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RESEARCH

Lipschitz Regularity for a Priori Bounded Minimizers of Integral Functionals with Nonstandard Growth

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Abstract
We establish the Lipschitz regularity of the a priori bounded local minimizers of integral functionals with non autonomous energy densities satisfying non standard growth conditions under a bound on the gap between the growth and the ellipticity exponent that is reminiscent of the sharp bound already found in [16].

Keywords Non-standard growth · Non-autonomous functional · Bounded minimizers.

Mathematics Subject Classification 35B45, 35B65, 35J60, 49J40, 49N60

1 Introduction

Since the pioneering papers by P. Marcellini [27, 28], the Lipschitz regularity for minimizers of integral functionals with non-standard growth and for weak solutions for the associated Dirichlet problem to the elliptic system has attracted a lot of attention (see e.g. [2, 5, 13, 14, 17–20, 30, 31]).

One of the main motivations comes from the applications, for instance to the theory of elasticity for strongly anisotropic materials (see Zhikov [34], and also [35]); to this aim, in recent years the integral of the Calculus of Variations

$$\int_\Omega |Du|^p + a(x)|Du|^q \, dx, \tag{1.1}$$

where the function $a = a(x)$ is Hölder continuous with exponent α and where $1 < p < q$, has been widely investigated from the point of view of the regularity of local minimizers. In particular M. Colombo and G. Mingione, ([8]), studied the regularity of minimizers for

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integrals of the type (1.1) under the sharp gap

$$\frac{q}{p} < 1 + \frac{\alpha}{n}. \tag{1.2}$$

On the other hand, M. Eleuteri, P. Marcellini and E. Mascolo ([17]) investigated more general integrals of the Calculus of Variations of the type

$$F(u) = \int_\Omega g(x, |Du|)\, dx \tag{1.3}$$

where

$$g(x, |Du|) = |Du|^p + a(x)\,|Du|^q \tag{1.4}$$

is just a model example, without therefore assuming the precise structure condition for the integrand as in (1.1); they proved the local Lipschitz continuity of the local minimizers and to the solutions to the corresponding elliptic systems assuming a $\mathrm{W}^{1,r}$ regularity on the coefficients and under the gap

$$\frac{q}{p} < 1 + \frac{1}{n} - \frac{1}{r}. \tag{1.5}$$

In the model case (1.4), the above condition Eq. 1.5 is equivalent to (1.2) by the Sobolev embedding with

$$\alpha = 1 - \frac{n}{r}. \tag{1.6}$$

On the other hand, it is well known that, when dealing with a priori bounded minimizers of functionals with non standard growth, the regularity can be obtained under a bound on the gap independent of the dimension n ([1, 3, 6, 7, 12, 21, 22, 26]), see also [29] in the case of functionals with quasi isotropic $(p, q)-$growth. In particular, for the double phase functional, in [9], the authors were able to prove that the a priori bounded local minimizers of integral functionals of kind Eq. 1.1 are $\mathcal{C}^{1,\beta}$-regular provided the sharp bound

$$q \le p + \alpha \tag{1.7}$$

holds.

It is natural to ask if the same phenomenon persists when the Lipschitz regularity of more general functionals of kind Eq. 1.3 is investigated under an analogous a priori sharp bound on the gap between the exponents p and q. The main motivation comes from the fact that there are several interesting examples of functionals with non-standard growth and with Uhlenbeck structure that are not covered by the double-phase functional Eq. 1.1 or Orlicz-type functionals such as $g(t) = t^p \log(1+t)$; for instance we refer to Remark 3.3 in [4] where an example of an integrand function exhibiting $p, q-$growth but not satisfying a Δ_2-condition is presented.

Our paper aims to answer this open question, by studying the local Lipschitz continuity of the a priori bounded solutions to a class of variational problems of the form

$$\min_{z \in \mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)} \int_\Omega F(x, Dz)\, dx, \tag{1.8}$$

where Ω is a bounded open set of $\mathbb{R}^n$, $n \ge 2$.
We shall consider Carathéodory integrands F such that $\xi \mapsto F(x,\xi)$ is $\mathcal{C}^2$ and there exists $f : \Omega \times \mathbb{R}^{nN} \mapsto [0, +\infty)$ such that $F(x,\xi) = f(x, |\xi|)$. Such an assumption simplifies the approximation procedure that, even in the scalar case, can be quite involved (see for instance [18]).

We shall assume the following set of conditions:

$$\ell(1+|\xi|^2)^{\frac{p}{2}} \le F(x,\xi) \le L(1+|\xi|^2)^{\frac{q}{2}} \tag{F1}$$

$$\nu(1+|\xi|^2)^{\frac{p-2}{2}}|\lambda|^2 \le \sum_{i,\ell,\alpha,\beta} F_{\xi_i^\alpha \xi_\ell^\beta}(x,\xi)\lambda_i^\alpha \lambda_\ell^\beta \tag{F2}$$

$$|F_{\xi_i^\alpha \xi_\ell^\beta}(x,\xi)| \le \tilde{L}(1+|\xi|^2)^{\frac{q-2}{2}} \tag{F3}$$

$$|F_{x\xi}(x,\xi)| \le h(x)(1+|\xi|^2)^{\frac{q-1}{2}} \tag{F4}$$

for almost all $x \in \Omega$, and all $\xi, \lambda \in \mathbb{R}^{nN}$, $\xi = \xi_i^\alpha$, $\lambda = \lambda_\ell^\beta$, $i, \ell = 1, \dots, n$, $\alpha, \beta = 1, \dots, N$, where $2 \le p \le q$ and $0 \le \nu \le \tilde{L}$ are fixed constants, and $h(x) \in \mathrm{L}^r_{\mathrm{loc}}(\Omega)$ is a fixed non negative function.

Before stating our main result, we recall the definition of local minimizer

Definition 1.1 A mapping $u \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ is a local minimizer of the integral functional (1.8) if $F(x, Du) \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ and

$$\int_{\mathrm{supp}\varphi} F(x, Du)\,dx \le \int_{\mathrm{supp}\varphi} F(x, Du + D\varphi)\,dx \tag{1.9}$$

for any $\varphi \in C_0^\infty(\Omega, \mathbb{R}^N)$.

The main result reads as follows.

Theorem 1.2 *Let* $u \in \mathrm{L}^\infty_{\mathrm{loc}}(\Omega; \mathbb{R}^N) \cap \mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$ *be a local minimizer of the functional Eq. 1.8 under the assumptions* (F1)–(F4). *Assume moreover that*

$$r > \max\{n, p+2\} \tag{1.10}$$

and

$$q < p + 1 - \max\left\{\frac{n}{r}, \frac{p+2}{r}\right\}. \tag{1.11}$$

Then u is locally Lipschitz continuous and the following estimate holds for any ball $B_{R_0} \Subset \Omega$

$$||Du||_{\mathrm{L}^\infty\left(B_{\frac{R_0}{2}};\mathbb{R}^{nN}\right)} \le C\left(1 + ||u||^\infty_{\mathrm{L}}(B_{R_0}; \mathbb{R}^N)\right)^{\hat{\chi}},$$

with $C \equiv C(n, N, \nu, \tilde{L}, ||h||_{L^r(\Omega)}, R_0)$ *and with a positive exponent* $\hat{\chi} = \hat{\chi}(p, q, r, n)$.

We observe that condition (1.11) not only reduces to (1.7) under the Sobolev embedding with Eq. 1.6 for $p < n - 2$, but also includes the case $p < n < p + 2$: indeed the a priori higher integrability L^{p+2} reveals to be crucial in order to weaken the assumption on the coefficients in the non-autonomous case.

The proof of this result goes along several steps. The first step is devoted to the construction of the approximating problems in Sect. 3.2 based on the approximation lemma stated in Sect. 3.1; the main feature here is that the approximating local minimizers have norm in a suitable Lebesgue space which is uniformly bounded by the L^∞ norm of the local minimizer u. This procedure, inspired by [6] and already used in a similar form in [23], is one of the main and delicate points of our arguments. Indeed, in the general vectorial setting, the a priori boundedness of the minimizer of the original functional does not imply the boundedness of the approximating minimizers. However, this construction complicates the form of the integrand function of the approximating functionals and, despite they satisfy standard growth conditions with respect to the gradient variable, the growth with respect to the u variable in our energy density yields the necessity to establish the Lipschitz regularity of the approximating minimizers in Sect. 3.3; the proof of this result relies on a classical Moser iteration argument and makes use of a preliminary higher differentiability and higher integrability result proven in [23]. The next step aims to prove, in Sect. 3.4, a second order Caccioppoli type inequality for the approximating minimizers; the main point here is that we are going to establish it with constants independent of the approximation parameters. In a further step, in Sect. 4, by using a Gagliardo-Nirenberg type inequality ([6]), we establish a uniform higher integrability result for the approximating minimizers, with constants independent of the parameter of the approximation. Finally we are ready to prove in Sect. 5 the main result of the paper, that will be divided in two steps. In the first one we establish an uniform a priori estimate for the L^∞ norm of the gradient of the minimizers of the approximating functionals while, in the second, we show that these estimates are preserved in passing to the limit.

We conclude by mentioning that, as a consequence of the Lipschitz regularity of the local minimizers, we are also able to obtain a second order regularity result. More precisely, we have the following:

Theorem 1.3 *Let $u \in \mathrm{L}^\infty_{\mathrm{loc}}(\Omega;\mathbb{R}^N) \cap \mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ be a local minimizer of the functional Eq. 1.8 under the assumptions* (F1)–(F4). *Assume moreover that Eq. 1.10 and (1.11) are in force. Then $u \in W^{2,2}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ and the following estimate holds for any ball $B_{R_0} \Subset \Omega$*

$$||D^2u||_{\mathrm{L}^2\left(B_{\frac{R_0}{2}};\mathbb{R}^{n^2N}\right)} \le C\left(1 + ||u||_{\mathrm{L}^\infty(B_{R_0};\mathbb{R}^N)}\right)^{\hat{\chi}}$$

with $C \equiv C(n, N, \nu, \tilde{L}, ||h||_{L^r(\Omega)}, R_0)$ and with a positive exponent $\hat{\chi} = \hat{\chi}(p, q, r, n)$.

2 Preliminary

In what follows, we shall denote by C a general positive constant that may vary on different occasions, even within the same line of estimates. Relevant dependencies will be suitably emphasized using parentheses or subscripts. The symbol $B(x,r) = B_r(x) = \{y \in \mathbb{R}^n : |y - x| < r\}$ will denote the ball centered at x of radius r.

We recall the following well known iteration lemma, whose proof can be found, e.g. in [24, Lemma 6.1, p.191].

Lemma 2.1 *For $0 < R_1 < R_2$, consider a bounded function $f : [R_1, R_2] \to [0,\infty)$ with*

$$f(r_1) \le \vartheta f(r_2) + \frac{A}{(r_2 - r_1)^\alpha} + \frac{B}{(r_2 - r_1)^\beta} + C \qquad \textit{for all } R_1 < r_1 < r_2 < R_2,$$

where A, B, C, and α, β denote nonnegative constants and $\vartheta \in (0, 1)$. Then we have

$$f(R_1) \le c(\alpha,\vartheta)\left(\frac{A}{(R_2 - R_1)^\alpha} + \frac{B}{(R_2 - R_1)^\beta} + C\right).$$

3 The approximation

3.1 An Approximation Lemma

In this subsection we will state a Lemma that will be the main tool in the approximation procedure. For the proof we refer to Proposition 4.1 in [11] (see also [10, Lemma 4.1] and the recent [15, Theorem 5.1]).

Lemma 3.1 *Let* $F : \Omega\times\mathbb{R}^{nN} \to [0,+\infty)$ *be a Carathéodory function satisfying assumptions* (F1)–(F4). *Then there exists a sequence of Carathéodory functions* $F^j : \Omega \times \mathbb{R}^{nN} \to [0,+\infty)$, *monotonically convergent to* F, *such that the following properties hold for a.e.* $x \in \Omega$ *and for every* $\xi \in \mathbb{R}^{nN}$*:*

$$F^j(x,\xi) \le F^{j+1}(x,\xi) \le F(x,\xi) \qquad \forall j \in \mathbb{N} \tag{3.1}$$

$$\begin{cases} K_0(|\xi|^p - 1) \le F^j(x,\xi) \le L(1+|\xi|)^q \\ F^j(x,\xi) \le K_1(j)(1+|\xi|)^p, \end{cases} \tag{3.2}$$

with positive constants $K_0 = K_0(\ell)$ *and* $K_1(j)$. *In addition for every* $\xi \in \mathbb{R}^{nN}$, *there hold*

$$F^j(x,\xi) = \tilde{F}^j(x,|\xi|), \qquad t \mapsto \tilde{F}^j(x,t) \quad \textit{nondecreasing}, \tag{3.3}$$

$$\sum_{i,\ell,\alpha,\beta} F^j_{\xi_i^\alpha \xi_\ell^\beta}(x,\xi)\lambda_i^\alpha \lambda_\ell^\beta \ge \overline{\nu}(1+|\xi|^2)^{\frac{p-2}{2}}|\lambda|^2 \qquad \forall \lambda, \xi \in \mathbb{R}^{nN}, \tag{3.4}$$

with $\overline{\nu} = \overline{\nu}(\nu, p) > 0$. *We also have*

$$\begin{cases} |F^j_{\xi\xi}(x,\xi)| \le C(j)(1+|\xi|^2)^{\frac{p-2}{2}} \\ |F^j_{\xi\xi}(x,\xi)| \le C(\tilde{L})(1+|\xi|^2)^{\frac{q-2}{2}}. \end{cases} \tag{3.5}$$

Moreover, the vector field $x \mapsto F^j_\xi(x,\xi)$ *is weakly differentiable and, for every* $\xi \in \mathbb{R}^{nN}$,

$$\begin{cases} |F^j_{x\xi}(x,\xi)| \le C(j)h(x)(1+|\xi|^2)^{\frac{p-1}{2}} \\ |F^j_{x\xi}(x,\xi)| \le Ch(x)(1+|\xi|^2)^{\frac{q-1}{2}}. \end{cases} \tag{3.6}$$

3.2 The approximating problems

Here we present the construction of the approximating problems that is inspired by the one in [6] and whose main feature is that the sequence of the approximating minimizers has norm in a suitable Lebesgue space uniformly bounded by the L^∞ norm of the minimizer u.

Fix a compact set $\Omega' \Subset \Omega$ and a real number $a \ge ||u||_{L^\infty(\Omega';\mathbb{R}^N)}$. For $m \in \mathbb{N}$, let $u_j \in W^{1,p}(\Omega';\mathbb{R}^N) \cap L^{2m}(\Omega';\mathbb{R}^N)$ be a minimizer to the functional

$$\mathfrak{F}^j(v,\Omega') = \int_{\Omega'} \Big(F^j(x,Dv) + \big(|v|^2 - a^2\big)_+^m\Big)\,dx \tag{3.7}$$

under the boundary condition

$$u_j = u \qquad \text{on} \qquad \partial\Omega',$$

and where F^j is the sequence of functions obtained applying Lemma 3.1 to the integrand F of the functional at (1.8).

The existence of u_j is easily established by the direct methods of the Calculus of Variation. We shall need the following

Lemma 3.2 *As $j \to +\infty$, we have that*

$$\int_{\Omega'} \Big((|u_j| - a)_+^{2m} \Big)\, dx \to 0, \qquad \int_{\Omega'} F^j(x, Du_j)\, dx \to \int_{\Omega'} F(x, Du)\, dx.$$

and

$$Du_j \to Du \quad \textit{strongly in} \quad \mathrm{L}^p(\Omega'; \mathbb{R}^{nN}).$$

Proof By the minimality of u_j, using u as test function in the minimality inequality at (1.9), we get

$$\int_{\Omega'} \Big(F^j(x, Du_j) + \big(|u_j|^2 - a^2\big)_+^m \Big)\, dx \le \int_{\Omega'} F^j(x, Du)\, dx, \tag{3.8}$$

since $|u| \le ||u||_{\mathrm{L}^\infty(\Omega';\mathbb{R}^N)} \le a$ a.e. in Ω'. Then, by virtue of the first inequality in (3.2), we have that

$$\begin{aligned} K_0 \int_{\Omega'} (|Du_j|^p - 1)\, dx &\le \int_{\Omega'} \Big(F^j(x, Du_j) + \big(|u_j|^2 - a^2\big)_+^m \Big)\, dx \\ &\le \int_{\Omega'} F^j(x, Du)\, dx \le \int_{\Omega'} F(x, Du)\, dx, \end{aligned} \tag{3.9}$$

where the last inequality is due to the monotonicity of the sequence F^j given by (3.1). Hence the sequence $(Du_j)_j$ is bounded in $\mathrm{L}^p(\Omega'; \mathbb{R}^{nN})$ and there exists $w \in \mathrm{W}^{1,p}(\Omega'; \mathbb{R}^N)$ such that

$$u_j \rightharpoonup w \qquad \text{weakly in} \quad \mathrm{W}^{1,p}(\Omega'; \mathbb{R}^N) \quad \text{as} \quad j \to +\infty.$$

Passing to the limit as $j \to +\infty$ in (3.8), using the last inequality in (3.9), we also have that

$$\limsup_{j \to +\infty} \int_{\Omega'} \Big(F^j(x, Du_j) + \big(|u_j|^2 - a^2\big)_+^m \Big)\, dx \le \int_{\Omega'} F(x, Du)\, dx. \tag{3.10}$$

On the other hand, for every fixed $j_0 \in \mathbb{N}$, the convexity of $\xi \to F^{j_0}(x, \xi)$, by lower semicontinuity, implies

$$\begin{aligned} \int_{\Omega'} F^{j_0}(x, Dw)\, dx &\le \liminf_{j \to +\infty} \int_{\Omega'} F^{j_0}(x, Du_j)\, dx \\ &\le \liminf_{j \to +\infty} \int_{\Omega'} F^j(x, Du_j)\, dx \\ &\le \liminf_{j \to +\infty} \int_{\Omega'} \Big(F^j(x, Du_j) + \big(|u_j|^2 - a^2\big)_+^m \Big)\, dx \\ &\le \int_{\Omega'} F(x, Du)\, dx, \end{aligned} \tag{3.11}$$

where we used again the monotonocity of the sequence F^j and (3.10). Taking the limit as $j_0 \to \infty$ in the previous estimate, using the monotone convergence Theorem, we obtain

$$\begin{aligned} \int_{\Omega'} F(x, Dw)\, dx &= \liminf_{j_0 \to +\infty} \int_{\Omega'} F^{j_0}(x, Dw)\, dx \\ &\le \int_{\Omega'} F(x, Du)\, dx \le \int_{\Omega'} F(x, Dw)\, dx, \end{aligned} \tag{3.12}$$

by the minimality of u and since $w = u$ on $\partial\Omega'$. This, by the strict convexity of F, yields that $w \equiv u$ in Ω'. Hence, we conclude that

$$u_j \rightharpoonup u \qquad \text{weakly in} \quad \mathrm{W}^{1,p}(\Omega'; \mathbb{R}^N).$$

Using (3.12) in (3.11), we have in particular that

$$\lim_{j\to+\infty} \int_{\Omega'} \big(|u_j|^2 - a^2\big)_+^m \, dx = 0 \tag{3.13}$$

which in turn implies

$$\sup_{j\in\mathbb{N}} \int_{\Omega'} |u_j|^{2m} \, dx \le 2^m (1 + |\Omega'| a^{2m}) \tag{3.14}$$

and also

$$\lim_{j\to+\infty} \int_{\Omega'} F^j(x, Du_j) \, dx = \int_{\Omega'} F(x, Du) \, dx, \tag{3.15}$$

i.e. the first conclusion of the Lemma. We also record that, by virtue of (3.4), we have

$$\begin{aligned} &\bar{\nu} \int_{\Omega'} (1 + |Du|^2 + |Du_j|^2)^{\frac{p-2}{2}} |Du - Du_j|^2 \, dx \\ &\le \int_{\Omega'} \Big(F^j(x, Du) - F^j(x, Du_j) + \langle D_\xi F^j(x, Du_j), Du_j - Du \rangle \Big) \, dx. \end{aligned}$$

Since the Euler Lagrange system of the functional $\mathfrak{F}^j$ reads as

$$\int_{\Omega'} \langle D_\xi F^j(x, Du_j), D\varphi \rangle \, dx + 2m \int_{\Omega'} (|u_j|^2 - a^2)^{m-1} u_j \cdot \varphi \, dx = 0$$

for all $\varphi = (\varphi^\alpha)_{\alpha=1,\dots,N} \in \mathrm{C}_0^1(\Omega', \mathbb{R}^N)$, testing it with $\varphi = u - u_j$, which is legitimate by density, we get

$$\begin{aligned} &\bar{\nu} \int_{\Omega'} (1 + |Du|^2 + |Du_j|^2)^{\frac{p-2}{2}} |Du - Du_j|^2 \, dx \\ &\le \int_{\Omega'} \Big(F^j(x, Du) - F^j(x, Du_j) + \langle D_\xi F^j(x, Du_j), Du_j - Du \rangle \Big) \, dx \\ &= \int_{\Omega'} \Big(F^j(x, Du) - F^j(x, Du_j) \Big) \, dx - 2m \int_{\Omega'} (|u_j|^2 - a^2)^{m-1} u_j (u - u_j) \, dx \\ &\le \int_{\Omega'} \Big(F(x, Du) - F^j(x, Du_j) \Big) \, dx - 2m \int_{\Omega'} (|u_j|^2 - a^2)^{m-1} u_j (u - u_j) \, dx. \end{aligned}$$

Therefore, by (3.13), (3.14) and (3.15), taking the limit as $j \to +\infty$ in previous inequality, we conclude that

$$\limsup_{j\to+\infty} \int_{\Omega'} (1 + |Du|^2 + |Du_j|^2)^{\frac{p-2}{2}} |Du - Du_j|^2 \, dx = 0$$

that is

$$u_j \to u \qquad \text{strongly in} \quad \mathrm{W}^{1,p}(\Omega'; \mathbb{R}^N) \tag{3.16}$$

which concludes the proof. □

The main tool in the proof of our main result is the following Gagliardo–Nirenberg type inequality that we state as a lemma and whose proof can be found in the Appendix A of [6] (see also [23]).

Lemma 3.3 *For $\eta \in C^1_c(\Omega')$ with $\eta \geq 0$ and C^2 maps $v\colon \Omega' \to \mathbb{R}^N$ we have*

$$\int_{\Omega'} \eta^2 |Dv|^{\frac{m}{m+1}(p+2)}\, dx \leq (p+2)^2 \left(\int_{\Omega'} \eta^2 |v|^{2m}\, dx\right)^{\frac{1}{m+1}}$$
$$\times \left[\left(\int_{\Omega'} \eta^2 |D\eta|^2 |Dv|^p\, dx\right)^{\frac{m}{m+1}} + nN\left(\int_{\Omega'} \eta^2 |Dv|^{p-2}|D^2 v|^2\, dx\right)^{\frac{m}{m+1}}\right]$$

where $p \in (1, \infty)$ and $m > 1$.

We conclude this subsection with a preliminary higher differentiability and a higher integrability result, that will be useful in the sequel.

Theorem 3.4 *Let $u_j \in W^{1,p}(\Omega'; \mathbb{R}^N) \cap L^{2m}(\Omega'; \mathbb{R}^N)$ be a local minimizer of $\mathfrak{F}^j(u, \Omega')$. Then*

$$(1 + |Du_j|^2)^{\frac{p-2}{4}} |Du_j| \in W^{1,2}_{\mathrm{loc}}(\Omega') \quad \textit{and} \qquad |Du_j| \in L^{\frac{m}{m+1}(p+2)}_{\mathrm{loc}}(\Omega').$$

For the proof we refer to [23].

3.3 The Lipschitz continuity of the approximating minimizers

Here, we establish the Lipschitz regularity of the approximating minimizers. Even tough such regularity is well known for minimizers of integral functionals satisfying standard growth conditions with respect to the gradient variable, the growth with respect to the u variable in our energy density doesn't seem to fit with the available literature. The proof, however, relies on the very classical Moser iteration argument. More precisely, we have the following

Theorem 3.5 *Let $u_j \in W^{1,p}(\Omega'; \mathbb{R}^N) \cap L^{2m}(\Omega'; \mathbb{R}^N)$ be a local minimizer of the functional (3.7). Then $u_j \in W^{1,\infty}_{\mathrm{loc}}(\Omega'; \mathbb{R}^N)$ with the estimate*

$$||Du_j||_{L^\infty(B_R;\mathbb{R}^N)} \leq M_j$$

for every ball $B_R \Subset \Omega'$ with a constant M_j depending on j.

Proof Testing the Euler–Lagrange system of the functional $\mathfrak{F}^j(v, \Omega)$ with the function $\psi^\alpha = D_{x_s}\varphi^\alpha$ with $s \in \{1, \dots, n\}$, $\alpha \in \{1, \dots, N\}$ we get

$$0 = \int_{\Omega'} \Big\langle \sum_{i,\alpha} F^j_{\xi^\alpha_i}(x, Du_j), D_{x_i x_s}\varphi^\alpha \Big\rangle dx$$
$$+2m \int_{\Omega'} \sum_{\alpha} \big(|u_j|^2 - a^2\big)_+^{m-1} u_j{}^\alpha \cdot D_{x_s}\varphi^\alpha\, dx,$$

for every $\varphi \in C^1_0(\Omega'; \mathbb{R}^N)$. By Theorem 3.4, we have that $u_j \in W^{2,2}_{\mathrm{loc}}(\Omega'; \mathbb{R}^N)$, therefore integrating by parts the integrals in previous identity, we get

$$\int_{\Omega'} \left(\sum_{i,\ell,\alpha,\beta} F^j_{\xi^\alpha_i \xi^\beta_\ell}(x, Du_j) D_{x_\ell x_s}(u_j{}^\beta)\varphi^\alpha_{x_i}\, dx + \sum_{i,\alpha} F^j_{\xi^\alpha_i x_s}(x, Du_j)\varphi^\alpha_{x_i}\right) dx$$
$$+2m \int_{\Omega'} \sum_{\alpha} D_{x_s}\left(\big(|u_j|^2 - a^2\big)_+^{m-1} u_j{}^\alpha\right)\varphi^\alpha\, dx = 0, \tag{3.17}$$

holds for all $s = 1, \ldots, n$ and for all $\varphi \in C_0^1(\Omega'; \mathbb{R}^N)$. For $\eta \in C_0^1(\Omega')$ and $\gamma \geq 0$, by density we can test (3.17) with the function $\varphi^\alpha = \eta^2 \left(\mathcal{D}_k u_j\right)^\gamma D_{x_s}(u_j{}^\alpha)$, where we used the notation

$$\mathcal{D}_k u_j := \left(1 + \min\left\{|Du_j|^2, k^2\right\}\right)^{\frac{1}{2}} \tag{3.18}$$

One can easily check that

$$\begin{aligned}\varphi^\alpha_{x_i} &= 2\eta\eta_{x_i}\left(\mathcal{D}_k u_j\right)^\gamma D_{x_s}(u_j{}^\alpha)\\ &\quad +\eta^2\gamma\left(\mathcal{D}_k u_j\right)^{\gamma-2}\chi_{\{|Du_j|\leq k\}}|Du_j|D_{x_i}(|Du_j|)D_{x_s}(u_j^\alpha)\\ &\quad +\eta^2\left(\mathcal{D}_k u_j\right)^\gamma D_{x_s x_i}(u_j{}^\alpha).\end{aligned}$$

Inserting in (3.17) we get:

$$\begin{aligned}0 &= 2\int_\Omega \eta\left(\mathcal{D}_k u_j\right)^\gamma \sum_{i,\ell,\alpha,\beta} F^j_{\xi_i^\alpha \xi_\ell^\beta}(x, Du_j) D_{x_\ell x_s}(u_j{}^\beta)\eta_{x_i} D_{x_s}(u_j{}^\alpha)\, dx\\ &\quad + \int_\Omega \eta^2\left(\mathcal{D}_k u_j\right)^\gamma \sum_{i,\ell,\alpha,\beta} F^j_{\xi_i^\alpha \xi_\ell^\beta}(x, Du_j) D_{x_\ell x_s}(u_j{}^\beta) D_{x_s x_i}(u_j{}^\alpha)\, dx\\ &\quad +\gamma\int_\Omega \eta^2\left(\mathcal{D}_k u_j\right)^{\gamma-2}\chi_{\{|Du_j|\leq k\}} \sum_{i,\ell,\alpha,\beta} F^j_{\xi_i^\alpha \xi_\ell^\beta}(x, Du_j) D_{x_\ell x_s}(u_j{}^\beta)\\ &\quad\quad \cdot|Du_j|D_{x_i}(|Du_j|)D_{x_s}(u_j{}^\alpha)\, dx\\ &\quad +2\int_\Omega \eta\left(\mathcal{D}_k u_j\right)^\gamma \sum_{i,\alpha} F^j_{\xi_i x_s}(x, Du_j)\eta_{x_i} D_{x_s}(u_j{}^\alpha)\, dx\\ &\quad + \int_\Omega \eta^2\left(\mathcal{D}_k u_j\right)^\gamma \sum_{i,\alpha} F^j_{\xi_i^\alpha x_s}(x, Du_j) D_{x_s x_i}(u_j{}^\alpha)\, dx\\ &\quad +\gamma\int_\Omega \eta^2\left(\mathcal{D}_k u_j\right)^{\gamma-2}\chi_{\{|Du_j|\leq k\}} \sum_{i,\alpha} F^j_{\xi_i^\alpha x_s}(x, Du_j)\\ &\quad\quad \cdot|Du_j|D_{x_i}(|Du_j|)D_{x_s}(u_j{}^\alpha)\, dx\\ &\quad +2m\int_{\Omega'} \eta^2\left(\mathcal{D}_k u_j\right)^\gamma \sum_\alpha D_{x_s}\left(\left(|u_j|^2 - a^2\right)_+^{m-1} u_j{}^\alpha\right)\cdot D_{x_s}(u_j{}^\alpha)\, dx\\ &=: I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7.\end{aligned}$$

Now we sum all the terms in the previous equation with respect to s from 1 to n, and we still denote for simplicity by $I_1 - I_7$ the corresponding integrals. Previous equality yields

$$I_2 + I_3 + I_7 \leq |I_1| + |I_4| + |I_5| + |I_6|. \tag{3.19}$$

Let us estimate the term I_3. First of all, we have that

$$F^j_{\xi_i^\alpha \xi_\ell^\beta}(x, \xi) = \left(\frac{\tilde{F}^j_{tt}(x, |\xi|)}{|\xi|^2} - \frac{\tilde{F}^j_t(x, |\xi|)}{|\xi|^3}\right)\xi_i^\alpha \xi_\ell^\beta + \frac{\tilde{F}^j_t(x, |\xi|)}{|\xi|}\delta_{\xi_i^\alpha \xi_\ell^\beta},$$

where $\tilde{F}^j$ is given by (3.3) of Lemma 3.1. Therefore

$$\sum_{i,\ell,\alpha,\beta,s} F^j_{\xi_i^\alpha \xi_\ell^\beta}(x, Du_j) D_{x_s}(u_j^\alpha) D_{x_\ell x_s}(u_j^\beta) D_{x_i}(|Du_j|)|Du_j|$$

$$= \left(\frac{\tilde{F}^j_{tt}(x,|Du_j|)}{|Du_j|^2} - \frac{\tilde{F}^j_t(x,|Du_j|)}{|Du_j|^3}\right) \sum_{i,\ell,\alpha,\beta,s} D_{x_i}(u_j^\alpha) D_{x_\ell}(u_j^\beta) D_{x_s}(u_j^\alpha) D_{x_\ell x_s}(u_j^\beta) D_{x_i}(|Du_j|)|Du_j|$$

$$+ \frac{\tilde{F}^j_t(x,|Du_j|)}{|Du_j|} \sum_{i,\alpha,s} D_{x_s}(u_j^\alpha) D_{x_s x_i}(u_j^\alpha) D_{x_i}(|Du_j|)|Du_j|$$

$$= \left(\frac{\tilde{F}^j_{tt}(x,|Du_j|)}{|Du_j|^2} - \frac{\tilde{F}^j_t(x,|Du_j|)}{|Du_j|^3}\right) \sum_{i,\alpha,s} D_{x_i}(u_j^\alpha) D_{x_s}(u_j^\alpha) D_{x_i}(|Du_j|) D_{x_s}(|Du_j|)|Du_j|^2$$

$$+ \tilde{F}^j_t(x,|Du_j|)|Du_j| \sum_i [D_{x_i}(|Du_j|)]^2$$

$$= \left(\tilde{F}^j_{tt}(x,|Du_j|) - \frac{\tilde{F}^j_t(x,|Du_j|)}{|Du_j|}\right) \sum_\alpha \left[\sum_i D_{x_i}(u_j^\alpha) D_{x_i}(|Du_j|)\right]^2$$

$$+ \frac{\tilde{F}^j_t(x,|Du_j|)}{|Du_j|} |Du_j|^2 |D(|Du_j|)|^2$$

where we used the fact that

$$D_{x_s}(|Du_j|)|Du_j| = \sum_{\ell,\beta} D_{x_\ell x_s}(u_j^\beta) D_{x_\ell}(u_j^\beta).$$

Now, by Cauchy-Schwarz' inequality, we have

$$\begin{aligned}\sum_\alpha \left[\sum_i D_{x_i}(u_j^\alpha) D_{x_i}(|Du_j|)\right]^2 &\le \sum_{i,\alpha} (D_{x_i}(u_j^\alpha))^2 \sum_i (D_{x_i}(|Du_j|))^2 \\ &\le |Du_j|^2 \, |D(|Du_j|)|^2\end{aligned}$$

therefore, by using Kato's inequality

$$|D(|Du_j|)|^2 \le |D^2 u_j|, \tag{3.20}$$

we obtain that

$$I_3 = \gamma \int_\Omega \eta^2 \left(\mathcal{D}_k u_j\right)^{\gamma-2} \chi_{\{|Du_j| \le k\}} \sum_{i,\ell,\alpha,\beta,s} F^j_{\xi_i^\alpha \xi_\ell^\beta}(x, Du_j) D_{x_\ell x_s}(u_j{}^\beta)|Du_j| D_{x_i}(|Du_j|) D_{x_s}(u_j{}^\alpha)\, dx$$

$$\ge \gamma \int_\Omega \eta^2 \left(\mathcal{D}_k u_j\right)^{\gamma-2} \chi_{\{|Du_j| \le k\}} \tilde{F}^j_{tt}(x,|Du_j|) \sum_\alpha \left[\sum_i D_{x_i}(u_j^\alpha) D_{x_i}(|Du_j|)\right]^2 \ge 0.$$

A simple calculation shows that

$$\begin{aligned} I_7 &= 2m \int_{\Omega'} \eta^2 \left(\mathcal{D}_k u_j\right)^\gamma \sum_{\alpha,s} \left(|u_j|^2 - a^2\right)_+^{m-1} \cdot |D_{x_s}(u_j{}^\alpha)|^2\, dx \\ &\quad + 4m(m-1) \int_{\Omega'} \eta^2 \left(\mathcal{D}_k u_j\right)^\gamma \sum_{\alpha,s} \left(|u_j|^2 - a^2\right)_+^{m-2} |u_j{}^\alpha|^2 \cdot |D_{x_s}(u_j{}^\alpha)|^2\, dx \ge 0. \end{aligned}$$

Therefore, estimate (3.19) implies

$$I_2 \leq |I_1| + |I_4| + |I_5| + |I_6|. \tag{3.21}$$

By Cauchy-Schwarz' inequality, Young's inequality and the right inequality in assumption (F2), we have

$$\begin{aligned}
|I_1| &= 2\left|\int_\Omega \eta\left(\mathcal{D}_k u_j\right)^\gamma \sum_{i,\ell,\alpha,\beta,s} F^j_{\xi_i^\alpha \xi_\ell^\beta}(x, Du_j) D_{x_\ell x_s}(u_j{}^\beta)\eta_{x_i} D_{x_s}(u_j{}^\alpha)\, dx\right| \\
&\leq 2\int_\Omega \eta\left(\mathcal{D}_k u_j\right)^\gamma \left\{\sum_{i,\ell,\alpha,\beta,s} F^j_{\xi_i^\alpha \xi_\ell^\beta}(x, Du_j)\eta_{x_i}\eta_{x_j} D_{x_s}(u_j{}^\alpha) D_{x_s}(u_j{}^\beta)\right\}^{1/2} \\
&\quad \times \left\{\sum_{i,\ell,\alpha,\beta,s} F^j_{\xi_i^\alpha \xi_\ell^\beta}(x, Du_j) D_{x_s x_i}(u_j{}^\alpha)\, D_{x_s x_\ell}(u_j{}^\beta)\right\}^{1/2} dx \\
&\leq C\int_\Omega \left(\mathcal{D}_k u_j\right)^\gamma \sum_{i,\ell,\alpha,\beta,s} F^j_{\xi_i^\alpha \xi_\ell^\beta}(x, Du_j)\eta_{x_i}\eta_{x_j} D_{x_s}(u_j{}^\alpha) D_{x_s}(u_j{}^\beta)\, dx \\
&\quad + \frac{1}{2}\int_\Omega \eta^2\left(\mathcal{D}_k u_j\right)^\gamma \sum_{i,\ell,\alpha,\beta,s} F^j_{\xi_i^\alpha \xi_\ell^\beta}(x, Du_j) D_{x_s x_i}(u_j{}^\alpha)\, D_{x_s x_\ell}(u_j{}^\beta)\, dx \\
&\leq C(j)\int_\Omega |D\eta|^2 \left(\mathcal{D}_k u_j\right)^\gamma (1+|Du_j|^2)^{\frac{p}{2}}\, dx \\
&\quad + \frac{1}{2}\int_\Omega \eta^2\left(\mathcal{D}_k u_j\right)^\gamma \sum_{i,\ell,\alpha,\beta,s} F^j_{\xi_i^\alpha \xi_\ell^\beta}(x, Du_j) D_{x_s x_i}(u_j{}^\alpha)\, D_{x_s x_\ell}(u_j{}^\beta)\, dx,
\end{aligned} \tag{3.22}$$

where the last bound is due to the first inequality in (3.5). We can estimate I_4 and I_5 by Cauchy-Schwartz' inequality together with the first inequality at (3.6) and Young's inequality, as follows

$$\begin{aligned}
|I_4| &\leq 2\int_\Omega \eta\left(\mathcal{D}_k u_j\right)^\gamma \sum_{i,\alpha,s}\left|F^j_{\xi_i^\alpha x_s}(x, Du_j)\eta_{x_i} D_{x_s}(u_j{}^\alpha)\right|\, dx \\
&\leq C(j)\int_\Omega \eta h(x)\left(\mathcal{D}_k u_j\right)^\gamma (1+|Du_j|^2)^{\frac{p-1}{2}}\sum_{i,\alpha,s}|\eta_{x_i} D_{x_s}(u_j{}^\alpha)|\, dx \\
&\leq C(j)\int_\Omega \eta|D\eta| h(x)\left(\mathcal{D}_k u_j\right)^\gamma (1+|Du_j|^2)^{\frac{p}{2}}\, dx \\
&\leq C(j)\int_\Omega |D\eta|^2\left(\mathcal{D}_k u_j\right)^\gamma (1+|Du_j|^2)^{\frac{p}{2}}\, dx \\
&\quad + C(j)\int_\Omega \eta^2 h^2(x)\left(\mathcal{D}_k u_j\right)^\gamma (1+|Du_j|^2)^{\frac{p}{2}}\, dx.
\end{aligned} \tag{3.23}$$

Moreover

$$\begin{aligned}
|I_5| &= \left|\int_\Omega \eta^2\left(\mathcal{D}_k u_j\right)^\gamma \sum_{i,\alpha,s} F^j_{\xi_i^\alpha x_s}(x, Du_j) D_{x_s x_i}(u_j{}^\alpha)\, dx\right| \\
&\leq C(j)\int_\Omega \eta^2 h(x)\left(\mathcal{D}_k u_j\right)^\gamma (1+|Du_j|^2)^{\frac{p-1}{2}}\left|\sum_{i,\alpha,s} D_{x_s x_i}(u_j{}^\alpha)\right| dx \\
&\leq C(j)\int_\Omega \eta^2 h(x)\left(\mathcal{D}_k u_j\right)^\gamma (1+|Du_j|^2)^{\frac{p-1}{2}}|D^2 u_j|\, dx
\end{aligned}$$

$$= C(j) \int_\Omega \left[\eta^2 \left(\mathcal{D}_k u_j \right)^\gamma (1 + |Du_j|^2)^{\frac{p-2}{2}} |D^2 u_j|^2 \right]^{\frac{1}{2}} \left[\eta^2 h^2(x) \left(\mathcal{D}_k u_j \right)^\gamma (1 + |Du_j|^2)^{\frac{p}{2}} \right]^{\frac{1}{2}} dx$$

$$\leq \varepsilon \int_\Omega \eta^2 \left(\mathcal{D}_k u_j \right)^\gamma (1 + |Du_j|^2)^{\frac{p-2}{2}} |D^2 u_j|^2 \, dx \tag{3.24}$$

$$+ C_\varepsilon(j) \int_\Omega \eta^2 h^2(x) \left(\mathcal{D}_k u_j \right)^\gamma (1 + |Du_j|^2)^{\frac{p}{2}} \, dx,$$

where $\varepsilon > 0$ will be chosen later. Finally, similar arguments give

$$|I_6| = \gamma \left| \int_\Omega \eta^2 \chi_{\{|Du_j| \leq k\}} \left(\mathcal{D}_k u_j \right)^{\gamma - 2} |Du_j| \sum_{i,\alpha,s} F^j_{\xi_i^\alpha x_s}(x, Du_j) D_{x_i}(|Du_j|) D_{x_s}(u_j{}^\alpha) \, dx \right|$$

$$\leq \gamma \int_\Omega \eta^2 \chi_{\{|Du_j| \leq k\}} \left(\mathcal{D}_k u_j \right)^{\gamma - 1}$$

$$\cdot \sum_{i,\alpha,s} \left| F^j_{\xi_i^\alpha x_s}(x, Du_j) D_{x_i}(|Du_j|) D_{x_s}(u_j{}^\alpha) \right| dx$$

$$\leq C(j) \gamma \int_\Omega \eta^2 \chi_{\{|Du_j| \leq k\}} \left(\mathcal{D}_k u_j \right)^{p-2+\gamma} h(x) \sum_{i,\alpha,s}^n D_{x_i}(|Du_j|) D_{x_s}(u_j^\alpha) \, dx$$

$$\leq C(j) \gamma \int_\Omega \eta^2 \chi_{\{|Du_j| \leq k\}} \left(\mathcal{D}_k u_j \right)^{p-1+\gamma} |D^2 u_j| \, h(x) \, dx$$

$$\leq \varepsilon \int_\Omega \eta^2 \left(\mathcal{D}_k u_j \right)^\gamma |D^2 u_j|^2 (1 + |Du_j|^2)^{\frac{p-2}{2}} \, dx \tag{3.25}$$

$$+ C_\varepsilon(j) \gamma^2 \int_\Omega \eta^2 h^2(x) \left(\mathcal{D}_k u_j \right)^\gamma (1 + |Du_j|^2)^{\frac{p}{2}} \, dx,$$

where the constants C and $C_\varepsilon(j)$ are independent of γ and where in the third inequality we used the Cauchy-Schwarz' inequality and (3.20). Plugging (3.22), (3.23), (3.24), (3.25) in (3.21) we obtain

$$\int_\Omega \eta^2 \left(\mathcal{D}_k u_j \right)^\gamma \sum_{i,\ell,\alpha,\beta,s} F_{\xi_i^\alpha \xi_\ell^\beta}(x, Du_j) D_{x_\ell x_s}(u_j{}^\beta) D_{x_s x_i}(u_j{}^\alpha) \, dx$$

$$\leq \frac{1}{2} \int_\Omega \eta^2 \left(\mathcal{D}_k u_j \right)^\gamma \sum_{i,\ell,\alpha,\beta,s} F_{\xi_i^\alpha \xi_\ell^\beta}(x, Du_j) D_{x_\ell x_s}(u_j{}^\beta) D_{x_s x_i}(u_j{}^\alpha) \, dx$$

$$+ 2\varepsilon \int_\Omega \eta^2 \left(\mathcal{D}_k u_j \right)^\gamma (1 + |Du_j|^2)^{\frac{p-2}{2}} |D^2 u_j|^2 \, dx$$

$$+ C_\varepsilon(j)(1 + \gamma^2) \int_\Omega \eta^2 h^2(x) \left(\mathcal{D}_k u_j \right)^\gamma (1 + |Du_j|^2)^{\frac{p}{2}} \, dx$$

$$+ C(j) \int_\Omega |D\eta|^2 \left(\mathcal{D}_k u_j \right)^\gamma (1 + |Du_j|^2)^{\frac{p}{2}} \, dx.$$

Reabsorbing the first integral in the right hand side by the left hand side, we get

$$\frac{1}{2} \int_\Omega \eta^2 \left(\mathcal{D}_k u_j \right)^\gamma \sum_{i,\ell,\alpha,\beta,s} F_{\xi_i^\alpha \xi_\ell^\beta}(x, Du_j) D_{x_\ell x_s}(u_j{}^\beta) D_{x_s x_i}(u_j{}^\alpha) \, dx$$

$$\leq 2\varepsilon \int_\Omega \eta^2 \left(\mathcal{D}_k u_j \right)^\gamma (1 + |Du_j|^2)^{\frac{p-2}{2}} |D^2 u_j|^2 \, dx$$

$$+ C_\varepsilon(j)(1 + \gamma^2) \int_\Omega \eta^2 h^2(x) \left(\mathcal{D}_k u_j \right)^\gamma (1 + |Du_j|^2)^{\frac{p}{2}} \, dx$$

$$+C(j)\int_{\Omega}|D\eta|^2\,\left(\mathcal{D}_k u_j\right)^{\gamma}\,(1+|Du_j|^2)^{\frac{p}{2}}\,dx.$$

Using (3.4) in the left hand side of previous estimate, we obtain

$$\begin{aligned}
&\bar{\nu}\int_{\Omega}\eta^2\left(\mathcal{D}_k u_j\right)^{\gamma}\,(1+|Du_j|^2)^{\frac{p-2}{2}}|D^2u_j|^2\,dx\\
&\le 2\varepsilon\int_{\Omega}\eta^2\left(\mathcal{D}_k u_j\right)^{\gamma}\,(1+|Du_j|^2)^{\frac{p-2}{2}}|D^2u_j|^2\,dx\\
&\quad+C_{\varepsilon}(j)(1+\gamma^2)\int_{\Omega}\eta^2h^2(x)\left(\mathcal{D}_k u_j\right)^{\gamma}\,(1+|Du_j|^2)^{\frac{p}{2}}\,dx\\
&\quad+C(j)\int_{\Omega}|D\eta|^2\left(\mathcal{D}_k u_j\right)^{\gamma}\,(1+|Du_j|^2)^{\frac{p}{2}}\,dx.
\end{aligned}$$

Choosing $\varepsilon=\frac{\bar{\nu}}{4}$, we can reabsorb the first integral in the right hand side by the left hand side thus getting

$$\begin{aligned}
&\int_{\Omega}\eta^2\left(\mathcal{D}_k u_j\right)^{\gamma}\,(1+|Du_j|^2)^{\frac{p-2}{2}}|D^2u_j|^2\,dx\\
&\le C(j)(1+\gamma^2)\int_{\Omega}\eta^2h^2(x)\left(\mathcal{D}_k u_j\right)^{\gamma}\,(1+|Du_j|^2)^{\frac{p}{2}}\,dx\\
&\quad+C(j)\int_{\Omega}|D\eta|^2\,\left(\mathcal{D}_k u_j\right)^{\gamma}\,(1+|Du_j|^2)^{\frac{p}{2}}\,dx,
\end{aligned}\tag{3.26}$$

with a constant C dependent on j but independent of m and γ. Let $0<r<R$, with $B_R\Subset\Omega'$ and fix $\eta\in C_0^1(B_R)$ such that $\eta=1$ on B_r and $|D\eta|\le\frac{C}{R-r}$ so that (3.26) implies

$$\begin{aligned}
&\int_{B_R}\eta^2\left(\mathcal{D}_k u_j\right)^{\gamma}\,(1+|Du_j|^2)^{\frac{p-2}{2}}|D^2u_j|^2\,dx\\
&\le C(j)(1+\gamma^2)\int_{B_R}h^2(x)\left(\mathcal{D}_k u_j\right)^{\gamma}\,(1+|Du_j|^2)^{\frac{p}{2}}\,dx\\
&\quad+\frac{C(j)}{(R-r)^2}\int_{B_R}\left(\mathcal{D}_k u_j\right)^{\gamma}\,(1+|Du_j|^2)^{\frac{p}{2}}\,dx.
\end{aligned}$$

The higher integrability result of Theorem 3.4, recalling the assumption $h\in L^r_{\mathrm{loc}}(\Omega)$ and choosing γ such that $(p+\gamma)\frac{r}{r-2}<\frac{m}{m+1}(p+2)$, i.e. $\gamma<\frac{r-2}{r}\frac{m}{m+1}(p+2)-p$, allows us to pass to the limit as $k\to+\infty$ in both sides of previous estimate thus getting

$$\begin{aligned}
&\int_{B_R}\eta^2(1+|Du_j|^2)^{\frac{p-2+\gamma}{2}}|D^2u_j|^2\,dx\\
&\le C(j)(1+\gamma^2)\int_{B_R}h^2(x)(1+|Du_j|^2)^{\frac{p+\gamma}{2}}\,dx\\
&\quad+\frac{C(j)}{(R-r)^2}\int_{B_R}(1+|Du_j|^2)^{\frac{p+\gamma}{2}}\,dx,
\end{aligned}$$

since the sequence $\mathcal{D}_k u_j$ converges monotonically to $(1+|Du_j|^2)^{\frac{1}{2}}$. Note that, by assumption (1.10), we may choose $m>\frac{rp}{2(r-p-2)}$ in order to have $\gamma>0$. Using the Sobolev inequality in the left hand side of the previous estimate, we get

$$\left(\int_{B_R}\eta^{2^*}(1+|Du_j|^2)^{\left(\frac{p+\gamma}{4}\right)2^*}\,dx\right)^{\frac{2}{2^*}}$$

$$\leq C(j)(1+\gamma^2)\int_{B_R}\eta^2 h^2(x)(1+|Du_j|^2)^{\frac{p+\gamma}{2}}\,dx$$
$$+\frac{C(j)}{(R-r)^2}\int_{\Omega'}(1+|Du_j|^2)^{\frac{p+\gamma}{2}}\,dx, \tag{3.27}$$

where we used the customary notation

$$2^* = \begin{cases} \frac{2n}{n-2} & \text{if } n>2 \\ \text{any finite exponent} & \text{if } n=2. \end{cases} \tag{3.28}$$

Since $h\in L^r_{\mathrm{loc}}(\Omega)$ with $r>n$, there exists $\vartheta\in(0,1)$ such that

$$\vartheta+\frac{2(1-\vartheta)}{2^*}+\frac{2}{r}=1 \iff \vartheta=1-\frac{2}{r}\frac{2^*}{2^*-2}$$

and therefore we use the interpolation inequality to estimate the first integral in the right hand side of (3.27), as follows

$$\int_{B_R}\eta^2 h^2(x)(1+|Du_j|^2)^{\frac{p+\gamma}{2}}\,dx$$
$$=\int_{B_R}\eta^2 h^2(x)(1+|Du_j|^2)^{\left(\frac{p+\gamma}{2}\right)\vartheta}(1+|Du_j|^2)^{\left(\frac{p+\gamma}{2}\right)(1-\vartheta)}\,dx$$
$$\leq\left(\int_{B_R}h^r\,dx\right)^{\frac{2}{r}}\left(\int_{B_R}\eta^2(1+|Du_j|^2)^{\frac{p+\gamma}{2}}\,dx\right)^{\vartheta}$$
$$\cdot\left(\int_{B_R}\eta^{2^*}(1+|Du_j|^2)^{(\frac{p+\gamma}{4})2^*}\,dx\right)^{\frac{2(1-\vartheta)}{2^*}}.$$

Inserting previous inequality in (3.27), by virtue of Young's inequality we obtain

$$\left(\int_{B_R}\eta^{2^*}(1+|Du_j|^2)^{\left(\frac{p+\gamma}{4}\right)2^*}\,dx\right)^{\frac{2}{2^*}}$$
$$\leq\frac{1}{2}\left(\int_{B_R}\eta^{2^*}(1+|Du_j|^2)^{\left(\frac{p+\gamma}{4}\right)2^*}\,dx\right)^{\frac{2}{2^*}}$$
$$+C(j,\vartheta)(1+\gamma^2)^{\frac{1}{\vartheta}}\left(\int_{B_R}h^r\,dx\right)^{\frac{2}{r\vartheta}}\int_{B_R}\eta^2(1+|Du_j|^2)^{\frac{p+\gamma}{2}}\,dx$$
$$+\frac{C(j)}{(R-r)^2}\int_{\Omega'}(1+|Du_j|^2)^{\frac{p+\gamma}{2}}\,dx.$$

Reabsorbing the first integral in the right hand side by the left hand side, we get

$$\left(\int_{B_R}\eta^{2^*}(1+|Du_j|^2)^{\left(\frac{p+\gamma}{4}\right)2^*}\,dx\right)^{\frac{2}{2^*}}$$
$$\leq C(j)(1+\gamma^2)^{\frac{1}{\vartheta}}\left(\int_{B_R}h^r\,dx\right)^{\frac{2}{r\vartheta}}\int_{B_R}\eta^2(1+|Du_j|^2)^{\frac{p+\gamma}{2}}\,dx$$
$$+\frac{C(j)}{(R-r)^2}\int_{\Omega'}(1+|Du_j|^2)^{\frac{p+\gamma}{2}}\,dx. \tag{3.29}$$

At this point it is quite standard to start the usual Moser iteration procedure to conclude with the desired Lipschitz continuity. □

3.4 A Caccioppoli type inequality

This subsection is devoted to the proof of a second order Caccioppoli type inequality for the approximating minimizers. It is worth mentioning that such inequality is available in [23] and it is also the first part of the proof of Theorem 3.5, but the main point here is that we are going to establish it with constants independent of the approximation parameters. More precisely, we have the following

Lemma 3.6 *Let $u_j \in \mathrm{W}^{1,p}(\Omega';\mathbb{R}^N) \cap \mathrm{L}^{2m}(\Omega';\mathbb{R}^N)$ be a local minimizer of the functional $\mathfrak{F}^j(u,\Omega')$. Then, the following second order Caccioppoli type inequality*

$$\begin{aligned}
&\int_{\Omega'} \eta^2 (1+|Du_j|^2)^{\frac{p-2+\gamma}{2}} |D^2 u_j|^2\, dx \\
&\le C(1+\gamma^2) \int_{\Omega'} \eta^2 h^2(x) (1+|Du_j|^2)^{\frac{2q-p+\gamma}{2}}\, dx \\
&\quad + C \int_{\Omega'} |D\eta|^2 \,(1+|Du_j|^2)^{\frac{q+\gamma}{2}}\, dx,
\end{aligned} \tag{3.30}$$

holds true for every $\gamma \ge 0$ and for every $\eta \in C^1_0(\Omega)$, with a constant C independent of j.

Proof For $\eta \in \mathcal{C}^1_0(\Omega')$ and $\gamma \ge 0$, by the Lipschitz regularity of u_j proven in Theorem 3.5 and the higher differentiability result of Theorem 3.4, we can test (3.17) with the function $\varphi^\alpha = \eta^2 \left(1+|Du_j|^2\right)^{\frac{\gamma}{2}} D_{x_s}(u_j{}^\alpha)$. Arguing exactly as done in Theorem 3.5 until inequality (3.21), we arrive at

$$|\tilde{I}_2| \le |\tilde{I}_1| + |\tilde{I}_4| + |\tilde{I}_5| + |\tilde{I}_6|, \tag{3.31}$$

where

$$\tilde{I}_2 = \int_\Omega \eta^2 \left(1+|Du_j|^2\right)^{\frac{\gamma}{2}} \sum_{i,\ell,\alpha,\beta} F^j_{\xi_i^\alpha \xi_\ell^\beta}(x,Du_j) D_{x_\ell x_s}(u_j{}^\beta) D_{x_s x_i}(u_j{}^\alpha)\, dx$$

$$\tilde{I}_1 = 2\int_\Omega \eta (1+|Du_j|^2)^{\frac{\gamma}{2}} \sum_{i,\ell,\alpha,\beta,s} F^j_{\xi_i^\alpha \xi_\ell^\beta}(x,Du_j) D_{x_\ell x_s}(u_j{}^\beta) \eta_{x_i} D_{x_s}(u_j{}^\alpha)\, dx$$

$$\tilde{I}_4 = 2\int_\Omega \eta (1+|Du_j|^2)^{\frac{\gamma}{2}} \sum_{i,\alpha,s} F^j_{\xi_i^\alpha x_s}(x,Du_j) \eta_{x_i} D_{x_s}(u_j{}^\alpha)\, dx$$

$$\tilde{I}_5 = \int_\Omega \eta^2 (1+|Du_j|^2)^{\frac{\gamma}{2}} \sum_{i,\alpha,s} F^j_{\xi_i^\alpha x_s}(x,Du_j) D_{x_s x_i}(u_j{}^\alpha)\, dx$$

and

$$\tilde{I}_6 = \gamma \int_\Omega (1+|Du_j|^2)^{\frac{\gamma}{2}-1} \sum_{i,\alpha,s} F^j_{\xi_i^\alpha x_s}(x,Du_j) \eta^2 |Du_j| D_{x_i}(|Du_j|) D_{x_s}(u_j{}^\alpha)\, dx.$$

By the Cauchy-Schwartz inequality, Young's inequality and the second inequality in (3.5), we have

$$|\tilde{I}_1| = 2\left| \int_\Omega \eta (1+|Du_j|^2)^{\frac{\gamma}{2}} \sum_{i,\ell,\alpha,\beta,s} F^j_{\xi_i^\alpha \xi_\ell^\beta}(x,Du_j) D_{x_\ell x_s}(u_j{}^\beta) \eta_{x_i} D_{x_s}(u_j{}^\alpha)\, dx \right|$$

$$
\begin{aligned}
&\leq 2\int_\Omega \eta(1+|Du_j|^2)^{\frac{\gamma}{2}}\left\{\sum_{i,\ell,\alpha,\beta,s} F^j_{\xi_i^\alpha\xi_\ell^\beta}(x,Du_j)\eta_{x_i}\eta_{x_j}D_{x_s}(u_j{}^\alpha)D_{x_s}(u_j{}^\beta)\right\}^{1/2}\\
&\quad\times\left\{\sum_{i,\ell,\alpha,\beta,s} F^j_{\xi_i^\alpha\xi_\ell^\beta}(x,Du_j)D_{x_sx_i}(u_j{}^\alpha)\,D_{x_sx_\ell}(u_j{}^\beta)\right\}^{1/2} dx\\
&\leq C\int_\Omega (1+|Du_j|^2)^{\frac{\gamma}{2}}\sum_{i,\ell,\alpha,\beta,s} F^j_{\xi_i^\alpha\xi_\ell^\beta}(x,Du_j)\eta_{x_i}\eta_{x_j}D_{x_s}(u_j{}^\alpha)D_{x_s}(u_j{}^\beta)\,dx\\
&\quad+\frac{1}{2}\int_\Omega \eta^2(1+|Du_j|^2)^{\frac{\gamma}{2}}\sum_{i,\ell,\alpha,\beta,s} F^j_{\xi_i^\alpha\xi_\ell^\beta}(x,Du_j)D_{x_sx_i}(u_j{}^\alpha)\,D_{x_sx_\ell}(u_j{}^\beta)\,dx\\
&\leq C(\tilde{L})\int_\Omega |D\eta|^2\,(1+|Du_j|^2)^{\frac{q+\gamma}{2}}\,dx\\
&+\frac{1}{2}\int_\Omega \eta^2(1+|Du_j|^2)^{\frac{\gamma}{2}}\sum_{i,\ell,\alpha,\beta,s} F^j_{\xi_i^\alpha\xi_\ell^\beta}(x,Du_j)D_{x_sx_i}(u_j{}^\alpha)\,D_{x_sx_\ell}(u_j{}^\beta)\,dx. \qquad (3.32)
\end{aligned}
$$

We can estimate the fourth and the fifth term by Cauchy-Schwartz' inequality together with the second inequality in (3.6), and Young's inequality, as follows

$$
\begin{aligned}
|\tilde{I}_4| &= \left|2\int_\Omega \eta(1+|Du_j|^2)^{\frac{\gamma}{2}}\sum_{i,\alpha,s} F^j_{\xi_i^\alpha x_s}(x,Du_j)\eta_{x_i}D_{x_s}(u_j{}^\alpha)\,dx\right|\\
&\leq C\int_\Omega \eta h(x)\,(1+|Du_j|^2)^{\frac{q-1+\gamma}{2}}\sum_{i,\alpha,s}|\eta_{x_i}D_{x_s}(u_j{}^\alpha)|\,dx\\
&\leq C\int_\Omega \eta|D\eta||Du_j|\,h(x)\,(1+|Du_j|^2)^{\frac{q-1+\gamma}{2}}\,dx\\
&\leq C\int_\Omega |D\eta|^2(1+|Du_j|^2)^{\frac{p+\gamma}{2}}\,dx \qquad (3.33)\\
&\quad+C\int_\Omega \eta^2h^2(x)(1+|Du_j|^2)^{\frac{2q-p+\gamma}{2}}\,dx.
\end{aligned}
$$

Moreover

$$
\begin{aligned}
|\tilde{I}_5| &= \left|\int_\Omega \eta^2(1+|Du_j|^2)^{\frac{\gamma}{2}}\sum_{i,\alpha,s} F^j_{\xi_i^\alpha x_s}(x,Du_j)D_{x_sx_i}(u_j{}^\alpha)\,dx\right|\\
&\leq C\int_\Omega \eta^2\,h(x)(1+|Du_j|^2)^{\frac{q-1+\gamma}{2}}\left|\sum_{i,\alpha,s} D_{x_sx_i}(u_j{}^\alpha)\right|dx\\
&\leq C\int_\Omega \eta^2h(x)(1+|Du_j|^2)^{\frac{q-1+\gamma}{2}}|D^2u_j|\,dx\\
&= C\int_\Omega \left[\eta^2(1+|Du_j|^2)^{\frac{p-2+\gamma}{2}}|D^2u_j|^2\right]^{\frac{1}{2}}\left[\eta^2(1+|Du_j|^2)^{\frac{2q-p+\gamma}{2}}h^2(x)\right]^{\frac{1}{2}}dx\\
&\leq \varepsilon\int_\Omega \eta^2(1+|Du_j|^2)^{\frac{p-2+\gamma}{2}}|D^2u_j|^2\,dx \qquad (3.34)\\
&\quad+C_\varepsilon\int_\Omega \eta^2h^2(x)(1+|Du_j|^2)^{\frac{2q-p+\gamma}{2}}\,dx,
\end{aligned}
$$

where $\varepsilon > 0$ will be chosen later. Finally, we have

$$
\begin{aligned}
|\tilde{I}_6| &= \gamma \left| \int_\Omega \sum_{i,\alpha,s} F^j_{\xi_i^\alpha x_s}(x, Du_j)\eta^2(1+|Du_j|^2)^{\frac{\gamma}{2}-1}|Du_j| D_{x_i}(|Du_j|) D_{x_s}({u_j}^\alpha)\, dx \right| \\
&\le \gamma \int_\Omega \eta^2(1+|Du_j|^2)^{\frac{\gamma-1}{2}} \\
&\qquad \cdot \sum_{i,\alpha,s} \left| F^j_{\xi_i^\alpha x_s}(x, Du_j) D_{x_i}(|Du_j|) D_{x_s}({u_j}^\alpha) \right| dx \\
&\le C\gamma \int_\Omega \eta^2\, h(x)\, (1+|Du_j|^2)^{\frac{q+\gamma-2}{2}} \sum_{i,\alpha,s}^{n} \left| D_{x_i}(|Du_j|) D_{x_s}(u_j^\alpha) \right| dx \\
&\le C\,\gamma \int_\Omega \eta^2\, h(x)\, (1+|Du_j|^2)^{\frac{q+\gamma}{2}} |D^2 u_j| dx \\
&\le \varepsilon \int_\Omega \eta^2 |D^2 u_j|^2 (1+|Du_j|^2)^{\frac{p-2+\gamma}{2}}\, dx \\
&\quad + C_\varepsilon \gamma^2 \int_\Omega \eta^2 h^2(x)(1+|Du_j|^2)^{\frac{2q-p+\gamma}{2}}\, dx,
\end{aligned} \tag{3.35}
$$

where all the constants C and C_ε are independent of γ, of j and m and where in the third inequality we used Cauchy-Schwarz' inequality and (3.20). Plugging (3.32), (3.33), (3.34), (3.35) in (3.31) we obtain

$$
\begin{aligned}
&\int_\Omega \eta^2(1+|Du_j|^2)^{\frac{\gamma}{2}} \sum_{i,\ell,\alpha,\beta,s} F_{\xi_i^\alpha \xi_\ell^\beta}(x, Du_j) D_{x_\ell x_s}({u_j}^\beta) D_{x_s x_i}({u_j}^\alpha)\, dx \\
&\le \frac{1}{2} \int_\Omega \eta^2(1+|Du_j|^2)^{\frac{\gamma}{2}} \sum_{i,\ell,\alpha,\beta,s} F_{\xi_i^\alpha \xi_\ell^\beta}(x, Du_j) D_{x_\ell x_s}({u_j}^\beta) D_{x_s x_i}({u_j}^\alpha)\, dx \\
&\quad + 2\varepsilon \int_\Omega \eta^2(1+|Du_j|^2)^{\frac{p-2+\gamma}{2}} |D^2 u_j|^2\, dx \\
&\quad + C_\varepsilon(1+\gamma^2) \int_\Omega \eta^2 h^2(x)(1+|Du_j|^2)^{\frac{2q-p+\gamma}{2}}\, dx \\
&\quad + C_\varepsilon \int_\Omega |D\eta|^2\, (1+|Du_j|^2)^{\frac{q+\gamma}{2}}\, dx.
\end{aligned}
$$

Reabsorbing the first integral in the right hand side by the left hand side we get

$$
\begin{aligned}
&\frac{1}{2} \int_\Omega \eta^2(1+|Du_j|^2)^{\frac{\gamma}{2}} \sum_{i,\ell,\alpha,\beta,s} F_{\xi_i^\alpha \xi_\ell^\beta}(x, Du_j) D_{x_\ell x_s}({u_j}^\beta) D_{x_s x_i}({u_j}^\alpha)\, dx \\
&\le 2\varepsilon \int_\Omega \eta^2(1+|Du_j|^2)^{\frac{p-2+\gamma}{2}} |D^2 u_j|^2\, dx \\
&\quad + C_\varepsilon(1+\gamma^2) \int_\Omega \eta^2 h^2(x)(1+|Du_j|^2)^{\frac{2q-p+\gamma}{2}}\, dx \\
&\quad + C_\varepsilon \int_\Omega |D\eta|^2\, (1+|Du_j|^2)^{\frac{q+\gamma}{2}}\, dx.
\end{aligned}
$$

Using Eq. 3.4 in the left hand side of previous estimate, we obtain

$$
\bar{\nu} \int_\Omega \eta^2(1+|Du_j|^2)^{\frac{p-2+\gamma}{2}} |D^2 u_j|^2\, dx\, dx
$$

$$\leq 2\varepsilon \int_{\Omega} \eta^2 (1+|Du_j|^2)^{\frac{p-2+\gamma}{2}} |D^2 u_j|^2 \, dx$$
$$+C_\varepsilon (1+\gamma^2) \int_{\Omega} \eta^2 h^2(x) (1+|Du_j|^2)^{\frac{2q-p+\gamma}{2}} \, dx$$
$$+C_\varepsilon \int_{\Omega} |D\eta|^2 \, (1+|Du_j|^2)^{\frac{q+\gamma}{2}} \, dx.$$

Choosing $\varepsilon = \frac{\bar{\nu}}{4}$, we can reabsorb the first integral in the right hand side by the left hand side thus getting

$$\int_{\Omega} \eta^2 (1+|Du_j|^2)^{\frac{p-2+\gamma}{2}} |D^2 u_j|^2 \, dx \, dx$$
$$\leq C(1+\gamma^2) \int_{\Omega} \eta^2 h^2(x) (1+|Du_j|^2)^{\frac{2q-p+\gamma}{2}} \, dx$$
$$+C \int_{\Omega} |D\eta|^2 \, (1+|Du_j|^2)^{\frac{q+\gamma}{2}} \, dx,$$

with a constant $C = C(\nu, \tilde{L}, n, N, p, q)$ independent of γ, j and m. This concludes the proof. □

4 The higher integrability

Here, we establish an higher integrability result for the approximating minimizers with constants independent of the parameter of the approximation. This is the main point in achieving the proof of our main result.

Lemma 4.1 *Let $u_j \in \mathrm{L}^{2m}(\Omega'; \mathbb{R}^N) \cap \mathrm{W}^{1,p}(\Omega'; \mathbb{R}^N)$ be a local minimizer of the functional $\mathfrak{F}^j$ in (3.7). Setting*

$$\mathfrak{m}_r := \frac{2rm}{2m+r}.$$

Then

$$|Du_j| \in \mathrm{L}^{\mathfrak{m}_r(p-q+1)}_{\mathrm{loc}}(\Omega)$$

with the following estimate

$$\int_{B_\rho} |Du_j|^{\mathfrak{m}_r(p-q+1)} \, dx \leq \frac{C \Theta_R^{\frac{\mathfrak{m}_r}{2}}}{(R-\rho)^r} \left(\int_{B_R} |u_j|^{2m} \, dx \right)^{\frac{\mathfrak{m}_r}{2m}} + C|B_R|, \tag{4.1}$$

for every balls $B_\rho \subset B_R \Subset \Omega'$ with a constant C depending at most on $K_0, \overline{\nu}, n, N, p, q, r$ but independent of j and of m and where we set

$$\Theta_R = \|1+h\|^2_{\mathrm{L}^r(B_R)}.$$

Proof By Theorem 3.5, we have that $u_j \in \mathrm{W}^{1,\infty}_{\mathrm{loc}}(\Omega')$ and the Caccioppoli inequality at Lemma 3.6 yields that

$$(1+|Du_j|^2)^{\frac{p-2}{2}+\gamma} |D^2 u_j|^2 \in \mathrm{L}^1_{\mathrm{loc}}(\Omega'),$$

for every $\gamma > 0$. Hence, we are legitimate to apply Lemma 3.3 with p replaced by $p + 2\gamma$, thus getting

$$\begin{aligned}&\int_{\Omega'} \eta^2 |Du_j|^{\frac{m}{m+1}(p+2+2\gamma)}\,dx\\ &\le (p+2+2\gamma)^2 \left(\int_{\Omega'} \eta^2 |u_j|^{2m}\,dx\right)^{\frac{1}{m+1}} \left(\int_{\Omega'} \eta^2 |D\eta|^2 |Du_j|^{p+2\gamma}\,dx\right)^{\frac{m}{m+1}}\\ &+nN(p+2+2\gamma)^2 \left(\int_{\Omega'} \eta^2 |u_j|^{2m}\,dx\right)^{\frac{1}{m+1}} \left(\int_{\Omega'} \eta^2 |Du_j|^{p-2+2\gamma} |D^2u_j|^2\,dx\right)^{\frac{m}{m+1}},\end{aligned}$$

for every non negative $\eta \in C^1_0(\Omega')$ such that $0 \le \eta \le 1$. Using Eq. 3.30 to estimate the last integral in the right hand side of the previous inequality, we obtain

$$\begin{aligned}&\int_{\Omega'} \eta^2 |Du_j|^{\frac{m}{m+1}(p+2+2\gamma)}\,dx\\ &\le (p+2+2\gamma)^2 \left(\int_{\Omega'} \eta^2 |u_j|^{2m}\,dx\right)^{\frac{1}{m+1}} \left(\int_{\Omega'} \eta^2 |D\eta|^2 |Du_j|^{p+2\gamma}\,dx\right)^{\frac{m}{m+1}}\\ &C(p+2+2\gamma)^4 \left(\int_{\Omega'} \eta^2 |u_j|^{2m}\,dx\right)^{\frac{1}{m+1}} \left(\int_{\Omega'} \eta^2 h^2(x)(1+|Du_j|)^{2q-p+2\gamma}\,dx\right)^{\frac{m}{m+1}}\\ &+C(p+2+2\gamma)^2 \left(\int_{\Omega'} \eta^2 |u_j|^{2m}\,dx\right)^{\frac{1}{m+1}} \left(\int_{\Omega'} |D\eta|^2 (1+|Du_j|)^{q+2\gamma}\,dx\right)^{\frac{m}{m+1}},\\ &\le C(p+2+2\gamma)^4 \left(\int_{\Omega'} \eta^2 |u_j|^{2m}\,dx\right)^{\frac{1}{m+1}}\\ &\quad\cdot \left(\int_{\Omega'} \left(\eta^2 + |D\eta|^2\right)\left(1+h^2(x)\right)(1+|Du_j|)^{2q-p+2\gamma}\,dx\right)^{\frac{m}{m+1}}\end{aligned}$$

where we used that $1+\gamma \le p+2+2\gamma$ and that $p+2\gamma \le q+2\gamma \le 2q-p+2\gamma$. By virtue of the assumption on $h(x)$, we use Hölder's inequality in the right hand side of the previous estimate thus getting

$$\begin{aligned}&\int_{\Omega'} \eta^2 |Du_j|^{\frac{m}{m+1}(p+2+2\gamma)}\,dx\\ &\le C(p+2+2\gamma)^4 \left(\int_{\Omega'} \eta^2 |u_j|^{2m}\,dx\right)^{\frac{1}{m+1}} \left(\int_{\Omega'} \left(\eta^2 + |D\eta|^2\right)\left(1+h(x)\right)^r\,dx\right)^{\frac{2m}{r(m+1)}}\\ &\quad\cdot \left(\int_{\Omega'} \left(\eta^2 + |D\eta|^2\right)(1+|Du_j|)^{\frac{r(2q-p+2\gamma)}{r-2}}\,dx\right)^{\frac{m(r-2)}{r(m+1)}}\end{aligned} \tag{4.2}$$

Fix concentric balls $B_\rho \subset B_s \subset B_t \subset B_R \Subset \Omega'$ and let $\eta \in C^1_0(B_t)$ be a standard cut off function between B_s and B_t i.e. $0 \le \eta \le 1$, $\eta = 1$ on B_s and $|D\eta| \le \frac{c(n)}{t-s}$. Without loss of generality, we shall suppose that $|B_R| \le 1$.

With such a choice, estimate (4.2) yields

$$\begin{aligned}&\int_{B_s} |Du_j|^{\frac{m}{m+1}(p+2+2\gamma)}\,dx\\ &\le \frac{C\Theta_R^{\frac{m}{m+1}}}{(t-s)^{\frac{2m}{m+1}}}(p+2+2\gamma)^4 \left(\int_{B_t} |u_j|^{2m}\,dx\right)^{\frac{1}{m+1}}\end{aligned}$$

$$\cdot \left(\int_{B_t} (1+|Du_j|)^{\frac{r(2q-p+2\gamma)}{r-2}}\, dx \right)^{\frac{m(r-2)}{r(m+1)}} \tag{4.3}$$

Note that we used the following

$$\eta^2 + |D\eta|^2 \le 1 + \frac{c}{(t-s)^2} \le \frac{c'}{(t-s)^2},$$

since $t - s \le 1$.Choose now γ such that

$$\frac{r(2q-p+2\gamma)}{r-2} = \frac{m}{m+1}(p+2+2\gamma) \iff 2\gamma = \frac{2mr(p-q+1) - 2m(p+2) - r(2q-p)}{2m+r},$$

which yields

$$\frac{m}{m+1}(p+2+2\gamma) = \frac{2rm}{2m+r}(p-q+1) = \mathfrak{m}_r(p-q+1)$$

Note that by virtue of (1.11), we have $\gamma > 0$. Indeed

$$\begin{aligned} \gamma > 0 &\iff 2mr(p-q+1) - 2m(p+2) - r(2q-p) > 0 \\ &\iff 2m[r(p-q+1) - (p+2)] > (2q-p)r \end{aligned}$$

The last inequality can be satisfied for a suitable $m \in \mathbb{N}$ if

$$r(p-q+1) - (p+2) > 0 \iff q < p+1 - \frac{p+2}{r},$$

that holds true by virtue of assumption (1.11).

With this choice of γ, observing that

$$\frac{m+1}{r+2m} \le \frac{1}{2}, \qquad \forall m \in \mathbb{N} \tag{4.4}$$

we have that

$$p+2+2\gamma = \frac{2r(m+1)}{2m+r}(p-q+1) \le r(p-q+1),$$

and so estimate (4.3) becomes

$$\begin{aligned} &\int_{B_s} |Du_j|^{\mathfrak{m}_r(p-q+1)}\, dx \\ &\le \frac{C\Theta_R^{\frac{m}{m+1}}}{(t-s)^{\frac{2m}{m+1}}} \left(\int_{B_t} |u_j|^{2m}\, dx \right)^{\frac{1}{m+1}} \left(\int_{B_t} (1+|Du_j|)^{\mathfrak{m}_r(p-q+1)}\, dx \right)^{\frac{m(r-2)}{r(m+1)}} \end{aligned} \tag{4.5}$$

with $C = C(K_0, \overline{\nu}, n, N, p, q, r)$ independent of j and m. Using Young's inequality with exponents

$$\left(\frac{r(m+1)}{m(r-2)}; \frac{r(m+1)}{r+2m} \right)$$

in the right hand side of the previous inequality, we obtain

$$\begin{aligned} \int_{B_s} |Du_j|^{\mathfrak{m}_r(p-q+1)}\, dx &\le \frac{1}{2} \int_{B_t} (1+|Du_j|)^{\mathfrak{m}_r(p-q+1)}\, dx \\ &+ \frac{2^{\frac{m(r-2)}{r+2m}} C^{\frac{r(m+1)}{r+2m}} \Theta_R^{\frac{\mathfrak{m}_r}{2}}}{(t-s)^{\frac{2rm}{r+2m}}} \left(\int_{B_t} |u_j|^{2m}\, dx \right)^{\frac{r}{r+2m}} \end{aligned}$$

$$\leq \frac{1}{2}\int_{B_t}|Du_j|^{\mathfrak{m}_r(p-q+1)}\,dx+|B_R| \\ +\frac{C\Theta_R^{\frac{\mathfrak{m}_r}{2}}}{(t-s)^r}\left(\int_{B_R}|u_j|^{2m}\,dx\right)^{\frac{r}{r+2m}}, \tag{4.6}$$

where, in order to control the constants, we used the bound at (4.4), that $C \geq 1$ and that $R-\rho \leq 1$. Since the previous estimate holds true for every $\rho < s < t < R$, we can apply Lemma 2.1 thus obtaining

$$\int_{B_\rho}|Du_j|^{\mathfrak{m}_r(p-q+1)}\,dx \leq \frac{C\Theta_R^{\frac{\mathfrak{m}_r}{2}}}{(R-\rho)^r}\left(\int_{B_R}|u_j|^{2m}\,dx\right)^{\frac{r}{r+2m}}+c|B_R|,$$

i.e. the conclusion. □

Corollary 4.2 Let $u_j \in \mathrm{L}^{2m}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)\cap \mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ be a local minimizer of the functional $\mathfrak{F}^j$ in (3.7). Then

$$|Du_j| \in \mathrm{L}^{\mathfrak{m}_r(p-q+1)}_{\mathrm{loc}}(\Omega)$$

with the following estimate

$$\int_{B_\rho}|Du_j|^{\mathfrak{m}_r(p-q+1)}\,dx \leq \frac{C\Theta_R^{\frac{\mathfrak{m}_r}{2}}}{(R-\rho)^r}\left(1+a^{2m}\right)^{\frac{r}{r+2m}},$$

for every balls $B_\rho \subset B_R \Subset \Omega'$, with $R \leq 1$ and with a constant C depending at most on K_0, p, q, r but independent of j and of m.

Proof It suffices to use (3.14) in the right hand side of estimate (4.1). □

5 Proof of Theorems 1.2 and (1.3)

We are now in position to establish the proof of our main result, that will be divided in two steps. In the first one we establish an uniform a priori estimate for the L^∞ norm of the gradient of the minimizers of the approximating functionals while in the second we show that these estimates are preserved in passing to the limit.

Proof of Theorem 1.2 Let us fix a ball $B_{R_0} \Subset \Omega$ and radii $\frac{R_0}{2} < \bar{\rho} < \rho < t_1 < t_2 < R < \bar{R} < R_0 \leq 1$ that will be needed in the three iteration procedures, constituting the essential steps in our proof.

Step 1. The uniform a priori estimate. Let us choose $\eta \in C^1_0(B_{t_2})$ such that $\eta = 1$ on B_{t_1} and $|D\eta| \leq \frac{C}{t_2-t_1}$, so that (3.30) implies

$$\int_{B_{t_2}}\eta^2(1+|Du_j|^2)^{\frac{p-2}{2}+\gamma}|D^2u_j|^2\,dx \\ \leq C\frac{(1+\gamma^2)}{(t_2-t_1)^2}\int_{B_{t_2}}(1+h^2(x))(1+|Du_j|^2)^{\frac{2q-p}{2}+\gamma}\,dx.$$

Using the assumptions on $h(x)$ and Hölder's inequality, we arrive at

$$\int_{B_{t_2}} \eta^2 (1+|Du_j|^2)^{\frac{p-2}{2}+\gamma} |D^2 u_j|^2 \, dx \leq (1+\gamma^2) \frac{C\Theta}{(t_2-t_1)^2} \left[\int_{B_{t_2}} (1+|Du_j|^2)^{\frac{(2\gamma+2q-p)\mathfrak{r}}{2}} \, dx \right]^{\frac{1}{\mathfrak{r}}} \tag{5.1}$$

for any $0 < t_1 < t_2$, where the constant C is independent of γ, of m and of ε, where $\Theta = \Theta_{R_0}$, and where we set

$$\mathfrak{r} = \frac{r}{r-2}.$$

The Sobolev inequality yields

$$\left(\int_{B_{t_2}} \eta^{2^*} (1+|Du_j|^2)^{(\frac{p}{4}+\frac{\gamma}{2})2^*} \, dx \right)^{\frac{2}{2^*}} \leq C \int_{B_{t_2}} |D(\eta (1+|Du_j|^2)^{\frac{p}{4}+\frac{\gamma}{2}})|^2 \, dx$$
$$\leq C(1+\gamma^2) \int_{B_{t_2}} \eta^2 (1+|Du_j|^2)^{\frac{p-2}{2}+\gamma} |D^2 u_j|^2 \, dx + C \int_{B_{t_2}} |D\eta|^2 \, (1+|Du_j|^2)^{\frac{p}{2}+\gamma} \, dx,$$

where 2^* is the exponent defined at (3.28). Using estimate (5.1) to control the first integral in the right hand side of the previous inequality, we obtain

$$\left(\int_{B_{t_2}} \eta^{2^*} (1+|Du_j|^2)^{(\frac{p}{4}+\frac{\gamma}{2})2^*} \, dx \right)^{\frac{2}{2^*}} \leq C \frac{\Theta(1+\gamma^4)}{(t_2-t_1)^2} \left[\int_{B_{t_2}} (1+|Du_j|^2)^{\frac{(2\gamma+2q-p)\mathfrak{r}}{2}} \, dx \right]^{\frac{1}{\mathfrak{r}}} + \frac{C}{(t_2-t_1)^2} \int_{B_{t_2}} (1+|Du_j|^2)^{\frac{p}{2}+\gamma} \, dx$$
$$\leq C \frac{\Theta(1+\gamma^4)}{(t_2-t_1)^2} \left[\int_{B_{t_2}} (1+|Du_j|^2)^{\frac{(2\gamma+2q-p)\mathfrak{r}}{2}} \, dx \right]^{\frac{1}{\mathfrak{r}}}, \tag{5.2}$$

where we used that $p \leq 2q-p$ and that $L^{\mathfrak{r}} \hookrightarrow L^1$.

Now, setting

$$V(Du_j) = (1+|Du_j|^2)^{\frac{1}{2}},$$

we can write (5.2) as follows

$$\left(\int_{B_R} \eta^{2^*} V(Du_j)^{(p+2\gamma)\frac{2^*}{2}} \, dx \right)^{\frac{2}{2^*}} \leq C \frac{\Theta(1+\gamma^4)}{(R-\rho)^2} \left[\int_{B_R} V(Du_j)^{[2(q-p)\mathfrak{r}+(p+2\gamma)\mathfrak{r}]} \, dx \right]^{\frac{1}{\mathfrak{r}}}$$
$$\leq C \frac{\Theta(1+\gamma^4)}{(R-\rho)^2} ||V(Du_j)||^{2(q-p)}_{L^\infty(B_R)} \left[\int_{B_R} V(Du_j)^{(p+2\gamma)\mathfrak{r}} \, dx \right]^{\frac{1}{\mathfrak{r}}}$$

and so

$$\left(\int_{B_\rho} V(Du_j)^{[\mathfrak{r}(p+2\gamma)]\frac{2^*}{2\mathfrak{r}}} \, dx \right)^{\frac{2\mathfrak{r}}{2^*}}$$

$$\leq C\left[\frac{\Theta(1+\gamma^4)}{(R-\rho)^2}\right]^{\mathfrak{r}} ||V(Du_j)||^{2\mathfrak{r}(q-p)}_{L^\infty(B_R)} \int_{B_R} V(Du_j)^{(p+2\gamma)\mathfrak{r}}\, dx, \tag{5.3}$$

where we also used that $\eta = 1$ on B_ρ.

With $\frac{R_0}{2} \leq \bar{\rho} < \bar{R} \leq R_0$ fixed at the beginning of the section, we define the decreasing sequence of radii by setting

$$\rho_i = \bar{\rho} + \frac{\bar{R}-\bar{\rho}}{2^i}.$$

Let us also define the following increasing sequence of exponents

$$p_0 = p\mathfrak{r} \qquad\qquad p_i = p_{i-1}\frac{2^*}{2\mathfrak{r}} = p_0\left(\frac{2^*}{2\mathfrak{r}}\right)^i$$

Noticing that, since $u_j \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$, estimate (5.3) holds true for $\gamma = 0$ and for every $\bar{\rho} < \rho < R < \bar{R}$, we may iterate it on the concentric balls B_{ρ_i} with exponents p_i, thus obtaining

$$\begin{aligned}
&\left(\int_{B_{\rho_{i+1}}} V(Du_j)^{p_{i+1}}\, dx\right)^{\frac{1}{p_{i+1}}} \\
&\leq \prod_{h=0}^{i}\left(C\frac{\Theta^{\mathfrak{r}} p_h^{4\mathfrak{r}}}{(\rho_h-\rho_{h+1})^{2\mathfrak{r}}}||V(Du_j)||^{2\mathfrak{r}(q-p)}_{L^\infty(B_R)}\right)^{\frac{1}{p_h}} \left(\int_{B_{\rho_0}} V(Du_j)^{p_0}\, dx\right)^{\frac{1}{p_0}} \\
&= \prod_{h=0}^{i}\left(C\frac{4^{h+1}\Theta^{\mathfrak{r}} p_h^{4\mathfrak{r}}}{(\bar{R}-\bar{\rho})^{2\mathfrak{r}}}||V(Du_j)||^{2\mathfrak{r}(q-p)}_{L^\infty(B_R)}\right)^{\frac{1}{p_h}} \left(\int_{B_{\rho_0}} V(Du_j)^{p_0}\, dx\right)^{\frac{1}{p_0}} \\
&= \prod_{h=0}^{i}\left(4^{h+1}p_h^{4\mathfrak{r}}\right)^{\frac{1}{p_h}} \prod_{h=0}^{i}\left(\frac{C\Theta^{\mathfrak{r}}}{(\bar{R}-\bar{\rho})^{2\mathfrak{r}}}||V(Du_j)||^{2\mathfrak{r}(q-p)}_{L^\infty(B_R)}\right)^{\frac{1}{p_h}} \\
&\qquad \cdot\left(\int_{B_{\rho_0}} V(Du_j)^{p_0} dx\right)^{\frac{1}{p_0}}
\end{aligned} \tag{5.4}$$

Since

$$\prod_{h=0}^{i}\left(4^{h+1}p_h^{4\mathfrak{r}}\right)^{\frac{1}{p_h}} = \exp\left(\sum_{h=0}^{i}\frac{1}{p_h}\log(4^{h+1}p_h^{4\mathfrak{r}})\right) \leq \exp\left(\sum_{h=0}^{+\infty}\frac{1}{p_h}\log(4^{h+1}p_h^{4\mathfrak{r}})\right) \leq c(r)$$

and

$$\begin{aligned}
&\prod_{h=0}^{i}\left(\frac{C\Theta^{\mathfrak{r}}}{(\bar{R}-\bar{\rho})^{2\mathfrak{r}}}||V(Du_j)||^{2\mathfrak{r}(q-p)}_{L^\infty(B_R)}\right)^{\frac{1}{p_h}} = \left(\frac{C\Theta^{\mathfrak{r}}}{(\bar{R}-\bar{\rho})^{2\mathfrak{r}}}||V(Du_j)||^{2\mathfrak{r}(q-p)}_{L^\infty(B_R)}\right)^{\sum_{h=0}^{i}\frac{1}{p_h}} \\
&\leq \left(\frac{C\Theta^{\mathfrak{r}}}{(\bar{R}-\bar{\rho})^{2\mathfrak{r}}}||V(Du_j)||^{2\mathfrak{r}(q-p)}_{L^\infty(B_R)}\right)^{\sum_{h=0}^{+\infty}\frac{1}{p_h}} = \left(\frac{C\Theta^{\mathfrak{r}}}{(\bar{R}-\bar{\rho})^{2\mathfrak{r}}}||V(Du_j)||^{2\mathfrak{r}(q-p)}_{L^\infty(B_R)}\right)^{\frac{2^*}{p_0(2^*-2\mathfrak{r})}},
\end{aligned}$$

we can let $i \to \infty$ in (5.4) thus getting

$$||V(Du_j)||_{L^\infty(B_{\bar{\rho}})} \leq C\left(\frac{\Theta}{(\bar{R}-\bar{\rho})^2}\right)^{\frac{2^*\mathfrak{r}}{p_0(2^*-2\mathfrak{r})}} ||V(Du_j)||^{\frac{2\cdot 2^*\mathfrak{r}(q-p)}{p_0(2^*-2m)}}_{L^\infty(B_{\bar{R}})} \left(\int_{B_{\bar{R}}} V(Du_j)^{p_0}\, dx\right)^{\frac{1}{p_0}},$$

since $\sum_{h=0}^{\infty}\frac{1}{p_h}=\frac{2^*}{p_0(2^*-2\mathfrak{r})}$. Therefore, using the fact that $p_0=p\mathfrak{r}$, we deduce

$$||V(Du_j)||_{L^\infty(B_{\bar\rho})}\le C\left(\frac{\Theta}{(\bar R-\bar\rho)^2}\right)^{\frac{2^*}{p(2^*-2\mathfrak{r})}}||V(Du_j)||_{L^\infty(B_{\bar R})}^{\frac{2\cdot 2^*(q-p)}{p(2^*-2\mathfrak{r})}}\left(\int_{B_{\bar R}}V(Du_j)^{p\mathfrak{r}}\,dx\right)^{\frac{1}{p\mathfrak{r}}}.$$

Using the higher integrability at Lemma 4.1 we deduce that

$$\begin{aligned}||V(Du_j)||_{L^\infty(B_{\bar\rho})}\le C\left(\frac{\Theta}{(\bar R-\bar\rho)^2}\right)^{\frac{2^*}{p(2^*-2\mathfrak{r})}}||V(Du_j)||_{L^\infty(B_{\bar R})}^{\frac{2\cdot 2^*(q-p)}{p(2^*-2\mathfrak{r})}-\frac{2m}{2m+r}\frac{r(p-q+1)}{p\mathfrak{r}}+1}\\ \cdot\left(\int_{B_{\bar R}}V(Du_j)^{\frac{2mr}{2m+r}(p-q+1)}\,dx\right)^{\frac{1}{p\mathfrak{r}}}.\end{aligned}\tag{5.5}$$

Now, we note that

$$\begin{aligned}&\frac{2\cdot 2^*(q-p)}{p(2^*-2\mathfrak{r})}+1-\frac{r(p-q+1)}{p\mathfrak{r}}<1\iff\frac{2\cdot 2^*(q-p)}{2^*-2\mathfrak{r}}<(r-2)(p-q+1)\\ &\iff(q-p)\left[\frac{2\cdot 2^*}{2^*-2\mathfrak{r}}+r-2\right]<r-2\iff q-p<\frac{r-2}{\frac{2\cdot 2^*}{2^*-2\mathfrak{r}}+r-2}\end{aligned}\tag{5.6}$$

where we used that $\mathfrak{r}=\frac{r}{r-2}$. Since

$$\frac{2\cdot 2^*}{2^*-2\mathfrak{r}}=\frac{2\cdot\frac{2n}{n-2}}{\frac{2n}{n-2}-\frac{2r}{r-2}}=\frac{\frac{2n}{n-2}}{\frac{n}{n-2}-\frac{r}{r-2}}=\frac{\frac{2n}{n-2}}{\frac{2(r-n)}{(n-2)(r-2)}}=\frac{n(r-2)}{r-n},$$

last inequality is equivalent to

$$q-p<\frac{r-2}{\frac{n(r-2)}{r-n}+r-2}=\frac{1}{\frac{n}{r-n}+1}=\frac{r-n}{r}=1-\frac{n}{r}$$

that is of course satisfied under our assumption on the gap at (1.11). Note that in case $n=2$, the bound (1.11) reads as

$$q<p+1-\max\left\{\frac{2}{r},\frac{p+2}{r}\right\},$$

and one can easily check that inequality (5.6) is fulfilled provided that we choose $2^*<2$. By the inequality

$$\frac{2\cdot 2^*(q-p)}{p(2^*-2\mathfrak{r})}<\frac{r(p-q+1)}{p\mathfrak{r}},$$

we can determine m large enough so that

$$\frac{2\cdot 2^*(q-p)}{p(2^*-2\mathfrak{r})}<\frac{2m}{2m+r}\frac{r(p-q+1)}{p\mathfrak{r}}.$$

Indeed it suffices to choose

$$m>\frac{2^*r\mathfrak{r}(q-p)}{r(2^*-2\mathfrak{r})(p-q+1)-2^*2\mathfrak{r}(q-p)}\tag{5.7}$$

For m satisfying the previous inequality, to simplify the notation we set

$$\chi_m=\frac{2m}{2m+r}\frac{r(p-q+1)}{p\mathfrak{r}}-\frac{2\cdot 2^*(q-p)}{p(2^*-2\mathfrak{r})}$$

so that estimate (5.5) can be expressed as

$$||V(Du_j)||_{L^\infty(B_{\bar\rho})} \le C\left(\frac{\Theta}{(\bar R-\bar\rho)^2}\right)^{\frac{2^*}{p(2^*-2\mathfrak{r})}} ||V(Du_j)||^{1-\chi_m}_{L^\infty(B_{\bar R})} \cdot\left(\int_{B_{\bar R}} V(Du_j)^{\frac{2mr}{2m+r}(p-q+1)}\,dx\right)^{\frac{1}{p\mathfrak{r}}}, \tag{5.8}$$

with $1-\chi_m \in (0,1)$. Hence, we can use Young's inequality with exponents

$$\frac{1}{1-\chi_m} \quad \text{and} \quad \frac{1}{\chi_m}$$

in the right hand side of (5.5), we get

$$||V(Du_j)||_{L^\infty(B_{\bar\rho})} \le \frac{1}{2}||V(Du_j)||_{L^\infty(B_{\bar R})} + \left(\frac{C\Theta}{(\bar R-\bar\rho)^2}\right)^{\frac{2^*}{\chi_m\, p(2^*-2\mathfrak{r})}} \left(\int_{B_{\bar R}} V(Du_j)^{\frac{2mr}{2m+r}(p-q+1)}\,dx\right)^{\frac{1}{p\mathfrak{r}\chi_m}}. \tag{5.9}$$

Since the previous estimate holds true for every $\frac{R_0}{2} < \bar\rho < \bar R < R_0$, by Lemma 2.1 we get

$$||V(Du_j)||_{L^\infty\left(B_{\frac{R_0}{2}}\right)} \le \left(\frac{C\Theta}{R_0^2}\right)^{\frac{2^*}{\chi_m\, p(2^*-2\mathfrak{r})}} \left(\int_{B_{R_0}} V(Du_j)^{\frac{2mr}{2m+r}(p-q+1)}\,dx\right)^{\frac{1}{p\mathfrak{r}\chi_m}}$$

and by Corollary 4.2

$$||V(Du_j)||_{L^\infty\left(B_{\frac{R_0}{2}}\right)} \le C\left(\frac{\Theta}{R_0^2}\right)^{\frac{2^*}{\chi_m\, p(2^*-2\mathfrak{r})}} \left(\frac{\Theta^{\frac{\mathfrak{m}r}{2}}}{R_0^r}\left(1+a^{2m}\right)^{\frac{r}{r+2m}}\right)^{\frac{1}{p\mathfrak{r}\chi_m}}$$

with a constant C independent of m and j.

Step 2. The passage to the limit. Recalling (3.16), taking the limit as $j\to\infty$ in the previous estimate, we have

$$||V(Du)||_{L^\infty\left(B_{\frac{R_0}{2}}\right)} \le \liminf_{j\to\infty} ||V(Du_j)||_{L^\infty\left(B_{\frac{R_0}{2}}\right)} \le C\left(\frac{\Theta}{R_0^2}\right)^{\frac{2^*}{\chi_m\, p(2^*-2\mathfrak{r})}} \left(\frac{\Theta^{\frac{\mathfrak{m}r}{2}}}{R_0^r}\left(1+a^{2m}\right)^{\frac{r}{r+2m}}\right)^{\frac{1}{p\mathfrak{r}\chi_m}}.$$

Now, we observe

$$\lim_{m\to\infty}\chi_m = \frac{r(p-q+1)}{p\mathfrak{r}} - \frac{2\cdot 2^*(q-p)}{p(2^*-2\mathfrak{r})} =: \chi_1(p,q,r,n)$$

$$\lim_{m\to\infty}\frac{2mr(p-q+1)}{(2m+r)} = r(p-q+1)$$

Since the previous estimate holds for every m large enough to satisfy Eq. 5.7 with constant independent of m, we now take the limit as $m \to \infty$ thus getting

$$||V(Du)||_{L^\infty\left(B_{\frac{R_0}{2}}\right)} \leq C(\Theta, R_0)^{\tilde{\chi}} (1+a)^{\hat{\chi}} \tag{5.10}$$

with constant Θ depending on $p, q, r, n, ||h||_{L^r}, R_0$ and positive exponents $\tilde{\chi}, \hat{\chi}$ depending on p, q, r, n. The conclusion follows taking the limit as $a \to ||u||_{L^\infty}$. □

We are now in position, using the Caccioppoli inequality of Lemma 3.6, to give the

Proof of Theorem 1.3 Using 5.10 in the right hand side in (3.30) with $\gamma = 0$, we have

$$\begin{aligned}
\int_{B_{\frac{R_0}{2}}} |D^2 u_j|^2 \, dx &\leq \int_{B_{\frac{R_0}{2}}} (1+|Du_j|^2)^{\frac{p-2}{2}} |D^2 u_j|^2 \, dx \\
&\leq C ||V(Du_j)||^{2q-p}_{\mathrm{L}^\infty(B_{R_0})} \int_{B_{R_0}} h^2(x) \, dx + \frac{C}{R_0^2} ||V(Du_j)||^q_{\mathrm{L}^\infty(B_{R_0})} \\
&\leq C \left(1 + ||u||_{\mathrm{L}^\infty(B_{R_0};\mathbb{R}^N)}\right)^{\hat{\chi}},
\end{aligned}$$

where $C \equiv C(n, N, \nu, \tilde{L}, ||h||_{L^r(\Omega)}, R_0)$. The conclusion now easily follows taking the limit as $j \to +\infty$ in the previous estimate and recalling that $u_j \to u$ in $\mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$. □

We conclude mentioning that the same argument leads to the following second order regularity result

$$\int_{B_{\frac{R_0}{2}}} (1+|Du_j|^2)^{\frac{p-2+\gamma}{2}} |D^2 u_j|^2 \, dx \leq C_\gamma \left(1 + ||u||_{\mathrm{L}^\infty(B_{R_0};\mathbb{R}^N)}\right)^{\chi_\gamma},$$

where now both the constant and the exponent depend on γ.

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