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NONLOCAL SOLUTIONS AND CONTROLLABILITY OF SCHRÖDINGER EVOLUTION EQUATION

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Abstract. The paper deals with semilinear evolution equations in complex Hilbert spaces. Nonlocal associated Cauchy problems are studied and the existence and uniqueness of classical solutions is proved. The controllability is investigated too and the topological structure of the controllable set discussed. The results are applied to nonlinear Schrödinger evolution equations with time dependent potential. Several examples of nonlocal conditions are proposed. The evolution system associated to the linear part is not compact and the theory developed in Okazawa-Yoshii [21] for its study is used. The proofs involve the Schauder-Tychonoff fixed point theorem and no strong compactness is assumed on the nonlinear part.
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1. Introduction

This paper concerns Schrödinger evolution equations with a time dependent potential. As it is known, these dynamics are models for the behaviour of the elementary particles and hence they are of great interest in applied sciences.

The initial value problem associated to the linear Schrödinger equation,

$$\begin{cases} i\dfrac{\partial}{\partial t}u(x,t) - \Delta u(x,t) + V(x,t)u(x,t) = f(x,t) & \text{for } (x,t) \in \mathbb{R}^3 \times I, \\ u(x,0) = u_0(x) & \text{for } x \in \mathbb{R}^3 \end{cases} \tag{LS}$$

where $I := [0,T]$, $V\colon \mathbb{R}^3 \times I \to \mathbb{R}$, $f\colon \mathbb{R}^3 \times I \to \mathbb{C}$ and $u_0\colon \mathbb{R}^3 \to \mathbb{C}$, was extensively investigated and several important contributions already appeared. They are frequently based on the semigroup theory introduced by Kato [13] [14] (see also Kato [15] and Vrabie [28]). We quote, in particular, the results by Acquistapace-Terreni [1], Neidhardt [18] (see also Tanabe [27, Chapter VI]), Okazawa [20] and Okazawa-Yoshii [21]. In [21] the existence of a unique solution to (LS) was proved, in the very general case of moving Coulomb potentials with multiple charges (see Section 2). The survey by

Yajima [29] accounts of the most important methods used for the study of (LS).
In abstract setting problem (LS) becomes

$$\begin{cases} \dfrac{d}{dt}u(t) + A(t)u(t) = f(t) \quad \text{for } t \in I, \\ u(0) = u_0. \end{cases} \tag{P$_0$}$$

where $\{A(t); t \in I\}$ is a family of closed linear operators in a complex Hilbert space X with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. The domain $D(A(t))$ may vary with t but it is assumed the existence of a subspace $Y \subset X$ such that $Y \subset D(A(t))$ for all $t \in I$ and $f\colon I \to Y$.

In this paper a nonlinear term is added, into the Schrödinger equation. We assume that the linear part has the very general behaviour discussed in [21] and briefly recall its properties in Theorems 2.2 and 2.4; the proofs are omitted and can be found in [21].

The paper splits into two parts. The first one is in Sections 4 and 5 and deals with the nonlocal problem

$$\begin{cases} i\dfrac{\partial}{\partial t}u(x,t) - \Delta u(x,t) + V(x,t)u(x,t) = f(x,t) \\ \qquad\qquad + \gamma\Big(t, \dfrac{1}{t}\displaystyle\int_0^t \int_{\mathbb{R}^3} a(y,s)\Delta u(y,s)\,dyds\Big) g(x,t) \quad \text{for } (x,t) \in \mathbb{R}^3 \times I, \\ u(x,0) = u_0(x) + \displaystyle\int_0^T b(s)u(x,s)\,ds \qquad \text{for } x \in \mathbb{R}^3, \end{cases} \tag{NLS}$$

with V and f as in (LS), $\gamma\colon I \times \mathbb{C} \to \mathbb{C}$, $a, g\colon \mathbb{R}^3 \times I \to \mathbb{C}$ and $b\colon I \to \mathbb{C}$. The existence of a unique solution for (NLS) is discussed in Section 5 (see Theorem 5.1). As usual problem (NLS) is treated in its abstract setting

$$\begin{cases} \dfrac{d}{dt}u(t) + A(t)u(t) = f(t) + \Gamma(t, K(t)u)g(t) \quad \text{for } t \in I, \\ u(0) = u_0 + Mu, \end{cases} \tag{P}$$

with $\{A(t); t \in I\}$ and f as in (P$_0$), $K(t)\colon C(I;Y) \to \mathbb{C}$ linear and bounded for all $t \in I$, $g\colon I \to Y$, $\Gamma\colon I \times \mathbb{C} \to \mathbb{C}$ continuous, $u_0 \in Y$, and $M\colon C(I;Y) \to Y$ linear and bounded.

Definition 1.1. (Nonlocal (classical) solution to (P)) A vector-valued function $u : I \to X$ is said to be a classical solution to (P) if $u \in C^1(I;X) \cap C(I;Y)$ and satisfies (P).

The existence and uniqueness of a classical solution to problem (P) is in Theorem 4.1. Its proof is based on the Schauder-Tychonoff fixed point theorem (see e.g. Theorem 3.5) applied to the solution operator Φ which is defined in (4.3) by a linearization device. In Remark 4.3 we show that, in the genuinely nonlinear case for the equation in (P), by Theorem 4.1 we never obtain the trivial solution $u \equiv 0$

Let V and W be arbitrary Banach spaces. The symbol $L(V,W)$ denotes the space of all bounded linear operators from V to W, with norm $\|\cdot\|_{L(V,W)}$. The abbreviations:

$L(V) := L(V,V)$, $L(W) := L(W,W)$ will be used. The symbol $C_*(I; L(V,W))$ stands for the space of all strongly continuous functions from W to V. More precisely, $F(\cdot) \in C_*(I; L(V,W))$ means that $F(t) \in L(V,W)$ is defined for all $t \in I$ and $F(\cdot)v \in C(I;W)$ for each $v \in V$ (for this notation see e.g. Kato [16]). Notice that, in Theorem 4.1, the function $K(\cdot)$ will be assumed strongly continuous from $C(I;Y)$ to $\mathbb{C}$.

Some concrete examples for the operators M and K are proposed now. They provide an idea of additional nonlocal problems which could be considered besides (NLS) (see also Remark 5.4).

Examples of $M : C(I;Y) \to Y$.

(1) $Mu = \displaystyle\int_0^T \mu(t)u(t)\,dt$ with $\mu \in L^1(I)$. It is possible to show that $\|M\| = \|\mu\|_{L^1(I)}$ (a proof is given in Lemma 7.2). In particular, when $\mu(t) = \dfrac{1}{T}$ for $t \in [0,T]$, then Mu is the mean value of u.

(2) $Mu = \displaystyle\sum_{i=1}^n \lambda_i u(t_i)$ with $\lambda_i \in \mathbb{C}$, $t_i \in I$, $i = 1,2,\dots,n$ and $0 < t_1 < t_2 < \cdots < t_n \le T$. Then we obtain $\|M\| = \displaystyle\sum_{i=1}^n |\lambda_i|$. In fact, it is clear that $\|M\| \le \displaystyle\sum_{i=1}^n |\lambda_i|$. On the other hand, set $y \in Y$ satisfying $\|y\|_Y = 1$, $\alpha \in C(I;\mathbb{C})$ such that $|\alpha(t)| \le 1$ and $\alpha(t_i) = \lambda_i^*/|\lambda_i|$, $\lambda_i \ne 0$; $\alpha(t_i) = 0$, $\lambda_i = 0$, where the symbol * stands for the complex conjugate. Then $u(t) := \alpha(t)y$ satisfies $\|u\|_{C(I;Y)} = 1$ and $\|Mu\|_Y = \displaystyle\sum_{i=1}^n |\lambda_i|$.

Examples of $K(\cdot) \in C_*(I; L(C(I;Y),\mathbb{C}))$.

(1) For all u and $\mu \in C(I;Y)$, $(u(\cdot),\mu(\cdot))_Y \in C(I;\mathbb{C})$. Therefore if we set

$$K(t)u := \begin{cases} \dfrac{1}{t}\displaystyle\int_0^t (u(s),\mu(s))_Y\,ds, & t \in (0,T]; \\ (u(0),\mu(0))_Y, & t = 0. \end{cases}$$

Then $K(\cdot) \in C_*(I; L(C(I;Y),\mathbb{C}))$, and $\|K(t)\| \le \|\mu\|_{C(I;Y)}$, for $t \in I$.

(2) Let $\{A_1(t); t \in I\}$ be a family of closed linear operators such that $Y \subset D(A_1(t)) \subseteq X$ for all $t \in I$ and $A_1(\cdot) \in C_*(I; L(C(I;Y),\mathbb{C}))$. Notice that $\|A_1(t)y\| \in C(I)$ for $y \in Y$, and thus it follows, from the uniform boundedness principle, that there exists a positive constant c_1 such that

$$c_1 := \max_{t\in I} \|A_1(t)\|_{L(Y,X)}.$$

As a consequence, $(A_1(\cdot)u(\cdot),\mu(\cdot)) \in C(I;\mathbb{C})$ for $\mu \in C(I;X)$ and $u \in C(I;Y)$. Therefore if we set

$$K(t)u := \begin{cases} \dfrac{1}{t}\displaystyle\int_0^t (A_1(s)u(s),\mu(s))\,ds, & t \in (0,T]; \\ (A_1(0)u(0),\mu(0)), & t = 0, \end{cases}$$

then, again, $K(\cdot) \in C_*(I; L(C(I;Y),\mathbb{C}))$, and $\|K(t)\| \le c_1\|\mu\|_{C(I;X)}$ for $t \in I$.

The use of an abstract framework is fairly common for the study of boundary value problems associated to differential dynamics. Usually, it is combined with the application of some fixed point theorem or with the use of an index invariance by homotopy. Except with the topological argument, the sole well posedness of the associated linear dynamics is always involved, and this explains the success of the technique. The seminal contributions go back to Hartman [10], Mawhin [17] and Schmitt-Thompson [26] (see also the references therein). The books by Pazy [24] and Kamenskii-Obukhovskii-Zecca [11] deal with this method. The technique is still actual and successfully used for the study of semilinear equations. We refer, in particular, to Paicu and Vrabie [22] and Papageorgiou [23], where the linear part does not depend on t and it generates a compact semigroup. We also mention Benedetti, Malaguti and Taddei [2] and Benedetti, Taddei and Väth [4] about nonlinear boundary conditions in a multivalued dynamic. The solution is always intended in integral form.

The domain of the solution operator in this paper is the space $C(I;Y)$ of continuous functions. Due to the special form which take both the nonlinear part and the nonlocal condition in (NLS), the weak topology in $C(I;Y)$ can be used, for proving the required regularities (see Lemmas 4.7 and 4.8); as a consequence, though the evolution system generated by $\{A(t);t\in I\}$ is no longer compact, the study of (NLS) can be lead with the only continuity of γ. Moreover, due to the properties of the evolution system, classical solutions are furnished. To the best of our knowledge (NLS) is the first study of a nonlocal problem associated to a nonlinear Schrödinger equation.

The second part of this paper is in Section 6 and treats the controllability of the nonlocal solutions for the Schrödinger equation, i.e. it is about the problem

$$\begin{cases} i\dfrac{\partial}{\partial t}u(x,t)-\Delta u(x,t)+V(x,t)u(x,t)=v(x,t)+f(x,t) \\ \qquad\qquad +\gamma\Big(t,\dfrac{1}{t}\displaystyle\int_0^t\int_{\mathbb{R}^3}a(y,s)\Delta u(y,s)\,dyds\Big)g(x,t), & (x,t)\in\mathbb{R}^3\times I, \\ u(x,0)=u_0(x)+\displaystyle\int_0^T b(s)u(x,s)\,ds, & x\in\mathbb{R}^3, \\ u(x,T)=u_1(x), & x\in\mathbb{R}^3, \end{cases} \tag{CPS}$$

and its abstract formulation, i.e.

$$\begin{cases} \dfrac{d}{dt}u(t)+A(t)u(t)=f(t)+\Gamma(t,K(t)u)g(t)+Bv(t) \quad \text{for } t\in I, \\ u(0)=u_0+Mu, \\ u(T)=u_1, \end{cases} \tag{CP}$$

where u_0, $u_1\in Y$, the control function $v(\cdot)$ is considered in a Banach space D and $B:D\to X$ is a bounded linear operator.

Definition 1.2. ((Classical) controllable solution to (CP)) A pair of vector-valued functions $u(\cdot):I\to X$ and $v(\cdot):I\to D$ is said to be a classical controllable solution to (CP) if $(u,Bv)\in\big(C^1(I;X)\cap C(I;Y)\big)\times C(I;X)$ and satisfies (CP).

The existence of a solution (u, v) satisfying (CP) for every given $u_0, u_1 \in Y$ is discussed in Section 6 (see Theorem 6.1). The application to problem (CPS) is in Example 6.5. The weak compactness in $C(I; Y)$ of the solutions to problem (CP) is proved in Theorem 6.4. It implies the possibility to find a solution u^*, and a corresponding control $v(u^*)$, which minimizes (or maximizes) any suitably regular cost function $J : C(I; Y) \to \mathbb{R}$. Again, the discussion is based on a topological argument and has some similarities with the recent contributions by Obukhovskii and Zecca [19] and by Benedetti, Obukhovskii and Taddei [3]. However, the abstract setting in [3] and [19] does not allow applications to Schrödinger evolution equations; further $M \equiv 0$ there, and the solutions are intended in integral form. The controllability of the Schrödinger equation by an additive control as in (CPS) was recently studied in Sarychev [25] whereas a multiplicative control was introduced in Chambrion, Mason, Sigalotti and Boscain [7], for the same purpose. In Section 6 we are able to discuss the exact controllability of the nonlinear Schrödinger equation; the solutions are classical and satisfy a nonlocal additional condition given by M; they form a compact set and the concrete formula for the associated control $Bv(\cdot)$ strategies is furnished.

Several preliminary theorems are contained in Section 3. Some calculations are confined in Section 7.

2. The linear abstract problem

This part is about the initial value problem (P_0). Sufficient conditions are proposed in Theorem 2.4 for its unique solvability. The result is based on [21].

In the case, as in this paper, that $D(A(t))$ depends on $t \in I$, the introduction of an auxiliary family of operators $\{S(t); t \in I\}$ with suitable properties can be very useful for the construction of the evolution system associated to $\{A(t); t \in I\}$ (see e.g. [21]). We follow this method here and hence we first introduce a family of operators $\{S(t); t \in I\}$.

Assumption on $\{S(t)\}$**.**

(**S1**) For every $t \in I$, $S(t)$ is positive selfadjoint in X and

$$(u, S(t)u) \geq \|u\|^2 \quad \text{for } u \in D(S(t)).$$

Let Y_t be the Hilbert space $D(S(t)^{1/2})$ with new inner product $(\cdot\,,\cdot)_{Y_t}$ and norm $\|\cdot\|_{Y_t}$ for $t \in I$ and $u, v \in Y_t$:

$$(u, v)_{Y_t} := (S(t)^{1/2}u, S(t)^{1/2}v), \qquad \|u\|_{Y_t} := (u, u)_{Y_t}^{1/2};$$

assume that Y_t is embedded continuously and densely in X and that $Y := Y_0$.

(**S2**) For $t \in I$, $Y_t = Y$ and $S(\cdot)^{1/2} \in C_*(I; L(Y, X))$.

(**S3**) There exists a nonnegative function $\sigma \in L^1(I)$ such that for $(t, s) \in \Delta_+ := \{(t, s); 0 \leq s \leq t \leq T\}$,

$$\exp\Big(-\int_s^t \sigma(r)\,dr\Big)\|S(s)^{1/2}v\| \leq \|S(t)^{1/2}v\| \leq \exp\Big(\int_s^t \sigma(r)\,dr\Big)\|S(s)^{1/2}v\|, \quad v \in Y.$$

Remark 2.1. (1) Under conditions (**S1**) and (**S2**), although domain $D(S(t)^{1/2})$ is independent of $t \in I$, both inner product $(\cdot,\cdot)_{Y_t}$ and norm $\|\cdot\|_{Y_t}$ depend on $t \in I$.

(2) Condition (**S3**) and following conditions are equivalent:
(**S3**)′ There exists a nonnegative function $\sigma' \in L^1(I)$ such that for $(t,s) \in \Delta_+$,

$$\big|\|S(t)^{1/2}v\| - \|S(s)^{1/2}v\|\big| \le \int_s^t \sigma'(r)\,dr \min_{r\in[s,t]} \|S(r)^{1/2}v\|, \quad v \in Y.$$

(**S3**)″ There exists a nonnegative function $\sigma'' \in L^1(I)$ such that for $(t,s) \in \Delta_+$,

$$\big|\|S(t)^{1/2}v\| - \|S(s)^{1/2}v\|\big| \le \int_s^t \sigma''(r)\,dr \max_{r\in[s,t]} \|S(r)^{1/2}v\|, \quad v \in Y.$$

The proof of this equivalency is given in Section 7.

Assumption on $\{A(t)\}$.
(**A1**) There exists a constant $\alpha \ge 0$ such that

$$|\operatorname{Re}(A(t)v, v)| \le \alpha\|v\|^2, \quad v \in D(A(t)),\ t \in I.$$

(**A2**) $Y \subset D(A(t))$, $t \in I$.

(**A3**) There exists a constant $\beta \ge \alpha$ such that

$$|\operatorname{Re}(A(t)u, S(t)u)| \le \beta\|S(t)^{1/2}u\|^2, \quad u \in D(S(t)) \subset Y,\ t \in I.$$

(**A4**) $A(\cdot) \in C_*(I; L(Y,X))$.

When the linear part $\{A(t)\}$ satisfies all conditions (**A1**)-(**A3**) and (**S1**)-(**S3**), then a unique evolution system exists and its main properties can be showed. This is discussed int the following result (see Theorem 2.2). Instead, we refer to Section 5 for a concrete example of linear part which satisfies all the quoted conditions.

Theorem 2.2. ([21, Theorem 1.2] see also [30]) *Suppose that* Assumptions on $\{A(t)\}$ *and* $\{S(t)\}$ *are satisfied. Then there exists a unique evolution operator*

$$\{U(t,s); (t,s) \in \Sigma := I \times I\}$$

for (P_0) *having the following properties*:

(**i**) $U(\cdot,\cdot) \in C_*(\Sigma; L(X))$, *with*

$$\|U(t,s)\|_{L(X)} \le e^{\alpha|t-s|}, \quad (t,s) \in \Sigma,$$

where α *is defined in* (**A1**).

(**ii**) $U(t,r)U(r,s) = U(t,s)$ *on* Σ *and* $U(s,s) = 1$ (*the identity*).

(**iii**) $U(t,s)Y \subset Y$ *and* $U(\cdot,\cdot) \in C_*(\Sigma; L(Y))$, *with*

$$\|U(t,s)\|_{L(Y_s,Y_t)} \le \exp\Big(\beta|t-s| + \Big|\int_s^t \sigma(r)\,dr\Big|\Big), \quad (t,s) \in \Sigma, \tag{2.1}$$

$$\|U(t,s)\|_{L(Y)} \le \exp\Big(\beta|t-s| + 2\int_0^{t\vee s} \sigma(r)\,dr\Big), \quad (t,s) \in \Sigma, \tag{2.2}$$

where $t \vee s := \max\{t,s\}$, β *and* $\gamma(\cdot)$ *are defined in* (**A3**) *and* (**S3**), *respectively.*

Furthermore, let $v \in Y$. *Then* $U(\cdot\,,\cdot)v \in C^1(\Sigma; X)$, *with*

(iv) $(\partial/\partial t)U(t,s)v = -A(t)U(t,s)v, \quad (t,s) \in \Sigma,$ *and*

(v) $(\partial/\partial s)U(t,s)v = U(t,s)A(s)v, \quad (t,s) \in \Sigma.$

Remark 2.3. We remark that in [21, 30] all the results in Theorem 2.2 are given only for $(t,s) \in \Delta_+ := \{(t,s); 0 \le s \le t \le T\}$. However, we can obtain the results on Σ. In fact, set

$$\widetilde{A}(t) := -A(T-t), \quad \widetilde{S}(t) := S(T-t), \quad \text{for } t \in I.$$

Then $\{\widetilde{A}(t)\}$ and $\{\widetilde{S}(t)\}$ also satisfy Assumptions $\{A(t)\}$ and $\{S(t)\}$. Therefore we can show that there exists a unique evolution operator $\{\widetilde{U}(t,s); (t,s) \in \Delta_+\}$. Define

$$U(t,s) := \widetilde{U}(T-t, T-s), \quad (t,s) \in \Delta_- := \{(t,s); 0 \le t \le s \le T\}.$$

This is an extension of $\{U(t,s); (t,s) \in \Delta_+\}$ to $\Sigma = \Delta_+ \cup \Delta_-$ and satisfies the properties in Theorem 2.2 on Σ. For instance, we can show that for $(t,s) \in \Delta_-$ and $v \in Y$,

$$\frac{\partial}{\partial t}U(t,s)v = -\frac{\partial}{\partial (T-t)}\widetilde{U}(T-t,T-s)v = \widetilde{A}(T-t)\widetilde{U}(T-t,T-s)v = -A(t)U(t,s)v$$

(see also [21, Remark 4] and [20, Remark 1.3 and Section 5.2]).

Theorem 2.4. ([21, Theorem 1.3]) *Let $\{U(t,s)\}$ be the evolution operator for* (P_0) *as in Theorem 2.2 above. For $u_0 \in Y$ and $f(\cdot) \in C(I;X) \cap L^1(I;Y)$ define $u(\cdot)$ as*

$$u(t) := U(t,0)u_0 + \int_0^t U(t,s)f(s)\,ds, \quad t \in I.$$

Then $u(\cdot) \in C^1(I;X) \cap C(I;Y)$ and $u(\cdot)$ is the unique (classical) solution to (P_0).

Remark 2.5. According to Theorem 2.2 it is easy to see that if $u_1 \in Y$ and

$$f \in C(I;X) \cap L^1(I;Y).$$

Then

$$u(t) := U(t,T)u_1 + \int_T^t U(t,s)f(s)\,ds$$

belongs to $C^1(I;X) \cap C(I;Y)$ and it is the unique (classical) solution to the final value problem

$$\begin{cases} \dfrac{d}{dt}u(t) + A(t)u(t) = f(t) & \text{for } t \in I, \\ u(T) = u_1. \end{cases}$$

3. Preliminary results

In this section we propose useful theorems for the study of the nonlinear problems (P) and (CP). Their proofs appear, for instance, in the quoted references.

Let $\{x_n\} \subset X$ be a sequence in the Banach space X and $x \in X$. If $\{x_n\}$ converges to x with respect to the weak topology then we write $x_n \rightharpoonup x$ in X, while $x_n \to x$ stands for the strong convergence in X.

Let $A \subset X$. Then $\overline{A}$ and $\overline{A}^{\mathrm{W}}$ denote the closure of A in X with respect to the strong and weak topology, respectively. The symbol A^c stands for the complementary of A in X.

We recall now the characterization of weak convergence in spaces of continuous functions.

Theorem 3.1. (Bochner and Taylor [5, Theorem 4.3]) *Let X be a Banach space. Set $\{f_n\} \subset C([a,b];X)$ and $f \in C([a,b];X)$. Then $f_n \rightharpoonup f$ in $C([a,b];X)$ weakly, if and only if*

(a) *there exists $M > 0$ such that $\|f_n\|_{C([a,b];X)} \leq M,\ n \in \mathbb{N}$,*

(b) *for $t \in [a,b]$, $f_n(t) \rightharpoonup f(t)$ in X.*

We recall now the Eberlein Šmulian theory about weak compactness in Banach spaces.

Theorem 3.2. (Eberlein Šmulian theory (see also Kantorovich and Akilov [12, Theorem 1, p.219])) *Let Ω be a subset of a Banach space. Then the following two statements are equivalent*:

(a) Ω *is weakly relatively compact*;

(b) Ω *is weakly relatively sequentially compact.*

Corollary 3.3. ([12, p.219]) *Let Ω be a subset of a Banach space. Then the following two statements are equivalent*:

(a) Ω *is weakly compact*;

(b) Ω *is weakly sequentially compact.*

The following result is about a sufficient condition for the weak compactness in L^1-space.

Theorem 3.4. (Diestel, etc.[8, Corollay 2.6]) *Let X be a Banach space. Assume that A is a bounded and uniformly integrable subset of $L^1([a,b];X)$ such that for $f \in A$, one has $f(t) \in B_t$ a.e. $t \in [a,b]$, where, for $t \in [a,b]$, $B_t \subset X$ is weakly relatively compact. Then A is weakly relatively compact in $L^1([a,b];X)$.*

Theorem 3.5. (Schauder-Tychonoff fixed point theorem, see e.g. Dunford and Schwartz [9, p. 458]) *Let E be a locally convex topological vector space. Let $Q \subset E$ be convex and closed. If $F : Q \to Q$ is continuous and compact then F has a fixed point.*

In a Banach space X endowed with its weak topology, the continuity condition of the map can be replaced by its weak sequential closure. This is showed in the following proposition. We recall that $\Phi\colon Q \to Q$ with $Q \subset X$ is weakly sequentially closed if $\{x_n\} \subset Q$ with $x_n \rightharpoonup x \in Q$ implies $\Phi(x_n) \rightharpoonup \Phi(x)$.

Proposition 3.6. *Let X be a Banach space. Assume that $Q \subset X$ is convex and closed. If $\Phi : Q \to Q$ is weakly sequentially closed and weakly compact, then Φ has a fixed point.*

The proof of previous proposition is a straightforward consequence of the following lemma.

Lemma 3.7. *In the same conditions of* Proposition 3.6 *there exists $\hat{C} \subseteq Q$ convex and weakly compact such that $\Phi(\hat{C}) \subseteq \hat{C}$ and $\Phi : \hat{C} \to \hat{C}$ is weakly continuous.*

Proof. **Step 1.** First we show the existence of a convex and weakly compact set $\hat{C} \subseteq Q$ satisfying $\Phi(\hat{C}) \subseteq \hat{C}$. Since Q is closed, we have $\overline{\Phi(Q)}^{\mathrm{W}} \subseteq Q$. Moreover $\overline{\mathrm{co}}\big(\overline{\Phi(Q)}^{\mathrm{W}}\big)$ is the smallest convex and closed subset of X which contains $\overline{\Phi(Q)}^{\mathrm{W}}$.

Notice that Q is also convex, closed and $\overline{\Phi(Q)}^{\mathrm{W}} \subseteq Q$, so define

$$\hat{C} := \overline{\mathrm{co}}\big(\overline{\Phi(Q)}^{\mathrm{W}}\big) \subseteq Q.$$

Then $\hat{C}$ is convex and closed. By the compactness of Φ, $\hat{C}$ is also weakly compact (see e.g. Dunford and Schwartz[9, p.434]). According to the above results we have

$$\Phi(\hat{C}) \subseteq \Phi(Q) \subseteq \overline{\Phi(Q)}^{\mathrm{W}} \subseteq \overline{\mathrm{co}}\big(\overline{\Phi(Q)}^{\mathrm{W}}\big) = \hat{C}.$$

Thus $\hat{C}$ is convex, weakly compact and $\Phi : \hat{C} \to \hat{C}$.

Step 2. We show that $\Phi : \hat{C} \to \hat{C}$ has weakly closed graph. Since $\hat{C}$ is weakly compact, $\hat{C} \times \hat{C}$ is also weakly compact in $X \times X$. By Corollary 3.3, $\hat{C} \times \hat{C}$ is weakly sequentially compact in $X \times X$. The graph $G(\Phi|_{\hat{C}}) = \{(x, \Phi(x)); x \in \hat{C}\}$ is weakly sequentially compact. In fact, let

$$\{(x_n, \Phi(x_n)); x_n \in \hat{C}\} \subset G(\Phi|_{\hat{C}}) \subset \hat{C} \times \hat{C}.$$

By weakly sequentially compactness of $\hat{C} \times \hat{C}$, there exists $\{(x_{n_k}, \Phi(x_{n_k}))\}_{k \geq 1}$ such that

$$(x_{n_k}, \Phi(x_{n_k})) \rightharpoonup (x_0, y_0) \quad \text{in } \hat{C} \times \hat{C}.$$

Since Φ has weakly sequentially closed graph, we obtain that $y_0 = \Phi(x_0)$. Therefore $(x_0, y_0) \in G(\Phi|_{\hat{C}})$, and then $G(\Phi|_{\hat{C}})$ is weakly sequentially compact. By Theorem 3.2 $G(\Phi|_{\hat{C}})$ is weakly compact. Thus $G(\Phi|_{\hat{C}})$ is weakly closed.

Step 3. We show that $\Phi : \hat{C} \to \hat{C}$ is weakly continuous. Fix $x \in \hat{C}$ and take $W \subset \hat{C}$ weakly open with $\Phi(x) \in W$. Take $y \in \Phi(\hat{C}) \setminus W$. It implies that $y \neq \Phi(x)$ and then $(x, y) \notin G(\Phi|_{\hat{C}})$. This means that $(x, y) \in (G(\Phi|_{\hat{C}}))^c$. Since $(G(\Phi|_{\hat{C}}))^c$ is weakly open in $\hat{C} \times \hat{C}$, there exist two weakly open sets $\widetilde{V}_y$ and $\widetilde{W}_y$ such that $(x, y) \in \widetilde{V}_y \times \widetilde{W}_y \subset (G(\Phi|_{\hat{C}}))^c$. And then we have

$$\Phi(\widetilde{V}_y) \cap \widetilde{W}_y = \emptyset. \tag{3.1}$$

Next we consider a set $\{\widetilde{W}_y : y \in \Phi(\hat{C}) \setminus W\}$. Since $\overline{\Phi(\hat{C})}^{\mathrm{W}}$ is weakly compact in $\hat{C}$, $\overline{\Phi(\hat{C}) \setminus W}^{\mathrm{W}}$ is also weakly compact in $\hat{C}$. Then, noting that $\{\widetilde{W}_y : y \in \Phi(\hat{C}) \setminus W\}$ is a weakly open covering of $\overline{\Phi(\hat{C}) \setminus W}^{\mathrm{W}}$, we can extract a finite sub-covering; that is there exist $y_1, \ldots, y_n \in \Phi(\hat{C}) \setminus W$ such that $\bigcup_{i=1}^{n} \widetilde{W}_{y_i} \supset \overline{\Phi(\hat{C}) \setminus W}^{\mathrm{W}}$. Set $V := \bigcap_{i=1}^{n} \widetilde{V}_{y_i}$. Then $x \in V$ and V is weakly open and $\Phi(V) \subset W$. In fact, if and only if $v \in V$, then $v \in \widetilde{V}_{y_i}$ for $i = 1, 2, \ldots, n$. By (3.1) it implies that $\Phi(v) \notin \widetilde{W}_{y_i}$ $\forall i = 1, \cdots, n$, so $\Phi(v) \notin \bigcup_{i=1}^{n} \widetilde{W}_{y_i}$. Therefore we have $\Phi(V) \subset W$. For the arbitrariness of x, we have showed that Φ is weakly continuous.

Proof of Proposition 3.6. It is sufficient to apply Schauder-Tychonoff fixed point theorem (see Theorem 3.5) to $\Phi|_{\hat{C}}$.

4. The abstract nonlocal problem

This part deals with problem (P). Its solvability is discussed in Theorem 4.1 and the proof involves a fixed point argument. The solution operator Φ is defined in (4.3) and its properties discussed in Lemmas 4.6, 4.7 and 4.8.

Theorem 4.1. *Suppose that* Assumptions on $\{A(t)\}$, $\{S(t)\}$ *are satisfied. Take* $K(\cdot) \in C_*(I; L(C(I;Y), \mathbb{C}))$, $g(\cdot) \in C(I;X) \cap L^1(I;Y)$. *Let*

($\mathbf{\Gamma M}$) Γ *and* M *satisfy* $\|M\|_{L(C(I;Y);Y)} + \liminf_{n\to\infty} \dfrac{\Gamma_n}{n} \|g\|_{L^1(I;Y)} < e^{-\beta T - 2\|\sigma\|_{L^1(I)}}$, *where*

$$k_0 := \sup_{t\in I} \|K(t)\|_{L(C(I;Y),\mathbb{C})}, \quad \Gamma_n := \max\{|\Gamma(t,h)|; t \in I, |h| \le nk_0\} \quad \text{for } n \in \mathbb{N},$$

the constant β *was introduced in* (**A3**) *and the function* $\sigma(\cdot)$ *in* (**S3**). *Then for* $u_0 \in Y$ *and* $f(\cdot) \in C(I;X) \cap L^1(I;Y)$, (P) *has a* (*classical*) *solution*

$$u(\cdot) \in C^1(I;X) \cap C(I;Y).$$

Further if the following condition is added

(**Lip**) *There exists a constant* $L > 0$ *such that*

$$|\Gamma(t,h_1) - \Gamma(t,h_2)| \le L|h_1 - h_2|, \quad t \in I, h_1, h_2 \in \mathbb{C} \tag{4.1}$$

and

$$\|M\|_{L(C(I;Y);Y)} + Lk_0\|g\|_{L^1(I;Y)} < e^{-\beta T - 2\|\sigma\|_{L^1(I)}},$$

then the solution $u(\cdot)$ *is unique.*

Remark 4.2. (1) By a similar reasoning as in Example (2) about $K(\cdot)$ in Section 1 it is possible to show that k_0 is well-defined.

(2) Condition (**Lip**) is stronger than condition ($\mathbf{\Gamma M}$). In fact, it follows from condition (**Lip**) that for $t \in I$ and $|h| \le nk_0$

$$|\Gamma(t,h)| \le L|h| + |\Gamma(t,0)| \le nLk_0 + \max_{t\in I} |\Gamma(t,0)|,$$

and then

$$\liminf_{n\to\infty} \frac{\Gamma_n}{n} \le Lk_0.$$

Remark 4.3. Assume that

$$f(t) + \Gamma(t,0)g(t) \not\equiv 0, \ t \in I,$$

and consider $u_0 = 0$. By the linearity of $K(t)$ for all $t \in I$, it is immediate to see that the solutions given by Theorem 4.1 are never the trivial solution $u \equiv 0$.

Now we introduce two special cases of Theorem 4.1. If, in particular, we consider the case $\Gamma \equiv 0$, then we obtain following

Corollary 4.4. *Suppose that* Assumptions on $\{A(t)\}$ *and* $\{S(t)\}$ *are satisfied. Assume that*

$$\|M\|_{L(C(I;Y);Y)} < e^{-\beta T - 2\|\sigma\|_{L^1(I)}}.$$

Then for $u_0 \in Y$ *and* $f(\cdot) \in C(I;X) \cap L^1(I;Y)$*, the abstract nonlocal Cauchy problem for linear evolution equation of the form*

$$\begin{cases} \dfrac{d}{dt}u(t) + A(t)u(t) = f(t) \quad \textit{for } t \in I, \\ u(0) = u_0 + Mu \end{cases}$$

has a unique (*classical*) *solution*

$$u(\cdot) \in C^1(I;X) \cap C(I;Y).$$

On the other hand, if we consider the case $M \equiv 0$, then we obtain

Corollary 4.5. *Suppose that* Assumptions on $\{A(t)\}$, $\{S(t)\}$ *are satisfied. Assume that* $K(\cdot) \in C_*(I; L(C(I;Y), \mathbb{C}))$, $g(\cdot) \in C(I;X) \cap L^1(I;Y)$ *and*

$$\liminf_{n\to\infty} \frac{\Gamma_n}{n} \|g\|_{L^1(I;Y)} < e^{-\beta T - 2\|\sigma\|_{L^1(I)}},$$

with Γ_n *defined in* (**ΓM**). *Then for* $u_0 \in Y$ *and* $f(\cdot) \in C(I;X) \cap L^1(I;Y)$*, the abstract Cauchy problem for nonlinear evolution equation of the form*

$$\begin{cases} \dfrac{d}{dt}u(t) + A(t)u(t) = f(t) + \Gamma(t, K(t)u)g(t) \quad \textit{for } t \in I, \\ u(0) = u_0 \end{cases}$$

has a (*classical*) *solution*

$$u(\cdot) \in C^1(I;X) \cap C(I;Y).$$

Further if (4.1) *is satisfied and*

$$Lk_0 \|g\|_{L^1(I;Y)} < e^{-\beta T - 2\|\sigma\|_{L^1(I)}}.$$

Then the solution $u(\cdot)$ *is unique.*

The case $\Gamma \equiv 0$ and $M \equiv 0$ was studied in [21, Theorem 1.3] (see also Theorem 2.4). Thus we can regard Theorem 4.1 as a generalization of [21, Theorem 1.3].

The proof of Theorem 4.1 is based on a fixed point argument. So, we first introduce a solution operator Φ (see (4.3) below) and discuss its main properties in the following Lemmas 4.6, 4.7 and 4.8.

For every $q \in C(I;Y)$ we consider the linearized problem

$$\begin{cases} \dfrac{d}{dt}u_q(t) + A(t)u_q(t) = f(t) + \Gamma(t, K(t)q)g(t) \quad \text{for } t \in I, \\ u_q(0) = u_0 + Mq. \end{cases} \tag{P$_q$}$$

Note that $u_0 + Mq \in Y$ and

$$\Gamma(\cdot, K(\cdot)q)g(\cdot) \in C(I;X) \cap L^1(I;Y). \tag{4.2}$$

In fact, by the assumption on $K(\cdot)$, we get that $K(\cdot)q \in C(I;\mathbb{C})$ and then, by the continuity of $\Gamma(\cdot,\cdot)$, we have that $\Gamma(\cdot, K(\cdot)q) \in C(I;\mathbb{C})$. Therefore we obtain (4.2). It follows from Theorem 2.4 that (P$_q$) has a unique solution $u_q(\cdot) \in C^1(I;X) \cap C(I;Y)$. So we can define the solution operator as follows:

$$\Phi : C(I;Y) \to C^1(I;X) \cap C(I;Y), \quad \Phi : q \mapsto u_q. \tag{4.3}$$

It is easy to see that every fixed point of Φ corresponds to a solution of (P). For this reason we need to investigate the properties of Φ. First we show the existence of $Q \subset C(I;Y)$ which satisfies $\Phi(Q) \subseteq Q$.

Lemma 4.6. *There exists $Q \subset C(I;Y)$ with Q bounded, closed and convex satisfying $\Phi(Q) \subseteq Q$.*

Proof. Set $Q_n := \{q \in C(I;Y); \sup_{t\in I} \|q(t)\|_Y \le n\}$; then Q_n is bounded, closed and convex. Assume that $q \in Q_n$. We see from Theorem 2.4 that

$$\Phi(q)(t) = u_q(t) = U(t,0)(u_0 + Mq) + \int_0^t U(t,s)(f(s) + \Gamma(s, K(s)q)g(s))\,ds. \quad (4.4)$$

We have from $|K(s)q| \le nk_0$ that

$$|\Gamma(t, K(t)q)| \le \Gamma_n, \quad t \in I,$$

with k_0 and Γ_n given in ($\mathbf{\Gamma M}$). Notice that

$$\|Mq\|_Y \le \|M\| \sup_{t\in I} \|q(t)\|_Y \le n\|M\|,$$

it follows from Theorem 2.4 that

$$\begin{aligned}
\|u_q(t)\|_Y \le\ & \|U(t,0)\|_{L(Y)}(\|u_0\|_Y + \|Mq\|_Y) \\
& + \int_0^t \|U(t,s)\|_{L(Y)}(\|f(s)\|_Y + |\Gamma(s,K(s)q)|\|g(s)\|_Y)\,ds \\
\le\ & e^{\int_0^t \beta + 2\sigma(r)\,dr}(\|u_0\|_Y + n\|M\|) \\
& + \int_0^t \exp\Big(\beta(t-s) + 2\int_0^t \sigma(r)\,dr\Big)(\|f(s)\|_Y + \Gamma_n\|g(s)\|_Y)\,ds,
\end{aligned}$$

and then

$$\|u_q(t)\|_Y \le e^{\beta T + 2\|\sigma\|_{L^1(I)}}(\|u_0\|_Y + \|f\|_{L^1(I;Y)}) + ne^{\beta T + 2\|\sigma\|_{L^1(I)}}\Big(\|M\| + \frac{\Gamma_n}{n}\|g\|_{L^1(I;Y)}\Big).$$

By condition ($\mathbf{\Gamma M}$), there exists $\bar{n} \in \mathbb{N}$ satisfying $\|u_q(t)\|_Y \le \bar{n}$ (details of this computation are in Lemma 7.3). It is clear that $Q_{\bar{n}}$ satisfies $\Phi(Q_{\bar{n}}) \subseteq Q_{\bar{n}}$.

In the following $Q := Q_{\bar{n}}$ and we denote by n_Q the upper bound of the norm of $q \in Q$. Next we show that the solution operator Φ is weakly sequentially closed.

Lemma 4.7. *Let Φ as in* (4.3) *and $Q \subset C(I;Y)$ be as in* Lemma 4.6*. Then $\Phi|_Q$ is weakly sequentially closed.*

Proof. Set $\{q_k\} \subset Q$ and $q \in C(I;Y)$ with $q_k \rightharpoonup q \in C(I;Y)$. Then

$$\|q\|_{C(I;Y)} \le \liminf_{n\to\infty} \|q_k\|_{C(I;Y)} \le n.$$

Therefore $q \in Q$. The proof is complete when showing that $\Phi(q_k) \rightharpoonup \Phi(q)$ in $C(I;Y)$. Since $K(t) : C(I;Y) \to \mathbb{C}$ is linear and bounded, then $K(t)q_k \to K(t)q$ in $\mathbb{C}$.
It implies that

$$\Gamma(t, K(t)q_k) \to \Gamma(t, K(t)q) \quad \text{in } \mathbb{C}$$

for $t \in I$. The convergence of $\{\Gamma(t, K(t)q_k)\}$ is dominated. In fact, by the assumption on $K(\cdot)$ we have that

$$|K(t)q_k| \le k_0 n_Q.$$

Then

$$|\Gamma(t, K(t)q_k)| \leq \max\{|\Gamma(t,h)|; t \in I, |h| \leq k_0 n_Q\} =: \Gamma_n,$$

with k_0 defined in ($\mathbf{\Gamma M}$). Therefore, by definition of g and the property (2.2), we have

$$U(t,s)\Gamma(s, K(s)q_k)g(s) \to U(t,s)\Gamma(s, K(s)q)g(s) \quad \text{in } Y, \text{ a.a. } s \in I \tag{4.5}$$

and the convergence is dominated, that is

$$\|U(t,s)\Gamma(s,K(s)q_k)g(s)\|_Y \leq e^{\beta T + 2\|\sigma\|_{L^1(I)}}\Gamma_n \|g(s)\|_Y \in L^1(I), \tag{4.6}$$

with Γ_n as in ($\mathbf{\Gamma M}$). Thus, by Lebesgue's dominated convergence theorem, we have

$$\int_0^t U(t,s)\Gamma(s,K(s)q_k)g(s)\,ds \to \int_0^t U(t,s)\Gamma(s,K(s)q)g(s)\,ds \quad \text{in } Y.$$

On the other hand,

$$Mq_k \rightharpoonup Mq \quad \text{in } Y \text{ as } k \to \infty,$$

because M is linear and bounded. According to previous estimates, it follows that

$$\Phi(q_k)(t) = U(t,0)(u_0 + Mq_k) + \int_0^t U(t,s)(f(s) + \Gamma(s,K(s)q_k)g(s))\,ds$$

weakly converges to

$$\Phi(q)(t) = U(t,0)(u_0 + Mq) + \int_0^t U(t,s)(f(s) + \Gamma(s,K(s)q)g(s))\,ds$$

in Y, for $t \in I$. Since $\{\Phi(q_k)\} \subset Q \subset C(I;Y)$ is bounded, we obtain from Theorem 3.1 that $\Phi(q_k) \rightharpoonup \Phi(q)$ in $C(I;Y)$.

We show that the solution operator Φ is weakly relatively compact.

Lemma 4.8. *Let $Q \subset C(I;Y)$ be as in* Lemma 4.6 *and set $\Phi : Q \to Q$, $\Phi(q) = u_q$ as in* (4.3). *Then $\Phi(Q)$ is weakly relatively compact in $C(I;Y)$.*

Proof. By Theorem 3.2, it is enough to show that $\Phi(Q)$ is weakly relatively sequentially compact. So let $\{q_k\} \subset Q$ for $k \in \mathbb{N}$. Then

$$\Phi(q_k)(t) = U(t,0)(u_0 + Mq_k) + \int_0^t U(t,s)(f(s) + \Gamma(s,K(s)q_k)g(s))\,ds, \quad t \in I.$$

(a) $\{Mq_k\} \subset Y$ is a bounded sequence. Y is a Hilbert space, so it is reflexive. By Kakutani's theorem (see e.g. Brezis [6, Theorem 3.17]) $\{Mq_k\}$ is weakly relatively compact in Y. By Theorem 3.2 $\{Mq_k\}$ weakly sequentially relatively compact. So there exist a subsequence $\{q_{k_h}\}$ of $\{q_k\}$ and $\hat{v} \in Y$;

$$Mq_{k_h} \rightharpoonup \hat{v} \in Y.$$

(b) Define $m_h \in L^1(I;Y)$;

$$m_h(t) := \Gamma(t, K(t)q_{k_h})g(t), \quad \text{a.a. } t \in I.$$

We show that $\{m_h\}$ satisfies all the assumptions in Theorem 3.4. By definition of m_h we have that

$$\|m_h(t)\|_Y \leq \Gamma_{n_Q}\|g(t)\|_Y, \quad \text{for a.a. } t \in I. \tag{4.7}$$

Hence $\{m_h\}$ is bounded in $L^1(I;Y)$. Take an arbitrary $\varepsilon > 0$. Since

$$t \mapsto \int_0^t \|g(s)\|_Y \, ds$$

is absolutely continuous, there exists $\delta > 0$:

$$\int_E \|g(t)\|_Y \, dt \leq \frac{\varepsilon}{\Gamma_n}$$

for every $E \subset I$, E measurable with Lebesgue measure $\mu(E) < \delta$. So

$$\left\| \int_E m_h(t) \, dt \right\|_Y \leq \varepsilon.$$

Hence $\{m_h\}$ is uniformly integrable. Let $B_t := \{y \in Y : \|y\|_Y \leq \Gamma_n \|g(t)\|_Y\}$ a.a.$t \in I$. B_t is bounded in Y. Y is a Hilbert space, then it is reflexive. By Kakutani's Theorem B_t is weakly relatively compact. Moreover notice that B_t is convex and strongly closed, since it is a closed ball. So B_t is also weakly closed. In conclusion B_t is weakly compact. By estimate (4.7) we have that $m_h(t) \in B_t$ for a.a. t, $h \in \mathbb{N}$, so we can apply Theorem 3.4. Then there exists a subsequence $\{m_{h_\ell}\}$ and $\hat{m} \in L^1(I;Y)$ such that $m_{h_\ell} \rightharpoonup \hat{m}$ in $L^1(I;Y)$. Fix $t \in I$. We claim that

$$U(t,\cdot)m_{h_\ell}(\cdot) \rightharpoonup U(t,\cdot)\hat{m}(\cdot) \quad \text{in } L^1([0,t];Y). \tag{4.8}$$

Notice that $L^\infty([0,t];Y')$ is the dual space of $L^1([0,t];Y)$ (see e.g. [6]).
Let $R \in (L^1([0,t];Y))'$, $R : L^1([0,t];Y) \to \mathbb{C}$. Then there exists $\rho \in L^\infty([0,t];Y')$ with

$$R\varphi = \int_0^t \langle \rho(s), \varphi(s) \rangle_{Y,Y'} \, ds,$$

where $\langle \cdot, \cdot \rangle_{Y,Y'}$ denotes the duality between Y and its dual space Y'. Let us define $\hat{R} : L^1([0,t];Y) \to \mathbb{C}$,

$$\hat{R}\varphi := \int_0^t \langle \rho(s), U(t,s)\varphi(s) \rangle_{Y,Y'} \, ds.$$

Notice that $\hat{R}$ is linear and bounded. Then $\hat{R} \in (L^1([0,t];Y))'$. Hence by $m_{h_\ell} \rightharpoonup \hat{m}$ in $L^1([0,t];Y)$ we have that $\hat{R}m_{h_\ell} \to \hat{R}\hat{m}$ in $\mathbb{C}$. This is equivalent as $R\,[U(t,\cdot)m_{h_\ell}(\cdot)] \to R\,[U(t,\cdot)\hat{m}(\cdot)]$ in $\mathbb{C}$. Since R is arbitrary in $(L^1([0,t];Y))'$, then the claim (4.8) is proved. By (4.8) we have that

$$\int_0^t U(t,s)m_{h_\ell}(s) \, ds \rightharpoonup \int_0^t U(t,s)\hat{m}(s) \, ds \quad \text{in } Y,\ t \in I.$$

Put

$$v(t) := U(t,0)(u_0 + \hat{v}) + \int_0^t U(t,s)(f(s) + \hat{m}(s)) \, ds,$$

then $v \in C(I;Y)$ and

$$\Phi(q_{k_{h_\ell}})(t) \rightharpoonup v(t) \quad \text{in } Y,\ t \in I.$$

Since $\{\Phi(q_k)\} \subset Q$ is bounded in $C(I;Y)$, we have $\Phi(q_{k_{h_\ell}}) \rightharpoonup v \quad$ in $C(I;Y)$.
Proof of Theorem 4.1. (Existence) Let us consider the solution operator Φ defined in (4.3). By Lemma 4.6, there exists Q closed and convex such that $\Phi(Q) \subset Q$. By

Lemmas 4.7 and 4.8, Φ is weakly sequentially closed and weakly relatively compact. By Proposition 3.6 Φ has a fixed point $q_0 \in C(I;Y)$. Clearly q_0 is a solution of (P), and $q_0 = \Phi(q_0) \in C^1(I;X)$.

(Uniqueness) Let u_1, u_2 be solutions to (P), hence fixed points of Φ. Then it follows from (4.1) that

$$|\Gamma(s,K(s)u_1) - \Gamma(s,K(s)u_2)| \le L|K(s)u_1 - K(s)u_2| \le Lk_0\|u_1 - u_2\|,$$

with k_0 as in ($\mathbf{\Gamma M}$). Therefore, with a similar computation as proof of Lemma 7.3, we obtain

$$\|u_1(t) - u_2(t)\|_Y \le e^{\beta T + 2\|\sigma\|_{L^1(I)}}(\|M\| + Lk_0\|g\|_{L^1(I;Y)})\|u_1 - u_2\|_{C(I;Y)}.$$

It follows from the latter half part of (**Lip**) that $\|u_1 - u_2\|_{C(I;Y)} = 0$. Thus we obtain $u_1 = u_2$.

5. Application to Schrödinger equation

In this section we shall apply Theorem 4.1 to the nonlocal Cauchy problem for the nonlinear Schrödinger equation (NLS). We assume that $u_0 : \mathbb{R}^3 \to \mathbb{C}$, $V : \mathbb{R}^3 \times I \to \mathbb{R}$, f, g, $a : \mathbb{R}^3 \times I \to \mathbb{C}$, $\gamma : I \times \mathbb{C} \to \mathbb{C}$ and $b : I \to \mathbb{C}$. Let $W^{m,p}(\mathbb{R}^3)$ be the usual Sobolev space and we set $H^2(\mathbb{R}^3) := W^{2,2}(\mathbb{R}^3)$. We define

$$\begin{aligned} &H_2(\mathbb{R}^3) := \{u \in L^2(\mathbb{R}^3); (1+|x|^2)u \in L^2(\mathbb{R}^3)\}, \qquad &&\|u\|_{H_2} := \|(1+|x|^2)u\|_{L^2},\\ &\Sigma^2(\mathbb{R}^3) := H^2(\mathbb{R}^3) \cap H_2(\mathbb{R}^3), &&\|u\|_{\Sigma^2} := \|u\|_{H^2} + \|u\|_{H_2}. \end{aligned}$$

We simply wrote H_2 and Σ^2 for denoting norms in the spaces $H_2(\mathbb{R}^3)$ and $\Sigma^2(\mathbb{R}^3)$, respectively and we will use this shorter notation also in the following.

By a solution of (NLS) we mean a function $u(x,t)$ such that

$$\bar{u}(\cdot) \in C^1(I;L^2(\mathbb{R}^3)) \cap C(I;\Sigma^2(\mathbb{R}^3)), \tag{5.1}$$

where $\bar{u}(t) := u(\cdot,t) \in L^2(\mathbb{R}^3)$ (or $\Sigma^2(\mathbb{R}^3)$) for $t \in I$.

We will prove

Theorem 5.1. *Let V satisfies*

$$V \in W^{1,1}(I;L^2(\mathbb{R}^3) + \langle x\rangle^2 L^\infty(\mathbb{R}^3)), \tag{5.2}$$

where

$$\langle x\rangle^2 L^\infty(\mathbb{R}^3) := \{f \in L^\infty_{\mathrm{loc}}(\mathbb{R}^3); (1+|x|^2)^{-1} f \in L^\infty(\mathbb{R}^3)\}$$

and $Z_1 + Z_2 := \{z_1 + z_2; z_1 \in Z_1, z_2 \in Z_2\}$, i.e., (5.2) *means that there exist*

$$\begin{aligned} &V_1 \in L^\infty(I;L^2(\mathbb{R}^3)), \quad V_2 \in L^\infty(I;\langle x^2\rangle L^\infty(\mathbb{R}^3)),\\ &W_1 \in L^1(I;L^2(\mathbb{R}^3)), \quad W_2 \in L^1(I;\langle x^2\rangle L^\infty(\mathbb{R}^3)) \end{aligned} \tag{5.3}$$

satisfying $V_1 + V_2 = V$ and $W_1 + W_2 = \dfrac{\partial}{\partial t}V$. Assume that

$g \in C(I;L^2(\mathbb{R}^3)) \cap L^1(I;\Sigma^2(\mathbb{R}^3))$, $a \in C(I;L^2(\mathbb{R}^3))$, $\gamma \in C(I \times \mathbb{C})$ and $b \in L^1(I;\mathbb{C})$.

Let $\gamma_n := \max\{|\gamma(t,h)|; t \in I, |h| \le n\|a\|_{C(I;L^2)}\}$ for $n \in \mathbb{N}$. The following cases occur

(i) *If*

$$\|b\|_{L^1(I)} + \liminf_{n\to\infty} \frac{\gamma_n}{n} \|g\|_{L^1(I;\Sigma^2)} < e^{-\beta T - 2\|\sigma\|},$$

where $\beta \in \mathbb{R}$ *and* $\sigma \in L^1(I)$ *depend only on* V (*concrete forms are given in the proof*), *then for every initial value* $u_0 \in \Sigma^2(\mathbb{R}^3)$ *and* $f \in C(I; L^2(\mathbb{R}^3)) \cap L^1(I; \Sigma^2(\mathbb{R}^3))$, *problem* (NLS) *has a* (*classical*) *solution* (5.1).

(ii) *If there exists a constant* $L \geq 0$ *satisfying*

$$|\gamma(t, h_1) - \gamma(t, h_2)| \leq L|h_1 - h_2|, \quad t \in I,\ h_1, h_2 \in \mathbb{C},$$
$$\|b\|_{L^1(I)} + L\|a\|_{C(I;L^2)} \|g\|_{L^1(I;\Sigma^2)} < e^{-\beta T - 2\|\sigma\|},$$

with β *and* $\sigma(\cdot)$ *as in* (i), *then for every initial value* $u_0 \in \Sigma^2(\mathbb{R}^3)$ *and* $f \in C(I; L^2(\mathbb{R}^3)) \cap L^1(I; \Sigma^2(\mathbb{R}^3))$, *problem* (NLS) *has a unique* (*classical*) *solution* (5.1).

Remark 5.2. Under the conditions in Theorem 5.1, it is clear that the function

$$t \mapsto \frac{1}{t} \int_0^t \Big(\int_{\mathbb{R}^3} a(y, s) \Delta u(y, s)\, dy \Big) ds$$

is well-defined on $t \in (0, T]$. We can consider the closed (continuous) extension of previous function, i.e., regard $0 \mapsto \displaystyle\int_{\mathbb{R}^3} a(y, 0) \Delta u(y, 0)\, dy$.

Before proving Theorem 5.1 we prepare a lemma.

Lemma 5.3. *Set* $S(t) := (c_{V_1} - \Delta + V(x, t) + c_{V_2}(1 + |x|^2))^2$, *where* V *satisfies* (5.2), $c_{V_1} \geq 0$ *depends on* $\|V_1\|_{L^\infty(I;L^2)}$ (*concrete definition is given at the end of the proof*) *and*

$$c_{V_2} := 1 + 2\|(1 + |x|^2)^{-1} V_2\|_{L^\infty(I;L^\infty)}.$$

Then $S(t)$ *is positive selfadjoint in* $L^2(\mathbb{R}^3)$ *and* $D(S(t)^{1/2}) = \Sigma^2(\mathbb{R}^3)$.

Proof. Define $H_0 := -\Delta + c_{V_2}(1 + |x|^2)$ with $D(H_0) := \Sigma^2(\mathbb{R}^3)$. Then $S(t) := H(t)^2$, where $H(t) := c_{V_1} + H_0 + V(x, t)$ with $D(H(t)) := D(H_0) \cap D(V(t))$. First $V(t)$ is H_0-bounded with H_0-bound $\dfrac{1}{\sqrt{2}}$ if we show that for $u \in \Sigma^2(\mathbb{R}^3)$,

$$\|V_1(t)u\|_{L^2} + \|V_2(t)u\|_{L^2} \leq \frac{1}{\sqrt{2}} \|H_0 u\|_{L^2} + \big(\sqrt{3c_{V_2}} + b_1\big) \|u\|_{L^2}, \tag{5.4}$$

where $b_1 \geq 0$. By the Gagliardo-Nirenberg interpolation inequality we have

$$\begin{aligned} \|V_1(t)u\|_{L^2} &\leq \|V_1\|_{L^\infty(I;L^2)} \|u\|_{L^\infty} \\ &\leq c_{GN} \|V_1\|_{L^\infty(I;L^2)} \|\Delta u\|_{L^2}^{3/4} \|u\|_{L^2}^{1/4} \\ &\leq \frac{1}{2} \|\Delta u\|_{L^2} + b_1 \|u\|_{L^2}, \end{aligned}$$

where c_{GN} is the constant given by the Gagliardo-Nirenberg interpolation inequality and $b_1 \geq 0$ depends on $\|V_1\|_{L^\infty(I;L^2)}$. On the other hand we see from the definition of c_{V_2} that

$$\|V_2(t)u\|_{L^2} \leq \|(1 + |x|^2)^{-1} V_2\|_{L^\infty(I;L^\infty)} \|(1 + |x|^2)u\|_{L^2} \leq \frac{c_{V_2}}{2} \|u\|_{H_2}.$$

Combining two inequalities, then we have

$$\|V_1(t)u\|_{L^2} + \|V_2(t)u\|_{L^2} \leq \frac{1}{\sqrt{2}}(\|\Delta u\|_{L^2}^2 + c_{V_2}^2\|u\|_{H_2}^2)^{1/2} + b_1\|u\|_{L^2}.$$

Noting that $\mathrm{Re}(-\Delta u, (1+|x|^2)u)_{L^2} = \|(1+|x|^2)^{1/2}\nabla u\|_{L^2}^2 - 3\|u\|_{L^2}^2$, we have

$$\begin{aligned}\|\Delta u\|_{L^2}^2 + c_{V_2}^2\|u\|_{H_2}^2 &\leq \|\Delta u\|_{L^2}^2 + 2c_{V_2}\|(1+|x|^2)^{1/2}\nabla u\|_{L^2}^2 + c_{V_2}^2\|u\|_{H_2}^2 \\ &= \|H_0 u\|_{L^2}^2 + 6c_{V_2}\|u\|_{L^2}^2.\end{aligned}$$

Therefore we obtain (5.4). It follows from the Kato-Rellich theorem that $H(t)$ is selfadjoint in L^2.

Next we show the positivity of $H(t)$. By the similar way, we can show that

$$\Big|\int_{\mathbb{R}^3} V_2(t,x)|u(x)|^2\,dx\Big| \leq \frac{c_{V_2}}{2}\|u\|_{H_1}^2$$

and

$$\begin{aligned}\Big|\int_{\mathbb{R}^3} V_1(t,x)|u(x)|^2\,dx\Big| &\leq \|V_1\|_{L^\infty(I;L^2)}\|u\|_{L^4}^2 \\ &\leq c_{GN}{}'\|V_1\|_{L^\infty(I;L^2)}\|u\|_{L^2}^{1/2}\|\nabla u\|_{L^2}^{3/2} \\ &\leq \frac{1}{4}\big(c_{GN}{}'\|V_1\|_{L^\infty(I;L^2)}\big)^4\|u\|_{L^2}^2 + \frac{3}{4}\|\nabla u\|_{L^2}^2,\end{aligned}$$

where $c_{GN}{}'$ is also the constant given by the Gagliardo-Nirenberg interpolation inequality. Therefore we obtain

$$\begin{aligned}(H(t)u,u)_{L^2} &= c_{V_1}\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + c_{V_2}\|u\|_{H_1}^2 + \int_{\mathbb{R}^3} V(t,x)|u(x)|^2\,dx \\ &\geq \Big(c_{V_1} - \frac{1}{4}\big(c_{GN}{}'\|V_1\|_{L^\infty(I;L^2)}\big)^4\Big)\|u\|_{L^2}^2.\end{aligned}$$

Thus, setting $c_{V_1} := 1 + \frac{1}{4}\big(c_{GN}{}'\|V_1\|_{L^\infty(I;L^2)}\big)^4$, then $H(t)$ is positive.

It follows from Lemma 5.3 that $\|\cdot\|_{\Sigma^2}$ and $\|S(t)^{1/2}\cdot\|_{L^2}$ are norm equivalent, such that, there exists a positive constant c satisfying

$$\|u\|_{H^2} + \|u\|_{H_2} \leq c\|S(t)^{1/2}u\|_{L^2}, \tag{5.5}$$

where $c \geq 0$ depends on only $\|V_1\|_{L^\infty(I;L^2)}$ and $\|(1+|x|^2)^{-1}V_2\|_{L^\infty(I;L^\infty)}$.

Proof of Theorem 5.1. Set $A(t) := i^{-1}(-\Delta + V(x,t))$ and $S(t)$ as above. Then $A(t)$ and $S(t)$ satisfy conditions (**S1**), (**S2**), (**S3**), (**A1**), (**A2**), (**A3**) and (**A4**) with $\alpha = 0$, $\beta = 2cc_{V_2}$ and

$$\sigma(t) := c\max\{c_{GN}\|W_1(\cdot,t)\|_{L^2}, \|(1+|\cdot|^2)^{-1}W_2(\cdot,t)\|_{L^\infty}\}, \tag{5.6}$$

where c is the same constant as in (5.5).

In fact, (**S1**) follows from Lemma 5.3. We show (**S2**) and (**S3**). By the definition of $S(t)$, we have

$$\begin{aligned} S(x,t)^{1/2}u(x) - S(x,t_0)^{1/2}u(x) &= V(x,t)u(x) - V(x,t_0)u(x) \\ &= \int_{t_0}^{t} (W_1(x,s) + W_2(x,s))u(x)\,ds \end{aligned}$$

for t, $t_0 \in I$. By the same way as in the proof of Lemma 5.3 we obtain

$$\|W_1(t)u\|_{L^2} \le c_{GN}\|W_1(t)\|_{L^2}\|u\|_{H^2}$$

and

$$\|W_2(t)u\|_{L^2} \le \|(1+|x|^2)^{-1}W_2(t)\|_{L^\infty}\|u\|_{H_2}.$$

Therefore we obtain

$$\begin{aligned} \|S(t)^{1/2}u - S(t_0)^{1/2}u\|_{L^2} &\le \Big|\int_{t_0}^{t} \|(W_1(s) + W_2(s))u\|_{L^2}\,ds\Big| \\ &\le \Big|\int_{t_0}^{t} \sigma(s)\|S(s)^{1/2}u\|_{L^2}\,ds\Big|, \end{aligned}$$

where $\sigma \in L^1(I)$ is given by (5.6). This inequality means (**S2**), and by the estimation of integral inequality, we obtain (**S3**) with (5.6) (see also Lemma 7.1).
(**A1**) with $\alpha = 0$ is clear because $A(t)$ is skew-symmetric. (**A2**) is shown in a way similar as in proof of Lemma 5.3. Let $u \in \Sigma^2(\mathbb{R}^3)$, then

$$\|A(t)u\|_{L^2} \le \|\Delta u\|_{L^2} + \|V_1(t)u\|_{L^2} + \|V_2(t)u\|_{L^2} \le \frac{3}{2}\|\Delta u\|_{L^2} + \frac{c_{V_2}}{2}\|u\|_{H_2} + b_1\|u\|_{L^2}.$$

Hence there exists a positive constant c_A satisfying

$$\|A(t)u\|_{L^2} \le c_A\|u\|_{\Sigma^2}.$$

This inequality implies $\Sigma^2(\mathbb{R}^3) \subset D(A(t))$.

To prove (**A3**) let $v \in D(S(t)) \subset Y$. Then we see from definitions of $A(t)$, $S(t)$ and $H(t)$ that

$$\mathrm{Re}(A(t)v, S(t)v)_{L^2} = -2c_{V_2}\,\mathrm{Im}(3u + 2x\cdot\nabla u, H(t)u)_{L^2}.$$

By simple computations we have

$$\|3u + 2x\cdot\nabla u\|_{L^2}^2 = 4\|x\cdot\nabla u\|^2 - 9\|u\|^2$$

and

$$\mathrm{Re}((1-\Delta)u,(1+|x|^2)u)_{L^2} = \||x|\nabla u\|^2 + \||x|u\|^2 + \|\nabla u\|^2 - 2\|u\|^2 \ge \||x|\nabla u\|^2 - 2\|u\|^2.$$

Thus we can show that

$$\|3u + 2x\cdot\nabla u\|_{L^2} \le \|u\|_{H^2} + \|u\|_{H_2} \le c\|S(t)^{1/2}u\|_{L^2}.$$

Therefore we obtain

$$|\,\mathrm{Re}(A(t)v, S(t)v)_{L^2}| = 2cc_{V_2}\|S(t)^{1/2}u\|_{L^2}^2.$$

(**A4**) follows from (5.2); note that $W^{1,1}(I) \subset C(I)$.

Set

$$K(t)u := \frac{1}{t}\int_0^t \int_{\mathbb{R}^3} a(y,s)\Delta u(y,s)\,dyds$$

and

$$Mu := \int_0^T b(s)u(x,s)\,ds.$$

Then $|K(t)u| \le \|a\|_{C(I;L^2)}\|u\|_{C(I;H^2)}$ and $\|M\| = \|b\|_{L^1(I)}$ (see also Example of $M(1)$ in Section 1). Thus we see from Theorem 4.1 that, in case (i), (NLS) has a (classical) solution $u(\cdot) \in C^1(I;L^2(\mathbb{R}^3)) \cap C(I;\Sigma^2(\mathbb{R}^3))$. Such a solution is also unique, when conditions *(ii)* occur.

Remark 5.4. Consider the equation in (NLS) associated to the Cauchy multicondition

$$u(0) = u_0 + \sum_{i=1}^n \lambda_i u(t_i)$$

with $u_0 \in L^1(I;\Sigma^2(\mathbb{R}^3))$ and λ_i, t_i, $i = 1, \dots n$ as in Example of $M(2)$ in Section 1. If, in the statement of Theorem 5.1, $\|b\|_{L^1(I)}$ is replaced by $\displaystyle\sum_{i=1}^n |\lambda_i|$ then the conclusions of Theorem 5.1 remains true, also in this case.

6. Controllability

This part is about problem (CPS), i.e. it deals with the controllability of nonlocal solutions of the Schrödinger equation. In Theorem 6.1 we prove that its abstract formulation (CP) has a classical controllable solution for every given $u_0, u_1 \in Y$. The proof exploits a topological method hence we introduce a solution operator Π (see (6.3) below) which is obtained by combining the operator Φ defined in (4.3) with the solution Ψ of the final value problem (6.2). The application to (CPS) is then straightforward and given in Example 6.5.

Theorem 6.1. Suppose that Assumptions on $\{A(t)\}$ and $\{S(t)\}$ are satisfied. Assume that $K(\cdot) \in C_*(I;L(C(I;Y),\mathbb{C}))$, $g(\cdot) \in C(I;X) \cap L^1(I;Y)$ and

$$\|M\|_{L(C(I;Y);Y)} + \frac{1}{2}\liminf_{n\to\infty}\frac{\Gamma_n}{n}\|g\|_{L^1(I;Y)} < e^{-\beta T - 2\|\sigma\|_{L^1(I)}} \qquad (\mathbf{\Gamma M})'$$

with Γ_n as in $(\mathbf{\Gamma M})$. Then for u_0, $u_1 \in Y$ and $f(\cdot) \in C(I;X) \cap L^1(I;Y)$, (CP) has a (classical) controllable solution (u,v) with

$$u \in C^1(I;X) \cap C(I;Y)$$

and $Bv(\cdot) \in C(I;X) \cap L^1(I;Y)$.

Set $q \in C(I;Y)$. Before constructing the solution operator of (CP), we introduce the operator

$$\Psi(q)(t) := U(t,T)u_1 + \int_T^t U(t,s)(f(s) + \Gamma(s,K(s)q)g(s))\,ds, \quad t \in I. \qquad (6.1)$$

where $u_1 \in Y$. By Remark 2.5, $\Psi : C(I;Y) \to C^1(I;X) \cap C(I;Y)$ and $\Psi(q)$ is the unique solution of the following problem:

$$\begin{cases} \dfrac{d}{dt}u_q(t) + A(t)u_q(t) = f(t) + \Gamma(t, K(t)q)g(t) \quad \text{for } t \in I, \\ u_q(T) = u_1. \end{cases} \tag{6.2}$$

Therefore, we expect that a solution to (CP) with some suitable $Bv(\cdot)$ corresponds to a fixed point of the following operator:

$$\Pi(q)(t) := (1 - \zeta(t))\Phi(q)(t) + \zeta(t)\Psi(q)(t), \tag{6.3}$$

where $\Phi(q)$ was defined in (4.3) and $\zeta \in C^\infty(I;\mathbb{R})$ satisfying $\zeta(0) = 0$, $\zeta(T) = 1$. The function $\Pi(q)(\cdot)$ is the unique solution of a two-point boundary value problem.

Lemma 6.2. *Let $\Pi(q)$ be defined as in* (6.3)*. Then $\Pi(q)$ is the unique classical solution of the two-point problem:*

$$\begin{cases} \dfrac{d}{dt}u_q(t) + A(t)u_q(t) = f(t) + \Gamma(t, K(t)q)g(t) + B\bar{v}_q(t) \quad \text{for } t \in I, \\ u_q(0) = u_0 + Mq, \\ u_q(T) = u_1, \end{cases} \tag{CP_q}$$

where

$$Bv_q(t) := -\frac{d\zeta}{dt}(t)(\Phi(q)(t) - \Psi(q)(t)).$$

Proof. First notice that, by Theorem 2.4, problem (CP_q) may have at most a classical solution $u_q(\cdot)$. Clearly, $\Pi(q)(0) = \Phi(q)(0) = u_0 + Mq$ and $\Pi(q)(T) = \Psi(T) = u_1$. We show that $\Pi(q)$ satisfies the differential equation in (CP_q).

Since $\Phi(q), \Psi(q) \in C^1(I;X) \cap C(I;Y)$, we obtain $\Pi(q) \in C^1(I;X) \cap C(I;Y)$. Noting that

$$\begin{aligned} \frac{d}{dt}\Phi(q)(t) &= -A(t)\Phi(q)(t) + f(t) + \Gamma(t, K(t)q)g(t), \\ \frac{d}{dt}\Psi(q)(t) &= -A(t)\Psi(q)(t) + f(t) + \Gamma(t, K(t)q)g(t) \end{aligned}$$

and by the definition of $Bv(\cdot)$, we have that

$$\begin{aligned} \frac{d}{dt}\Pi(q)(t) &= (1 - \zeta(t))\frac{d}{dt}\Phi(q)(t) + \zeta(t)\frac{d}{dt}\Psi(q)(t) - \frac{d\zeta}{dt}(t)\big(\Phi(q)(t) - \Psi(q)(t)\big) \\ &= -A(t)\Pi(q)(t) + f(t) + \Gamma(t, K(t)q)g(t) + Bv_q(t). \end{aligned}$$

Therefore $\Pi(q)$ is the classical solution to (CP_q).

We see from $g \in L^1(I;Y)$ that the function $t \mapsto \displaystyle\int_0^t \|g(s)\|_Y \, ds$ is continuous and increasing. Therefore there exists $t_0 \in (0,T)$ such that

$$\int_0^{t_0} \|g(s)\|_Y \, ds = \int_{t_0}^T \|g(s)\|_Y \, ds = \frac{1}{2}\int_0^T \|g(s)\|_Y \, ds. \tag{6.4}$$

Let

$$F := \Big\{\varphi \in C(I;\mathbb{R}); \varphi(0) = 0,\ \varphi(t_0) = \frac{1}{2},\ \varphi(T) = 1,\ \text{monotone increasing}\Big\}. \tag{6.5}$$

We choose in the following a function $\zeta \in C^\infty(I;\mathbb{R}) \cap F$ with t_0 satisfying (6.4). Consequently (see Lemma 7.4), for $t \in I$,

$$(1-\zeta(t))\int_0^t \|g(s)\|_Y\,ds + \zeta(t)\int_t^T \|g(s)\|_Y\,ds \le \frac{1}{2}\|g\|_{L^1(I;Y)}. \tag{6.6}$$

Proof of Theorem 6.1 Let ζ be as above. We prove the existence of a fixed point of Π.

Step 1. First we show that there exists $Q \subset C(I;Y)$ with Q bounded, closed and convex satisfying $\Pi(Q) \subseteq Q$. Set Q_n as in the proof of Lemma 4.6 and assume that $q \in Q_n$. We see from (4.4) and (6.1) that

$$\|\Phi(q)(t)\|_Y \le e^{\beta T + 2\|\sigma\|_{L^1(I)}}\left(\|u_0\|_Y + n\|M\| + \int_0^t \|f(s)\|_Y + \Gamma_n\|g(s)\|_Y\,ds\right),$$
$$\|\Psi(q)(t)\|_Y \le e^{\beta T + 2\|\sigma\|_{L^1(I)}}\left(\|u_1\|_Y + \int_t^T \|f(s)\|_Y + \Gamma_n\|g(s)\|_Y\,ds\right).$$

Therefore we have by (6.6) that

$$\begin{aligned} & e^{-\beta T - 2\|\sigma\|_{L^1(I)}}\|\Pi(q)(t)\|_Y \\ \le\, & e^{-\beta T - 2\|\sigma\|_{L^1(I)}}\big((1-\zeta(t))\|\Phi(q)(t)\|_Y + \zeta(t)\|\Psi(q)(t)\|_Y\big) \\ \le\, & \max\{\|u_0\|_Y, \|u_1\|_Y\} + n\|M\| + \|f\|_{L^1(I;Y)} \\ & + \Gamma_n\Big((1-\zeta(t))\int_0^t \|g(s)\|_Y\,ds + \zeta(t)\int_t^T \|g(s)\|_Y\,ds\Big) \\ \le\, & \max\{\|u_0\|_Y, \|u_1\|_Y\} + \|f\|_{L^1(I;Y)} + n\Big(\|M\| + \frac{\Gamma_n}{2n}\|g\|_{L^1(I;Y)}\Big). \end{aligned}$$

By the same way as in the proof of Lemma 4.6 (see also Lemma 7.3), it follows from condition $(\mathbf{\Gamma M})'$ that there exists $\bar{n} \in \mathbb{N}$ satisfying $\Pi(Q_{\bar{n}}) \subseteq Q_{\bar{n}}$. In next steps, we set $Q := Q_{\bar{n}}$ and denote by n_Q the upper bound of the norm of $q \in Q$.

Step 2. Next we show that Π is weakly sequentially close. Set $\{q_k\} \subset Q$ and $q \in C(I;Y)$ with $q_k \rightharpoonup q$ in $C(I;Y)$. We already proved in Lemma 4.8 that $q \in Q$ and $\Phi(q_k) \rightharpoonup \Phi(q)$ in $C(I;Y)$. By (4.5) and (4.6) it is easy to show that $\Psi(q_k) \to \Psi(q)$ in $C(I;Y)$. Therefore we obtain $\Pi(q_k) \rightharpoonup \Pi(q)$ in $C(I;Y)$.

Step 3. Third we prove that Π is weakly relatively sequentially compact. Let $\{q_k\} \subset Q$. By Lemma 4.8, there exists a subsequence $\{q_{k_h}\}$ and $v_\Phi \in Q$ satisfying $\Phi(q_{k_h}) \rightharpoonup v_\Phi$ in $C(I;Y)$. We define the sequence $\{m_h\}$ as in part (b) of the proof of Lemma 4.8; with a similar reasoning as there we can find a subsequence $\{m_{h_\ell}\}$ and a function $\hat{m} \in L^1(I;Y)$ such that

$$\int_T^t U(t,s)m_{h_\ell}(s)\,ds \rightharpoonup \int_T^t U(t,s)\hat{m}(s)\,ds$$

in Y, $t \in I$. It implies that $\Psi(q_{k_{h_\ell}}) \rightharpoonup v_\Psi$ in $C(I;Y)$ where

$$v_\Psi(t) := U(t,T)u_1 + \int_T^t U(t,s)\hat{m}(s)\,ds.$$

Therefore, $\Pi(q_{k_{h_\ell}})(t) \rightharpoonup v(t) := (1-\zeta(t))v_\Phi(t) + \zeta(t)v_\Psi(t)$ in Y for $t \in I$. Since $\{\Pi(q_k)\} \subset Q$ and Q is bounded, by Theorem 3.1 we obtain that $\Pi(q_{k_{h_\ell}}) \rightharpoonup v$ in $C(I;Y)$. By above properties we see from Proposition 3.6 that Π has a fixed point $u(\cdot)$ which is a solution of (CP) with

$$Bv(t) := -\frac{d\zeta}{dt}(t)(\Phi(u)(t) - \Psi(u)(t)), \quad t \in I \tag{6.7}$$

and $u(\cdot) \in C^1(I;X) \cap C(I;Y)$.

Remark 6.3. It is easy to see that every solution $u(\cdot) \in C^1(I;X) \cap C(I;Y)$ to (CP) with $Bv(\cdot)$ satisfying (6.7) is indeed a fixed point of Π, i.e. $u(\cdot) = \Pi(u)(\cdot)$.

We investigate the topological structure of the solution set to (CP). Put

$$\mathcal{S} := \{u \in C^1(I;X) \cap C(I;Y); (u,v) \text{ is a solution of (CP) with } Bv \text{ as in (6.7)}\}.$$

Theorem 6.4. *Suppose that* Assumptions on $\{A(t)\}$ *and* $\{S(t)\}$ *are satisfied. Assume that* $K(\cdot) \in C_*(I; L(C(I;Y),\mathbb{C}))$ *and*

$$\|M\|_{L(C(I;Y);Y)} + \frac{1}{2}\limsup_{n\to\infty}\frac{\Gamma_n}{n}\|g\|_{L^1(I;Y)} < e^{-\beta T - 2\|\sigma\|_{L^1(I)}}, \tag{$\mathbf{\Gamma M}$)$''$}$$

u_0, $u_1 \in Y$ *and* $f(\cdot)$, $g(\cdot) \in C(I;X) \cap L^1(I;Y)$. *Then* $\mathcal{S}$ *is weakly compact in* $C(I;Y)$.

Proof. The proof splits into two parts.

(*Boundedness of* $\mathcal{S}$). Let $u(\cdot)$ satisfy (CP) with $Bv(\cdot)$ given by (6.7). We show the boundedness of $u(\cdot)$. Put $b := \limsup_{n\to\infty}\dfrac{\Gamma_n}{2n}\|g\|_{L^1(I;Y)}$. Then for $\varepsilon > 0$ there exists $\bar{n} = \bar{n}(\varepsilon) \in \mathbb{N}$ such that $\dfrac{\Gamma_n}{2n}\|g\|_{L^1(I;Y)} \le b + \varepsilon$ for $n \ge \bar{n}$. By condition $(\mathbf{\Gamma M})''$, we can choose $\varepsilon = \varepsilon_0$ satisfying

$$\|M\| + b + \varepsilon_0 < e^{-\beta T - 2\|\sigma\|_{L^1(I)}}.$$

For $u \in C(I;Y)$, there exists $n \in \mathbb{N}$ such that $n - 1 < \|u\|_{C(I;Y)} \le n$. If $n \le \bar{n}$, then $\|u\|_{C(I;Y)} \le \bar{n}$. If $n > \bar{n}$, then we obtain from $|\Gamma(s, K(s)u)| \le \Gamma_n$ that

$$\Gamma_n\|g\|_{L^1(I)} \le 2n(b+\varepsilon_0) \le 2(1 + \|u\|_{C(I;Y)})(b + \varepsilon_0),$$

and therefore by Remark 6.3

$$\begin{aligned}
\|u(t)\|_Y &\le (1-\zeta(t))\|\Phi(u)(t)\|_Y + \zeta(t)\|\Psi(u)(t)\|_Y \\
&\le e^{\beta T + 2\|\sigma\|_{L^1(I)}}\Big(\|u_0\| \vee \|u_1\| + \|M\|\|u\|_{C(I;Y)} + \|f\|_{L^1(I;Y)} + \frac{\Gamma_n}{2}\|g\|_{L^1(I;Y)}\Big) \\
&\le e^{\beta T + 2\|\sigma\|_{L^1(I)}}\big(\|u_0\| \vee \|u_1\| + \|M\|\|u\|_{C(I;Y)} + \|f\|_{L^1(I;Y)} \\
&\qquad\qquad + (1 + \|u\|_{C(I;Y)})(b+\varepsilon_0)\big) \\
&\le e^{\beta T + 2\|\sigma\|_{L^1(I)}}\big(\|u_0\| \vee \|u_1\| + \|f\|_{L^1(I;Y)} + b + \varepsilon_0\big) \\
&\quad + e^{\beta T + 2\|\sigma\|_{L^1(I)}}\big(\|M\| + b + \varepsilon_0\big)\|u\|_{C(I;Y)}.
\end{aligned}$$

Thus we have

$$\|u\|_{C(I;Y)} \leq \frac{\|u_0\| \vee \|u_1\| + \|f\|_{L^1(I;Y)} + b + \varepsilon_0}{e^{-\beta T - 2\|\sigma\|_{L^1(I)}} - \|M\| - b - \varepsilon_0} =: m.$$

In conclusion, we obtain $\|u\|_{C(I;Y)} \leq \bar{n} \vee m$.

(*Weak compactness of* $\mathcal{S}$). By the Eberlein-Šmulian theory (see e.g. Corollary 3.3) it is sufficient to prove that $\mathcal{S}$ is weakly sequentially compact. So, let $\{u_k\} \subset \mathcal{S}$. Since $\mathcal{S}$ is bounded, then $\{Mu_k\}$ is also bounded in Y. Then we can find $\hat{v} \in Y$ and a subsequence $\{u_{k_h}\}$ such that $Mu_{k_h} \rightharpoonup \hat{v} \in Y$. Let

$$m_h(t) := \Gamma(t, K(t)u_{k_h}(t))g(t), \qquad \text{a.a.}\, t \in I.$$

With a similar reasoning as in the proof of Lemma 4.8 we can find $\hat{m} \in L^1(I;Y)$ and a subsequence $\{m_{h_\ell}\}$ satisfying $m_{h_\ell} \rightharpoonup \hat{m}$ in $L^1(I;Y)$. Hence (see the the proof of Lemma 4.8)

$$U(t,\cdot)m_{h_\ell}(\cdot) \rightharpoonup U(t,\cdot)\hat{m}(\cdot) \qquad \text{in } L^1([0,t];Y) \tag{6.8}$$

and

$$U(t,\cdot)m_{h_\ell}(\cdot) \rightharpoonup U(t,\cdot)\hat{m}(\cdot) \qquad \text{in } L^1([t,T];Y) \tag{6.9}$$

for $t \in I$. Put

$$\begin{aligned} \hat{u}(t) := (1-\zeta(t))\Big[U(t,0)(u_0+\hat{v}) + \int_0^t U(t,s)(f(s)+\hat{m}(s))\,ds\Big] \\ + \zeta(t)\Big[U(t,T)u_1 + \int_T^t U(t,s)(f(s)+\hat{m}(s))\,ds\Big] \end{aligned}$$

for $t \in I$ with ζ defined in (6.6). By (6.8), (6.9) and the definition of $\hat{v}$, it follows that $\Pi(u_{k_{h_\ell}})(t) \rightharpoonup \hat{u}(t)$ in Y for $t \in I$. Since $\{u_k\} \subset \mathcal{S}$, we have that $u_k = \Pi(u_k)$ for all k (see Remark 6.3). By the boundedness of $\mathcal{S}$ we obtain that

$$u_{k_{h_\ell}} = \Pi(u_{k_{h_\ell}}) \rightharpoonup \hat{u} \tag{6.10}$$

in $C(I;Y)$. Notice that $\Phi(u_{h_{k_\ell}}) \rightharpoonup \Phi(\hat{u})$ according to Lemma 4.7 and $\Psi(u_{h_{k_\ell}}) \rightharpoonup \Psi(\hat{u})$ by Step 2 in the proof of Theorem 6.1 and both convergences are in $C(I;Y)$. Consequently, as in Step 3 of the proof of Theorem 6.1 we get that

$$u_{h_{k_\ell}} := (1-\zeta)\Phi(u_{h_{k_\ell}}) + \zeta\Psi(u_{h_{k_\ell}}) \rightharpoonup (1-\zeta)\Phi(\hat{u}) + \zeta\Psi(\hat{u}) = \Pi(\hat{u}) \tag{6.11}$$

in $C(I;Y)$. Thus we obtain from (6.10) and (6.11) that $\hat{u} = \Pi(\hat{u})$ and then $\hat{u} \in \mathcal{S}$.

Example 6.5. Let V satisfy (5.2). Assume that $g \in C(I;L^2(\mathbb{R}^3)) \cap L^1(I;\Sigma^2(\mathbb{R}^3))$, $a \in C(I;L^2(\mathbb{R}^3))$, $\gamma \in C(I \times \mathbb{C})$, $b \in L^1(I;\mathbb{C})$ and they satisfy

$$\begin{gathered} C_\gamma := \liminf_{n\to\infty} \frac{\max\{|\gamma(t,h)|; t \in I, |h| \leq n\|a\|_{C(I;L^2)}\}}{n} < \infty, \\ \|b\|_{L^1(I)} + \frac{C_\gamma}{2}\|g\|_{L^1(I;\Sigma^2)} < e^{-\beta T - 2\|\sigma\|}, \end{gathered}$$

where $\beta \in \mathbb{R}$ and $\sigma \in L^1(I)$ are defined in Theorem 5.1. Then for every initial value u_0, $u_1 \in \Sigma^2(\mathbb{R}^3)$ and $f \in C(I; L^2(\mathbb{R}^3)) \cap L^1(I; \Sigma^2(\mathbb{R}^3))$, controllable problem for Schrödinger equation

$$\begin{cases} i\dfrac{\partial}{\partial t}u(x,t) - \Delta u(x,t) + V(x,t)u(x,t) = v(x,t) + f(x,t) \\ \qquad\qquad + \gamma\Big(t, \dfrac{1}{t}\displaystyle\int_0^t \int_{\mathbb{R}^3} a(y,s)\Delta u(y,s)\,dyds\Big)g(x,t), & (x,t) \in \mathbb{R}^3 \times I, \\ u(x,0) = u_0(x) + \displaystyle\int_0^T b(s)u(x,s)\,ds, & x \in \mathbb{R}^3, \\ u(x,T) = u_1(x), & x \in \mathbb{R}^3 \end{cases} \tag{CPS}$$

has a controllable solution

$$u \in C^1(I; L^2(\mathbb{R}^3)) \cap C(I; \Sigma^2(\mathbb{R}^3))$$

with $v(\cdot, t)$ as in (6.7).

By the same way as in the proof of Theorem 5.1, we can verify the conditions of Theorem 6.1.

7. Appendix for some calculations

We collect in this part some technical results which are useful in the proofs of main results.

Lemma 7.1. *Let $f \in C(I)$ be a nonnegative function. Then following properties are equivalent*:

(a) *There exists a nonnegative function $\sigma \in L^1(I)$ such that*

$$f(t) \le \exp\Big(\int_s^t \sigma(r)\,dr\Big)f(s) \quad (t,s) \in \Delta_+.$$

(b) *There exists a nonnegative function $\sigma' \in L^1(I)$ such that*

$$f(t) - f(s) \le \int_s^t \sigma'(r)\,dr \min_{r\in[s,t]} f(r), \quad (t,s) \in \Delta_+.$$

(c) *There exists a nonnegative function $\sigma'' \in L^1(I)$ such that*

$$f(t) - f(s) \le \int_s^t \sigma''(r)\,dr \max_{r\in[s,t]} f(r), \quad (t,s) \in \Delta_+.$$

Proof. (b) $\Rightarrow$ (c) is clear. First we show that (a) $\Rightarrow$ (b). Fix $(t,s) \in \Delta_+$. Then there exists $r_m \in [s,t]$;

$$f(r_m) = \min_{r\in[s,t]} f(r).$$

We see from (a) that

$$f(t) \le \exp\Big(\int_{r_m}^t \sigma(r)\,dr\Big)f(r_m), \quad f(s) \ge \exp\Big(-\int_s^{r_m} \sigma(r)\,dr\Big)f(r_m).$$

Therefore we obtain

$$
\begin{aligned}
f(t)-f(s) &\le \Big(\exp\Big(\int_{r_m}^{t}\sigma(r)\,dr\Big)-\exp\Big(\int_{r_m}^{s}\sigma(r)\,dr\Big)\Big)f(r_m)\\
&=\int_s^t \sigma(r)\exp\Big(\int_{r_m}^{r}\sigma(\tau)\,d\tau\Big)\,dr f(r_m)\le \int_s^t \sigma(r)e^{\|\sigma\|_{L^1(I)}}\,dr f(r_m).
\end{aligned}
$$

Thus we obtain (b) with $\sigma' := \sigma e^{\|\sigma\|_{L^1(I)}}$.

Next we show that (c) $\Rightarrow$ (a). Fix $(t,s)\in\Delta_+$ and take $n\in\mathbb{N}$ grater than $\displaystyle\int_s^t \sigma''(r)\,dr$. Then we can find a finite sequence $\{t_i\}\subset I$ $(i=0,1,2,\dots,n)$ satisfying

$$
s=t_0<t_1<\cdots<t_n=t
$$

and

$$
\int_{t_{i-1}}^{t_i}\sigma''(r)\,dr=\frac{1}{n}\int_s^t\sigma''(r)\,dr\ (<1)\quad \text{for } i=1,2,\dots,n.
$$

There exist $r_i\in[t_{i-1},t_i]$;

$$
f(r_i)=\max\{f(r);r\in[t_{i-1},t_i]\}\quad \text{for } i=1,2,\dots,n.
$$

We see from (c) that for $i=1,2,\dots,n$,

$$
f(t_i)-f(r_i)\le\int_{r_i}^{t_i}\sigma''(r)\,dr f(r_i),\quad f(r_i)-f(t_{i-1})\le\int_{t_{i-1}}^{r_i}\sigma''(r)\,dr f(r_i).
$$

Therefore we obtain for $i=1,2,\dots,n$,

$$
f(t_i)\le\Big(1+\int_{r_i}^{t_i}\sigma''(r)\,dr\Big)f(r_i),\quad f(r_i)\le\Big(1-\int_{t_{i-1}}^{r_i}\sigma''(r)\,dr\Big)^{-1}f(t_{i-1}).
$$

Noting that $(1+a)^{-1}(1-b)\ge(1-a-b)$ for $a,b\ge 0$, we have for $i=1,2,\dots,n$,

$$
\begin{aligned}
f(t_i) &\le\Big(1+\int_{r_i}^{t_i}\sigma''(r)\,dr\Big)\Big(1-\int_{t_{i-1}}^{r_i}\sigma''(r)\,dr\Big)^{-1}f(t_{i-1})\\
&\le\Big(1-\int_{t_{i-1}}^{t_i}\sigma''(r)\,dr\Big)^{-1}f(t_{i-1}),
\end{aligned}
$$

and then

$$
f(t)\le\Big(1-\frac{1}{n}\int_s^t\sigma''(r)\,dr\Big)^{-n}f(s).
$$

Passing to the limit as $n\to\infty$, we obtain (a) with $\sigma=\sigma''$.

Lemma 7.2. *Let X be a Banach space with norm $\|\cdot\|$ and $\mu\in L^1(I;\mathbb{C})$. Set $M : C(I;X)\to X$,*

$$
Mu := \int_0^T \mu(t)u(t)\,dt.
$$

Then $\|M\|_{L(C(I;X);X)}=\|\mu\|_{L^1(I)}$.

Proof. ($\leq$) Set $u \in C(I;X)$ satisfying $\|u\|_{C(I;X)} := \max_{t\in I} \|u(t)\| = 1$. Then

$$\|Mu\| \leq \int_0^T |\mu(t)|\|u(t)\|\,dt \leq \int_0^T |\mu(t)|\,dt\,\|u\|_{C(I;X)}.$$

($\geq$) Put $\varepsilon > 0$ and $x \in X$ satisfying $\|x\| = 1$ and set $\mu_\varepsilon(t) := (\bar{\mu} * \rho_\varepsilon)(t)$, where $\bar{\mu}$ is 0-extension of μ and ρ_ε is a mollifier. Define

$$u_\varepsilon := x\Big(\frac{\mu_\varepsilon(\cdot)^*}{|\mu_\varepsilon(\cdot)| + \varepsilon} * \check{\rho}_\varepsilon\Big)\Big|_I(t),$$

where * is complex conjugate, $\check{\rho}(t) = \rho(-t)$ ($t \in \mathbb{R}$) and $|_I$ is the restriction to I. Then $u_\varepsilon \in C(I;X)$ and $\|u_\varepsilon\|_{C(I;X)} \leq \|x\| = 1$. By properties of the convolution, we have

$$\begin{aligned} Mu_\varepsilon &= x\int_{\mathbb{R}} \bar{\mu}(t)\Big(\frac{\mu_\varepsilon(\cdot)^*}{|\mu_\varepsilon(\cdot)| + \varepsilon} * \check{\rho}_\varepsilon\Big)(t)\,dt = x\int_{\mathbb{R}} (\bar{\mu} * \rho_\varepsilon)(t)\frac{\mu_\varepsilon(t)^*}{|\mu_\varepsilon(t)| + \varepsilon}\,dt \\ &= x\int_{\mathbb{R}} \frac{|\mu_\varepsilon(t)|^2}{|\mu_\varepsilon(t)| + \varepsilon}\,dt, \end{aligned}$$

and then

$$\begin{aligned} \|Mu_\varepsilon\| &= \int_{\mathbb{R}} \frac{|\mu_\varepsilon(t)|^2}{|\mu_\varepsilon(t)| + \varepsilon}\,dt = \int_{\mathbb{R}} |\mu_\varepsilon(t)|\,dt - \varepsilon\int_{\mathbb{R}} \frac{|\mu_\varepsilon(t)|}{|\mu_\varepsilon(t)| + \varepsilon}\,dt \\ &\geq \int_{\mathbb{R}} |\mu_\varepsilon(t)|\,dt - \varepsilon|\mathrm{supp}\,(\mu_\varepsilon)|. \end{aligned}$$

Since $\mu_\varepsilon \to \bar{\mu}$ in $L^1(\mathbb{R})$ and $\|\bar{\mu}\|_{L^1(\mathbb{R})} = \|\mu\|_{L^1(I)}$, therefore we have

$$\|M\|_{L(C(I;X);X)} = \sup_{\|u\|_{C(I;X)}\leq 1} \|Mu\| \geq \liminf_{\varepsilon\to 0} \|Mu_\varepsilon\| \geq \|\mu\|_{L^1(I)}.$$

Thus we obtain the desired equality.

Lemma 7.3. *Let $\{a_n\} \subset \mathbb{R}$ satisfy* $\liminf_{n\to\infty} a_n < 1$. *Then for every $C > 0$ there exists $\bar{n} \in \mathbb{N}$ such that $C + \bar{n}a_{\bar{n}} < \bar{n}$.*

Proof. Set $b := \liminf\limits_{n\to\infty} a_n$. Then we can find a subsequence $\{a_{n_k}\}$ such that

$$a_{n_k} < (b+1)/2.$$

Choose $\bar{n} \in \{n_k\}$ grater than $\dfrac{2C}{1-b}$. Then $C + \bar{n}a_{\bar{n}} < \dfrac{1-b}{2}\bar{n} + \dfrac{b+1}{2}\bar{n} = \bar{n}$.

Lemma 7.4. *Let $t_0 \in (0,T)$ and φ, $\psi \in F$, with F defined in* (6.5). *Then for $t \in I$,*

$$(1-\varphi(t))\psi(t) + \varphi(t)(1-\psi(t)) \leq \frac{1}{2}.$$

Proof. Since φ, $\psi \in F$, we have

$$\begin{aligned} t \in [0,t_0] &\Rightarrow 1 - 2\varphi(t) \geq 0,\ 1 - 2\psi(t) \geq 0, \\ t \in [t_0,T] &\Rightarrow 1 - 2\varphi(t) \leq 0,\ 1 - 2\psi(t) \leq 0 \end{aligned}$$

and therefore obtain for $t \in I$,

$$(1-\varphi(t))\psi(t) + \varphi(t)(1-\psi(t)) = \frac{1}{2} - \frac{1}{2}(1-2\varphi(t))(1-2\psi(t)) \leq \frac{1}{2}.$$

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