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On d -dimensional nowhere-zero r -flows on a graph

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Abstract

A d -dimensional nowhere-zero r -flow on a graph G , an (r, d) -NZF from now on, is a flow where the value on each edge is an element of \mathbb{R}^d whose (Euclidean) norm lies in the interval $[1, r - 1]$. Such a notion is a natural generalization of the well-known concept of circular nowhere-zero r -flow (i.e. $d = 1$). For every bridgeless graph G , the 5-flow Conjecture claims that $\phi_1(G) \leq 5$, while a conjecture by Jain suggests that $\phi_d(G) = 1$, for all $d \geq 3$. Here, we address the problem of finding a possible upper-bound also for the remaining case $d = 2$. We show that, for all bridgeless graphs, $\phi_2(G) \leq 1 + \sqrt{5}$ and that the oriented 5-cycle double cover Conjecture implies $\phi_2(G) \leq \tau^2$, where τ is the Golden Ratio. Moreover, we propose a geometric method to describe an $(r, 2)$ -NZF of a cubic graph in a compact way, and we apply it in some instances. Our results and some computational evidence suggest that τ^2 could be a promising upper bound for the parameter $\phi_2(G)$ for an arbitrary bridgeless graph G . We leave that as a relevant open problem which represents an analogous of the 5-flow Conjecture in the 2-dimensional case (i.e. complex case).

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1 Introduction

Let $r \geq 2$ be a real number and d a positive integer, a d -dimensional nowhere-zero r -flow on a graph G , denoted by (r, d) -NZF on G , is an orientation of G together with an assignment $\varphi: E(G) \rightarrow \mathbb{R}^d$ such that, for all $e \in E(G)$, the (Euclidean) norm of $\varphi(e)$ lies in the interval $[1, r - 1]$ and, for every vertex, the sum of the inflow and outflow is the zero vector in \mathbb{R}^d . The d -dimensional flow number of a bridgeless graph G , denoted by $\phi_d(G)$, is defined as the infimum of the real numbers r such that G admits an (r, d) -NZF. Note that, by Seymour's 6-flow theorem [11] we have that $\phi_d(G) \leq 6$ for every d . Actually, $\phi_d(G)$ is a minimum: due to the above upper bound, it suffices to consider only the set of feasible d -dimensional nowhere-zero r -flows with $r \leq 6$, which can be represented as a compact subset of $\mathbb{R}^{d \cdot |E(G)|}$, and the function that assigns to every feasible flow the maximum norm among its components, that are elements of \mathbb{R}^d , is continuous.

In the above definitions it is not restrictive to assume any graph G to be connected. So we only consider connected graphs in the rest of the paper.

The notion of (r, d) -NZF includes some parameters already considered in the literature. First of all, the 1-dimensional case, that is $\phi_1(G)$, is nothing but the classical circular flow number of a graph (see [6]). A famous conjecture by Tutte can be stated according to our notation as follows.

Conjecture 1 (5-flow Conjecture). *Let G be a bridgeless graph. Then, $\phi_1(G) \leq 5$.*

An upper bound for ϕ_d is also conjectured for $d \geq 3$. Indeed, Jain suggested (see [17]) that every bridgeless graph admits a nowhere-zero flow with flow values taken on the unitary sphere S^2 , that is the set of unit vectors of \mathbb{R}^3 . Clearly, such a conjecture can be stated in our terminology as follows.

Conjecture 2 (S^2 -flow Conjecture). *Let G be a bridgeless graph. Then, $\phi_d(G) = 2$ for every $d > 2$.*

Let us remark that, along the paper, we will use the term cycle in its largest acception of subgraph with all vertices of even degree. Such a use is quite common in this context and permits to simplify the presentation. It is already observed in [13] that $\phi_7(G) = 2$ for every bridgeless graph G . This is a consequence of a covering result by Bermond, Jackson and Jaeger [3], claiming that every bridgeless graph G has seven cycles such that every edge

of G is contained in exactly four of them. Moreover, the Berge-Fulkerson conjecture (see [5, 9]), if it holds true, implies that every bridgeless cubic graph has six cycles such that every edge is in exactly four of them. As noted in [13], this would imply that $\phi_6(G) = 2$ for any bridgeless cubic graph G . In a similar way, if the conjecture of Celmins and Preissmann on the existence, for every bridgeless graph, of five cycles covering each edge twice is true, then $\phi_5(G) = 2$ for any bridgeless graph G .

Now, it is natural to ask what is a general upper bound for the 2-dimensional case. Indeed, Conjecture 1 and Conjecture 2 do not address the case $d = 2$. As far as we know, such a question is not considered in the literature yet, and one of the main goals of this paper is proposing a general upper bound for $\phi_2(G)$, see Corollary 5, Theorem 6 and Problem 1.

Let us note that a 2-dimensional nowhere-zero r -flow can be viewed as a generalization of a 1-dimensional nowhere-zero flow where flow values are taken in the complex field \mathbb{C} . This notion is already considered in [13] in relation with Conjecture 2. Among other results, it is proved that $\phi_2(G) = 2$ if G is 6-edge-connected, but no discussion about a general upper bound for ϕ_2 is proposed by the author. Some other results on 2-dimensional nowhere-zero r -flows are obtained in [15], where the special case of flow values taken in the 2-dimensional unit sphere S^1 is considered.

2 Possible upper bounds for $\phi_2(G)$

A cycle double cover of a graph G is a collection of cycles that together include each edge of G exactly twice. Notice that a subgraph with no edges is also considered a cycle of G . The existence of a cycle double cover for each bridgeless graph is a famous unsolved problem, posed by Seymour and Szekeres [10, 12], and known as the cycle double cover conjecture. There are many variations on the cycle double cover conjecture (see [16] for a comprehensive survey).

Here we will consider one of the strongest formulations, known as the *oriented 5-cycle double cover conjecture*. In order to introduce it, we need to recall some terminology.

If G is a graph and O is an orientation of the edges of G , we denote by $O(G)$ the directed graph so obtained and, for every edge $e \in E(G)$, we denote by $O(e)$ its orientation with respect to O . A subgraph H of $O(G)$ is a *directed cycle* of $O(G)$ if for each vertex v of H , the indegree of v equals

the outdegree of v .

The collection $\mathcal{C} = \{O_1(C_1), \dots, O_k(C_k)\}$ of directed cycles of a graph G is said to be an *oriented cycle double cover* of G if every edge e of G belongs to exactly two cycles C_i and C_j and the directions of $O_i(C_i)$ and $O_j(C_j)$ are opposite on e .

If we would like to stress the number of cycles in \mathcal{C} we will write that \mathcal{C} is an oriented k -cycle double cover of G .

The oriented 5-cycle double cover conjecture, which is due to Archdeacon and Jaeger [1, 8], states the following.

Conjecture 3 (Oriented 5-cycle double cover Conjecture). *Each bridgeless graph G has an oriented 5-cycle double cover.*

Now, we show that if Conjecture 3 holds true, then we can deduce a general upper bound for the parameter ϕ_2 . We shall obtain such a relation by the following more general result.

Theorem 4. *Let G be a bridgeless graph and $k \in \{2, 3, 4, 5\}$. If G admits an oriented k -cycle double cover, then*

- $\phi_2(G) = 2$, if $k \leq 3$;
- $\phi_2(G) \leq 1 + \sqrt{2}$, if $k = 4$;
- $\phi_2(G) \leq \tau^2$, if $k = 5$.

where τ denotes the Golden Ratio $\frac{1+\sqrt{5}}{2}$ ¹.

Proof. Let $\mathcal{C} = \{O_1(C_1), \dots, O_k(C_k)\}$ be an oriented k -cycle double cover of G . We construct a 2-dimensional flow on G as follows. Choose an arbitrary orientation O of G and k elements p_1, \dots, p_k in \mathbb{R}^2 . For every $i \in \{1, \dots, k\}$, we add a flow value equal to p_i to all edges $e \in C_i$ such that $O_i(e) = O(e)$, while we add $-p_i$ to all edges $e \in C_i$ such that $O_i(e) \neq O(e)$.

Observe that this procedure generates a 2-dimensional flow, where every edge which belongs to $C_i \cap C_j$ receives one of the two vectors $\pm(p_i - p_j)$. In order to obtain an $(r, 2)$ -NZF, we need the norm of each flow value $p_i - p_j$ to be at least one. Then, we choose p_1, \dots, p_k pointing at the k vertices of

¹To our knowledge, the Greek letter τ represented the Golden Ratio for hundreds of years, up to the early 20th century. This ancient notation is used along the paper for the sake of a better distinction from the flow number.

a regular k -gon of side length 1. If $k \in \{2, 3\}$, since $|p_i - p_j| = 1$ for every $i \neq j$, then $\phi_2(G) = 2$. If $k = 4$, since $|p_i - p_j|$ for every $i \neq j$ is either 1 or $\sqrt{2}$, then $\phi_2(G) \leq 1 + \sqrt{2}$. Finally, if $k = 5$, the diagonals of a regular pentagon are in the golden ratio to its sides. Hence $|p_i - p_j|$ is equal to either 1 or τ for every $i \neq j$, then $\phi_2(G) \leq \tau^2 (= 1 + \tau)$. \square

Note that our choice of the vectors p_1, \dots, p_k in each of the three cases of the proof of Theorem 4 is known to be optimal in order to minimize the ratio between the maximum and the minimum length of k points in the Euclidean plane (see [2]).

Corollary 5. *The oriented 5-cycle double cover conjecture (Conjecture 3) implies $\phi_2(G) \leq \tau^2$ for every bridgeless graph G .*

In Section 3, we will discuss the problem of finding a graph G such that $\phi_2(G)$ is close to τ^2 .

The upper bound of τ^2 is obtained by assuming true a well-known conjecture. Now, we complete this section by proving a general upper bound for $\phi_2(G)$ as a consequence of the proof of the 6-flow theorem of Seymour.

Theorem 6. *If G is a bridgeless graph, then $\phi_2(G) \leq 1 + \sqrt{5}$.*

Proof. In the proof of the 6-flow theorem [11, p. 133] Seymour showed that each bridgeless graph G has an integer 2-flow φ_2 and an integer 3-flow φ_3 such that $\varphi_2(e) \neq 0$ or $\varphi_3(e) \neq 0$ for each edge $e \in E(G)$. Let φ be a 2-dimensional flow on $O(G)$, for an arbitrary orientation O , such that $\varphi(e) = (\varphi_2(e), \varphi_3(e))$ for each $e \in E(O(G))$. Since φ_2 and φ_3 are 2-flow and 3-flow, respectively, we have $\sqrt{\varphi_2(e)^2 + \varphi_3(e)^2} \leq \sqrt{1^2 + 2^2} = \sqrt{5}$. Also, one of the values $\varphi_2(e)$ and $\varphi_3(e)$ is nonzero, so $\sqrt{\varphi_2(e)^2 + \varphi_3(e)^2} \geq 1$. Thus, φ is indeed a $(1 + \sqrt{5}, 2)$ -NZF of G . \square

3 2-dimensional flows on cubic graphs

In the case of nowhere-zero circular flows (i.e. 1-dimensional flows) it is well known that every bridgeless graph has a nowhere-zero r -flow if and only if every bridgeless cubic graph has a nowhere-zero r -flow. Following the same proof, one can get the following result.

Proposition 7. *For all positive integers d and real numbers $r \geq 2$, the following statements are equivalent:*

- every bridgeless graph has a d -dimensional nowhere-zero r -flow;
- every bridgeless cubic graph has a d -dimensional nowhere-zero r -flow.

By Proposition 7, there is a fixed constant k such that $\phi_2(G) \leq k$ for all bridgeless graphs G if and only if the same holds for all bridgeless cubic graphs.

Recall that Thomassen [13] proved that a cubic graph is bipartite if and only if it has an S^1 -flow, that is a 2-dimensional nowhere-zero 2-flow. In particular, up to a rotation, one can assume that the flow values are the three cube roots of unity, that is the complex solutions of the equation $z^3 = 1$.

As a further step in studying the 2-dimensional flow numbers of cubic graphs we consider those being 3-edge-colourable. Observe that any 3-edge-colourable cubic graph has an oriented 4-cycle double cover (see for instance [16]), hence the following proposition follows from Theorem 4.

Proposition 8. *Let G be a 3-edge-colourable cubic graph. Then $\phi_2(G) \leq 1 + \sqrt{2}$.*

The above inequality is the best possible as one can directly check that $\phi_2(K_4) = 1 + \sqrt{2}$. However, we can obtain it as a special case (i.e. $n = 3$) of the following more general result which gives an exact value for $\phi_2(W_n)$, where W_n is the wheel graph of order $n + 1$.

The proof of Theorem 9 is quite long and technical and it will appear in another paper [7].

Theorem 9. *Let W_n be the wheel graph of order $n + 1$, for $n \geq 3$. Then*

$$\phi_2(W_n) = \begin{cases} 2 & \text{if } n \text{ is even,} \\ 1 + 2 \sin\left(\frac{\pi}{6} \cdot \frac{n}{n-1}\right) & \text{if } n \equiv 1, 3 \pmod{6}, \\ 1 + 2 \sin\left(\frac{\pi}{6} \cdot \frac{n+1}{n}\right) & \text{if } n \equiv 5 \pmod{6}. \end{cases}$$

The next lemma is an immediate consequence of Theorem 9.

Lemma 10. *Let G be a cubic graph containing a chordless cycle C of length k . Then $\phi_2(G) \geq \phi_2(W_k)$.*

Proof. Suppose to the contrary that $\phi_2(G) < \phi_2(W_k)$. Then G has an $(r, 2)$ -NZF φ with $r < \phi_2(W_k)$. Contract all the vertices of G that are not in C to a unique vertex v . The obtained graph is W_k and φ induces on W_k an $(r', 2)$ -NZF with $r' \leq r < \phi_2(W_k)$, a contradiction. \square

Lemma 10 combined with the values given in Theorem 9, gives the following.

Corollary 11. *Let G be a cubic graph with odd-girth equal to g . Then $\phi_2(G) \geq \phi_2(W_g)$.*

Using Corollary 11 we prove the following proposition.

Proposition 12. *Let n be odd and let P_n be the prism graph of order $2n$. Then $\phi_2(P_n) = \phi_2(W_n)$.*

Proof. Since P_n has odd girth n , we have $\phi_2(P_n) \geq \phi_2(W_n)$.

Moreover, each flow on W_n can be easily extended to a flow on P_n using the same vectors: for every 4-cycle $uu'v'v$ where uu' and vv' are spokes of P_n , we set the flow from u to v to be the same as the flow from v' to u' . Thus we have $\phi_2(P_n) = \phi_2(W_n)$. \square

Also, Corollary 11, together with Proposition 8 and Theorem 9, implies the following result.

Proposition 13. *Let G be a 3-edge-colourable cubic graph with a triangle. Then $\phi_2(G) = 1 + \sqrt{2}$.*

Up to now, the unique infinite classes of non-bipartite cubic graphs for which we are able to determine the exact value of ϕ_2 are the ones considered in Proposition 12 and Proposition 13.

In the rest of the paper we provide upper bounds on the 2-dimensional flow number of certain cubic graphs. To make our descriptions of 2-dimensional flows more compact, we show that they can be equivalently represented in a geometric way. The main idea of this approach is that, by the Kirchhoff's law, the three vectors assigned to three edges incident with the same vertex correspond to a triangle in the Euclidean plane. Thus we can represent a 2-dimensional flow as a suitable collection of triangles.

By a *triangle* we mean a subset of the Euclidean plane consisting of its three sides and interior points. Let s_1 and s_2 be sides of triangles T_1 and T_2 , respectively. We say that s_1 and s_2 are *attachable* if we can translate T_1 to T'_1 in such a way that the image of s_1 coincides with s_2 and T'_1 and T_2 have no common internal points. In other words, attachable sides need to be parallel, of the same length and they need to have their triangles on mutually opposite sides. An *r -flow triangulation* of a bridgeless cubic graph G is a collection \mathcal{T} containing a triangle T_v for each vertex v of G such that

- (i) for each $v \in V(G)$, each edge incident to v corresponds to a unique side of T_v ;
- (ii) lengths of sides of all triangles from \mathcal{T} are from the interval $[1, r - 1]$;
- (iii) for each edge $uv \in E(G)$, the sides of the triangles T_u and T_v corresponding to uv are attachable.

Proposition 14. *Let G be a bridgeless cubic graph. Then G has an r -flow triangulation if and only if G has an $(r, 2)$ -NZF.*

Proof. We start with the only if part. Let O be an arbitrary orientation of the edges of G . We construct a 2-dimensional flow on G as follows. Consider an oriented edge uv of $O(G)$ and let a and b be the vectors corresponding to the attachable sides of triangles T_u and T_v , respectively, which are oriented in such a way that T_u is on the right side of a and T_v is on the left side of b . Due to the definition of attachable sides, the vectors a and b have the same direction, so they are equal. We set to a the flow value of the edge uv . Note that if we orient uv in the opposite direction, it receives the opposite vector, thus we do not need any specific orientation of G .

We prove that this assignment is an $(r, 2)$ -NZF. Consider a vertex v and orient all three edges incident with v as incoming. The vectors assigned to these edges form a triangle T_v and since all of them have T_v on the left side, they sum up to zero.

Now for the if part, assume that G has an $(r, 2)$ -NZF. For each vertex v of G , let e_1, e_2 and e_3 be the oriented edges of $O(G)$ incident with v . For each $i \in \{1, 2, 3\}$, let a_i be the flow value of e_i , if v is the tail of e_i , and let a_i be the opposite of flow value of e_i otherwise. Then, the vectors a_1, a_2 and a_3 sum up to zero. Moreover, we can arrange them to form an oriented triangle T_v that is on the left side of each of a_1, a_2 and a_3 .

We prove that the triangles T_v for each $v \in V(G)$ form an r -flow triangulation. Properties (i) and (ii) are trivially satisfied. Let uv be an oriented edge of $O(G)$ with flow value a . Since u is the tail and v is the head of uv , the triangle T_u lies on the left side of a and T_v lies on the right. Thus the sides of T_u and T_v corresponding to a are attachable. Hence Property (iii) also holds. \square

For a bridgeless cubic graph G , finding the representation of a 2-dimensional flow through a flow triangulation is, in general, only a reformulation of the

original problem. However, in the following examples we present flow triangulations in some “nice” way. The term nice can be understood in several ways, but perhaps the most basic one requires that the intersection of every two different triangles T_1 and T_2 , if not empty, consists either of one vertex, or of two coinciding sides s_1 and s_2 of T_1 and T_2 , respectively. In the latter case, s_1 and s_2 correspond to the same edge of G and the set of all such edges induces a connected spanning subgraph of G . Examples of such nice flow triangulations of K_4 and $K_{3,3}$ are depicted in Figure 1. In all our figures, the graph is grey with its vertices placed in their corresponding triangles. Bold sides are of length 1 and dashed ones are always the sides with maximum length. Nevertheless, we do not know whether such a “nice” flow triangulation exists for every 2-dimensional flow.

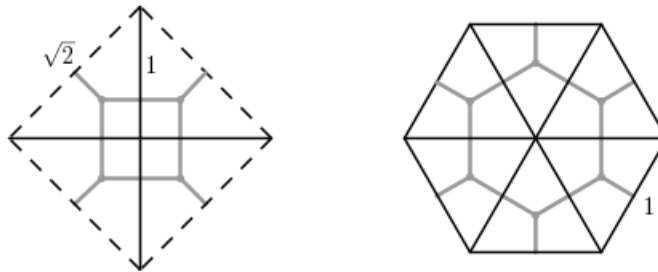


Figure 1: A 2-flow triangulation of K_4 (left) and $K_{3,3}$ (right).

We have already seen an upper bound on the 2-dimensional flow number of 3-edge-colourable cubic graphs in Proposition 8. As usual, in order to prove a general bound on $\phi_2(G)$ for every bridgeless cubic graph G , the hard case is when G is not 3-edge-colourable. Therefore, we are naturally interested in the 2-dimensional flow number of the Petersen graph, which is the smallest such graph. Let us say that determining this value appears to be a hard problem. Here we propose an upper bound by constructing a suitable flow triangulation.

Proposition 15. *The 2-dimensional flow number of the Petersen graph is at most $1 + \sqrt{7/3}$.*

Proof. Throughout this proof, we take all the indices modulo 3. Consider, in the real Euclidean plane, an equilateral triangle $p_1p_2p_3$ with side length 1.

For $i \in \{1, 2, 3\}$, let $q_i p_i$ be the reflection of $p_{i-1} p_i$ through p_i and let q'_1, q'_2 and q'_3 be the points such that $q_1 q'_3 q_2 q'_1 q_3 q'_2$ is a regular hexagon. By adding the segments $q'_i p_{i+1}$ and $q'_i p_{i+2}$, for each $i \in \{1, 2, 3\}$, we obtain 10 triangles as depicted in Figure 2. The solid, dash-dotted and dashed lines have lengths 1, $\sqrt{4/3}$ and $\sqrt{7/3}$, respectively. It is easy to check that these triangles form a $(1 + \sqrt{7/3})$ -flow triangulation of the Petersen graph. \square

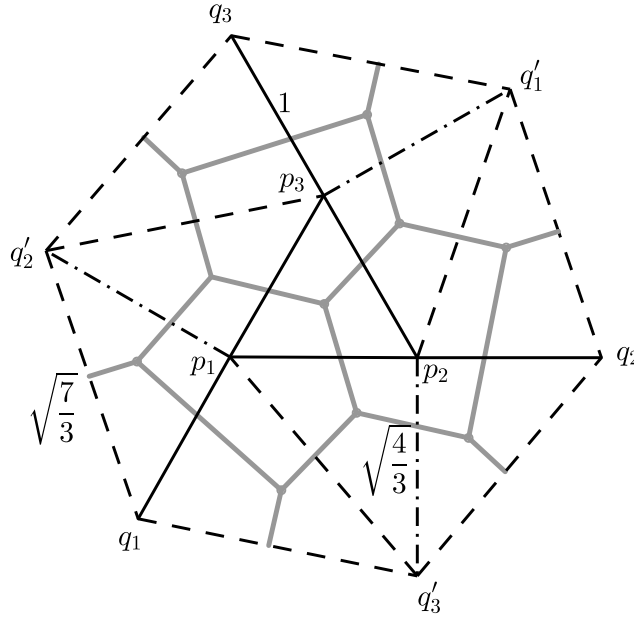


Figure 2: A $(1 + \sqrt{7/3})$ -flow triangulation of the Petersen graph.

Supported by computational results we believe that this is the exact 2-dimensional flow number of the Petersen graph. Since we currently know no tools for proving such high lower bounds on 2-dimensional flow numbers, we propose the following conjecture.

Conjecture 16. *The 2-dimensional flow number of the Petersen graph is $1 + \sqrt{7/3}$.*

The 1-dimensional flow number can distinguish 3-edge-colourable cubic graphs, which have 1-dimensional flow number at most 4, from the non-3-edge-colourable bridgeless cubic graphs having 1-dimensional flow number greater than 4 (see for instance [14]). However, the 2-dimensional flow number does not serve for this purpose. One of the counterexamples is the Isaacs

snark J_5 (see Figure 3) for which we show that $\phi_2(J_5) < 1 + \sqrt{2} = \phi_2(K_4)$. We have found an $(r, 2)$ -NZF of J_5 for $r = 1 + 1.387893647$ with the help of a computer.

Proposition 17. $\phi_2(J_5) \leq 2.387893647 < 1 + \sqrt{2}$.

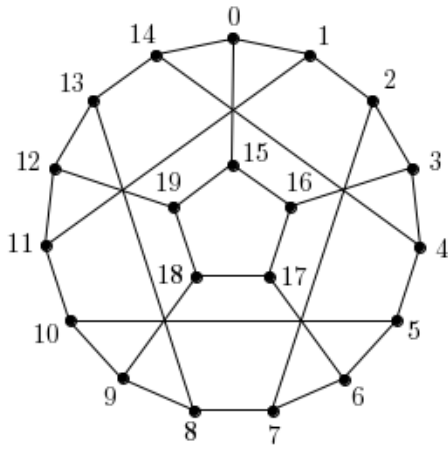


Figure 3: Isaacs snark J_5 .

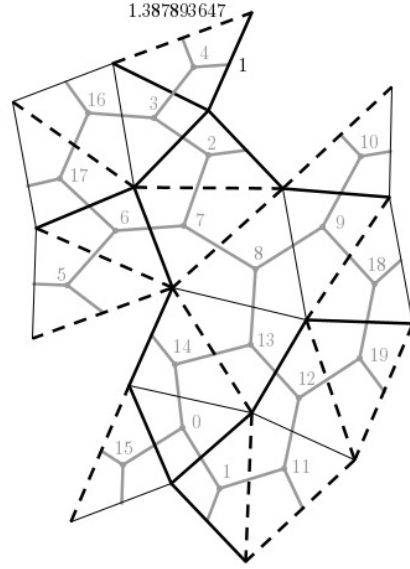


Figure 4: A 2-dimensional flow on J_5 .

Figure 4 depicts an approximation of the flow triangulation corresponding to the found flow. We emphasise only the sides with minimum (bold) and maximum (dashed) length.

The Petersen graph is the worst case for many other problems in this area. Surprisingly, this seems not to be the case here. Indeed if we replace every vertex of P with a triangle, denoting the resulting graph by P_Δ , we are not able to extend our $(1 + \sqrt{7/3}, 2)$ -NZF on P to a $(1 + \sqrt{7/3}, 2)$ -NZF on P_Δ . The best $(r, 2)$ -NZF flow on P_Δ we have up to now is for $r \approx 2.59$, also found by a computer.

We wonder if $\tau^2 \approx 2.618$ is the upper bound on the 2-dimensional flow number of all bridgeless graphs and also whether this bound is reached by some graph. Therefore, we propose the following problems.

Problem 1. Determine if $\phi_2(G) \leq \tau^2$ for every bridgeless graph G .

Problem 2. Establish the existence (or not) of a bridgeless cubic graph G with $\phi_2(G) = \tau^2$.

We would also like to note that flow triangulations can be represented in a topological way. For instance, the $(1 + \sqrt{7/3})$ -flow triangulation of the Petersen graph can be described as a dual of P embedded on a torus. Similarly, the aforementioned flow triangulations for K_4 , $K_{3,3}$ and J_5 can be also obtained from embeddings on some orientable surfaces. However, since we need to measure Euclidean distance, we avoid mentioning other surfaces, where the notion of distance is not clear.

Also, we noted that it is not clear if every 2-dimensional flow on a cubic graph can be represented through a nice flow triangulation. We do not know the answer even for bipartite cubic graphs, which are perhaps the most simple family of cubic graphs for this problem, since each 2-flow triangulation consists of equilateral triangles with side length 1. Therefore, we leave it as a further open problem.

Declarations

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