nu\psi^3 = E\psi \, , \; x \in (-\pi, +\pi) \, . \tag{36}$$

The general solution to such an equation has the form [10]

$$\psi(x) = \chi \, \mathrm{sn}[\zeta(x - x_0), k] \, , \; \chi \in \mathbb{R},$$

where x_0 and ζ are arbitrary constants and where

$$k^2 = \frac{E - \zeta^2}{\zeta^2} \text{ and } \chi^2 = 2\frac{E - \zeta^2}{\nu} .$$

The Dirichlet boundary conditions imply that $x_0 = -\pi$ and that $2\zeta\pi$ is a zero of the Jacobian Elliptic function $\mathrm{sn}(x,k)$, i.e.:

$$2\zeta\pi = 2K(k)m, \; m = 1, 2, \ldots,$$

where $K(k)$ and $E(k)$ are the complete elliptic integral of first and second kind. The norm of the wavefunction ψ is given by

$$\begin{aligned} \|\psi\|_{L^2}^2 &= \chi^2 \int_{-\pi}^{+\pi} \mathrm{sn}^2[\zeta(x+\pi), k] \, \mathrm{d}x = 2m\frac{\chi^2}{\zeta}\frac{K(k) - E(k)}{k^2} \\ &= 2\pi\chi^2 \frac{K(k) - E(k)}{K(k)k^2} . \end{aligned}$$

Hence

$$\chi^2 = \frac{1}{2\pi}\frac{K(k)k^2}{K(k) - E(k)}\|\psi\|_{L^2}^2 .$$

In order to find the stationary solutions to (36) the quantisation conditions read

$$2\frac{E-\zeta^2}{\nu} = \frac{1}{2\pi}\frac{K(k)k^2}{K(k)-E(k)}\|\psi\|_{L^2}^2\,,\ k^2 = \frac{E-\zeta^2}{\zeta^2} \text{ and } \zeta = \frac{K(k)}{\pi}m\,,\ m=1,2,\dots\,.$$

5.3.1. Defocusing nonlinearity: $\nu > 0$. When $\nu > 0$ then stationary solutions there exist provided that $E-\zeta^2 \geqslant 0$ and k is a real-valued solution to the equation

$$K(k)\left[K(k)-E(k)\right] = \frac{\nu\pi}{4m^2}\|\psi\|_{L^2}^2\,. \tag{37}$$

If we remark that the function $K(k)[K(k)-E(k)]$ is a monotone increasing function such that

$$\lim_{k\to 0^+} K(k)\left[K(k)-E(k)\right] = 0 \text{ and } \lim_{k\to 1^-} K(k)\left[K(k)-E(k)\right] = +\infty$$

then the equation above (37) has a unique solution $k_m \in (0,1)$, for any $m = 1,2,\dots$ fixed, and then there exists a family of values of the parameter E:

$$E = \left[\frac{K(k_m)m}{\pi}\right]^2\left[1+k_m^2\right]\,,\ m=1,2,\dots\,. \tag{38}$$

5.3.2. Focusing nonlinearity: $\nu < 0$. On the other hands if $\nu < 0$ then stationary solutions there exist provided that $E-\zeta^2 \leqslant 0$ and $k = i\kappa$, $\kappa \in \mathbb{R}$, is a purely imaginary complex number; in such a case we recall that

$$\mathrm{sn}(x,i\kappa) = k_1'\mathrm{sd}\left(x\sqrt{1+\kappa^2},k_1\right) \text{ where } \mathrm{sd}(x,k_1) = \frac{\mathrm{sn}(x,k_1)}{\mathrm{dn}(x,k_1)}$$

and where

$$k_1 = \kappa/\sqrt{1+\kappa^2} \text{ and } k_1' = \sqrt{1-k_1^2} = \frac{1}{\sqrt{1+\kappa^2}}\,.$$

Hence, equation (37) becomes

$$K(i\kappa)\left[K(i\kappa)-E(i\kappa)\right] = \frac{\nu\pi}{4m^2}\|\psi\|_{L^2}^2\,. \tag{39}$$

If we remark that the function $K(i\kappa)[K(i\kappa)-E(i\kappa)]$ is a monotone real-valued decreasing function for $\kappa \in [0,+\infty)$ such that

$$\lim_{\kappa\to 0^+} K(i\kappa)\left[K(i\kappa)-E(i\kappa)\right] = 0 \text{ and } \lim_{\kappa\to +\infty} K(i\kappa)\left[K(i\kappa)-E(i\kappa)\right] = -\infty$$

then the equation above (39) has a unique solution $\kappa_m \in (0,+\infty)$ for any $m = 1,2,\dots$ fixed, and also in this case there exists a family of values of the parameter E:

$$E = \left[\frac{K(i\kappa_m)m}{\pi}\right]^2\left[1-\kappa_m^2\right]\,,\ m=1,2,\dots\,. \tag{40}$$

5.4. Comparison between the perturbative result and the exact one

From table 1 the perturbative result gives that the stationary solution to (1) for $\nu = 0.1$ and $N = 6$ has energy

$$E_6 = 0.273780 \tag{41}$$

with associated wavefunction with norm

$$\|\psi\|_{L^2} \approx \|\psi_6\|_{L^2} = 1.000007730\,. \tag{42}$$

The value of the solution k to (37), where $m = 1$ and where the value of $\|\psi\|_{L^2}$ is the one in (42), is given by

$$k = 0.2474031338$$

and the associated energy E is given by (38)

$$E = 0.273780$$

in full agreement with (41). Similarly, For $\nu = 1$ and $N = 6$ then table 1 gives that

$$E_6 = 0.480827 \tag{43}$$

with associated wavefunction with norm

$$\|\psi\|_{L^2} \approx \|\psi_6\|_{L^2} = 1.000632410\,.$$

From (37) and (38) then (where $m = 1$)

$$k = 0.6682383718 \text{ and } E = 0.480829$$

in good agreement with (43).

From table 2 the perturbative result gives that the stationary solution for $\nu = -0.1$ and $N = 6$ has energy

$$E_6 = 0.226030 \tag{44}$$

with associated wavefunction with norm

$$\|\psi\|_{L^2} \approx \|\psi_6\|_{L^2} = 1.000008108\,.$$

From (39) and (40) then (where $m = 1$)

$$\kappa = 0.2574471610 \text{ and } E = 0.226030$$

in full agreement with (44). Similarly, For $\nu = -1$ and $N = 6$ then table 2 gives that

$$E_6 = -0.000537 \tag{45}$$

with associated wavefunction with norm

$$\|\psi\|_{L^2} \approx \|\psi_6\|_{L^2} = 1.001022099\,.$$

From (39) and (40) then (where $m = 1$)

$$\kappa = 1.001546564 \text{ and } E = -0.000540$$

in very good agreement with (45).

6. Harmonic oscillator

Let us consider, in dimension one, the harmonic oscillator with potential

$$V(x) = x^2\,,\ x \in \mathbb{R}\,.$$

That is, the linear operator H is defined as follows:

$$H\psi = -\psi'' + x^2\psi\,,\ x \in \mathbb{R},\ \psi \in L^2(\mathbb{R})\,.$$

Table 3. Harmonic oscillator potential—table of values corresponding to the case of defocusing nonlinearities when $\nu = 0.1$ and $\nu = +1$.

	$\nu = +0.1$			$\nu = +1$		
N	E_N	$\|\psi_N\|_{L^2}$	$\|r_N\|_{L^2}$	E_N	$\|\psi_N\|_{L^2}$	$\|r_N\|_{L^2}$
0	1	1	$0.43 \cdot 10^{-1}$	1	1	$0.43 \cdot 10^{0}$
1	1.039 894 228	1.000 006 539	$0.24 \cdot 10^{-3}$	1.398 942 280	1.000 653 715	$0.23 \cdot 10^{-1}$
2	1.039 728 699	1.000 006 483	$0.25 \cdot 10^{-5}$	1.382 389 419	1.000 599 885	$0.24 \cdot 10^{-2}$
3	1.039 730 376	1.000 006 484	$0.24 \cdot 10^{-7}$	1.384 066 368	1.000 602 159	$0.23 \cdot 10^{-3}$
4	1.039 730 361	1.000 006 484	$0.22 \cdot 10^{-9}$	1.383 909 162	1.000 602 036	$0.21 \cdot 10^{-4}$
5	1.039 730 361	1.000 006 484	$0.21 \cdot 10^{-11}$	1.383 923 548	1.000 602 043	$0.18 \cdot 10^{-5}$
6	1.039 730 361	1.000 006 484	$0.43 \cdot 10^{-12}$	1.383 922 248	1.000 602 043	$0.17 \cdot 10^{-6}$

It is well known that the spectrum of H is purely discrete and it is given by simple eigenvalues

$$\lambda_j = 2j - 1\,,\; j = 1, 2, \ldots\,,$$

with associated normalised eigenvectors

$$q_j(x) = \frac{1}{\sqrt{2^{j-1}(j-1)!}} \left(\frac{1}{\pi}\right)^{1/4} e^{-x^2/2} H_{j-1}(x)\,,$$

where

$$H_j(x) = (-1)^j e^{x^2} \frac{d^j}{dx^j} e^{-x^2}$$

are the Hermite's polynomials.

By making use of the perturbation formula we compute now the coefficients of the formal power series (7) associated to the first unperturbed eigenvalue

$$e_0 = \lambda_1 = 1$$

with associated unperturbed normalised eigenvector

$$\phi_0(x) = q_1(x) = \frac{1}{\sqrt[4]{\pi}} e^{-x^2/2}.$$

In such a case the perturbative procedure gives that

$$e_1 = \frac{\|\phi_0^2\|_{L^2}^2}{\|\phi_0\|_{L^2}^2} = \frac{1}{\sqrt{2\pi}}.$$

Furthermore,

$$\phi_1 = [H - e_0]^{-1} \varphi_1 = \sum_{j=2}^{+\infty} \frac{1}{\lambda_j - e_0} q_j(x) \langle q_j, \varphi_1 \rangle_{L^2}$$
$$\approx \sum_{j=2}^{N_2} \frac{1}{\lambda_j - e_0} q_j(x) \langle q_j, \varphi_1 \rangle_{L^2}\,,$$

where $\varphi_1 = e_1\phi_0 - \phi_0^3$ and where the resolvent operator is given by a with infinitely many terms. In numerical calculation we truncate the series for j up to a some large enough positive integer N_2; in numerical experiments we observe that $N_2 = 60$ is a suitable value. Iterating such a procedure we can obtain in tables 3 and 4 the numerical values of E_N, $\|\psi_N\|_{L^2}$ and $\|r_N\|_{L^2}$ for $N = 1, 2, \ldots, 6$, where r_N is the remainder term defined by (10), for $\nu = \pm 0.1$ and $\nu = \pm 1$.

Table 4. Harmonic oscillator potential—table of values corresponding to the case of focusing nonlinearities when $\nu = -0.1$ and $\nu = -1$.

	$\nu = -0.1$			$\nu = -1$		
N	E_N	$\lVert\psi_N\rVert_{L^2}$	$\lVert r_N\rVert_{L^2}$	E_N	$\lVert\psi_N\rVert_{L^2}$	$\lVert r_N\rVert_{L^2}$
0	1	1	$0.43 \cdot 10^{-1}$	1	1	$0.43 \cdot 10^{0}$
1	0.9601 057 720	1.000 006 539	$0.24 \cdot 10^{-3}$	0.6010 577 196	1.0006 537 148	$0.25 \cdot 10^{-1}$
2	0.9599 402 433	1.000 006 596	$0.25 \cdot 10^{-5}$	0.5845 048 581	1.0007 119 732	$0.26 \cdot 10^{-2}$
3	0.9599 385 664	1.000 006 596	$0.24 \cdot 10^{-7}$	0.5828 279 087	1.0007 147 627	$0.25 \cdot 10^{-3}$
4	0.9599 385 507	1.000 006 596	$0.22 \cdot 10^{-9}$	0.5826 707 021	1.0007 149 164	$0.24 \cdot 10^{-4}$
5	0.9599 385 505	1.000 006 596	$0.20 \cdot 10^{-11}$	0.5826 563 161	1.0007 149 254	$0.22 \cdot 10^{-5}$
6	0.9599 385 505	1.000 006 596	$0.28 \cdot 10^{-12}$	0.5826 550 168	1.0007 149 260	$0.20 \cdot 10^{-6}$

As in the toy model discussed in section 5 it turns out that the formal power series seems to rapidly converges for $|\nu| \leqslant 1$.

7. Conclusions

In this paper we have applied the Rayleigh–Schrödinger perturbation theory when the unperturbed linear operator has an isolated nondegenerate eigenvalue and where the nonlinear term plays the role of the perturbation. The power series has coefficients that can be iteratively obtained and such a series is proved to be convergent when the strength ν of the nonlinear term has absolute value less than a threshold value $\nu^\star$, for some $\nu^\star > 0$.

From the numerical experiments resumed in tables 1, 2 and tables 3, 4 one has evidence that the formal power series (7) rapidly converges for $|\nu|$ small enough when N goes to infinity. In particular, from figure 1 one can see that $|e_n|$ and $\|\phi_n\|_{L^2}$ behaves like C^n for some positive constant $C > 0$ that can be numerically estimated, and then one can conclude that the power series (7) converges when $|\nu| < \nu^\star := C^{-1}$.

For instance, concerning the convergence of the power series for E_N in the model with an infinite well potential we observe that

$$e_n = (-1)^n 4 \frac{a_n}{\pi^n 8^n}$$

where

$$a_n \sim a^n$$

for large n and where

$$a \leqslant 4 .$$

Thus, we expect that the power series E_N is absolutely convergent for any ν such that $|\nu| < \nu^\star$ where $\nu^\star = \frac{8\pi}{a} \geqslant 2\pi$. Similarly, a numerical estimate of the radius of convergence for the harmonic potential case could be obtained.

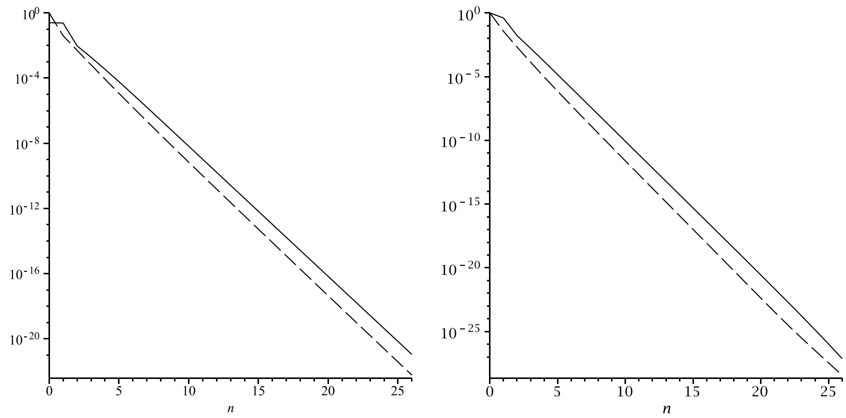

Figure 1. In the two figures are respectively plotted the values of $|e_n|$ (full lines) and $\|\phi_n\|_{L^2}$ (broken lines) for the infinite well potential case (left hand side plot) and for the harmonic potential case (right hand side plot).

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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Conflict of interest

The author has no competing interests to declare that are relevant to the content of this article.

Appendix. A simple estimate

Let

$$J := J(n) = \sum_{m=1}^{n-1} \frac{1}{(m+1)^2 (n-m+1)^2}\,,\ n > 2\,.$$

A simple inequality gives that

$$J(n) \leqslant 2\int_1^{(n+1)/2} \frac{1}{x^2 (n+1-x)^2} \mathrm{d}x = 2\frac{n^2-1+2n\ln(n)}{(n+1)^3 n} = \frac{f(n)}{(n+1)^2}$$

where

$$f(n) := 2\frac{n^2-1+2n\ln(n)}{(n+1)n} \leqslant 2.70.$$

In fact, such an estimate is not optimal. A simple numerical experiment shows that

$$J(n) = \frac{g(n)}{(n+1)^2} \text{ where } g(n) \leqslant g(19) = 1.517\,106\,786\,.$$

Furthermore, a closed expression for $J(n)$ could be given by means of Polygamma functions; however, we do not dwell here on this detail.

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