

PROFINITE GROUPS WITH AN AUTOMORPHISM WHOSE FIXED POINTS ARE RIGHT ENGEL

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ABSTRACT. An element g of a group G is said to be right Engel if for every $x \in G$ there is a number $n = n(g, x)$ such that $[g, {}_n x] = 1$. We prove that if a profinite group G admits a coprime automorphism φ of prime order such that every fixed point of φ is a right Engel element, then G is locally nilpotent.

1. INTRODUCTION

Let G be a profinite group, and φ a (continuous) automorphism of G of finite order. We say for short that φ is a *coprime automorphism* of G if its order is coprime to the orders of elements of G (understood as Steinitz numbers), in other words, if G is an inverse limit of finite groups of order coprime to the order of φ . Coprime automorphisms of profinite groups have many properties similar to the properties of coprime automorphisms of finite groups. In particular, if φ is a coprime automorphism of G , then for any (closed) normal φ -invariant subgroup N the fixed points of the induced automorphism (which we denote by the same letter) in G/N are images of the fixed points in G , that is, $C_{G/N}(\varphi) = C_G(\varphi)N/N$. Therefore, if φ is a coprime automorphism of prime order q such that $C_G(\varphi) = 1$, Thompson's theorem [18] implies that G is pronilpotent, and Higman's theorem [7] implies that G is nilpotent of class bounded in terms of q .

In this paper we consider profinite groups admitting a coprime automorphism of prime order all of whose fixed points are right Engel elements. Recall that the n -Engel word $[y, {}_n x]$ is defined recursively by $[y, {}_0 x] = y$ and $[y, {}_{i+1} x] = [[y, {}_i x], x]$. An element g of a group G is said to be right Engel if for any $x \in G$ there is an integer $n = n(g, x)$ such that $[g, {}_n x] = 1$. If all elements of a group are right Engel (therefore also left Engel), then the group is called an Engel group. By a theorem of Wilson and Zelmanov [20] based on Zelmanov's results [21, 22, 23] on Engel Lie algebras, an Engel profinite group is locally nilpotent. Recall that a group is said to be locally nilpotent if every finite subset generates a nilpotent subgroup. In our main result the right Engel condition is imposed on the fixed points of a coprime automorphism of prime order.

Theorem 1.1. *Suppose that φ is a coprime automorphism of prime order of a profinite group G . If every element of $C_G(\varphi)$ is a right Engel element of G , then G is locally nilpotent.*

The proof of Theorem 1.1 begins with the observation that a group G satisfying the hypothesis is pronilpotent. Indeed, right Engel elements of a finite group are contained in the hypercentre by the well-known theorem of Baer [1]. Therefore every finite quotient of G by a φ -invariant open normal subgroup is nilpotent by Thompson's theorem [18], since

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φ acts fixed-point-freely on the quotient by the hypercentre. Assuming in addition that G is finitely generated, it remains to prove that all Sylow p -subgroups S_p of G are nilpotent with a uniform upper bound for the nilpotency class. This is achieved in two stages. First a bound for the nilpotency class of S_p depending on p is obtained for all p . Then a bound independent of p is obtained for all sufficiently large primes p . At both stages we apply Lie ring methods and the crucial tool is Zelmanov's theorem [21, 22, 23] on Lie algebras and some of its consequences. Other important ingredients include criteria for a pro- p group to be p -adic analytic in terms of the associated Lie algebra due to Lazard [11], and in terms of bounds for the rank due to Lubotzky and Mann [13], and a theorem of Bahturin and Zaicev [2] on Lie algebras admitting a group of automorphisms whose fixed-point subalgebra is PI.

2. PRELIMINARIES

Lie rings and algebras. Products in Lie rings and algebras are called commutators. We use simple commutator notation for left-normed commutators $[x_1, \dots, x_k] = [\dots[x_1, x_2], \dots, x_k]$, and the short-hand for Engel commutators $[x, {}_n y] = [x, y, y, \dots, y]$ with y occurring n times. An element a of a Lie ring or a Lie algebra L is said to be ad-nilpotent if there exists a positive integer n such that $[x, {}_n a] = 0$ for all $x \in L$. If n is the least integer with this property, then we say that a is ad-nilpotent of index n .

The next theorem is a deep result of Zelmanov [21, 22, 23].

Theorem 2.1. *Let L be a Lie algebra over a field and suppose that L satisfies a polynomial identity. If L can be generated by a finite set X such that every commutator in elements of X is ad-nilpotent, then L is nilpotent.*

An important criterion for a Lie algebra to satisfy a polynomial identity is provided by the next theorem, which was proved by Bahturin and Zaicev for soluble group of automorphisms [2] (and later extended by Linchenko to the general case [12]). We use the centralizer notation for the fixed point subring $C_L(A)$ of a group of automorphisms A of L .

Theorem 2.2. *Let L be a Lie algebra over a field K . Assume that a finite group A acts on L by automorphisms in such a manner that $C_L(A)$ satisfies a polynomial identity. Assume further that the characteristic of K is either 0 or coprime with the order of A . Then L satisfies a polynomial identity.*

Both Theorems 2.1 and 2.2 admit respective quantitative versions (see for example [16]). For our purposes, we shall need the following proposition for Lie rings proved in [17], which combines both versions. As usual, $\gamma_i(L)$ denotes the i -th term of the lower central series of L .

Proposition 2.3. *Let L be a Lie ring and A a finite group of automorphisms of L such that $C_L(A)$ satisfies a polynomial identity $f \equiv 0$. Suppose that L is generated by an A -invariant set of m elements such that every commutator in these elements is ad-nilpotent of index at most n . Then there exist positive integers e and c depending only on $|A|$, f , m , and n such that $e\gamma_c(L) = 0$.*

We also quote the following useful result proved in [10, Lemma 5] (although it was stated for Lie algebras in [10], the proof is the same for Lie rings).

Lemma 2.4. *Let L be a Lie ring, and M a subring of L generated by m elements such that all commutators in these elements are ad-nilpotent in L of index at most n . If M is nilpotent of class c , then for some number $\varepsilon = \varepsilon(m, n, c)$ bounded in terms of m, n, c we have $[L, \underbrace{M, M, \dots, M}_\varepsilon] = 0$.*

Associated Lie rings and algebras. We now remind the reader of one of the ways of associating a Lie ring with a group. A series of subgroups of a group G

$$G = G_1 \geq G_2 \geq \dots \quad (2.1)$$

is called a *filtration* (or an *N -series*, or a *strongly central series*) if

$$[G_i, G_j] \leq G_{i+j} \quad \text{for all } i, j. \quad (2.2)$$

For any filtration (2.1) we can define an associated Lie ring $L(G)$ with additive group

$$L(G) = \bigoplus_i G_i/G_{i+1},$$

where the factors $L_i = G_i/G_{i+1}$ are additively written. The Lie product is defined on homogeneous elements $xG_{i+1} \in L_i, yG_{j+1} \in L_j$ via the group commutators by

$$[xG_{i+1}, yG_{j+1}] = [x, y]G_{i+j+1} \in L_{i+j}$$

and extended to arbitrary elements of $L(G)$ by linearity. Condition (2.2) ensures that this Lie product is well-defined, and group commutator identities imply that $L(G)$ with these operations is a Lie ring. If all factors G_i/G_{i+1} of a filtration (2.1) have prime exponent p , then $L(G)$ can be viewed as a Lie algebra over the field of p elements \mathbb{F}_p . If all terms of (2.1) are invariant under an automorphism φ of the group G , then φ naturally induces an automorphism of $L(G)$.

We shall normally indicate which filtration is used for constructing an associated Lie ring. One example of a filtration (2.1) is given by the lower central series, the terms of which are denoted by $\gamma_1(G) = G$ and $\gamma_{i+1}(G) = [\gamma_i(G), G]$. It is worth noting that the corresponding associated Lie ring $L(G)$ is generated by the homogeneous component $L_1 = G/\gamma_2(G)$.

Another example, for a fixed prime number p , is the *Zassenhaus p -filtration* (also called the *p -dimension series*), which is defined by

$$G_i = \langle g^{p^k} \mid g \in \gamma_j(G), jp^k \geq i \rangle.$$

The factors of this filtration are elementary abelian p -groups, so the corresponding associated Lie ring $D_p(G)$ is a Lie algebra over \mathbb{F}_p . We denote by $L_p(G)$ the subalgebra generated by the first factor G/G_2 . It is well known that the homogeneous components of $D_p(G)$ of degree s coincide with the homogeneous components of $L_p(G)$ for all s that are not divisible by p . In particular, $L_p(G)$ is nilpotent if and only if $D_p(G)$ is nilpotent. (Sometimes, the notation $L_p(G)$ is used for $D_p(G)$.)

A group G is said to satisfy a *coset identity* if there is a group word $w(x_1, \dots, x_m)$ and cosets a_1H, \dots, a_mH of a subgroup $H \leq G$ of finite index such that $w(a_1h, \dots, a_mh) = 1$ for any $h \in H$. Wilson and Zelmanov [20] proved that if a group G satisfies a coset identity, then the Lie algebra $L_p(G)$ constructed with respect to the Zassenhaus p -filtration satisfies a polynomial identity. In fact, the proof of Theorem 1 in [20] can be slightly modified to become valid for any filtration (2.1) with abelian factors of prime exponent p and the corresponding associated Lie algebra.

Profinite groups. We always consider a profinite group as a topological group. A subgroup of a topological group will always mean a closed subgroup, all homomorphisms are continuous, and quotients are by closed normal subgroups. This also applies to taking commutator subgroups, normal closures, subgroups generated by subsets, etc. We say that a subgroup is generated by a subset X if it is generated by X as a topological group. Note that if φ is a continuous automorphism of a topological group G , then the fixed-point subgroup $C_G(\varphi)$ is closed.

Recall that a pronilpotent group is a pro-(finite nilpotent) group, that is, an inverse limit of finite nilpotent groups. For a prime p , a pro- p group is an inverse limit of finite p -groups. The Frattini subgroup $P'P^p$ of a pro- p group P is the product of the derived subgroup P' and the subgroup generated by all p -th powers of elements of P . A subset generates P (as a topological group) if and only if its image generates the elementary abelian quotient $P/(P'P^p)$. See, for example, [19] for these and other properties of profinite groups.

3. LOCAL NILPOTENCY OF SYLOW p -SUBGROUPS

In this section we prove the local nilpotency of a pro- p group satisfying the hypotheses of the main Theorem 1.1. We shall use without special references the fact that fixed points $C_{G/N}(\varphi)$ of an automorphism φ of finite coprime order in a quotient by a φ -invariant normal open subgroup N are covered by the fixed points in the group: $C_{G/N}(\varphi) = C_G(\varphi)N/N$.

Theorem 3.1. *Let p be a prime and suppose that a finitely generated pro- p group G admits an automorphism φ of prime order $q \neq p$. If every element of $C_G(\varphi)$ is a right Engel element of G , then G is nilpotent.*

We begin with constructing a normal subgroup with nilpotent quotient that will be the main focus of the proof. Recall that $h(q)$ is a function bounding the nilpotency class of a nilpotent group admitting a fixed-point-free automorphism of prime order q by Higman's theorem [7].

Lemma 3.2. *There is a finite set $S \subseteq C_G(\varphi)$ of fixed points of φ such that the quotient G/H by its normal closure $H = \langle S^G \rangle$ is nilpotent of class $h(q)$.*

Proof. In the nilpotent quotient $G/\gamma_{h(q)+2}(G)$ of the finitely generated group G every subgroup is finitely generated. Therefore there is a finite set S of elements of $C_G(\varphi)$ whose images cover all fixed points of φ in $G/\gamma_{h(q)+2}(G)$. Let $H = \langle S^G \rangle$ be the normal closure of S . Then the quotient of G by $H\gamma_{h(q)+2}(G)$ is nilpotent of class $h(q)$ by Higman's theorem, which means that $\gamma_{h(q)+1}(G) \leq H\gamma_{h(q)+2}(G)$. Since the group G/H is pronilpotent, it follows that $\gamma_{h(q)+1}(G) \leq H$, as required. \square

We fix the notation for the subgroup $H = \langle S^G \rangle$ and the finite set $S \subseteq C_G(\varphi)$ given by Lemma 3.2. We aim at an application of Zelmanov's Theorem 2.1 to the associated Lie algebra of H , verifying the requisite conditions in a number of steps. The first step is to show that the quotient G/H' is nilpotent, which is achieved by the following lemma.

Lemma 3.3. *Suppose that L is a finitely generated pro- p group, M is an abelian normal subgroup equal to the normal closure $M = \langle T^L \rangle$ of a finite set T consisting of right Engel elements of L , and L/M is nilpotent. Then L is nilpotent.*

Proof. We proceed by induction on the nilpotency class of $L/C_L(M)$. The base of induction is the case where $L/C_L(M)$ is abelian, and the corresponding proof follows from the arguments in the step of induction.

Let $T = \{t_1, \dots, t_k\}$. Let Z be the inverse image of the centre $Z(L/C_L(M))$ of $L/C_L(M)$ (possibly, $Z = L$ in the base of induction). We claim that Z is nilpotent. For any fixed $z \in Z$ there are positive integers n_i such that $[t_i, n_i z] = 1$. Set $n = \max_i n_i$; then $[t_i, n z] = 1$ for all i . Moreover, for any $g \in L$ we have $[t_i^g, n z] = [t_i, n z]^g = 1$ since $[z, g] \in C_L(M)$. Since $M = \langle T^L \rangle$ is abelian, this implies that $[m, n z] = 1$ for any finite product m of the elements t_i^g , $g \in G$. Since these finite products form a dense subset of M , we obtain

$$[m, n z] = 1 \quad \text{for any } m \in M. \quad (3.1)$$

Since L/M is nilpotent and finitely generated, Z/M is nilpotent and finitely generated. Together with (3.1) this implies that Z is nilpotent. Indeed, let $Z = \langle M, z_1, \dots, z_s \rangle$. Any sufficiently long simple commutator in the elements of M and z_1, \dots, z_s has an initial segment that belongs to M because Z/M is nilpotent. Since $Z/C_Z(M)$ is abelian, the remaining elements (which can all be assumed to be among the z_i) can be arbitrarily re-arranged without changing the value of the commutator. If the commutator is sufficiently long, one of the z_i will appear sufficiently many times in a row making the commutator trivial by (3.1).

We now consider L/Z' , denoting by the bar the corresponding images of subgroups and elements. Clearly, \bar{L} , \bar{M} , and \bar{T} satisfy the hypotheses of the lemma. But now $\bar{Z} \leq C_{\bar{L}}(\bar{M})$, so the nilpotency class of $\bar{L}/C_{\bar{L}}(\bar{M})$ is less than that of $L/C_L(M)$ (unless $Z = L$ when the proof is complete). By the induction hypothesis, L/Z' is nilpotent. Together with the nilpotency of Z proved above, this implies that L is nilpotent by Hall's theorem [5]. \square

Lemma 3.4. *The subgroup H is generated by finitely many right Engel elements.*

Proof. By Lemma 3.3 applied with $L = G/H'$, $M = H/H'$, and $T = S$, the quotient G/H' is nilpotent. Then H/H' is finitely generated as a subgroup of a finitely generated nilpotent group. The Frattini quotient $H/(H'H^p)$ is a finite elementary abelian p -group. Since H is generated by a set of right Engel elements, conjugates of elements of $C_G(\varphi)$, we can choose a finite subset of these elements whose images generate $H/(H'H^p)$. Then this finite set also generates the pro- p group H . \square

Let $L_p(H)$ be the associated Lie algebra of H over \mathbb{F}_p constructed with respect to the Zassenhaus p -filtration of H .

Proposition 3.5. *The Lie algebra $L_p(H)$ is nilpotent.*

Proof. This will follow from Zelmanov's Theorem 2.1 if we show that $L_p(H)$ satisfies a polynomial identity and is generated by finitely many elements such that all commutators in these elements are ad-nilpotent.

Lemma 3.6. *The Lie algebra $L_p(H)$ satisfies a polynomial identity.*

Proof. As a profinite Engel group, $C_H(\varphi) = H \cap C_G(\varphi)$ is locally nilpotent by the Wilson–Zelmanov theorem [20]. It follows that $C_H(\varphi)$ satisfies a coset identity on cosets of an open subgroup of $C_H(\varphi)$. For example, in the group $C_H(\varphi) \times C_H(\varphi)$ the subsets

$$E_i = \{(x, y) \in C_H(\varphi) \times C_H(\varphi) \mid [x, iy] = 1\}$$

are closed in the product topology, and

$$C_H(\varphi) \times C_H(\varphi) = \bigcup_{i=1}^{\infty} E_i.$$

Hence by the Baire category theorem [8, Theorem 34], one of these subsets E_n contains an open subset of $C_H(\varphi) \times C_H(\varphi)$, which means that there are cosets x_0K_1, y_0K_2 of open subgroups $K_1, K_2 \leq C_H(\varphi)$ such that $[x, {}_ny] = 1$ for all $x \in x_0K_1$ and $y \in y_0K_2$, and therefore for all $x \in x_0(K_1 \cap K_2)$ and $y \in y_0(K_1 \cap K_2)$. Thus, $C_H(\varphi)$ satisfies a coset identity.

The intersections $C_i = C_H(\varphi) \cap H_i$ with the terms H_i of the Zassenhaus p -filtration for H form a filtration of $C_H(\varphi)$, since obviously, $[C_i, C_j] \leq C_{i+j}$. Let $\hat{L}_p(C_H(\varphi))$ be the Lie algebra constructed for $C_H(\varphi)$ with respect to the filtration $\{C_i\}$. Since φ is a coprime automorphism, the fixed-point subalgebra $C_{L_p(H)}(\varphi)$ is isomorphic to $\hat{L}_p(C_H(\varphi))$. We apply a version of the Wilson–Zelmanov result [20, Theorem 1], by which a coset identity on $C_H(\varphi)$ implies that $\hat{L}_p(C_H(\varphi))$ satisfies some polynomial identity. Indeed, the proof of Theorem 1 in [20] only uses the filtration property $[F_i, F_j] \leq F_{i+j}$ for showing that the homogeneous Lie polynomial constructed from a coset identity on a group F is an identity of the Lie algebra constructed with respect to the filtration $\{F_i\}$.

Thus, the fixed-point subalgebra $C_{L_p(H)}(\varphi)$ satisfies a polynomial identity. Hence the Lie algebra $L_p(H)$ also satisfies a polynomial identity by the Bahturin–Zaicev Theorem 2.2. \square

Lemma 3.7. *The Lie algebra $L_p(H)$ is generated by finitely many elements such that all commutators in these elements are ad-nilpotent.*

Proof. By Lemma 3.4 the group H is generated by finitely many right Engel elements, say, h_1, \dots, h_m . Their images $\bar{h}_1, \dots, \bar{h}_m$ in the first factor H/H_2 of the Zassenhaus p -filtration of H generate the Lie algebra $L_p(H)$. Let \bar{c} be some commutator in these generators \bar{h}_i , and c the same group commutator in the elements h_i . For every j , since $[h_j, {}_{k_j}c] = 1$ for some $k_j = k_j(c)$, we also have $[\bar{h}_j, {}_{k_j}\bar{c}] = 0$ in $L_p(H)$. We choose a positive integer s such that $p^s \geq \max\{k_1, \dots, k_m\}$. Then $[\bar{h}_j, {}_{p^s}\bar{c}] = 0$ for all j . In characteristic p this implies that

$$[\varkappa, {}_{p^s}\bar{c}] = 0 \tag{3.2}$$

for any commutator \varkappa in the \bar{h}_i . This easily follows by induction on the weight of \varkappa from the formula

$$[[u, v], {}_{p^s}w] = [[u, {}_{p^s}w], v] + [u, [v, {}_{p^s}w]]$$

that holds in any Lie algebra of characteristic p . This formula follows from the Leibnitz formula

$$[[u, v], {}_nw] = \sum_{i=0}^n \binom{n}{i} [[u, {}_iw], [v, {}_{n-i}w]]$$

(where $[a, {}_0b] = a$), because the binomial coefficient $\binom{p^s}{i}$ is divisible by p unless $i = 0$ or $i = p^s$.

Since any element of $L_p(H)$ is a linear combination of commutators in the \bar{h}_i , equation (3.2) by linearity implies that \bar{c} is ad-nilpotent of index at most p^s . \square

We can now finish the proof of Proposition 3.5. Lemmas 3.6 and 3.7 show that $L_p(H)$ satisfies the hypotheses of Zelmanov’s Theorem 2.1, by which $L_p(H)$ is nilpotent. \square

Proof of Theorem 3.1. By Lemma 3.2 the quotient G/H is nilpotent. Being finitely generated, then G/H is a group of finite rank. Here, the rank of a pro- p group is the supremum of the minimum number of (topological) generators over all open subgroups.

The nilpotency of the Lie algebra $L_p(H)$ of the finitely generated pro- p group H established in Proposition 3.5 implies that H is a p -adic analytic group. This result goes back to Lazard [11]; see also [15, Corollary D]. By the Lubotzky–Mann theorem [13], being a p -adic analytic group is equivalent to being a pro- p group of finite rank. Thus, both H and G/H have finite rank, and therefore the whole pro- p group G has finite rank. Hence G is a p -adic analytic group and therefore a linear group. By Gruenberg’s theorem [4], right Engel elements of a linear group are contained in the hypercentre. Since H is generated by right Engel elements, we obtain that H is contained in the hypercentre of G , and since G/H is nilpotent, the whole group G is hypercentral. Being also finitely generated, then G is nilpotent (see [14, 12.2.4]). \square

4. UNIFORM BOUND FOR THE NILPOTENCY CLASS

In the main Theorem 1.1, we need to prove that if a finitely generated profinite group G admits a coprime automorphism φ of prime order q all of whose fixed points are right Engel in G , then G is nilpotent. We already know that G is pronilpotent, and every Sylow p -subgroup of G is nilpotent by Theorem 3.1. This would imply the nilpotency of G if we had a uniform bound for the nilpotency class of Sylow p -subgroups independent of p . However, the nilpotency class furnished by the proof of Theorem 3.1 depends on p .

In this section we prove that for large enough primes p the nilpotency classes of Sylow p -subgroups of G are uniformly bounded above in terms of certain parameters of the group G . Together with bounds depending on p given by Theorem 3.1, this will complete the proof of the nilpotency of G . In the proof, we do not specify the conditions on p beforehand, but proceed with our arguments noting along that our conclusions hold for all large enough primes p .

One of the aforementioned parameters is the finite number of generators of G , say, d . Clearly, every Sylow p -subgroup of G can also be generated by d elements, being a homomorphic image of G by the Cartesian product of all other Sylow subgroups.

Lemma 4.1. *There are positive integers n and N_1 such that for every $p > N_1$ all fixed points of φ in the Sylow p -subgroup P of G are right n -Engel elements of P .*

Proof. In the group $C_G(\varphi) \times G$, the subsets

$$E_i = \{(x, y) \in C_G(\varphi) \times G \mid [x, {}_i y] = 1\}$$

are closed in the product topology. By hypothesis,

$$\bigcup_i E_i = C_G(\varphi) \times G.$$

Hence, by the Baire category theorem [8, Theorem 34], one of these subsets E_n contains an open subset of $C_G(\varphi) \times G$, which means that there are cosets x_0K and y_0L of open subgroups $K \leq C_G(\varphi)$ and $L \leq G$ such that $[x, {}_n y] = 1$ for all $x \in x_0K$ and $y \in y_0L$. Since the indices $|C_G(\varphi) : K|$ and $|G : L|$ are finite, for all large enough primes $p > N_1$ the Sylow p -subgroups of $C_G(\varphi)$ and G are contained in the subgroups K and L , respectively. Then for every prime $p > N_1$, in the Sylow p -subgroup P the centralizer $C_P(\varphi)$ consists of right n -Engel elements of P . \square

Lemma 4.2. *There are positive integers c and N_2 such that for every $p > N_2$ in the Sylow p -subgroup P the fixed-point subgroup $C_P(\varphi)$ is nilpotent of class c .*

Proof. By Lemma 4.1, for $p > N_1$ in the Sylow p -subgroup P the subgroup $C_P(\varphi)$ is an n -Engel group. By a theorem of Burns and Medvedev [3], then $C_P(\varphi)$ has a normal subgroup N_p of exponent $e(n)$ such that the quotient $C_P(\varphi)/N_p$ is nilpotent of class $c(n)$, for some numbers $e(n)$ and $c(n)$ depending only on n . Clearly, $N_p = 1$ for all large enough primes $p > N_2 \geq N_1$. Thus, for every prime $p > N_2$ the subgroup $C_P(\varphi)$ is nilpotent of class $c = c(n)$. \square

The following proposition will complete the proof of the main Theorem 1.1 in view of Lemmas 4.1 and 4.2.

Proposition 4.3. *There are functions $N_3(d, q, n, c)$ and $f(d, q, n, c)$ of four positive integer variables d, q, n, c with the following property. Let p be a prime, and suppose that P is a d -generated pro- p group admitting an automorphism φ of prime order $q \neq p$ such that $C_P(\varphi)$ is nilpotent of class c and consists of right n -Engel elements of P . If $p > N_3(d, q, n, c)$, then the group P is nilpotent of class at most $f(d, q, n, c)$.*

Proof. It is sufficient to obtain a bound for the nilpotency class in terms of d, q, n, c for every finite quotient T of P by a φ -invariant open normal subgroup. Consider the associated Lie ring $L(T)$ constructed with respect to the filtration consisting of the terms $\gamma_i(T)$ of the lower central series of T :

$$L(T) = \bigoplus \gamma_i(T)/\gamma_{i+1}(T).$$

As is well known, this Lie ring is nilpotent of exactly the same nilpotency class as T (see, for example, [9, Theorem 6.9]). Therefore it is sufficient to obtain a required bound for the nilpotency class of $L(T)$. We set $L = L(T)$ for brevity. Let $\tilde{L} = L \otimes_{\mathbb{Z}} \mathbb{Z}[\omega]$ be the Lie ring obtained by extending the ground ring by a primitive q -th root of unity ω . We regard L as $L \otimes 1$ embedded in \tilde{L} . The automorphism of L and of \tilde{L} induced by φ is denoted by the same letter. Since the order of the automorphism φ is coprime to the orders of elements of the additive group of \tilde{L} , which is a p -group, we have the decomposition into analogues of eigenspaces

$$\tilde{L} = \bigoplus_{i=0}^{q-1} L_j, \quad \text{where } L_j = \{x \in \tilde{L} \mid x^\varphi = \omega^j x\}.$$

For clarity we call the additive subgroups L_j *eigenspaces*, and their elements *eigenvectors*. This decomposition can also be viewed as a $(\mathbb{Z}/q\mathbb{Z})$ -grading of \tilde{L} , since

$$[L_i, L_j] \subseteq L_{i+j \pmod{q}}.$$

Note that $L_0 = C_L(\varphi) \otimes_{\mathbb{Z}} \mathbb{Z}[\omega]$.

Lemma 4.4. *The fixed-point subring $C_{\tilde{L}}(\varphi)$ is nilpotent of class at most c .*

Proof. Since φ is a coprime automorphism of T , we have

$$C_L(\varphi) = \bigoplus_i (C_T(\varphi) \cap \gamma_i(T))\gamma_{i+1}(T)/\gamma_{i+1}(T).$$

Since the fixed-point subgroup $C_T(\varphi)$ is nilpotent of class c , the definition of the Lie products implies that the same is true for $C_L(\varphi)$ and therefore also for $C_{\tilde{L}}(\varphi) = C_L(\varphi) \otimes_{\mathbb{Z}} \mathbb{Z}[\omega]$. \square

Our main aim is to enable an application to \tilde{L} of the effective version of Zelmanov's theorem given by Proposition 2.3. For that we need a φ -invariant set of generators of \tilde{L} such that all commutators in these generators are ad-nilpotent of bounded index.

Let $L_{(k)} = \gamma_k(T)/\gamma_{k+1}(T)$ denote the homogeneous component of weight k of the Lie ring L , and let $\tilde{L}_{(k)} = L_{(k)} \otimes_{\mathbb{Z}} \mathbb{Z}[\omega]$. For clarity we say that elements of $\tilde{L}_{(k)}$ or $L_{(k)}$ are *homogeneous*. The component $L_{(1)}$ generates the Lie ring L , and $\tilde{L}_{(1)}$ generates \tilde{L} . If elements t_1, \dots, t_d generate the group T , then their images $\bar{t}_1, \dots, \bar{t}_d$ in $L_{(1)} = T/\gamma_2(T)$ generate the Lie ring L , as well as \tilde{L} (over the extended ground ring). Writing $\bar{t}_i = \sum_{j=0}^{q-1} t_{ij}$, where $t_{ij} \in \tilde{L}_{(1)} \cap L_j$ we obtain a φ -invariant set of generators of \tilde{L}

$$\{\omega^k t_{ij} \mid i = 1, \dots, d; j = 0, \dots, q-1; k = 0, \dots, q-1\}.$$

We claim that for $p > n$ all commutators in these generators are ad-nilpotent of index bounded in terms of q, n, c .

We set for brevity $\tilde{L}_{(v)k} = \tilde{L}_{(v)} \cap L_k$ for any weight v . A commutator of weight v in the eigenvectors t_{ij} is an eigenvector belonging to $\tilde{L}_{(v)k}$, where k is the modulo q sum of the second indices of the t_{ij} involved. We actually prove that for $p > n$ any homogeneous eigenvector $l_k \in \tilde{L}_{(v)k}$ is ad-nilpotent of index s bounded in terms of q, n, c . It is clearly sufficient to show that $[x_j, {}_s l_k] = 0$ for any homogeneous eigenvector $x_j \in \tilde{L}_{(u)j}$, for any weights u, v and any indices $j, k \in \{0, 1, \dots, q-1\}$. (Here we use indices j, k for elements x_j, l_k only to indicate the eigenspaces they belong to.) First we consider the case where $j = 0$.

Lemma 4.5. *If $p > n$, then for any weights u, v , for any eigenvector $x_0 \in \tilde{L}_{(u)0}$ and any homogeneous element $l \in \tilde{L}_{(v)}$ we have $[x_0, {}_n l] = 0$.*

Proof. Since φ is an automorphism of coprime order, for $x_0 \in \tilde{L}_{(u)0}$ there are elements $y_i \in C_T(\varphi) \cap \gamma_u(T)$ such that $x_0 = \sum_{i=0}^{q-2} \omega^i \bar{y}_i$, where \bar{y}_i is the image of y_i in $\gamma_u(T)/\gamma_{u+1}(T)$ (here the indices of the y_i are used for numbering). For any $\bar{h} \in L_{(v)}$, there is an element $h \in T \cap \gamma_v(T)$ such that \bar{h} is the image of h in $L_{(v)} = \gamma_v(T)/\gamma_{v+1}(T)$. Since $[y_i, {}_n h] = 1$ in the group T by the hypothesis of Proposition 4.3, we have $[\bar{y}_i, {}_n \bar{h}] = 0$ in L for every i . Hence, by linearity,

$$[x_0, {}_n \bar{h}] = 0 \tag{4.1}$$

in \tilde{L} . Note, however, that $\tilde{L}_{(v)}$ does not consist only of $\mathbb{Z}[\omega]$ -multiples of elements of $L_{(v)}$. Nevertheless, (4.1) looks like the n -Engel identity, which implies its linearization, which in turn survives extension of the ground ring, and then implies the required property due to the condition $p > n$ making $n!$ an invertible element of the ground ring. However, we cannot simply make a reference to these well-known facts, since this is not exactly an identity, so we reproduce these familiar arguments in our specific situation (jumping over one of the steps).

We substitute $a_1 + \dots + a_n$ for \bar{h} in (4.1) with arbitrary homogeneous elements $a_i \in L_{(v)}$ (the indices of the a_i are used for numbering). Thus,

$$[x_0, {}_n(a_1 + \dots + a_n)] = 0$$

for any elements $a_i \in L_{(v)}$, some of which may also be equal to one another. After expanding all brackets, we obtain the equation

$$0 = [x_0, {}_n(a_1 + \cdots + a_n)] = \sum_{\substack{i_1 \geq 0, \dots, i_n \geq 0 \\ i_1 + \cdots + i_n = n}} \mathfrak{X}_{i_1, \dots, i_n}, \quad (4.2)$$

where $\mathfrak{X}_{i_1, \dots, i_n}$ denotes the sum of all commutators of degree i_j in a_j . Replacing a_1 with 0 (only this formal occurrence, keeping intact all other a_i even if some are equal to a_1) shows that

$$0 = \sum_{\substack{i_1=0, i_2 \geq 0, \dots, i_n \geq 0 \\ i_1 + \cdots + i_n = n}} \mathfrak{X}_{i_1, \dots, i_n}.$$

Hence we can remove from the right-hand side of (4.2) all terms not involving a_1 as a formal entry (keeping the other a_i even if some are equal to a_1). We obtain

$$0 = [x_0, {}_n(a_1 + \cdots + a_n)] = \sum_{\substack{i_1 \geq 1, i_2 \geq 0, \dots, i_n \geq 0 \\ i_1 + \cdots + i_n = n}} \mathfrak{X}_{i_1, \dots, i_n}.$$

Then we do the same with a_2 for the resulting equation, and so on, consecutively with all the a_i . As a result we obtain

$$0 = \sum_{\substack{i_1 \geq 1, \dots, i_n \geq 1 \\ i_1 + \cdots + i_n = n}} \mathfrak{X}_{i_1, \dots, i_n} = \mathfrak{X}_{1, \dots, 1},$$

that is,

$$0 = \sum_{\pi \in S_n} [x_0, a_{\pi(1)}, \dots, a_{\pi(n)}], \quad (4.3)$$

where the right-hand side is the desired linearization. Every element $l \in \tilde{L}_{(v)}$ can be written as a linear combination $l = m_0 + \omega m_1 + \omega^2 m_2 + \cdots + \omega^{q-2} m_{q-2}$, where $m_i \in L_{(v)}$. Then

$$\begin{aligned} [x_0, {}_n l] &= [x_0, {}_n(m_0 + \omega m_1 + \omega^2 m_2 + \cdots + \omega^{q-2} m_{q-2})] \\ &= \sum_{i=0}^{n(q-2)} \omega^i \sum_{j_1 + 2j_2 + \cdots + (q-2)j_{q-2} = i} \lambda_{j_0, j_1, \dots, j_{q-2}}, \end{aligned}$$

where $\lambda_{j_0, j_1, \dots, j_{q-2}}$ denotes the sum of all commutators in the expansion of the left-hand side with weight j_s in m_s . But each of these sums is clearly symmetric and therefore is equal to 0 as a consequence of (4.3), where, if an element a_i is required to be repeated n_i times, then the coefficient $n_i!$ appears, which is invertible in the ground ring, since $n_i < p$ and the additive group is a p -group. The lemma is proved. \square

Lemma 4.6. *If $p > n$, then for any v and k , any homogeneous eigenvector $l_k \in \tilde{L}_{(v)k}$ is ad-nilpotent of index bounded in terms of q, n, c .*

Proof. First consider the case $k = 0$. Then $l_0 = \sum_{i=0}^{q-2} \omega^i \bar{y}_i$, where \bar{y}_i is the image of an element $y_i \in C_T(\varphi) \cap \gamma_v(T)$ in $\gamma_v(T)/\gamma_{v+1}(T)$ (the indices of the y_i are used for numbering). For each i , since y_i^{-1} is a right n -Engel element of T by hypothesis, y_i is a left $(n+1)$ -Engel element by a result of Heineken [6] (see also [14, 12.3.1]). For any homogeneous element $\bar{h} \in L_{(u)}$ there is an element $h \in T \cap \gamma_u(T)$ such that \bar{h} is the image of h in $\gamma_u(T)/\gamma_{u+1}(T)$. Since $[h, {}_{n+1}y_i] = 1$ in the group T , we have $[\bar{h}, {}_{n+1}\bar{y}_i] = 0$ in L for every i . Hence, by

linearity, each \bar{y}_i is ad-nilpotent in L of index at most $n + 1$. Let M be the subring of L generated by $\bar{y}_0, \bar{y}_1, \dots, \bar{y}_{q-2}$. Since $M \leq C_L(\varphi)$, the subring M is nilpotent of class at most c by Lemma 4.4. We can now apply Lemma 2.4, by which

$$[L, \underbrace{M, M, \dots, M}_\varepsilon] = 0$$

for some $\varepsilon = \varepsilon(q - 1, n + 1, c)$ bounded in terms of q, n, c . This equation remains valid after extension of the ground ring. In particular, $l_0 = \sum_{i=0}^{q-2} \omega^i \bar{y}_i$ is ad-nilpotent in \tilde{L} of index at most $\varepsilon = \varepsilon(q - 1, n + 1, c)$, as required.

Now suppose that $k \neq 0$. For a homogeneous eigenvector $x_j \in \tilde{L}_{(w)j}$, the commutator

$$[x_j, {}_{q+n-1}l_k] = [x_j, \underbrace{l_k, \dots, l_k}_s, l_k, \dots, l_k] \quad (4.4)$$

has an initial segment of length $s + 1 \leq q$ that is a homogeneous eigenvector $x_0 = [x_j, {}_s l_k] \in \tilde{L}_{(w)0}$ (for some weight w). Indeed, the congruence $j + sk \equiv 0 \pmod{q}$ has a solution $s \in \{0, 1, \dots, q - 1\}$ since $k \not\equiv 0 \pmod{q}$. There remain at least n further entries of l_k in (4.4), so that we have a subcommutator of the form $[x_0, {}_n l_k]$, which is equal to 0 by Lemma 4.5. Thus, by linearity, l_k is ad-nilpotent of index at most $q + n - 1$. \square

We now finish the proof of Proposition 4.3. By Lemma 4.6, for $p > n$ every commutator in the generators t_{ij} of the Lie ring \tilde{L} is ad-nilpotent of index bounded in terms of q, n, c . The same is true for the generators in the φ -invariant set

$$\{\omega^k t_{ij} \mid i = 1, \dots, d; j = 0, \dots, q - 1; k = 0, \dots, q - 1\},$$

which consists of $q^2 d$ elements. The fixed-point subring $C_{\tilde{L}}(\varphi)$ is nilpotent of class at most c by Lemma 4.4. Thus, for $p > n$ the Lie ring \tilde{L} and its group of automorphisms $\langle \varphi \rangle$ satisfy the hypotheses of Proposition 2.3. By this proposition, there exist positive integers e and r depending only on d, q, n, c such that $e\gamma_r(\tilde{L}) = 0$. The additive group of \tilde{L} is a p -group. Therefore, if $p > e$, then e is invertible in the ground ring, so that we obtain $\gamma_r(L) = 0$. It remains to put $N_3(d, q, n, c) = \max\{n, e\}$ and $f(d, q, n, c) = r - 1$.

We thus proved that for $p > N_3(d, q, n, c)$ every finite quotient of P by a φ -invariant normal open subgroup is nilpotent of class at most $f(d, q, n, c)$. Therefore P is nilpotent of class at most $f(d, q, n, c)$ if $p > N_3(d, q, n, c)$. \square

We finally combine all the results in the proof of the main theorem.

Proof of Theorem 1.1. Recall that G is a profinite group admitting a coprime automorphism φ of prime order q all of whose fixed points are right Engel in G ; we need to prove that G is locally nilpotent. Any finite set $S \subseteq G$ is contained in the φ -invariant finite set $S^{\langle \varphi \rangle} = \{s^{\varphi^k} \mid s \in S, k = 0, 1, \dots, q - 1\}$. Therefore we can assume that the group G is finitely generated, say, by d elements, and then need to prove that G is nilpotent. As noted in the Introduction, the group G is pronilpotent, so we only need to prove that all Sylow p -subgroups of G are nilpotent of class bounded by some number independent of p .

Let n and N_1 be the numbers given by Lemma 4.1, and c and N_2 the numbers given by Lemma 4.2. Further, let $N_3(d, q, n, c)$ be the number given by Proposition 4.3. Then for every prime $p > \max\{N_1, N_2, N_3(d, q, n, c)\}$ the Sylow p -subgroup of G is nilpotent of class at most $f(d, q, n, c)$ for the function given by Proposition 4.3. Since every Sylow p -subgroup is nilpotent by Theorem 3.1, we obtain a required uniform bound for the nilpotency classes of Sylow p -subgroups independent of p . \square

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