Electronic Journal of Differential Equations, Vol. 2003(2003), No. 118, pp. 1–21.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

EXISTENCE AND MULTIPLICITY OF HETEROCLINIC SOLUTIONS FOR A NON-AUTONOMOUS BOUNDARY EIGENVALUE PROBLEM

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Abstract. In this paper we investigate the boundary eigenvalue problem

$$x'' - \beta(c,t,x)x' + g(t,x) = 0$$
$$x(-\infty) = 0, \quad x(+\infty) = 1$$

depending on the real parameter c. We take β continuous and positive and assume that g is bounded and becomes active and positive only when x exceeds a threshold value $\theta \in]0,1[$. At the point θ we allow $g(t,\cdot)$ to have a jump. Additional monotonicity properties are required, when needed. Our main discussion deals with the non-autonomous case. In this context we prove the existence of a continuum of values c for which this problem is solvable and we estimate the interval of such admissible values. In the autonomous case, we show its solvability for at most one c^*. In the special case when β reduces to $c + h(x)$ with h continuous, we also give a non-existence result, for any real c. Our methods combine comparison-type arguments, both for first and second order dynamics, with a shooting technique. Some applications of the obtained results are included.

1. Introduction

This paper concerns the boundary value problem

$$\begin{aligned} x'' - \beta(c,t,x)x' + g(t,x) = 0 \\ x(-\infty) = 0, \quad x(+\infty) = 1 \end{aligned} \tag{1.1}$$

depending on a real parameter c. Our aim is to study the solvability of (1.1) when c varies in a given open interval $J \subset \mathbb{R}$.

We consider $\beta : J \times \mathbb{R}^2 \to \mathbb{R}$ continuous with $\beta(\cdot,t,u)$ increasing for all $(t,u) \in \mathbb{R} \times [0,1]$ and such that with

$$m_c := \inf_{(t,u)\in\mathbb{R}\times[0,1]} \beta(c,t,u), \quad M_c := \sup_{(t,u)\in\mathbb{R}\times[0,1]} \beta(c,t,u)$$

we have

$$0 < m_c \le M_c < +\infty \quad \text{for all } c \in J. \tag{1.2}$$

2000 *Mathematics Subject Classification.* 34B40, 34B18, 34C37.
Key words and phrases. Boundary eigenvalue problems, positive bounded solutions, shooting method.

Submitted April 15, 2003. Published November 28, 2003.

In several applications it is possible to choose the interval J in such a way that (1.2) is satisfied. Moreover note that the functions m_c, M_c of the parameter $c \in J$ are increasing, due to the monotonicity of $\beta(\cdot, t, x)$.

In the paper we are interested in models for which the nonlinear term g is active only for x greater than a fixed threshold value $\theta \in]0,1[$. So we take $g : \mathbb{R}^2 \to \mathbb{R}$ bounded, continuous in $\mathbb{R} \times [\theta, 1]$ and satisfying the following conditions

$$\begin{gathered} g(t,x) = 0 \quad \text{whenever } x \in]-\infty, \theta[\cup [1, +\infty[, \text{ for every } t \in \mathbb{R} \\ g(t,x) \geq \tilde{g}(x) > 0 \quad \text{whenever } x \in]\theta, 1[, \text{ for every } t \in \mathbb{R} \end{gathered} \tag{1.3}$$

where $\tilde{g} \in C^0([\theta, 1])$ is a given function and $\tilde{g}(1) = 0$.

The above hypotheses allow $g(t, \cdot)$ to have a jump at the threshold value θ. When this occurs, each solution of (1.1) will be of class $C^1(\mathbb{R})$ and twice continuously differentiable on every interval where it does not attain the value θ.

In correspondence to the values x where g is not active we take β increasing, that is we also require

$$\beta(c, t, \cdot) \quad \text{increasing in } [0, \theta] \text{ for all } (c, t) \in J \times \mathbb{R}. \tag{1.4}$$

As a consequence of our assumptions, we prove that any solution x of (1.1) necessarily satisfies $x'(t) \geq 0$ for all t. Therefore problem (1.1) can be regarded as a search of monotone heteroclinic connections between the stationary states $x \equiv 0$ and $x \equiv 1$ of the differential equation in (1.1). Since our discussion is mainly centered on the role of the parameter c, this research can be viewed as an investigation of a boundary eigenvalue problem.

There are several models, arising from different sciences, where it is important to find a positive solution of a second order dynamic satisfying suitable boundary conditions. This problem has received lot of attention in the last decades and we provide some progress also in this direction. We refer, in particular, to [1] (Chapter 5) for the investigation of bounded positive solutions in a half-line when the second order equation does not depend on x'. Moreover we refer to [9, Theorem 5.1] for an existence result of positive solutions on all $\mathbb{R}$ for the differential equation in (1.1), but under different assumptions on g.

Problem (1.1) can also be seen as the investigation of non-trivial (i.e. non-constant in ξ) stationary solutions of the parabolic equation

$$u_\tau = u_{\xi\xi} + cu_\xi + g(\tau, u), \quad \tau \geq 0, \ \xi \in \mathbb{R}$$

having limits $x(\pm\infty)$ at infinity. We refer to [15] for such an analysis but under different conditions on the non-linear term g.

Finally, note that by applying [12, Theorem 4.3], which requires very few regularity assumptions on β and g, we could be able to obtain, for each $c \in J$, the existence of a solution x of the differential equation in (1.1) satisfying $0 \leq x(t) \leq 1$ for all $t \in \mathbb{R}$. However, under the assumptions (1.3) on g, this is not enough to guarantee the asymptotic properties of x at $\pm\infty$ required in (1.1).

Under suitable constraints on the values of M_c, m_c in the interval J, in this paper we show the existence of a range of values of c for which (1.1) is solvable. More accurate conclusions follow with additional monotonicity properties.

Throughout the paper we denote

$$g_1(x) := \inf_{t \in \mathbb{R}} g(t,x) \quad \text{and} \quad g_2(x) := \sup_{t \in \mathbb{R}} g(t,x). \tag{1.5}$$

Of course, g_1, g_2 are bounded with $g_1(x) = g_2(x) = 0$ in $]-\infty, \theta[\cup[1, +\infty[$, $g_2(x) \geq g_1(x) \geq \tilde{g}(x) > 0$ in $]\theta, 1[$.

Our main result is the following.

Theorem 1.1. *Let $\beta : J \times \mathbb{R} \times [0,1] \to \mathbb{R}$ be a continuous function, satisfying conditions* (1.2) *and* (1.4). *Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a bounded function, continuous in $\mathbb{R} \times [\theta, 1]$, satisfying condition* (1.3). *Assume*

$$\lim_{c \to \inf J} M_c < \sqrt{2 \int_\theta^1 g_1(s)\, ds} \tag{1.6}$$

$$\lim_{c \to \sup J} m_c > \frac{1}{\theta} \sqrt{2 \int_\theta^1 g_2(s)\, ds}. \tag{1.7}$$

Then, for all $\tau \in \mathbb{R}$ there exists a non-empty set $C_\tau \subset J$ of values of the parameter c for which problem (1.1) *has a solution x_τ satisfying $x_\tau(\tau) = \theta$. Moreover, if we further assume*

$$\begin{array}{ll} (i) & \beta(\cdot, t, x) \text{ strictly increasing for all } (t, x) \in \mathbb{R} \times [0,1], \\ (ii) & \beta(c, \cdot, x) \text{ decreasing for each } (c, x) \in J \times [0,1] \\ (iii) & g(\cdot, x) \text{ increasing for all } x \in [0,1] \end{array} \tag{1.8}$$

then the set C_τ contains a unique element c_τ, for all $\tau \in \mathbb{R}$.

This theorem can be compared to related recent papers. First, we refer to [4], [5] and [14] for boundary eigenvalue theories developed in different contexts. In particular, in [14] $\beta(c, t, x) = cq(t)$ where t varies in a compact interval and the boundary conditions depend polynomially on the spectral parameter c. In [4] and [5] the nonlinear eigenvalue problem for the generalized p-Laplacian equation $-\mathrm{div}(a(t)|\nabla x|^{p-2}\nabla x) = cf(t, x)$ is considered in an unbounded domain, with $x \in \mathbb{R}^n$, $p > 1$ and $c > 0$. Notice the non-variational nature of the differential equation appearing in (1.1).

Even when C_τ is a singleton, say $C_\tau = \{c_\tau\}$, the values of the parameter c_τ corresponding to different times τ, are in general distinct, that is there is a range of values of the parameter c for which (1.1) is solvable, as the following result states.

Theorem 1.2. *Let all the assumptions of Theorem 1.1 be valid and take in addition $\beta(c, \cdot, x)$ strictly decreasing in t for all $(c, x) \in J \times [0, 1]$. Then the map $\tau \longmapsto c_\tau$ is an injective continuous function from $\mathbb{R}$ to J. Moreover, the image set $C := \{c_\tau : \tau \in \mathbb{R}\}$ is a bounded open interval, with* $\inf C > \inf J$.

The existence of a range of values of the parameter c is typical of the non-autonomous case. When the problem is autonomous, that is $\beta(c, t, x) = \beta(c, x)$ and $g(t, x) = g(x)$, assumptions (1.8 *ii-iii*) are trivially fulfilled. Hence, if all the other conditions of Theorem 1.1 hold, there exists a unique $c = c^*$ for which problem (1.1) is solvable. In fact, by Theorem 1.1 we know that the set C_τ contains the unique element c_τ. Moreover, if x_τ is a solution for $c = c_\tau$ satisfying $x_\tau(\tau) = \theta$, then for every $\tau \neq \tau'$ the shifted function $x_{\tau'}(t) := x_\tau(t + \tau - \tau')$ is a solution of (1.1), again for $c = c_\tau$, satisfying $x_{\tau'}(\tau') = \theta$. Hence, in the autonomous case we necessarily have $c_\tau = c'_\tau = c^*$ for every $\tau, \tau' \in \mathbb{R}$. In addition it is possible to prove that the solution of (1.1) corresponding to $c = c^*$ is unique, up to a time-shift.

The consequences of Theorem 1.1 for autonomous problems are summarized by the following result.

Corollary 1.3. *Take $\beta = \beta(c, x)$ and $g = g(x)$ satisfying all the assumptions of Theorem 1.1, with $\beta(\cdot, x)$ strictly increasing. Then, there is a unique $c^* \in J$ such that* (1.1) *is solvable and the solution is unique up to a time-shift.*

Typical examples of autonomous functions β in our analysis are

(a) $\beta(c, t, x) = c + h(x)$
(b) $\beta(c, t, x) = ck(x)$

where h and k denote real continuous functions.

The case (a) appears in the investigation of front-type solutions with wave speed c, for reaction-diffusion equations with convective effects, that is equations of the type

$$\frac{\partial u}{\partial t} + \frac{\partial H(u)}{\partial x} = \frac{\partial^2 u}{\partial x^2} + g(u) \quad t \geq 0, x \in \mathbb{R} \tag{1.9}$$

where $h(u) = \frac{dH}{du}$ denotes the convective speed. We recall that a solution $u(t, x)$ of (1.9) is said to be a travelling wave (or front-type) solution (see e.g. [6]) whenever there exist a function $v \in C^2(\mathbb{R})$ and a real constant c satisfying $u(t, x) = v(x + ct)$ for all $t \geq 0$ and $x \in \mathbb{R}$. This problem, when the function H is constant, was extensively studied mainly in combustion and population genetics models (see e.g. [2], [3] and [6]). In particular, in the special case when g is lipschitzian in $[\theta, 1]$, Berestycki-Nicolaenko-Scheurer (see [3]) proved the existence of a unique positive c^* for which (1.1) is solvable. Note that in this case we have $m_c = M_c = c$ and we can choose $J =]0, +\infty[$ in such a way that (1.6) and (1.7) are trivially satisfied. Hence, our result can be seen as a generalization of the one in [3, Theorem 3.1].

On the other hand, few results are available up to now regarding equation (1.9) with a non-constant convective effect H; see to this purpose [11] for linear convective terms, [8] and [10] for the nonlinear case. This situation presents an interesting dynamic, since the presence of a convective effect may cause the disappearance of front-type solutions.

More in detail, observe that condition (1.4) is satisfied whenever $h(x)$ is increasing in $[0, \theta]$. Moreover, (1.7) is trivially fulfilled if J is unbounded, while (1.6) holds whenever

$$M - m < \sqrt{2 \int_\theta^1 g_1(s) ds} \tag{1.10}$$

where $M := \max_{x \in [0,1]} h(x)$ and $m := \min_{x \in [0,1]} h(x)$. In fact, in order to have $m_c = c + m > 0$ for every $c \in J$ we should take $J =] - m, +\infty[$, so that $\lim_{c \to \inf J} M_c = M - m$ and $\lim_{c \to \sup J} m_c = +\infty$.

Condition (1.10) is essentially a constraint on the growth of $h(u)$. We remark that if $h(u)$ grows too much on $[0, 1]$, it may happen that the boundary value problem (1.1) has no solutions for any value of the parameter c, that is the reaction-diffusion equation (1.9) does not admit travelling wave solutions, even when the convective speed h is linear. In fact, the following non-existence result holds.

Theorem 1.4. *Let $\beta(c, t, x) := c + h(x)$ where h is a continuous function, and let g be as in Theorem 1.1. Assume*

$$\int_0^\theta h(s) ds - \theta h(0) \geq \sqrt{2 \int_\theta^1 g_2(s) ds}. \tag{1.11}$$

Then, problem (1.1) *has no solution, whatever the value $c \in \mathbb{R}$ may be.*

On the other hand, when (1.10) holds, we prove the following existence result, which also provides an estimate for the value c^*.

Corollary 1.5. *Let $\beta(c,t,x) := c + h(x)$ where h is a continuous function. If (1.10) holds, then there exists a unique value c^* for which problem* (1.1) *is solvable and we have*

$$\begin{aligned}\sqrt{2\int_\theta^1 g_1(s)ds} - \int_0^\theta h(s)ds - (1-\theta)M \\ < c^* < \frac{1}{\theta}\Big[\sqrt{2\int_\theta^1 g_2(s)ds} - \int_0^\theta h(s)ds\Big].\end{aligned} \tag{1.12}$$

In the particular case $h(x) \equiv 0$, the previous estimate becomes

$$\sqrt{2\int_\theta^1 g_1(s)\,ds} \le c \le \frac{1}{\theta}\sqrt{2\int_\theta^1 g_2(s)\,ds} \quad \text{for every } c \in C_\tau \text{ and every } \tau \in \mathbb{R}.$$

Our approach consists in reducing the problem (1.1) to an equivalent one on the half line $[0,+\infty[$, which is tackled by a shooting technique.

More in detail, if x is a solution of (1.1) for a given c in J, we denote $\tau_x := \min\{t : x(t) = \theta\}$. Due to the boundary condition $x(-\infty) = 0$, the value τ_x is well-defined and $x(t) < \theta$ for every $t < \tau_x$. Note that in the special case when $\beta(c,t,u) \equiv c$, every solution x satisfies $x'' - cx' = 0$ for all $t \le \tau_x$. Hence, as it is easy to see, the boundary condition $x(-\infty) = 0$ is equivalent to the tangential condition $x'(\tau_x) = c\theta$ at the point $t = \tau_x$. Also in our general setting we replace the boundary condition at $-\infty$ with a suitable tangential one at $t = \tau_x$. This is possible (see Section 2, Theorem 2.4) for example in the case when the trajectories having at $t = \tau_x$ different slopes do not intersect each other in the negative half-line. Lemma 2.2 shows that such a behavior is guaranteed by condition (1.4).

In Section 3 we prove some asymptotic properties of the solutions. We then combine these results with a shooting method developed in Section 4 for studying the asymptotic behavior when $t \to +\infty$. Section 5 is devoted to a relative compactness result for families of solutions. The proofs of all these results are presented in Section 6. This section also contains an example of an autonomous problem of the type (a) with a linear function $h(x) = kx$, for which the solvability and the non-solvability depend on the value of the slope k.

2. A comparison type approach for negative times

The first part of this study is devoted to the investigation of the behavior of the solutions for negative times, restricting our study to those solutions x for which $\tau_x := \min\{t : x(t) = \theta\} = 0$, that is from now on we investigate solutions x of the terminal value problem

$$\begin{aligned} x'' - \beta(c,t,x)x' &= 0 \quad \text{for } t \le 0 \\ x(0) &= \theta. \end{aligned} \tag{2.1}$$

The following preliminary results concern properties of the solutions of (2.1) which will be used to replace the boundary condition $x(-\infty) = 0$ with a tangential one at $t = 0$.

Lemma 2.1. *Given $c \in J$, let x be a non-constant solution of (2.1). Then $x'(t) \neq 0$, for all $t \leq 0$.*

Proof. First observe that if $x'(0) = 0$ then x is constant and equal to θ in $]-\infty, 0]$. In fact, if $x'(t_0) \neq 0$ for some $t_0 < 0$, put $t_1 := \sup\{\tau \leq 0 : x'(t) \neq 0 \text{ for every } t \in [t_0, \tau]\}$, of course $x'(t_1) = 0$. Since $\frac{x''(t)}{x'(t)} = \beta(c, t, x(t))$ in $[t_0, t_1[$, we deduce

$$x'(t_1) = x'(t_0) e^{\int_{t_0}^{t_1} \beta(c,s,x(s))\, ds} \neq 0,$$

a contradiction. Hence, if x is not constant, we have $x'(0) \neq 0$. Therefore, put $\tau_0 := \inf\{\tau < 0 : x'(t) \neq 0 \text{ for every } t \in]\tau, 0]\}$, for every $t \in]\tau_0, 0[$ we have $x'(t) = x'(0) e^{-\int_t^0 \beta(c,s,x(s))\, ds}$, hence $\tau_0 = -\infty$. □

We are now able to show that, under condition (1.4), the solutions of (2.1) having different positive slopes at $t = 0$, do not intersect each other on the negative half-line.

Lemma 2.2. *Assume conditions* (1.2) *and* (1.4). *Given $c \in J$, let x_1 and x_2 be solutions of* (2.1). *Then, if $x_1'(0) > x_2'(0) > 0$, we have*

$$x_1(t) < x_2(t) \quad \text{for all } t < 0.$$

Proof. Given $c \in J$, take $\tau < 0$ satisfying $x_1(t) < x_2(t) < \theta$ for each $t \in]\tau, 0[$. Such a value τ exists due to the tangential conditions at $t = 0$ and Lemma 2.1. Assumption (1.4) then implies

$$e^{-\int_t^0 \beta(c,s,x_1(s))\, ds} \geq e^{-\int_t^0 \beta(c,s,x_2(s))\, ds}$$

for any $t \in [\tau, 0]$ and this yields

$$x_1'(t) = x_1'(0) e^{-\int_t^0 \beta(c,s,x_1(s))\, ds} > x_2'(0) e^{-\int_t^0 \beta(c,s,x_2(s))\, ds} = x_2'(t).$$

Hence $x_1(\tau) < x_2(\tau)$. □

For the sake of completeness we recall now the comparison type result that we shall employ. Let I be a real interval and denote by I^0 its interior. Given a continuous function $f : I \times \mathbb{R}^2 \to \mathbb{R}$, consider the second order equation

$$x'' = f(t, x, x'). \tag{2.2}$$

We shall say that a function $\varphi \in C^0(I) \cap C^2(I^0)$ is a lower solution of (2.2) on I if $\varphi''(t) \geq f(t, \varphi(t), \varphi'(t))$ for all $t \in I^0$. In a similar way a function $\psi \in C^0(I) \cap C^2(I^0)$ satisfying $\psi''(t) \leq f(t, \psi(t), \psi'(t))$ for all $t \in I^0$ will be called an upper solution for (2.2) on I (see e.g. [13]). The following result, which is a slightly modified version of Theorem 4.1 in [13], holds.

Proposition 2.3. *Let $\varphi, \psi \in C^1]-\infty, 0] \cap C^2]-\infty, 0[$ be respectively lower and upper solutions for* (2.2), *with $\varphi(t) \leq \psi(t)$ for all $t \in]-\infty, 0]$. Assume that for every compact subinterval I of $]-\infty, 0]$ there exist two positive continuous functions h and k, defined for $s \geq 0$, satisfying*

$$\int_0^{+\infty} \frac{s}{h(s)} ds = \int_0^{+\infty} \frac{s}{k(s)} ds = +\infty$$

such that

$$\begin{aligned} f(t,x,y) \leq k(y) \quad & \text{whenever } y \geq 0, t \in I \text{ and } \varphi(t) \leq x \leq \psi(t) \\ f(t,x,y) \geq -h(-y) \quad & \text{whenever } y \leq 0, t \in I \text{ and } \varphi(t) \leq x \leq \psi(t). \end{aligned} \tag{2.3}$$

Then, for every $\alpha \in [\varphi(0), \psi(0)]$ *equation* (2.2) *admits a solution* x *such that* $x(0) = \alpha$ *and* $\varphi(t) \le x(t) \le \psi(t)$ *for* $t < 0$*.*

Proof. For $(t, x, y) \in [0, +\infty[\times \mathbb{R}^2$ define $f_1(t, x, y) := f(-t, x, -y)$. For $t \ge 0$, define moreover $\varphi_1(t) := \varphi(-t)$ and $\psi_1(t) := \psi(-t)$. Then φ_1 and ψ_1 are respectively a lower and an upper solution of $x'' = f_1(t, x, x')$ on $[0, +\infty[$ satisfying $\varphi_1(t) \le \psi_1(t)$ for all $t \ge 0$. Finally, according to assumption (2.3), the following growth conditions hold on each compact interval I contained in $[0, +\infty[$,

$$f_1(t, x, y) \ge -h(y) \quad \text{whenever } y \ge 0, t \in I \text{ and } \varphi_1(t) \le x \le \psi_1(t)$$
$$f_1(t, x, y) \le k(-y) \quad \text{whenever } y \le 0, t \in I \text{ and } \varphi_1(t) \le x \le \psi_1(t).$$

Hence [13, Theorem 4.1] can be applied and for all $\alpha \in [\varphi_1(0), \psi_1(0)]$ a solution x_1 of $x'' = f_1(t, x, x')$ exists on $[0, +\infty[$ satisfying $x_1(0) = \alpha$ and $\varphi_1(t) \le x_1(t) \le \psi_1(t)$ for $t \ge 0$. As it is easy to see, the function $x(t) := x_1(-t)$ is a solution of (2.2) with the required properties. □

We are now able to state our main result concerning the behavior of the solutions of (1.1) in the negative half-line.

Theorem 2.4. *Assume* (1.2) *and* (1.4)*. Then for all* $c \in J$ *the following boundary value problem on* $]-\infty, 0]$

$$\begin{gathered} x'' - \beta(c, t, x)x' = 0 \\ x(0) = \theta, \quad x(-\infty) = 0 \end{gathered} \tag{2.4}$$

is solvable. Moreover, all the solutions of (2.4) have the same slope $\lambda = \lambda(c)$ *at* $t = 0$*, which is a continuous increasing function of the parameter* c*, satisfying* $\theta m_c \le \lambda(c) \le \theta M_c$*.*

Proof. i) Solvability of (2.4). Given $c \in J$, consider the functions $\varphi(t) := \theta e^{M_c t}$ and $\psi(t) := \theta e^{m_c t}$ defined for $t \le 0$. According to (1.2), it is easy to see that φ and ψ are respectively a lower and an upper solution of problem (2.1) in $]-\infty, 0]$, satisfying

$$\varphi(t) \le \psi(t) \ \text{ for all } t < 0 \quad \text{and} \quad \varphi(0) = \psi(0) = \theta.$$

In addition, again by (1.2), it follows that

$$\beta(c, t, x)y \le M_c y \quad \text{for } y \ge 0 \text{ and } (t, x) \in]-\infty, 0] \times \mathbb{R}$$
$$\beta(c, t, x)y \ge -M_c(-y) \quad \text{for } y \le 0 \text{ and } (t, x) \in]-\infty, 0] \times \mathbb{R}.$$

Therefore, assumption (2.3) of Proposition 2.3 is satisfied taking $h(y) = k(y) := M_c y$. Hence, a solution $x(t)$ of (2.1) exists on $]-\infty, 0]$ such that

$$\theta e^{M_c t} \le x(t) \le \theta e^{m_c t} \quad \text{for all } t \le 0. \tag{2.5}$$

This implies, in particular, $x(0) = \theta$ and $x(-\infty) = 0$.
ii) Uniqueness of $\lambda(c)$*.* Given $c \in J$, let $x(t)$ be a solution of (2.4). Taking Lemma 2.1 into account, we have $x'(t) > 0$ for every $t \le 0$. Since $\frac{x''(t)}{x'(t)} = \beta(c, t, x(t))$ for $t \le 0$, we obtain

$$x(-\infty) = \theta - x'(0) \int_{-\infty}^{0} e^{-\int_s^0 \beta(c,\sigma,x(\sigma))\, d\sigma}\, ds.$$

Consider now an initial positive slope $\eta < x'(0)$ and let y be a solution of the Cauchy problem

$$\begin{gathered} y'' - \beta(c,t,y)y' = 0 \\ y(0) = \theta, \quad y'(0) = \eta. \end{gathered}$$

According to Lemma 2.1 we have $y'(t) > 0$ for all $t < 0$. On the other hand, Lemma 2.2 ensures $y(t) > x(t)$ for all $t < 0$. As a consequence of the monotonicity property (1.4) of β with respect to x, we have

$$\int_{-\infty}^{0} e^{-\int_s^0 \beta(c,\sigma,y(\sigma))\,d\sigma}\,ds \leq \int_{-\infty}^{0} e^{-\int_s^0 \beta(c,\sigma,x(\sigma))\,d\sigma}\,ds,$$

implying

$$y(-\infty) = \theta - \eta \int_{-\infty}^{0} e^{-\int_s^0 \beta(c,\sigma,y(\sigma))\,d\sigma}\,ds > x(-\infty) = 0.$$

Similarly from $y'(0) > x'(0)$ it follows $y(-\infty) < 0$. Hence, the boundary condition $x(-\infty) = 0$ implies a unique tangential condition at $t = 0$, which only depends on the parameter c.

iii) Monotonicity of $\lambda(c)$. Given $c_1 < c_2$, consider a solution $x_1(t)$ of the boundary value problem

$$\begin{gathered} x'' - \beta(c_1,t,x)x' = 0 \\ x(0) = \theta, \quad x(-\infty) = 0 \end{gathered}$$

lying between the functions $\theta e^{M_{c_1} t}$ and $\theta e^{m_{c_1} t}$; such a solution exists by the proof of part *i)*. According to the monotonicity of β with respect to c and since $x_1'(t) > 0$ for all $t \leq 0$ (see *ii)*), we have $x_1'' = \beta(c_1,t,x_1(t))x_1'(t) \leq \beta(c_2,t,x_1(t))x_1'(t)$. Hence x_1 is an upper solution of the equation $x'' - \beta(c_2,t,x)x' = 0$ on $]-\infty,0]$. On the other hand, recall that $\theta e^{M_{c_2} t}$ is a lower solution of the same equation in $]-\infty,0]$ satisfying

$$\theta e^{M_{c_2} t} \leq \theta e^{M_{c_1} t} \leq x_1(t) \quad \text{for } t \leq 0.$$

Hence, by applying Proposition 2.3, the equation $x'' - \beta(c_2,t,x)x' = 0$ admits a solution $x_2(t)$ satisfying $x_2(0) = \theta$ and $\theta e^{M_{c_2} t} \leq x_2(t) \leq x_1(t)$ for all $t \leq 0$, in particular $x_2(-\infty) = 0$. Since $x_2'(0) \geq x_1'(0)$, by the uniqueness of $\lambda(c_2)$ we have $x_2'(0) = \lambda(c_2)$ implying

$$\lambda(c_1) \leq \lambda(c_2).$$

iv) Continuity of $\lambda(c)$. Fixed $c_0 \in J$, let $(c_n)_n$ be a monotone sequence of values in J converging to c_0 as $n \to +\infty$. Let $(x_n)_n$ be a corresponding sequence of solutions of the boundary value problems

$$\begin{gathered} x'' - \beta(c_n,t,x)x' = 0 \\ x(0) = \theta, \quad x(-\infty) = 0 \end{gathered} \tag{2.6}$$

satisfying $\theta e^{M_{c_n} t} \leq x_n(t) \leq \theta e^{m_{c_n} t}$ for all $t \leq 0$. According to *i)* such solutions exist and by *ii)* they satisfy $x_n'(0) = \lambda(c_n)$. Moreover we have $0 < x_n(t) \leq \theta$ for all $t \leq 0$ and from Lemma 2.1 we deduce

$$x_n'(t) > 0 \quad \text{for all } t \leq 0 \text{ and } n \in N,$$

implying $x_n''(t) = \beta(c_n,t,x_n(t))x_n'(t) > 0$; hence

$$0 < x_n'(t) \leq x_n'(0) = \lambda(c_n) \quad \text{for } t \leq 0 \text{ and } n \in N.$$

Let $\bar{c} := \sup_{n\in N} c_n$. According to the monotonicity of both λ and β, we obtain $0 \le x_n''(t) \le M_{c_n} x_n'(t) \le M_{\bar{c}}\lambda(\bar{c})$. Therefore, the set $(x_n)_n$ is relatively compact in the Fréchet space $C^1(]-\infty,0])$. It is then possible to extract a subsequence $(x_{n_k})_k$, which converges to a function $x \in C^1(]-\infty,0])$ uniformly on the compact subsets of $]-\infty,0]$ and such that also $(x'_{n_k})_k$ uniformly converges to x' on the compact subsets of $]-\infty,0]$. Consequently, x is a solution of $x'' - \beta(c_0,t,x)x' = 0$ on $]-\infty,0]$. Moreover, note that $x_n(t) \le \theta e^{m_{\tilde{c}} t}$ for all $t \le 0$, where $\tilde{c} := \inf c_n$. Hence, $x(-\infty) = 0$ and then, by the uniqueness of $\lambda(c_0)$, we get $x'(0) = \lambda(c_0)$. Therefore, $\lambda(c_{n_k}) = x'_{n_k}(0) \to x'(0) = \lambda(c_0)$ when $k \to +\infty$. Taking the monotonicity of function λ into account, this implies that $\lambda(c_0^+) = \lambda(c_0^-) = \lambda(c_0)$. □

3. Some asymptotic properties

This part deals with the asymptotic behavior of the solutions of the second order differential equation

$$x'' - \beta(c,t,x,)x' + g(t,x) = 0 \tag{3.1}$$

subject to conditions (1.2) and (1.3). We shall need such properties in the next section for developing our shooting method.

Since the solutions we are looking for take values in $[0,1]$, the behavior of function $\beta(c,t,\cdot)$ outside the interval $[0,1]$ is not relevant for the aims of this investigation; so, we can assume, without loss of generality,

$$\beta(c,t,x) = \begin{cases} \beta(c,t,1) & \text{for } x \ge 1 \\ \beta(c,t,0) & \text{for } x \le 0. \end{cases} \tag{3.2}$$

Lemma 3.1. *Given $c \in J$, let x be a solution of* (3.1) *satisfying conditions* (1.2) *and* (1.3). *Assume that $x'(t) \ge 0$ [or $x'(t) \le 0$] for each sufficiently large t. Then there exists $x'(+\infty)$.*

Proof. For some fixed $c \in J$, let x be a solution of (3.1) and assume there exists t_0 such that $x'(t) \ge 0$ for all $t \ge t_0$. For $\xi, t \in \mathbb{R}$, define the functions

$$G_2(\xi) := \int_0^\xi g_2(s)\,ds \quad \text{and} \quad H_2(t) := \frac{1}{2}x'^2(t) + G_2(x(t))$$

with the function g_2 defined by (1.5). Since

$$\begin{aligned} H_2'(t) &= x'(t)x''(t) + g_2(x(t))x'(t) \\ &= x'(t)\big[\beta(c,t,x(t))x'(t) - g(t,x(t)) + g_2(x(t))\big] \\ &= \beta(c,t,x(t))x'^2(t) + \big[g_2(x(t)) - g(t,x(t))\big]x'(t), \end{aligned}$$

we have $H_2'(t) \ge 0$ for all $t \ge t_0$. Hence, there exists $\lim_{t\to+\infty} H_2(t) \in [0,+\infty]$. On the other hand, since $x'(t) \ge 0$ for all $t \ge t_0$, there exists also $\lim_{t\to+\infty} x(t) \in \mathbb{R} \cup \{+\infty\}$. Therefore, since the function $G_2(\xi)$ is bounded and increasing, there exists finite $\lim_{t\to+\infty} G_2(x(t))$. Consequently, also $\lim_{t\to+\infty} x'(t)$ exists in $[0,+\infty]$.

The case when $x'(t) \le 0$ for each t sufficiently large can be treated in a similar way, introducing the functions

$$G_1(\xi) := \int_0^\xi g_1(s)\,ds \quad \text{and} \quad H_1(t) := \frac{1}{2}x'^2(t) + G_1(x(t))$$

with g_1 defined by (1.5). Also in this case, it is easy to show that $H_1'(t) \ge 0$ for all $t \ge t_0$ and this easily leads to the conclusion. □

Lemma 3.2. *Fix $c \in J$ and let x be a solution of* (3.1) *subject to conditions* (1.2) *and* (1.3)*. Then:*

(i) *If $x(t_0) \geq 1$ and $x'(t_0) > 0$ for some t_0, then $x(+\infty) = +\infty$ and $x'(t) > 0$ for every $t > t_0$.*
(ii) *If $x(t_0) \leq \theta$ and $x'(t_0) < 0$ for some t_0, then $x(+\infty) = -\infty$ and $x'(t) < 0$ for every $t > t_0$.*

Proof. Fix $c \in J$.
(i) According to (1.3) and (3.2), we have

$$x''(t_0) = \beta(c, t_0, x(t_0))x'(t_0) = \beta(c, t_0, 1)x'(t_0) > 0.$$

Take $\bar{t} > t_0$ such that $x''(t) > 0$ for all $t \in [t_0, \bar{t}[$. Then $x'(t) > x'(t_0)$, implying $x(t) > 1$ and

$$x''(t) = \beta(c, t, 1)x'(t) \geq m_c x'(t_0) > 0 \quad \text{in } [t_0, \bar{t}[$$

so the same relation holds also for $x''(\bar{t})$. Hence, $x''(t) > 0$ for all $t \geq t_0$ and this yields $x'(t) \geq x'(t_0) > 0$ and $x(+\infty) = +\infty$.
(ii) Since $x''(t_0) = \beta(c, t_0, x(t_0))x'(t_0) < 0$, reasoning as in (i), one finds $x''(t) < 0$ for all $t > t_0$. This implies $x'(t) < x'(t_0) < 0$ and $x(+\infty) = -\infty$. □

Lemma 3.3. *Given $c \in J$, let x be a solution of* (3.1) *subject to conditions* (1.2) *and* (1.3)*. If there exists $t_0 \geq 0$ such that $x'(t_0) = 0$ and $\theta < x(t_0) < 1$, then $x'(t) < 0$ for all $t > t_0$ and $x(+\infty) = -\infty$. Moreover, if there exists $t_0 \geq 0$ such that $x'(t_0) < 0$ then $x'(t) < 0$ for all $t > t_0$ and $x(+\infty) = -\infty$.*

Proof. As a consequence of (1.3), if $x'(t_0) = 0$ and $\theta < x(t_0) < 1$, then $x''(t_0) = -g(t_0, x(t_0)) < 0$, hence $x'(t) < 0$ for t in a right neighborhood of t_0. Similarly, if $x'(t_0) < 0$, then by (1.2) we have $x''(t_0) = \beta(c, t_0, x(t_0))x'(t_0) - g(t_0, x(t_0)) < 0$, hence again $x'(t) < 0$ for t in a right neighborhood of t_0, where $x''(t) < 0$. This implies that $x'(t) < 0$ for all $t > t_0$ and $x(+\infty) = -\infty$. □

In the next section, in order to apply a shooting method, we shall need to introduce the following Cauchy problem on the positive half-line

$$\begin{gathered} x'' - \beta(c, t, x)x' + g(t, x) = 0, \quad t \geq 0 \\ x(0) = \theta, \quad x'(0) = a \end{gathered} \tag{3.3}$$

where a denotes a positive real number.

We can sum up the results of the present section in the following statement.

Corollary 3.4. *Let x be a solution of problem* (3.3)*, satisfying conditions* (1.2) *and* (1.3)*. Then only one of the following four situations may occur:*

(a) *There exists $t_0 > 0$ such that $x'(t_0) = 0$ and $\theta < x(t_0) < 1$, implying $x(+\infty) = -\infty$.*
(b) *There exists $t_0 > 0$ such that $x(t_0) = 1$ and $x'(t_0) = 0$; in this case also the function*

$$y(t) = \begin{cases} x(t) & \text{for } t \leq t_0 \\ 1 & \text{for } t \geq t_0 \end{cases}$$

is a solution of (3.3).
(c) *There exists $t_0 > 0$ such that $x(t_0) = 1$ and $x'(t_0) > 0$, implying $x(+\infty) = +\infty$ and $x'(t) > 0$ for all $t \geq 0$.*

(d) $x'(t) > 0$ *and* $x(t) < 1$ *for all positive* t*, implying* $x'(+\infty) = 0$*; therefore also* $x''(+\infty) = 0$ *and* $x(+\infty) = 1$.

Proof. In the case (d), by Lemma 3.1 we have $x'(+\infty) = 0$. So, by (1.3) we have $\limsup_{t\to+\infty} x''(t) \le -\lim_{t\to+\infty}\tilde g(x(t)) = -\lim_{\xi\to x(+\infty)}\tilde g(\xi)$. Since $\tilde g(x) > 0$ in $]\theta, 1[$, we get $x(+\infty) = 1$ and $x''(+\infty) = 0$. □

4. A shooting method approach for positive times

Given $c \in J$, for each $a > 0$ let us consider the boundary value problem (3.3) on the positive half-line and define the following subsets of $]0, +\infty[$.

$$A_c = \{a > 0 : \text{ each solution } x_a \text{ of (3.3) satisfies } x_a(+\infty) = -\infty\}$$
$$B_c = \{a > 0 : \text{ each solution } x_a \text{ of (3.3) satisfies } x_a(+\infty) = +\infty\}.$$

By means of a shooting technique and taking Corollary 3.4 into account we shall provide now sufficient conditions implying that the sets A_c and B_c are non-empty for some $c \in J$ (see Theorems 4.1 and 4.2), finding estimates, dependent on the parameter c, for $\sup A_c$ and $\inf B_c$.

Theorem 4.1. *Consider equation* (3.1) *subject to conditions* (1.2) *and* (1.3). *Let* $c \in J$ *be given such that*

$$M_c < \frac{\sqrt{2\int_\theta^1 g_1(s)\,ds}}{1-\theta}. \tag{4.1}$$

Then A_c *is non-empty and* $A_c \supseteq \Big]0, -M_c(1-\theta) + \sqrt{2\int_\theta^1 g_1(s)\,ds}\,\Big]$.

Proof. Let $c \in J$ be fixed. Given $a > 0$, assume that $a \notin A_c$. Hence there exists at least a solution y_a of problem (3.3) such that $y_a(+\infty) \ne -\infty$. Therefore, according to Corollary 3.4, we have $y_a(+\infty) = 1$ or $y_a(+\infty) = +\infty$. In order to simplify notations, we shall omit, during this proof the dependence on a of y.

First assume $y(+\infty) = 1$. Applying Lemma 3.1, we get $y'(+\infty) = y''(+\infty) = 0$. Integrating the equation (3.1) in $[0, +\infty[$ we then obtain

$$a + \int_0^{+\infty} \beta(c, s, y(s))y'(s)\,ds - \int_0^{+\infty} g(s, y(s))\,ds = 0. \tag{4.2}$$

Since $y' \in L^1(0, +\infty)$ and $\beta(c, \cdot, y(\cdot)) \in L^\infty(0, +\infty)$, we have $\beta(c, \cdot, y(\cdot))y'(\cdot) \in L^1(0, +\infty)$, so also $g(\cdot, y(\cdot)) \in L^1(0, +\infty)$; hence the integrals in (4.2) are finite.

Now let us multiply (3.1) by y and integrate by parts on $[0, +\infty[$. Since $y'(+\infty) = 0$, we obtain

$$\theta a + \int_0^{+\infty} {y'}^2(s)\,ds + \int_0^{+\infty} \beta(c, s, y(s))y(s)y'(s)\,ds - \int_0^{+\infty} g(s, y(s))y(s)\,ds = 0. \tag{4.3}$$

Since $y(t) < 1$, we have $\beta(c, s, y(s))y(s)y'(s) \le \beta(c, s, y(s))y'(s)$ and $g(s, y(s))y(s) \le g(s, y(s))$ for every $s \ge 0$, then we get $\beta(c, \cdot, y(\cdot))y(\cdot)y'(\cdot)$, $g(\cdot, y(\cdot))y(\cdot) \in L^1(0, +\infty)$. Hence, all the integrals appearing in (4.3) are finite.

Finally, multiply (3.1) by y' and integrate on $[0, +\infty[$; we find

$$\frac{1}{2}a^2 + \int_0^{+\infty} \beta(c, s, y(s)){y'}^2(s)\,ds - \int_0^{+\infty} g(s, y(s))y'(s)\,ds = 0. \tag{4.4}$$

Note that, according to Corollary 3.4 we have $y'(t) > 0$ for all $t \geq 0$. Consequently, we have $g(t, y(t))y'(t) \leq g_2(y(t))y'(t)$ for all $t \geq 0$, hence

$$\int_0^{+\infty} g(s, y(s))y'(s)\, ds \leq \int_\theta^1 g_2(\xi)\, d\xi < +\infty$$

and this ensures that also the integrals appearing in (4.4) are finite. Subtracting (4.2) from (4.3) we obtain

$$\begin{aligned}(\theta - 1)a + \int_0^{+\infty} {y'}^2(s)\, ds - \int_0^{+\infty} \beta(c, s, y(s))\big[1 - y(s)\big]y'(s)\, ds \\ - \int_0^{+\infty} g(s, y(s))\big[y(s) - 1\big]\, ds = 0.\end{aligned}$$

According to (1.2), this implies

$$\int_0^{+\infty} {y'}^2(s)\, ds < a(1 - \theta) + M_c \int_0^{+\infty} \big[1 - y(s)\big]y'(s)\, ds = a(1 - \theta) + \frac{M_c}{2}(1 - \theta)^2.$$

Therefore, by (4.4), since $y'(t) > 0$ for all positive t, we get

$$\begin{aligned}\int_\theta^1 g_1(\xi)\, d\xi &\leq \int_0^{+\infty} g(s, y(s))\, y'(s)\, ds = \frac{1}{2}a^2 + \int_0^{+\infty} \beta(c, s, y(s))\, {y'}^2(s)\, ds \\ &\leq \frac{1}{2}a^2 + M_c \int_0^{+\infty} {y'}^2(s)\, ds < \frac{1}{2}a^2 + M_c(1 - \theta)a + \frac{M_c^2}{2}(1 - \theta)^2.\end{aligned}$$

Finally, given $a > 0$ and $c \in J$ with $a \notin A_c$, assuming $y(+\infty) = 1$ for at least one solution of problem (3.3), we obtain the following relation between the parameters of the dynamic

$$a^2 + 2M_c(1 - \theta)a + M_c^2(1 - \theta)^2 - 2\int_\theta^1 g_1(s)\, ds > 0; \tag{4.5}$$

that is

$$a > -M_c(1 - \theta) + \sqrt{2\int_\theta^1 g_1(s)\, ds}.$$

Hence, if (4.1) holds, the set A_c is nonempty and the assertion follows.

Consider now the remaining case when $y(+\infty) = +\infty$ for a solution y of (3.3). Obviously there exists a positive value t_1 such that $y(t_1) = 1$ and $y(t) < 1$ for $0 \leq t \leq t_1$. According to Corollary 3.4 we have $y'(t) > 0$ for all $t \in [0, t_1[$. Integrating the equation (3.1) in $[0, t_1]$ we obtain

$$a - y'(t_1) + \int_0^{t_1} \beta(c, s, y(s))y'(s)\, ds - \int_0^{t_1} g(s, y(s))\, ds = 0. \tag{4.6}$$

Again multiplying (3.1) by y and integrating by parts on $[0, t_1]$ we have

$$\theta a - y'(t_1) + \int_0^{t_1} {y'}^2(s)\, ds + \int_0^{t_1} \beta(c, s, y(s))y(s)y'(s)\, ds - \int_0^{t_1} g(s, y(s))y(s)\, ds = 0. \tag{4.7}$$

Consequently, subtracting (4.6) from (4.7) we obtain

$$\int_0^{t_1} y'^2(s)ds < a(1 - \theta) + M_c \int_0^{t_1} [(1 - y(s)]y'(s)ds \leq a(1 - \theta) + \frac{M_c}{2}(1 - \theta)^2.$$

Multiplying now (3.1) by y' and integrating on $[0, t_1]$ we have

$$\frac{1}{2}a^2 - \left[y'(t_1)\right]^2 + \int_0^{t_1} \beta(c, s, y(s)) y'^2(s)\, ds - \int_0^{t_1} g(s, y(s)) y'(s)\, ds = 0.$$

Reasoning as before we again arrive to relation (4.5). Hence the conclusion holds also in this case. □

Theorem 4.2. *Consider equation* (3.1) *subject to conditions* (1.2) *and* (1.3). *For every $c \in J$, the set B_c is nonempty. In particular we have*

$$B_c \supseteq \left[\sqrt{2\int_\theta^1 g_2(s)\, ds}, +\infty\right[.$$

Proof. Fix $c \in J$. Given $a > 0$, assume that $a \notin B_c$. Then, according to Corollary 3.4, a solution y_a of problem (3.3) exists such that $y_a(+\infty) = 1$ or $y_a(+\infty) = -\infty$. In both cases there exists $t_0 \in]0, +\infty]$ such that $y_a'(t_0) = 0$; in fact, when $y_a(+\infty) = 1$ we have $y'(+\infty) = 0$ by Corollary 3.4. In addition, Lemma 3.3 implies that $y_a'(t) > 0$ for all $0 \le t < t_0$. To simplify notation, as in the proof of Theorem 4.1, we shall denote y_a and y_a' respectively by y and y'.

Let us multiply (3.1) by y' and integrate on $[0, t_0]$; we obtain

$$\frac{1}{2}a^2 + \int_0^{t_0} \beta(c, s, y(s)\, y'^2(s)\, ds - \int_0^{t_0} g(s, y(s)) y'(s)\, ds = 0. \tag{4.8}$$

Since

$$\int_0^{t_0} g(s, y(s)) y'(s)\, ds \le \int_0^{t_0} g_2(y(s)) y'(s)\, ds \le \int_\theta^1 g_2(\xi)\, d\xi < +\infty,$$

even if $t_0 = +\infty$ both the integrals in (4.8) are finite. Moreover, since

$$\int_0^{t_0} \beta(c, s, y(s))\, y'^2(s)\, ds > 0,$$

we have

$$\frac{1}{2}a^2 < \int_0^{t_0} g(s, y(s)) y'(s)\, ds \le \int_\theta^1 g_2(\xi)\, d\xi$$

implying $a < \sqrt{2\int_\theta^1 g_2(s)\, ds}$. □

5. Compactness of solution sets

In this section we shall consider suitable families of solutions of (3.3) obtained when a and c vary in bounded sets. Our aim is to prove their relative compactness in the Fréchet space $C^1([0, +\infty[)$. To this purpose, given a real interval $I \subset \mathbb{R}$, recall that a subset A of $C^1(I)$ is bounded if and only if there exists a positive continuous function $\Phi : I \to \mathbb{R}$ such that

$$|x(t)| + |x'(t)| \le \Phi(t) \quad \text{for all } x \in A \text{ and } t \in I.$$

Moreover, according to Ascoli's theorem, the relative compactness of A in $C^1(I)$ is guaranteed by the boundedness combined with the equicontinuity, at each $t \in I$, of the derivatives of all $x \in A$.

Hence, the relative compactness of a family A of functions in $C^1(I)$ is ensured by the existence of a function $\Phi \in C^0(I)$ such that

$$|x(t)| + |x'(t)| + |x''(t)| \le \Phi(t) \quad \text{for all } x \in A \text{ and } t \in I.$$

Indeed, since $x(0) = \theta$ for every solution of (3.3), in this case it suffices to prove that

$$|x'(t)| + |x''(t)| \le \phi(t) \quad \text{for all } x \in A \text{ and } t \in I$$

for some function $\phi \in C^0([0, +\infty[)$, since we have $|x(t)| \le \theta + \int_0^t \phi(\tau)d\tau$ for every $t > 0$.

Proposition 5.1. *Let $C \subset J$ and $I \subset]0, +\infty[$ be two bounded intervals, with $\inf C > \inf J$. Then, each family $X = (x_{c,a})_{c,a}$ of solutions of (3.3) with $c \in C$ and $a \in I$ is relatively compact in $C^1([0, +\infty[)$.*

Proof. Let $\bar{c} = \sup C$, $\bar{a} = \sup I$, and $\bar{g_2} = \sup_{x\in[0,1]} g_2(x)$. Moreover, let

$$X_+ := \{x_{c,a} \in X : x_{c,a}(+\infty) = +\infty\},$$
$$X_- := \{x_{c,a} \in X : x_{c,a}(+\infty) = -\infty\}, X_1 := \{x_{c,a} \in X : x_{c,a}(+\infty) = 1\}.$$

By virtue of Corollary 3.4, we have $X = X_+ \cup X_- \cup X_1$, so it suffices to prove that these three subfamilies are relatively compact.

Note that for every $x_{c,a} \in X_+$ we have $x'_{c,a}(t) \ge 0$ for every $t \ge 0$ (see Lemma 3.3), hence from (1.3) we deduce

$$x''_{c,a}(t) \le \beta(c, t, x_{c,a}(t))x'_{c,a}(t) \le M_c x'_{c,a}(t) \le M_{\bar{c}} x'_{c,a}(t), \quad \text{for every } t \ge 0.$$

Therefore, $0 < x'_{c,a}(t) \le \bar{a}e^{M_{\bar{c}}t}$. Then, for every $t \ge 0$ we have $-\bar{g_2} \le x''_{c,a}(t) \le M_{\bar{c}}\bar{a}e^{M_{\bar{c}}t}$. Hence, X_+ is relatively compact.

Observe now that for every $x_{c,a} \in X_1$ we have $x'_{c,a}(+\infty) = 0$, therefore $x'_{c,a}$ has a maximum on $[0, +\infty[$ attained at a point t_0 which obviously depends on both c and a. We have two possibilities, either $t_0 = 0$ and $x'_{c,a}(t) \le x'_{c,a}(0) = a$ for all $t \ge 0$, or $t_0 > 0$ and $x''_{c,a}(t_0) = 0$. In the latter case, since $\tilde{c} := \inf C > \inf J$, and consequently $m_{\tilde{c}} > 0$, we have

$$x'_{c,a}(t_0) = \frac{g(t_0, x_{c,a}(t_0))}{\beta(c, t_0, x_{c,a}(t_0))} \le \frac{\bar{g_2}}{m_{\tilde{c}}}.$$

So, put $H := \frac{\bar{g_2}}{m_{\tilde{c}}}$, we deduce $0 < x'_{c,a}(t) \le \max\{\bar{a}, H\} \le \bar{a} + H$, implying

$$-\bar{g_2} \le x''_{c,a}(t) \le M_{\bar{c}}(\bar{a} + H) \quad \text{for every } t \ge 0.$$

Hence, also X_1 is relatively compact.

Finally, let us consider the family X_-. Similarly to what done above, it is easy to show that $x'_{c,a}(t) \le \bar{a}+H$, for $t \ge 0$, for every $x_{c,a} \in X_-$. Moreover, in the half-lines $[t_0, +\infty[$ where $x'_{c,a}$ is negative and $x'_{c,a}(t_0) = 0$, we have $x_{c,a}''(t) \ge M_{\bar{c}}x'_{c,a}(t) - \bar{g_2}$, implying $x'_{c,a}(t) \ge \frac{\bar{g_2}}{M_{\bar{c}}}\big(1 - e^{M_{\bar{c}}t}\big)$. We have then obtained

$$\frac{\bar{g_2}}{M_{\bar{c}}}\big(1 - e^{M_{\bar{c}}t}\big) \le x'_{c,a}(t) \le \bar{a} + H \quad \text{for all } t \ge 0.$$

Consequently we have

$$-\bar{g_2}e^{M_{\bar{c}}t} \le x_{c,a}''(t) \le M_{\bar{c}}(\bar{a} + H) \quad \text{for } t \ge 0.$$

Then, also X_- is relatively compact in $C^1([0, +\infty[)$. □

6. Proofs of the main results

We are now ready to provide the proofs of the results stated in Introduction.

Proof of Theorem 1.1. First we prove, under conditions (1.6) and (1.7), the existence of at least a value c^* for which problem (1.1) has a solution x satisfying $x(0) = \theta$. To this aim, according to Theorem 2.4, it suffices to prove the existence of at least a value c^* for which the Cauchy problem on $[0, +\infty[$

$$\begin{gathered} x'' - \beta(c, t, x)x' + g(t, x) = 0, \quad t \geq 0 \\ x(0) = \theta, \quad x'(0) = \lambda(c) \end{gathered} \tag{6.1}$$

admits a solution x satisfying $x(+\infty) = 1$. Set

$$\hat{A} := \{c \in J : \lambda(c) \in A_c\}, \quad \hat{B} := \{c \in J : \lambda(c) \in B_c\}.$$

Note that assumption (1.6) easily implies that condition (4.1) is satisfied, for all c sufficiently close to $\inf J$. Moreover, being $\lambda(c) \leq \theta M_c$, we have $c \in \hat{A}$ for every c sufficiently close to $\inf J$. Similarly, by Theorem 4.2, being $\lambda(c) \geq m_c$ condition (1.7) implies that $c \in \hat{B}$ for every c sufficiently close to $\sup J$. Hence, both sets $\hat{A}$, $\hat{B}$ are nonempty. Let us now show that they are open.

Assume, by contradiction, the existence of a point $c_0 \in \hat{A}$ and a sequence $(c_n)_n$ converging to c_0 such that $c_n \notin \hat{A}$ for every $n \in \mathbb{N}$. It is then possible to find a corresponding sequence $(x_n)_n$ of solutions of problems

$$\begin{gathered} x'' - \beta(c_n, t, x)x' + g(t, x) = 0, \quad t \geq 0 \\ x(0) = \theta, \quad x'(0) = \lambda(c_n) \end{gathered} \tag{6.2}$$

satisfying $x_n(+\infty) \neq -\infty$.

Of course, the set $C = \{c_n : n \in N\}$ is bounded with $\inf C > \inf J$. Moreover, owing to the continuity and monotonicity of the function $\lambda(c)$ also $I = \{\lambda(c_n) : n \in N\}$ is bounded. Hence, by applying Proposition 5.1 we deduce that $(x_n)_n$ is a relatively compact subset of the Fréchet space $C^1([0, +\infty[)$. It is then possible to extract a subsequence, again denoted $(x_n)_n$, which converges in $C^1([0, +\infty[)$ to a function x. Therefore, x is a solution of the Cauchy problem (6.1) with $c = c_0$ and since $c_0 \in \hat{A}$ we have $x(+\infty) = -\infty$. On the other hand, by Corollary 3.4 we have that $x_n'(t) \geq 0$ for all $n \in N$ and $t \geq 0$. Therefore $x'(t) \geq 0$ for all $t > 0$, a contradiction.

Similarly, assume by contradiction the existence of a point $c_0 \in \hat{B}$ and a sequence $(c_n)_n$ converging to c_0 such that $c_n \notin \hat{B}$ for every $n \in \mathbb{N}$. It is then possible to find a corresponding sequence $(x_n)_n$ of solutions of problems (6.1) with $c = c_n$ satisfying $x_n(+\infty) \neq +\infty$.

By the relative compactness of the set $(x_n)_n$, we can extract a subsequence, again denoted $(x_n)_n$, which converges in $C^1([0, +\infty[)$ to a function x. Therefore, x is a solution of the Cauchy problem (6.1) with $c = c_0$ and since $c_0 \in \hat{B}$ we have $x(+\infty) = +\infty$. Hence, we have $x(t) > 1$ for all t sufficiently large. On the other hand, by Lemma 3.3 and Corollary 3.4 we have that $x_n(t) \leq 1$ for all $t \geq 0$, and this is a contradiction.

Therefore, since $\hat{A}$ and $\hat{B}$ are disjoint, nonempty and open, there exists a value $c^* \notin \hat{A} \cup \hat{B}$. We will now prove that problem (1.1) is solvable for $c = c^*$. Let us assume, by contradiction, that for every solution x of problem (6.1) with $c = c*$ we have $x(+\infty) \neq 1$. Set

$$X^+ := \{x \text{ is a solution of (6.1) with } c = c^* \text{ and } x(+\infty) = +\infty\}$$
$$X^- := \{x \text{ is solution of (6.1) with } c = c^* \text{ and } x(+\infty) = -\infty\}.$$

Since $c^* \notin \hat{A} \cup \hat{B}$, $X^+ \neq \emptyset$ and $X^- \neq \emptyset$. Let

$$\tau^+ := \sup\{t : x(t) = 1 \text{ for some } x \in X^+\} \in]0, +\infty],$$
$$\tau^- := \sup\{t : x(t) = 0 \text{ for some } x \in X^-\} \in]0, +\infty].$$

Note that $\tau^+ = \tau^- = +\infty$. In fact, let us consider the associate differential system

$$\begin{aligned} y_1'(t) &= y_2(t) \\ y_2'(t) &= \beta(c^*, t, y_1(t))y_2(t) - g(t, y_1(t)) \\ y_1(0) &= \theta, \quad y_2(0) = \lambda(c^*) \end{aligned}$$

and consider, for every $t > 0$ the sections

$$S_t := \{(\bar{y}_1, \bar{y}_2) \in \mathbb{R}^2 : y_1(t) = \bar{y}_1, y_2(t) = \bar{y}_2 \text{ for some solution } (y_1, y_2) \text{ of (6)}\}.$$

By classical results, each section S_t is a continuum, that is a nonempty, compact, connected set. Hence, if $\tau^+ \in \mathbb{R}$, for every $t > \tau^+$ the section S_t is not a continuum. In fact, each point (α, y_2) coming from a solution $x \in X^+$ necessarily has $\alpha > 1$. Moreover all points $(\gamma, \tilde{y}_2)$ deriving from solutions $x \in X^-$ must have $\gamma < 1$, since otherwise by virtue of Corollary 3.4 (b) problem (6.1) with $c = c^*$ also admits a solution satisfying $x(+\infty) = 1$, while we have just assumed $x(+\infty) \neq 1$ for every solution x of problem (6.1) with $c = c^*$. Hence S_t does not contain points $(1, y)$ for any $y \in \mathbb{R}$ and this implies S_t is not a continuum.

Therefore, $\tau^+ = +\infty$ and by means of an analogous argument we can show that also $\tau^- = +\infty$. Let us take now a diverging sequence $(t_n)_n$ and a sequence of solutions $(x_n)_n$ of problem (6.1) with $c = c^*$, such that $x_n(t_n) = 1$ for every $n \in \mathbb{N}$. By the relative compactness of the sets of solutions, we deduce the existence of a subsequence, again denoted $(x_n)_n$, converging to a function x in $C^1([0, +\infty[)$. Hence, also x is a solution of (6.1) with $c = c^*$, and it satisfies $x(+\infty) = 1$.

Thus, we have proved the existence of a solution x of problem (1.1), for $c = c^*$, satisfying $x(0) = \theta$.

Now, for every $\tau \in \mathbb{R}$, let us consider the functions

$$\tilde{\beta}(c, t, x) := \beta(c, t + \tau, x)\,, \quad \tilde{g}(t, x) := g(t + \tau, x).$$

As it is easy to verify, $\tilde{\beta}$ and $\tilde{g}$ satisfy all the assumptions on β and g, i.e. conditions (1.2), (1.4), (1.6) and (1.7). Hence, a value $c_\tau \in J$ exists such that

$$\begin{gathered} x'' - \tilde{\beta}(c_\tau, t, x) + \tilde{g}(t, x) = 0 \\ x(-\infty) = 0,\ x(+\infty) = 1 \end{gathered}$$

admits a solution $\tilde{x}$ with $\tilde{x}(0) = \theta$. Therefore, the shifted function $x_\tau(t) := \tilde{x}(t - \tau)$ is a solution of problem (1.1), with the same c_τ, satisfying $x_\tau(\tau) = \theta$.

Under the additional monotonicity conditions (1.8), now it remains to prove the uniqueness of c_τ for any given real τ. To this aim we reason by contradiction and we assume that for a fixed $\tau \in \mathbb{R}$ there exist two parameters $c_1 < c_2$ in J as well as two corresponding solutions $x_1(t)$ and $x_2(t)$ of problem (1.1) satisfying $x_1(\tau) = x_2(\tau) = \theta$. Since the functions $\tilde{x}_i(t)$ with $i = 1, 2$, defined as before, are solutions of the same boundary value problem with $\tilde{\beta}$ and $\tilde{g}$ satisfying all the required conditions and $\tilde{x}_i(0) = \theta$, we can apply Lemma 2.1 and Corollary 3.4 in order to obtain $x_i'(t) > 0$ for all $t \in \mathbb{R}$ such that $x_i(t) < 1$ and $i = 1, 2$. Hence the inverse functions $t_i :]0, 1[\to \mathbb{R}$ $i = 1, 2$ exist and satisfy

$$\lim_{x \to 0^+} t_i(x) = -\infty, \quad \lim_{x \to 1^-} t_i(x) = T_i \in]0, +\infty]$$

with $x_i(T_i) = 1$. We put now, for $i = 1, 2$ and $x \in [0, 1]$,

$$\beta_i(x) := \beta(c_i, t_i(x), x) \quad g_i(x) := g(t_i(x), x) \quad \text{and} \quad z_i(x) := x_i'(t_i(x)).$$

It is easy to see that z_i satisfies

$$\dot{z} = \beta_i(x) - \frac{g_i(x)}{z}, \quad x \in]0, 1[\quad \big(\dot{} = \frac{d}{dx} \big). \tag{6.3}$$

Moreover, according to Lemma 3.1, we have $x_i'(\pm\infty) = 0$, implying $z_i(0) = z_i(1) = 0$, $i = 1, 2$. As a consequence of Theorem 2.4 and condition (1.8(i)) one has $z_2(\theta) - z_1(\theta) = \lambda(c_2) - \lambda(c_1) \geq 0$ and

$$\dot{z}_2(\theta) - \dot{z}_1(\theta) = \beta(c_2, \tau, \theta) - \beta(c_1, \tau, \theta) - \frac{g(\tau, \theta)}{\lambda(c_2)} + \frac{g(\tau, \theta)}{\lambda(c_1)} > 0.$$

Moreover, since $z_2(x) > z_1(x)$ we have

$$t_2(x) - \tau = \int_\theta^x t_2'(\xi)\, d\xi = \int_\theta^x \frac{d\xi}{z_2(\xi)} < \int_\theta^x \frac{d\xi}{z_1(\xi)} = \int_\theta^x t_1'(\xi)\, d\xi = t_1(x) - \tau,$$

hence, according to (1.2), (1.8(ii)) and (1.8(iii)) we get

$$\dot{z}_2(x) = \beta(c_2, t_2(x), x) - \frac{g(t_2(x), x)}{z_2(x)} \geq \beta(c_1, t_1(x), x) - \frac{g(t_1(x), x)}{z_1(x)} = \dot{z}_1(x).$$

Consequently, $z_1(x) > z_2(x)$ for all $x \in]\theta, 1]$ in contradiction with $z_1(1) = z_2(1) = 0$. □

Proof of Theorem 1.2. Take $\tau_1 < \tau_2$ and assume there are two solutions x_1 and x_2 of problem (1.1) corresponding to the same parameter c and such that $x_1(\tau_1) = x_2(\tau_2) = \theta$. Reasoning as in the proof of Theorem 1.1, we introduce $t_i(x)$, with $t_i(\theta) = \tau_i$, $\beta_i(x)$ with $c_1 = c_2 = c$, $g_i(x)$ and $z_i(x)$ for $i = 1, 2$. We recall that $z_i(x)$ satisfies (6.3) on $]0, 1[$ as well as $z_i(0) = z_i(1) = 0$. Moreover, by conditions (1.3) we also have

$$\dot{z}_i(x) = \beta_i(x) \quad \text{for all } x \in]0, \theta[.$$

First we show that $z_1(\theta) < z_2(\theta)$ leads to a contradiction. Indeed, since $\tau_1 < \tau_2$ and according to the strict monotonicity of β with respect to t, we obtain $\dot{z}_1(\theta^-) = \beta(c, \tau_1, \theta) > \beta(c, \tau_2, \theta) = \dot{z}_2(\theta^-)$. Moreover, assuming $\dot{z}_1(x) > \dot{z}_2(x)$ for all $x \in]\bar{x}, \theta[$ with $0 < \bar{x} < \theta$, we get $z_1(x) < z_2(x)$ in the same interval and

$$\tau_1 - t_1(x) = \int_x^\theta t_1'(\xi)\, d\xi = \int_x^\theta \frac{d\xi}{z_1(\xi)} > \int_x^\theta \frac{d\xi}{z_2(\xi)} = \int_x^\theta t_2'(\xi)\, d\xi = \tau_2 - t_2(x)$$

implying $t_2(x) > t_1(x)$ for all $x \in [\bar{x}, \theta]$. Therefore $\dot{z}_1(\bar{x}) = \beta(c, t_1(\bar{x}), \bar{x}) > \beta(c, t_2(\bar{x}), \bar{x}) = \dot{z}_2(\bar{x})$. Hence $\dot{z}_1(x) > \dot{z}_2(x)$ for all $x \in]0, \theta[$ but this is in contradiction with $z_1(0) = z_2(0) = 0$. We have then proven that $z_1(\theta) \geq z_2(\theta)$.

Consequently we have $\dot{z}_1(\theta^+) - \dot{z}_2(\theta^+) > 0$, and assuming $\dot{z}_1(x) - \dot{z}_2(x) > 0$ for all $x \in [\theta, \bar{x}[$, with $\theta < \bar{x} < 1$, we can reason as before and obtain $t_1(x) < t_2(x)$ in $[\theta, \bar{x}[$. Proceeding as in the proof of Theorem 1.1 this leads to a contradiction with $z_1(1) = z_2(1) = 0$ and we have proven that $\tau \to c_\tau$ is an injective function. Now it remains to show that it is also continuous.

First notice that $c_\tau > \inf J$ for all $\tau \in \mathbb{R}$. In fact, consider again the functions $\tilde{\beta}$, $\tilde{g}$ and $\tilde{x}$ introduced in the proof of Theorem 1.1. Since also $\tilde{\beta}$ and $\tilde{g}$ respectively satisfy conditions (1.2), (1.3) and (1.4), then Theorem 2.4 implies $\theta m_{c_\tau} \le \tilde{x}'(0) = x'(\tau) = \lambda(c_\tau) \le \theta M_{c_\tau}$. Moreover, according to Theorems 4.1 and 4.2, we get

$$M_{c_\tau} > \sqrt{2\int_\theta^1 g_1(s)\,ds}, \quad m_{c_\tau} < \frac{\sqrt{2\int_\theta^1 g_2(s)\,ds}}{\theta}.$$

Consequently, conditions (1.6), (1.7) and the monotonicity of both m_c and M_c imply that the image set $C = \{c_\tau : \tau \in \mathbb{R}\}$ is bounded with $\inf C > \inf J$.

Given $\tau_0 \in \mathbb{R}$, suppose now the existence of $\tau_n \to \tau_0$ as $n \to +\infty$ and such that $c_{\tau_n} \not\to c_{\tau_0}$. Since c_{τ_n} is bounded, with no loss of generality we can assume that $c_{\tau_n} \to \bar{c} \in J$ with $\bar{c} \neq c_{\tau_0}$. Let $(x_n)_n$ be a sequence of solutions of (1.1) with $c = c_{\tau_n}$ and $x_n(\tau_n) = \theta$. For all $n \in \mathbb{N}$, define $\tilde{x}_n(t) := x_n(t + \tau_n)$. As it is easy to see, each $\tilde{x}_n$ is a solution of the problem

$$\begin{gathered} x'' - \beta(c_n, t + \tau_n, x)x' + g(t + \tau_n, x) = 0 \\ x(-\infty) = 0, \quad x(+\infty) = 1, \quad x(0) = \theta. \end{gathered}$$

Moreover, by Theorems 2.4 and 4.2 which are valid also when $\beta(c, t, x)$ and $g(t, x)$ are respectively replaced by $\beta(c, t + \tau_n, x)$ and $g(t + \tau_n, x)$, we have

$$0 < \tilde{x}_n'(0) < \sqrt{2\int_\theta^1 g_2(s)\,ds}.$$

Since, in addition, $c_{\tau_n} > \inf J$ and c_{τ_n} is bounded, we can reason as in the proof of Proposition 5.1 in order to obtain that $(\tilde{x}_n)_n$ is relatively compact in the Fréchet space $C^1([0, +\infty[)$.

It is then possible to extract a subsequence, again denoted $(\tilde{x}_n)_n$, which converges to $\tilde{x}$. According to the continuity of the function λ, as shown in Theorem 2.4, and since $\tau_n \to \tau_0$ when $n \to +\infty$, then $\tilde{x}(t)$ is a solution of

$$\begin{gathered} x'' - \beta(\bar{c}, t + \tau_0, x)x' + g(t + \tau_0, x) = 0 \\ x(0) = \theta, \quad x'(0) = \lambda(\bar{c}). \end{gathered}$$

Moreover $0 \le \tilde{x}(t) \le 1$ and $\tilde{x}'(t) \ge 0$ for all $t \in \mathbb{R}$. Therefore by Lemma 3.1 and condition (1.3), we obtain $\tilde{x}(+\infty) = 1$. In addition, by the definition of λ, we also have $\tilde{x}(-\infty) = 0$. Hence the associated function $x(t) := \tilde{x}(t - \tau_0)$ is a solution of problem (1.1) with $c = \bar{c}$ and $x(\tau_0) = \theta$ in contradiction with the uniqueness of c_{τ_0}. □

Proof of Corollary 1.3. By virtue of what observed in Introduction, we have only to prove the uniqueness of the solution of (1.1) for $c = c^*$, up to a time-shift. Take in fact $x_1(t)$ and $x_2(t)$ satisfying (1.1), with $c = c^*$, as well as $x_1(\tau) = x_2(\tau) = \theta$. Then $x_i'(t) > 0$ for all $t \in \mathbb{R}$ such that $x_i(t) < 1$ and $i = 1, 2$ (see Lemma 2.1 and Corollary 3.4). Therefore, reasoning as in the proof of Theorem 1.1 we can introduce two functions $z_i(x) = (x_i'(t_i(x)))$, where $t_i(x)$ is the inverse function of x_i for $i = 1, 2$, and they are both solutions of the problem

$$\begin{gathered} \dot{z} = \beta(c^*, x) - \frac{g(x)}{z} \\ z(0) = z(1) = 0. \end{gathered}$$

Moreover $z_1(\bar{x}) < z_2(\bar{x})$ for some $\bar{x} \in]0,1[$ implies $\dot{z}_1(\bar{x}) = \beta(c^*, \bar{x}) - \frac{g(\bar{x})}{z_1(\bar{x})} < \beta(c^*, \bar{x}) - \frac{g(\bar{x})}{z_2(\bar{x})} = \dot{z}_2(\bar{x})$ and this yields to the contradictory conclusion $0 = z_2(1) - z_1(1) > 0$. Therefore $z_1(x) = z_2(x)$ for all $x \in [0,1]$. Hence

$$\begin{aligned} t_1(x) - \tau &= \int_\theta^x t_1'(\xi)\, d\xi = \int_\theta^x \frac{d\xi}{x_1'(t_1(\xi))} = \int_\theta^x \frac{d\xi}{z_1(\xi)} \\ &= \int_\theta^x \frac{d\xi}{z_2(\xi)} = \int_\theta^x \frac{d\xi}{x_2'(t_2(\xi))} = \int_\theta^x t_2'(\xi)\, d\xi = t_2(x) - \tau \end{aligned}$$

and $x_1(t) = x_2(t)$ for all $t \in \mathbb{R}$. □

Proof of Theorem 1.4. First of all, observe that if problem (1.1) is solvable for some c, then $c \geq -h(0)$. In fact, if $c + h(0) < 0$, then $c + h(x) < 0$ for every positive x sufficiently small. Therefore, there exists a value $\bar{t}$ such that $x''(t) < 0$ for every $t < \bar{t}$, and this is a contradiction being $x(-\infty) = 0$.

Since the problem is autonomous, in order to show the existence of a unique $\lambda(c)$, we do not need any monotonicity assumption on h. In fact, being $\dot{z}(x) = c + h(x) - \frac{g(x)}{z(x)}$, the slope $\lambda(c)$ can be computed explicitly: $\lambda(c) = c\theta + \int_0^\theta h(s)ds$. Hence, by (1.11) we have

$$\lambda(c) \geq \int_0^\theta h(s)ds - \theta h(0) \geq \sqrt{2\int_\theta^1 g_2(s)ds} \quad \text{for every } c \geq -h(0)$$

and the assertion is an immediate consequence of Theorem 4.2. □

Proof of Corollary 1.5. First observe that we used the monotonicity assumption (1.4) on $[0,\theta]$ only in the analysis for negative times, in order to prove Theorem 2.4. But if the problem is autonomous, as we just noted in the previous proof, the slope $\lambda(c)$ can be computed explicitly and it trivially satisfies all the properties proved in Theorem 2.4. So, we can avoid the monotonicity assumption on function $h(x)$.

Now, note that all the other assumptions of Theorem 1.1 are satisfied taking $J =]-m, +\infty[$ (see Introduction); hence a $c^* > -m$ exists for which (1.1) is solvable. Since $M_{c^*} = c^* + M$, by Theorems 4.1 and 4.2 we deduce that

$$-(c^* + M)(1-\theta) + \sqrt{2\int_\theta^1 g_1(s)ds} < \lambda(c^*) < \sqrt{2\int_\theta^1 g_2(s)ds}.$$

But in this case we get $\lambda(c^*) = c^*\theta + \int_0^\theta h(s)ds$, hence the assertion immediately follows. □

We conclude this paper with an application of our results to the case when $\beta(c,t,x) = c + kx$, with $k > 0$, and g is a suitable autonomous function. Such a situation occurs when studying the existence of travelling wavefronts for equation (1.9), with $\frac{\partial H}{\partial u} = ku$, i.e. with a linear convective speed.

Example 6.1 Let $h(x) = kx$, with $k > 0$ constant, and let

$$g(t,x) = g(x) = \begin{cases} 0 & \text{for } 0 \leq x \leq \frac{1}{2} \\ -2x^2 + 3x - 1 & \text{for } \frac{1}{2} \leq x \leq 1. \end{cases}$$

Note that function g satisfies (1.3) for $\theta = 1/2$. Moreover, $\int_{1/2}^{1} g(s)ds = 1/24$. Condition (1.11) becomes $k/8 \geq 1/(2\sqrt{3})$. Hence, by applying Theorem 1.4, we deduce that if $k \geq 4/\sqrt{3}$ the problem

$$\begin{gathered} x'' - (c + kx)x' + g(x) = 0 \\ x(-\infty) = 0, \quad x(+\infty) = 1, \quad 0 \leq x(t) \leq 1 \end{gathered} \tag{6.4}$$

has no solutions for any $c \in \mathbb{R}$. Instead, condition (1.10) becomes $k < 1/(2\sqrt{3})$ and in this case problem (6.4) is solvable for $c = c^*$ with

$$\frac{1}{2\sqrt{3}} - \frac{5}{8}k < c^* < \frac{1}{\sqrt{3}} - \frac{k}{4}.$$

Put $K := \{k \in \mathbb{R} : \text{ problem (6.4) is solvable}\}$. The continuous dependence for the differential equation in (6.4) on the parameters c and k implies that K is an open set. Moreover by classical comparison-type techniques, as employed in [8] and [9], applied to the associated first order problem

$$\begin{gathered} \dot{z} = c + kx - \frac{g(x)}{z} \\ z(0^+) = z(1^-) = 0 \end{gathered}$$

one can show that K is a connected set. Consequently, there exists a threshold value k^*, with $1/(2\sqrt{3}) < k^* \leq 4/\sqrt{3}$, such that (6.4) is solvable if and only if $k < k^*$.

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