TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 355, Number 6, Pages 2453–2476
S 0002-9947(03)02929-5
Article electronically published on February 7, 2003

ON L^p CONTINUITY OF SINGULAR FOURIER INTEGRAL OPERATORS

ANDREW COMECH AND SCIPIO CUCCAGNA

Abstract. We derive L^p continuity of Fourier integral operators with one-sided fold singularities. The argument is based on interpolation of (asymptotics of) L^2 estimates and $\mathrm{H}^1 \to L^1$ estimates. We derive the latter estimates elaborating arguments of Seeger, Sogge, and Stein's 1991 paper.

We apply our results to the study of the L^p regularity properties of the restrictions of solutions to hyperbolic equations onto timelike hypersurfaces and onto hypersurfaces with characteristic points.

0. Introduction and results

The standard Fourier integral operators $\mathfrak{F} : \mathcal{E}'(Y) \to \mathcal{D}'(X)$ treated by Hörmander [Ho71] are associated with (local) symplectomorphisms from T^*X to T^*Y. The graph $\mathbf{C} \subset (T^*X\backslash 0) \times (T^*Y\backslash 0)$ of this symplectomorphism is referred to as the canonical relation. The continuity of such operators in standard L^2-based Sobolev spaces already follows from [Ho71]. The L^p estimates have been obtained in [SeSoSt91]: Let $\dim X = \dim Y = n$ and let $\mathfrak{F} \in I^\mu(X, Y, \mathbf{C})$ be the Fourier integral operator of order μ, with its integral kernel vanishing away from a compact set in $X \times Y$. Given $1 < p < \infty$, the mapping

$$\mathfrak{F} : \ L^p_\alpha(Y) \to L^p_\beta(X) \tag{0.1}$$

is continuous if $\mu \le \alpha - \alpha_p - \beta$, where $\alpha_p = (n-1)\left|\frac{1}{p} - \frac{1}{2}\right|$. This continuity is obtained by Fefferman-Stein interpolation between L^2-L^2 continuity of Fourier integral operators of order 0 and H^1-L^1 continuity (where H^1 is the Hardy space) of operators of order $-(n-1)/2$.

In the present paper, we consider singular Fourier integral operators: the associated canonical relation $\mathbf{C}$ is again a smooth Lagrangian submanifold in $T^*X \times T^*Y$, but, contrary to the standard case, the projections $\pi_L : \mathbf{C} \to T^*X$, $\pi_R : \mathbf{C} \to T^*Y$ are allowed to have singularities.

The simplest singularities are Whitney folds. The L^2 Sobolev continuity for operators associated to canonical relations with Whitney folds on both sides was derived in [MeTa85]. Such operators "lose a sixth of a derivative", versus operators associated to (local) symplectomorphisms. L^p continuity of such operators was derived in [SmSo94] (for some special cases see [PhSt91] and [Se93]). Even though there is a certain loss of smoothness near $p = 2$ (as we pointed out, it is a sixth

Received by the editors September 4, 1998 and, in revised form, June 3, 2001.
2000 *Mathematics Subject Classification.* Primary 35S30.
Both authors were partially supported by grants from the National Science Foundation.

of a derivative at $p = 2$), for p outside the interval $3/2 \le p \le 3$ Fourier integral operators with two-sided fold singularities have the same continuity as operators associated to symplectomorphisms. In particular, operators of order $-(n-1)/2$ are continuous from H^1 to L^1. We are going to generalize these results for operators with higher order singularities.

First, let us formulate the following straightforward generalization of [SeSoSt91] to singular Fourier integral operators, which motivates our paper. Let $X = Y = \mathbb{R}^n$, and let $\mathfrak{F} : C^\infty_{\mathrm{comp}}(Y) \to C^\infty(X)$ be a Fourier integral operator associated to a singular canonical relation $\mathbf{C}$. We assume that $\mathfrak{F}$ has the form as in [SeSoSt91]:

$$\mathfrak{F}u(x) = \iint e^{i(\langle x,\xi\rangle - \varphi(y,\xi))} b(x,\xi,y) u(y)\, d\xi\, dy, \qquad x,\, y,\, \xi \in \mathbb{R}^n. \tag{0.2}$$

Theorem 1. *Let $\mathfrak{F} \in I^{-\frac{n-1}{2}}(X, Y, \mathbf{C})$, where $X = Y = \mathbb{R}^n$, $n > 1$. We assume that $\mathfrak{F}$ is of the form (0.2) and that $b(x, \xi, y)$ vanishes for all x, y away from a compact set in $X \times Y$. If the projection $\pi_R : \mathbf{C} \to T^*Y$ has at most fold singularities, then the following action is continuous:*

$$\mathfrak{F} :\ \mathsf{H}^1(Y) \to L^1(X).$$

Here H^1 is the Hardy space.

Proof. We follow [SeSoSt91]. Let $a_{\mathsf{Q}}(y)$ be an atom supported in a box $\mathsf{Q} \subset Y$ with the sidelength r, so that $|\mathsf{Q}| = r^n$, $\|a_{\mathsf{Q}}\|_{L^\infty} \le r^{-n}$, $\|a_{\mathsf{Q}}\|_{L^1} \le 1$. Let $\mathcal{N}_{\mathsf{Q}} \subset X$ be the exceptional set associated with Q (as in [SeSoSt91]; see also Section 4 below), $|\mathcal{N}_{\mathsf{Q}}| \le \mathrm{const}\, r$. We need to prove the uniform boundedness for

$$\|\mathfrak{F}a_{\mathsf{Q}}\|_{L^1(\mathbb{R}^n)} = \|\mathfrak{F}a_{\mathsf{Q}}\|_{L^1(\mathcal{N}_{\mathsf{Q}})} + \|\mathfrak{F}a_{\mathsf{Q}}\|_{L^1(\mathbb{R}^n\setminus\mathcal{N}_{\mathsf{Q}})}.$$

For the first term, we employ Cauchy-Schwarz:

$$\|\mathfrak{F}a_{\mathsf{Q}}\|_{L^1(\mathcal{N}_{\mathsf{Q}})} \le |\mathcal{N}_{\mathsf{Q}}|^{\frac12} \cdot \|\mathfrak{F}a_{\mathsf{Q}}\|_{L^2} \le |\mathcal{N}_{\mathsf{Q}}|^{\frac12} \cdot \|a_{\mathsf{Q}}\|_{L^p} \cdot \|\mathfrak{F}\|_{L^p \to L^2}. \tag{0.3}$$

We take $p = p_n \equiv \frac{2n}{2n-1}$; then $\|a_{\mathsf{Q}}\|_{L^{p_n}} \le r^{-\frac12}$. Since π_R is a Whitney fold, $\mathfrak{F}$ is bounded from L^{p_n} to L^2, see [GrSe94]. We conclude that (0.3) is bounded uniformly in r.

The argument used in [SeSoSt91] to prove the boundedness of $\|\mathfrak{F}a_{\mathsf{Q}}\|_{L^1(\mathbb{R}^n\setminus\mathcal{N}_{\mathsf{Q}})}$ can be repeated verbatim. □

Remark 0.1. The lesson here is that off the exceptional set the regularity properties of the projections from $\mathbf{C}$ are not important. The $L^2 \to L^2$-estimates used in [SeSoSt91] at the exceptional set can and will be replaced by some estimates which are not so sensitive to the regularity of the projections.

We will characterize the singularities of the projections from the canonical relation in terms of the type conditions. Let M and N be smooth manifolds of the same dimension and let $\pi : M \to N$ be a smooth map. Let Σ be the critical variety of the map π:

$$\Sigma = \{p \in M \mid \det d\pi|_p = 0\}. \tag{0.4}$$

Here $d\pi$ denotes the Jacobi matrix of π (in certain local coordinates).

Definition 0.1. Assume that π drops rank simply by 1:

$$\dim\ker d\pi \le 1, \qquad\qquad d(\det d\pi)|_\Sigma \ne 0.$$

We will say that at a point $p \in \Sigma$ the map $\pi : M \to N$ is of type k if it is the smallest integer such that

$$V^k(\det d\pi)|_p \neq 0, \tag{0.5}$$

where V is an arbitrary smooth vector field over M which generates $\ker d\pi$:

$$V|_\Sigma \in \ker d\pi, \qquad V|_\Sigma \neq 0.$$

If p is a regular point of π, so that $\det d\pi|_p \neq 0$, then $k = 0$.

Remark 0.2. If π is of type at most 1 at the critical points, then it is a Whitney fold.

Definition 0.2. Let $\mathcal{V} \subset C^\infty(\Gamma(TM))$ be a module over $C^\infty(M)$ which is also a subalgebra of the Lie algebra of smooth vector fields over M. We will say that at a point $p \in M$ the map $\pi : M \to N$ is of type w relative to $\mathcal{V}$ if w is the smallest integer such that

$$V_1 V_2 \dots V_w(\det d\pi)|_p \neq 0, \qquad V_i \in \mathcal{V}. \tag{0.6}$$

If p is a regular point of π, so that $\det d\pi|_p \neq 0$, then $w = 0$.

Conditions related to Definition 0.2 appeared in [Se98] and in [GrSe98]. For example, if the projection from the canonical relation $\pi_L : \mathbf{C} \to T^*X$ has a *strong Morin S_{1_k}-singularity*, in the sense of [GrSe98] (where such singularities are denoted by $S^+_{1_k}$), then in our terminology π_L is of type at most 1 relative to $\ker d(\pi_X \circ \pi_L)$.

We consider Fourier integral operators of the form

$$\mathfrak{F}u(x) = \iint e^{i\Phi(x,\theta,y)} b(x,\theta,y) u(y)\, d\theta\, dy, \tag{0.7}$$

with a non-degenerate phase function $\Phi \in C^\infty(X \times \mathbb{R}^N \times Y)$ of degree 1 in θ, and with a symbol $b \in S^d_{1,0}(X \times \mathbb{R}^N \times Y)$ of order d, which vanishes away from a compact set in $X \times Y$. We assume that $\dim X = \dim Y = n$, so that $\mathfrak{F} \in I^{d+\frac{N-n}{2}}$.

We will assume that one of the projections from the associated canonical relation

$$\mathbf{C} = \{(x, d_x\Phi(x,\theta,y)),\, (y, -d_y\Phi(x,\theta,y)) \mid \Phi_\theta(x,\theta,y) = 0\}$$

(specifically, $\pi_L : \mathbf{C} \to T^*X$) is a Whitney fold, and denote by $\Sigma \subset \mathbf{C}$ the common critical variety of π_L, π_R:

$$\Sigma = \{p \in \mathbf{C} \mid \det d\pi_L|_p = 0, \quad \det d\pi_R|_p = 0\}. \tag{0.8}$$

Remark 0.3. If either of the projections from the canonical relation is a Whitney fold (or, more generally, if either of them has a Morin S_{1_k}-singularity), then both projections from the canonical relation drop rank simply by 1:

$$\dim\ker d\pi_\alpha \leq 1, \qquad d(\det d\pi_\alpha)|_\Sigma \neq 0, \qquad \text{where } \alpha = L,\, R. \tag{0.9}$$

Theorem 2 (Main result). *Let $\mathfrak{F} \in I^\mu(X, Y, \mathbf{C})$, where $\dim X = \dim Y = n \geq 2$, and assume that the integral kernel of $\mathfrak{F}$ vanishes away from a compact set in $X \times Y$. We also assume that the projections $\mathbf{C} \to X$, $\mathbf{C} \to Y$ are submersions.*

If $\pi_L : \mathbf{C} \to T^*X$ is a Whitney fold and if $\pi_R : \mathbf{C} \to T^*Y$ is of type at most k, and of type at most $w \leq k$ relative to $\ker d(\pi_Y \circ \pi_R)$, then the action

$$\mathfrak{F} :\ L^p_\alpha(Y) \to L^p_\beta(X), \qquad p \in (1, (w+2)/(w+1)) \cup (3, \infty), \tag{0.10}$$

is continuous if $\mu \leq \alpha - \alpha_p - \beta$, where $\alpha_p = (n-1)\left|\frac{1}{p} - \frac{1}{2}\right|$.

For p between $\frac{w+2}{w+1}$ and 3, the estimates are obtained by interpolating (0.10) *with the Sobolev L^2 estimates,*

$$\mathfrak{F}:\ L^2_\alpha(Y) \to L^2_{\alpha-\mu-1/(4+2k^{-1})}(X). \tag{0.11}$$

More precisely, the action

$$\mathfrak{F}:\ L^p_\alpha(Y) \to L^p_\beta(X), \qquad p \in ((w+2)/(w+1), 2) \cup (2,3), \tag{0.12}$$

is continuous if $\mu < \alpha - \alpha_p - \beta - \delta_p(1,k)$, where

$$\delta_p(1,k) = \begin{cases} \left(\frac{w+1}{w} - \frac{w+2}{pw}\right)\frac{k}{2k+1}, & (w+2)/(w+1) \le p < 2, \\ \left(\frac{3}{p} - 1\right)\frac{k}{2k+1}, & 2 < p \le 3. \end{cases}$$

The L^p continuity of Fourier integral operators with π_L being of type at most k and π_R a Whitney fold is obtained from (0.10) *and* (0.12) *by duality.*

Remark 0.4. The asymmetry of the boundary points $p = (w+2)/(w+1)$ and $p = 3$ is caused by different assumptions on π_L and π_R.

According to [SeSoSt91], the estimate (0.10) of Theorem 2 is sharp for elliptic Fourier integral operators with the maximal singular support (when the natural projection $\mathbf{C} \to X \times Y$ has full rank $2n-1$ somewhere). We expect that (0.12) is almost sharp and remains true if $\mu = \alpha - \alpha_p - \beta - \delta_p(1,k)$, for $(w+2)/(w+1) < p < 3$. At the boundary points $p = (w+2)/(w+1)$ and $p = 3$ the continuity is probably not sharp; see [Se93] and [Ch95] for the case $k = w = 1$.

Theorem 2 overlaps with already known results: For $k = 1$ the sharp result is in [SmSo94]; for $n = 2$ and $\mathbf{C}$ being a conormal bundle, the optimal results are in [Se98]. The L^2-continuity (0.11) follows from [Co99].

We need to mention that the vector fields in the kernel of a differential of a map constitute a subalgebra:

Lemma 0.1. *Let M and N be smooth manifolds and let $\pi : M \to N$ be a smooth map. Then the vector space $\mathcal{V} = \{V \in C^\infty(\Gamma(TM)) \mid V \in \ker d\pi\}$ is a subalgebra of the Lie algebra of smooth vector fields over M.*

Proof. Let $V_1, V_2 \in \mathcal{V}$. Then, for any $f \in C^\infty(N)$, we have $V_1(\pi^* f) = V_2(\pi^* f) = 0$, and hence $d\pi([V_1, V_2])f = [V_1, V_2]\pi^* f = 0$. □

Remark 0.5. If $\pi :\ M \to N$ is a submersion, then $\ker d\pi \subset TM$ is a vector bundle over M.

Remark 0.6. If $\pi_Y \circ \pi_R$ is a submersion, then, since $\ker d(\pi_Y \circ \pi_R) \to \mathbf{C}$ is a vector bundle, any vector field $V \in \Gamma(T_\Sigma \mathbf{C})$, $V \in \ker d\pi_R \subset \ker d(\pi_Y \circ \pi_R)|_\Sigma$, can be extended to a smooth vector field $\tilde{V}$ on an open neighborhood in $\mathbf{C}$ so that $\tilde{V} \in \ker d(\pi_Y \circ \pi_R)$. Since the type of π_R is at most k, $\tilde{V}^k(\det d\pi_R) \neq 0$, and we conclude that the type of π_R with respect to $\ker d(\pi_Y \circ \pi_R)$ is smaller than the type of π_R: $w \le k$. It is convenient to think of w as of "weak type".

Lemma 0.2. *If at a point $p \in M$ the map $\pi :\ M \to N$ is of type w relative to the subalgebra $\mathcal{V} \subset C^\infty(\Gamma(TM))$, then there is a smooth vector field $W \in \mathcal{V}$ such that*

$$W^w(\det d\pi)|_p \neq 0.$$

Proof. We define a tensor $\mathfrak{t} : \underbrace{\mathcal{V} \otimes_{C^\infty(M)} \cdots \otimes_{C^\infty(M)} \mathcal{V}}_{w} \to \mathbb{R}$ by the relation

$$\mathfrak{t}(V_1, \dots, V_w) = V_1 \dots V_w (\det d\pi)\big|_p.$$

The tensor $\mathfrak{t}$ is defined up to a nonzero factor, which depends on the local coordinates where $\det d\pi$ is evaluated. The consistency of this definition follows from the identity

$$\mathfrak{t}(V_1, \dots, \varphi V_i, \dots, V_w) = \varphi(p)\mathfrak{t}(V_1, \dots, V_i, \dots, V_w), \qquad \forall V_i \in \mathcal{V}, \quad \forall \varphi \in C^\infty(M),$$

which is due to the assumption that $V_1 \dots V_{w-1}(\det d\pi)\big|_p = 0$, for any $V_i \in \mathcal{V}$.

The following identity implies that $\mathfrak{t}$ is symmetric:

$$\mathfrak{t}(V_1, \dots, V_w) - \mathfrak{t}(V_1, \dots, V_{j+1}, V_j, \dots, V_w) = \underbrace{V_1 \dots [V_j, V_{j+1}] \dots V_w}_{w-1}(\det d\pi)\big|_p = 0.$$

Now assume that $V_1, \dots, V_w \in \mathcal{V}$ are such that $V_1 \dots V_w(\det d\pi)\big|_p \neq 0$. Since $\mathfrak{t}$ is symmetric, there is a linear combination $W = \sum_{j=1}^{w} a_j V_j \in \mathcal{V}$, for some scalars a_j, such that $\mathfrak{t}(W, \dots, W) \neq 0$ (hint: use induction in w). This proves the lemma. $\square$

In Section 1, we reduce the Fourier integral operators of the form (0.7) to a more convenient form (similar to (0.2)). The scheme of the proof and the partitions of $\mathfrak{F}$ into pieces are described in Section 2. The estimates on the pieces of $\mathfrak{F}$ are obtained in Sections 3 and 4.

In Section 5, we will apply Theorem 2 to the study of the L^p regularity properties of the restrictions of solutions to hyperbolic equations onto timelike hypersurfaces ("trace regularity"). In Section 6, we consider the case when the hypersurfaces have characteristic points.

1. Reduction to a model case

Since the discussion is local, we replace both X and Y by $\mathbb{R}^n$. We consider a Lagrangian in the cotangent of $\mathbb{R}^n \times \mathbb{R}^n$ parameterized by a non-degenerate phase function $\Phi(x, \theta, y)$, with $\theta \in \mathbb{R}^N$:

$$\Sigma_\Phi = \{(x, \theta, y) \mid \Phi_\theta(x, \theta, y) = 0\} \xrightarrow{\cong} \mathbf{C} = \{x, \Phi_x, y, -\Phi_y \mid (x, \theta, y) \in \Sigma_\Phi\}.$$

Let Π be the natural projection $\Pi : \Sigma_\Phi \to \mathbb{R}^n \times \mathbb{R}^n$. By Theorem 3.1.4 of [Ho71], p. 137, we have

$$N - \operatorname{rank} \Phi_{\theta\theta}(x_0, \theta_0, y_0) = 2n - \operatorname{rank} d\Pi\big|_{(x_0, \theta_0, y_0)}; \tag{1.1}$$

we denote this number by m, $1 \leq m \leq n$.

Following [Ho71], p. 142, we reparameterize the Lagrangian with a new non-degenerate phase function $\Psi(x, \tau, y)$ with $\tau \in \mathbb{R}^m$. From the above we conclude that $\Psi''_{\tau\tau}(x_0, \tau_0, y_0) = 0$ if (x_0, τ_0, y_0) and (x_0, θ_0, y_0) correspond to the same point in $\mathbf{C}$.

Lemma 1.1. *Assume that $\pi_Y \circ \pi_R$ is a submersion. Then, in an open neighborhood of (x_0, τ_0, y_0), the condition $\Psi_\tau(x, \tau, y) = 0$ is equivalent to $x' = G(x'', \tau, y)$, where $(x', x'') \in \mathbb{R}^m \times \mathbb{R}^{n-m}$ are some local coordinates in an open neighborhood of x_0.*

Proof. Consider the map $\varrho:\ \mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^n\to\mathbb{R}^m$, $(x,\tau,y)\mapsto\Psi_\tau(x,\tau,y)$. Since Ψ is non-degenerate, ϱ is of rank m, so that its derivative $(X,T,Y)\mapsto\Psi_{\tau x}X+\Psi_{\tau\tau}T+\Psi_{\tau y}Y$ is surjective. At (x_0,τ_0,y_0), where $\Psi_{\tau\tau}=0$, we have

$$d\varrho\big|_{(x_0,\tau_0,y_0)}:\ (X,T,Y)\mapsto\Psi_{\tau x}X+\Psi_{\tau y}Y,\qquad \operatorname{rank}d\varrho\big|_{(x_0,\tau_0,y_0)}=m. \tag{1.2}$$

We claim that near (x_0,τ_0,y_0) the following map is surjective:

$$T_x\mathbb{R}^n\ni X\mapsto\Psi_{\tau x}X. \tag{1.3}$$

It is enough to consider the point (x_0,τ_0,y_0) itself. If $X\mapsto\Psi_{\tau x}X$ had rank less than m, then we could pick $\alpha\in T^*_{\tau_0}\mathbb{R}^m$, $\alpha\neq0$, such that $\langle\alpha,\Psi_{\tau x}X\rangle=0$ for any $X\in T_{x_0}\mathbb{R}^n$. We notice that $(X,T,Y)\in T_{(x_0,\tau_0,y_0)}\Sigma_\Psi$ if and only if

$$\Psi_{\tau x}X+\Psi_{\tau y}Y=0. \tag{1.4}$$

Due to the assumption that $\pi_Y\circ\pi_R$ is a submersion, for any $Y\in T_{y_0}\mathbb{R}^n$ there exists $Z=(X,T,Y)\in T_{(x_0,\tau_0,y_0)}\Sigma_\Psi$. Then (1.4) would imply that $\langle\alpha,\Psi_{\tau y}Y\rangle=0$ for any Y, and hence $\langle\alpha,d\varrho(Z)\rangle=0$ for any $Z\in T_{(x_0,\tau_0,y_0)}\Sigma_\Psi$. Then (1.2) would have rank less than m, which is a contradiction. Hence (1.3) is surjective, and we conclude that $\Psi_{\tau x}$ is of rank m. This proves the lemma. $\square$

Remark 1.1. Since $\Psi(x,\tau,y)$ is homogeneous in τ of degree 1, $\Psi_\tau(x,\tau,y)$ is homogeneous in τ of degree 0, and so is $G(x'',\tau,y)$.

By [Ho71], Lemma 1.1 proves that (locally) $\mathfrak{F}$ has the form

$$\mathfrak{F}u(x)=\iint e^{i\langle x'-G(x'',\tau,y),\tau\rangle}b(x,\tau,y)u(y)\,d\tau\,dy,\qquad x,\,y\in\mathbb{R}^n,\quad\tau\in\mathbb{R}^m, \tag{1.5}$$

where $G(x'',\tau,y)$ is homogeneous in τ of degree 0. We will denote the order of the symbol by d: $b(x,\tau,y)\in S^d_{1,0}(\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^n)$, so that $\mathfrak{F}\in I^{d+\frac{m-n}{2}}(X,Y,\mathbf{C})$.

The canonical relation $\mathbf{C}\subset T^*X\times T^*Y$ is parameterized by (x'',τ,y):

$$\{(\partial_\tau\langle G(x'',\tau,y),\tau\rangle,\,x'';\,\tau,\,\langle\partial_{x''}G(x'',\tau,y),\tau\rangle)\,;\ \ (y;\,\langle\partial_yG(x'',\tau,y),\tau\rangle)\}. \tag{1.6}$$

Representing the projections π_L and π_R as

$$\pi_L:\ (x'',\tau,y)\mapsto(\partial_\tau\langle G(x'',\tau,y),\tau\rangle,\,x'';\,\tau,\,\langle\partial_{x''}G(x'',\tau,y),\tau\rangle)\,, \tag{1.7}$$

$$\pi_R:\ (x'',\tau,y)\mapsto(y;\,\langle\partial_yG(x'',\tau,y),\tau\rangle)\,, \tag{1.8}$$

we conclude that the determinants of the Jacobi matrices of both π_L and π_R are proportional to $\det\partial_{x'',\tau}\partial_y\langle G(x'',\tau,y),\tau\rangle$. We multiply this expression by $|\tau|^{-(n-m)}$, to make it homogeneous of degree 0 in τ, and denote it by $h(x'',\tau,y)$:

$$h(x'',\tau,y)=|\tau|^{-(n-m)}\det\partial_{x'',\tau}\partial_y\langle G(x'',\tau,y),\tau\rangle. \tag{1.9}$$

We need to reformulate the type conditions which enter Theorem 2 in terms of the phase in (1.5). From the explicit form (1.8) for π_R we see that $\pi_Y\circ\pi_R$: $(x'',\,\tau,\,y)\mapsto y$, and therefore

$$\ker d(\pi_Y\circ\pi_R)=\operatorname{span}(\partial_{x''},\partial_\tau). \tag{1.10}$$

Now, for example, the condition that π_R is of type 1 relative to $\ker d(\pi_Y\circ\pi_R)$ (this corresponds to $w=1$ in Theorem 2) can be expressed as $d_{x'',\tau}h(x'',\tau,y)\neq0$.

According to (1.10), $\ker d(\pi_Y\circ\pi_R)$ is a vector bundle over $\mathbf{C}$, which is a consequence of $\pi_Y\circ\pi_R$ being a submersion (see Remark 0.5).

2. Scheme of the proof

Let $b(x,\tau,y) \in S^{-\frac{m-1}{2}}(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n)$, so that $\mathfrak{F} \in I^{-\frac{n-1}{2}}(X,Y,\mathbf{C})$. Let us localize the integral kernel of $\mathfrak{F}$ with respect to the values of $|\tau|$ and $h(x'',\tau,y) = |\tau|^{-(n-m)} \det \partial_{x'',\tau}\partial_y \langle G(x'',\tau,y),\tau\rangle$:

$$\mathfrak{F}^{\hbar}_{\lambda} u(x) = \int\limits_{\mathbb{R}^m\times\mathbb{R}^n} e^{i\langle x'-G(x'',\tau,y),\tau\rangle} b(x,\tau,y)\beta(\frac{|\tau|}{\lambda})\beta(\hbar^{-1}h)u(y)\,d\tau\,dy, \tag{2.1}$$

$$\bar{\mathfrak{F}}^{\hbar}_{\lambda} u(x) = \int\limits_{\mathbb{R}^m\times\mathbb{R}^n} e^{i\langle x'-G(x'',\tau,y),\tau\rangle} b(x,\tau,y)\beta(\frac{|\tau|}{\lambda})\bar{\beta}(\hbar^{-1}h)u(y)\,d\tau\,dy, \tag{2.2}$$

where $\beta \in C^\infty_{\mathrm{comp}}([\frac{1}{2},2])$, $\bar{\beta} \in C^\infty_{\mathrm{comp}}([-2,2])$ satisfy $\sum_{\pm}\sum_{j\in\mathbb{N}} \beta(\pm 2^{-j}t) + \bar{\beta}(t) = 1$.

We use the following decompositions of $\mathfrak{F}$:

$$\begin{aligned} \mathfrak{F} = \mathfrak{F}_0 &+ \sum_{\lambda=2^l,\, l\in\mathbb{N}} \; \sum_{\substack{\hbar=2^{-j},\, j\in\mathbb{Z} \\ 2\hbar_o(\lambda)\le\hbar\le\sup|h|}} \; \sum_{\pm} \mathfrak{F}^{\pm\hbar}_{\lambda} \\ &+ \sum_{\lambda=2^l,\, l\in\mathbb{N}} \; \sum_{\substack{\hbar=2^{-j},\, j\in\mathbb{Z} \\ \hbar_o(\lambda)\le\hbar<2\hbar_o(\lambda)}} \bar{\mathfrak{F}}^{\hbar}_{\lambda}. \end{aligned} \tag{2.3}$$

We have used dyadic partitions with respect to the magnitude of momenta, $|\tau| \approx \lambda = 2^l$, $l \in \mathbb{N}$, and with respect to the distance from the critical variety (which is the set $\Sigma \subset \mathbf{C}$ where the projections from $\mathbf{C}$ are singular): $|h| \approx \hbar = 2^{-j}$, $j \in \mathbb{Z}$. $\mathfrak{F}_0$ corresponds to the part of $\mathfrak{F}$ with $|\tau| \le 2$; it is an infinitely smoothing operator. The cut-off value $\hbar_o(\lambda)$ is chosen to be $\hbar_o(\lambda) = \lambda^{-\frac{k}{2k+1}}$ (as in [Co99]). The second summation in (2.3) contains only one operator $\bar{\mathfrak{F}}^{\hbar}_{\lambda}$ for each particular value of $\lambda = 2^l$ (there is only one value of $\hbar = 2^{-j}$, $j \in \mathbb{Z}$, such that $\hbar_o(\lambda) \le \hbar < 2\hbar_o(\lambda)$).

In (2.3), we separate the terms into groups with a fixed value of $\hbar = 2^{-j}$, $j \in \mathbb{Z}$:

$$\mathfrak{F}^{\hbar} = \sum_{\lambda:\, \hbar_o(\lambda)\le\hbar<2\hbar_o(\lambda)} \bar{\mathfrak{F}}^{\hbar}_{\lambda} + \sum_{\lambda:\, \hbar\ge 2\hbar_o(\lambda)} \sum_{\pm} \mathfrak{F}^{\pm\hbar}_{\lambda}, \tag{2.4}$$

where λ takes values $\lambda = 2^l$, $l \in \mathbb{N}$. Since $\hbar_o(\lambda) = \lambda^{-\frac{1}{2k+1}}$, the first sum in the right-hand side of (2.4) only contains finitely many terms (at most three). We have

$$\mathfrak{F} = \mathfrak{F}_0 + \sum_{\hbar\le\sup|h|} \mathfrak{F}^{\hbar}, \qquad \text{where} \quad \hbar = 2^{-j},\ j \in \mathbb{Z}. \tag{2.5}$$

According to the results of [GrSe94], [Cu97], and [Co99], we know the following L^2-continuity of Fourier integral operators $\mathfrak{F}^{\hbar}_{\lambda}$, $\bar{\mathfrak{F}}^{\hbar}_{\lambda}$ (the argument for $\pm\hbar$ is the same, independent of the sign):

Proposition 2.1. *Let $\mathfrak{F} \in I^0(X,Y,\mathbf{C})$. Assume that one of the projections π_L, π_R from $\mathbf{C}$ is a Whitney fold, while the other is of type at most k. Then*

$$\|\mathfrak{F}^{\hbar}_{\lambda}\|_{L^2\to L^2} \le \mathrm{const}\,\hbar^{-\frac{1}{2}}, \tag{2.6}$$

$$\|\bar{\mathfrak{F}}^{\hbar}_{\lambda}\|_{L^2\to L^2} \le \mathrm{const}\,\lambda^{\frac{1}{2}}\hbar^{\frac{1}{2}+\frac{1}{2k}}. \tag{2.7}$$

Using standard orthogonality arguments for the operators $\mathfrak{F}^{\hbar}_{\lambda}$ with different values of λ, as in [Se93], we obtain

Corollary. *For $\mathfrak{F} \in I^0(X,Y,\mathbf{C})$, with $\mathbf{C}$ as in Proposition 2.1,*

$$\|\mathfrak{F}^\hbar\|_{L^2\to L^2} \le \operatorname{const}\hbar^{-\frac{1}{2}}. \tag{2.8}$$

Now we would like to establish the $\mathsf{H}^1 \to L^1$ continuity of these operators.

Proposition 2.2. *Let $\mathfrak{F} \in I^{-\frac{n-1}{2}}(X,Y,\mathbf{C})$. Assume that $\pi_R : \mathbf{C} \to T^*Y$ is of type at most k, and of type at most $w \le k$ relative to $\ker d(\pi_Y \circ \pi_R)$. Then for any atom a_{Q} supported in a box Q with the sidelength r there is the following bound on $\mathfrak{F}^\hbar_\lambda a_{\mathsf{Q}}$ and $\bar{\mathfrak{F}}^\hbar_\lambda a_{\mathsf{Q}}$:*

$$\|\mathfrak{F}^\hbar_\lambda a_{\mathsf{Q}}\|_{L^1},\ \|\bar{\mathfrak{F}}^\hbar_\lambda a_{\mathsf{Q}}\|_{L^1} \le \operatorname{const}\hbar^{\frac{1}{w}}. \tag{2.9}$$

We will prove this proposition in Section 3.

This estimate does not contain any improvement for the situations when λ is very large or instead very small compared to r^{-1}, thus giving the same bounds on all the terms in

$$\|\mathfrak{F}^\hbar a_{\mathsf{Q}}\|_{L^1} = \sum_{\lambda:\ \hbar_o(\lambda)\le\hbar<2\hbar_o(\lambda)} \|\bar{\mathfrak{F}}^\hbar_\lambda a_{\mathsf{Q}}\|_{L^1} + \sum_{\lambda:\ \hbar\ge 2\hbar_o(\lambda)} \sum_{\pm} \|\mathfrak{F}^{\pm\hbar}_\lambda a_{\mathsf{Q}}\|_{L^1}. \tag{2.10}$$

Remark 2.1. If we took $\mathfrak{F} \in I^\mu(X,Y,\mathbf{C})$ with $\mathbf{C}$ as in Theorem 2 with $\mu < -\frac{n-1}{2}$, then the summation in (2.10) would converge (due to the extra negative power of λ in each term), yielding the estimate $\|\mathfrak{F}^\hbar\|_{\mathsf{H}^1\to L^1} \le \|\mathfrak{F}^\hbar a_{\mathsf{Q}}\|_{L^1} \le \operatorname{const}\hbar^{\frac{1}{w}}$. The Fefferman-Stein interpolation of this estimate with the $L^2 \to L^2$ estimate (2.8) would show that the estimates $\|\mathfrak{F}^\hbar\|_{L^p\to L^p}$, where $\mathfrak{F} \in I^\mu(X,Y,\mathbf{C})$ with $\mu < (n-1)\left|\frac{1}{p}-\frac{1}{2}\right|$, decrease together with $\hbar$ if $1 < p < \frac{w+2}{w+1}$, thus proving an almost sharp version of the result stated in Theorem 2.

To enable the summation in (2.10), we need to consider several different cases.

First, let us consider the case when $r \ge 1$.

Proposition 2.3. *Let $\mathfrak{F} \in I^{-\frac{n-1}{2}}(X,Y,\mathbf{C})$. Assume that both $\pi_L : \mathbf{C} \to T^*X$ and $\pi_R : \mathbf{C} \to T^*Y$ are of finite type, and that at least one of them is a Whitney fold. Then*

$$\|\mathfrak{F}^\hbar_\lambda a_{\mathsf{Q}}\|_{L^1},\ \|\bar{\mathfrak{F}}^\hbar_\lambda a_{\mathsf{Q}}\|_{L^1} \le \operatorname{const}\lambda^{-\frac{n}{2}+\frac{3}{4}} \qquad \text{when} \quad r \ge 1. \tag{2.11}$$

Proof. Since the integral kernel of $\mathfrak{F}$ is compactly supported,

$$\left\|\mathfrak{F}^\hbar_\lambda a_{\mathsf{Q}}\right\|_{L^1} \le \operatorname{const}\left\|\mathfrak{F}^\hbar_\lambda a_{\mathsf{Q}}\right\|_{L^2} \le \operatorname{const}\|a_{\mathsf{Q}}\|_{L^2}\left\|\mathfrak{F}^\hbar_\lambda\right\|_{L^2\to L^2}.$$

Due to Proposition 2.1, $\left\|\mathfrak{F}^\hbar_\lambda\right\|_{L^2\to L^2} \le \operatorname{const}\lambda^{-\frac{n-1}{2}}\hbar^{-\frac{1}{2}} \le \operatorname{const}\lambda^{-\frac{n}{2}+\frac{3}{4}}$, while $\|a_{\mathsf{Q}}\|_{L^2} \le r^{-\frac{n}{2}} \le 1$. This proves (2.11). □

Now we assume that $r \le 1$.

The summation in (2.10) when $\lambda < r^{-1}$ is performed in the same way as in [SeSoSt91]:

Proposition 2.4. *Under the assumptions of Proposition 2.2, if we further assume that the atom a_{Q} is supported in a box Q with the sidelength r so small that $\lambda < r^{-1}$, then both $\mathfrak{F}^\hbar_\lambda a_{\mathsf{Q}}$ and $\bar{\mathfrak{F}}^\hbar_\lambda a_{\mathsf{Q}}$ are bounded by (2.9) with an extra factor λr:*

$$\|\mathfrak{F}^\hbar_\lambda a_{\mathsf{Q}}\|_{L^1},\ \|\bar{\mathfrak{F}}^\hbar_\lambda a_{\mathsf{Q}}\|_{L^1} \le \operatorname{const}\lambda r\hbar^{\frac{1}{w}}. \tag{2.12}$$

Proof. Let us denote the integral kernel of $\mathfrak{F}^\hbar_\lambda$ by $K^\hbar_\lambda(x,\tau,y)$:

$$K^\hbar_\lambda(x,\tau,y) = e^{i\langle x' - G(x'',\tau,y),\tau\rangle} b(x,\tau,y)\beta\left(\frac{|\tau|}{\lambda}\right)\beta(\hbar^{-1}h(x'',\tau,y)). \tag{2.13}$$

We fix some point $\bar{y} \in \mathrm{Q}$. Since $\int a_{\mathrm{Q}}(y)dy = 0$, we can write

$$\begin{aligned}
\mathfrak{F}^\hbar_\lambda a_{\mathrm{Q}}(x) &= \int \left[K^\hbar_\lambda(x,\tau,y) - K^\hbar_\lambda(x,\tau,\bar{y})\right] a_{\mathrm{Q}}(y) d\tau\, dy \\
&= \int_0^1 dt \partial_t \left(\int K^\hbar_\lambda(x,\tau,\bar{y} + (y-\bar{y})t) a_{\mathrm{Q}}(y) d\tau\, dy \right) \\
&= \lambda r \int \left\{ \int_0^1 dt\, \frac{y-\bar{y}}{r}\, \lambda^{-1}\partial_y K^\hbar_\lambda(x,\tau,\bar{y}+(y-\bar{y})t) \right\} a_{\mathrm{Q}}(y) d\tau\, dy.
\end{aligned} \tag{2.14}$$

The expression in the curly brackets can be treated as an integral kernel of another Fourier integral operator of the same order μ associated to $\mathbf{C}$. Let us mention that $|\frac{y-\bar{y}}{r}| \le \mathrm{const}$ and that the increase in the order of the symbol due to the derivative ∂_y is compensated by λ^{-1}. When the derivative ∂_y acts on $\beta(\hbar^{-1}h(x'',\tau,y))$ (which is hidden inside $K^\hbar_\lambda$), the contribution is bounded by $\mathrm{const}\,\hbar^{-1}$ and is also compensated by λ^{-1}. The integration in t is irrelevant. □

Proposition 2.4 means that the sum of all the terms in (2.10) with $\lambda < r^{-1}$ can be estimated by the same quantity as individual terms with $\lambda \sim r^{-1}$, so that we do not have to think about them.

Proposition 2.5. *Let $\mathfrak{F} \in I^{-\frac{n-1}{2}}(X,Y,\mathbf{C})$. Assume that both $\pi_L : \mathbf{C} \to T^*X$ and $\pi_R : \mathbf{C} \to T^*Y$ are of finite type, and that at least one of them is a Whitney fold. Then*

$$\|\mathfrak{F}^\hbar_\lambda a_{\mathrm{Q}}\|_{L^1},\ \|\bar{\mathfrak{F}}^\hbar_\lambda a_{\mathrm{Q}}\|_{L^1} \le \mathrm{const}\,\hbar^{-\frac{1}{2}}(\lambda r)^{-\frac{n-1}{2}} \qquad \textit{when} \quad \lambda^{-1} \le r \le 1. \tag{2.15}$$

We will prove (2.15) in Section 4.

Now let us summarize how Propositions 2.3, 2.4, and 2.5 to enable the summation in (2.10).

Corollary. *Let $\mathfrak{F} \in I^{-\frac{n-1}{2}}(X,Y,\mathbf{C})$. Assume that $\pi_L : \mathbf{C} \to T^*X$ is a Whitney fold and that $\pi_R : \mathbf{C} \to T^*Y$ is of type at most k, and of type at most $w \le k$ relative to $\ker d(\pi_Y \circ \pi_R)$. Then, for $\mathfrak{F}^\hbar$ defined as above,*

$$\|\mathfrak{F}^\hbar\|_{\mathrm{H}^1 \to L^1} \le C_\epsilon \hbar^{\frac{1}{w}-\epsilon}, \tag{2.16}$$

for any $\epsilon > 0$.

Proof of the Corollary. Consider an atom a_{Q} supported in a cube Q with a side of length r. We need to show that the L^1-norm of $\mathfrak{F}^\hbar a_{\mathrm{Q}}$ is bounded by (2.16). We have

$$\|\mathfrak{F}^\hbar a_{\mathrm{Q}}\|_{L^1} \le \sum_{\lambda:\ \hbar_o(\lambda) \le \hbar < 2\hbar_o(\lambda)} \|\bar{\mathfrak{F}}^\hbar_\lambda a_{\mathrm{Q}}\|_{L^1} + \sum_{\lambda:\ \hbar \ge 2\hbar_o(\lambda)} \sum_{\pm} \|\mathfrak{F}^{\pm\hbar}_\lambda a_{\mathrm{Q}}\|_{L^1}, \qquad \lambda = 2^N.$$

We use Propositions 2.2 and 2.5. If $r \le 1$, we apply the estimate (2.11) for the terms with $\lambda \le r^{-1}$ and the weighted geometric mean of (2.9) and (2.15) for the terms with $\lambda^{-1} \le r$, and obtain

$$\|\mathfrak{F}^\hbar a_Q\|_{L^1} \le \text{const}\left[\sum_{\substack{\lambda=2^l,\, l\in\mathbb{N}\\ \lambda\le r^{-1}}} \hbar^{\frac{1}{w}}\lambda r + \sum_{\substack{\lambda=2^l,\, l\in\mathbb{N}\\ \lambda> r^{-1}}} \hbar^{\frac{1}{w}-\epsilon}(\lambda r)^{-O(\epsilon)}\right] \le C_\epsilon \hbar^{\frac{1}{w}-\epsilon}, \tag{2.17}$$

for any $\epsilon > 0$.

If $r \ge 1$, the weighted geometric mean of (2.9) and (2.11) leads to the same bound. □

The Fefferman-Stein interpolation theorem applied to (2.16) and (2.8) yields the following:

Corollary. *If $\mathfrak{F} \in I^{-\alpha_p}(X, Y, \mathbf{C})$, $\alpha_p = (n-1)|\frac{1}{p} - \frac{1}{2}|$, $1 < p \le 2$, and $\mathfrak{F}^\hbar$ is defined as above, then*

$$\|\mathfrak{F}^\hbar\|_{L^p\to L^p} \le C_\epsilon \hbar^{(\frac{1}{w}-\epsilon)\left(1-\frac{2}{p'}\right)-\frac{1}{p'}}, \tag{2.18}$$

for any $\epsilon > 0$.

If $p < \frac{w+2}{w+1}$ (so that $\frac{1}{w}(1-\frac{2}{p'}) - \frac{1}{p'} > 0$), then in (2.17) we can take ϵ small enough for the exponent in (2.18) to be positive. Then the series $\sum_{\hbar\le 2\sup|h|}\|\mathfrak{F}^\hbar\|_{L^p\to L^p}$, $\hbar = 2^{-l}$, $l \in \mathbb{Z}$, converges, and hence $\|\mathfrak{F}\|_{L^p\to L^p}$ is bounded. This proves Theorem 2 for $1 < p < \frac{w+2}{w+1}$. The continuity of $\mathfrak{F}$ in L^p for $3 < p < \infty$ is obtained by duality from the case with $w = 1$.

3. Asymptotics for H^1-L^1 estimates

In this section, we are going to prove Proposition 2.2. We denote the integral kernels of $\mathfrak{F}^\hbar_\lambda$, $\bar{\mathfrak{F}}^\hbar_\lambda$ by $K^\hbar_\lambda(x,\tau,y)$ and $\bar{K}^\hbar_\lambda(x,\tau,y)$. We will decompose and bound these kernels following the discussion on pp. 238-241 in [SeSoSt91]. For a particular λ, we introduce unit vectors τ^ν_λ, with $1 \le \nu \le N(\lambda^{-1/2}) \approx \lambda^{\frac{m-1}{2}}$, equidistributed on the unit sphere in the τ-space $\mathbb{R}^m$, so that $|\tau^\nu_\lambda - \tau^{\nu'}_\lambda| \ge \text{const}\,\lambda^{-\frac{1}{2}}$ for $\nu \ne \nu'$.

We introduce a corresponding partition of unity:

$$1 = \sum_{\nu=1}^{N(\lambda^{-1/2})} \chi^\nu_\lambda(\tau), \tag{3.1}$$

where the functions χ^ν_λ are homogeneous of degree 0 and supported in the spherical angles Ω^ν_λ with the span $\sim \lambda^{-1/2}$, centered at τ^ν_λ:

$$\chi^\nu_\lambda(\tau) \ne 0 \quad \text{only if} \quad \left|\frac{\tau}{|\tau|} - \tau^\nu_\lambda\right| \le \text{const}\,\lambda^{-\frac{1}{2}}. \tag{3.2}$$

We assume that $|\partial^\alpha_\tau \chi^\nu_\lambda(\tau)| \le \text{const}\,\lambda^{\frac{|\alpha|}{2}}|\tau|^{-|\alpha|}$.

We introduce $\mathfrak{F}^{\hbar,\nu}_\lambda$ by

$$\mathfrak{F}^{\hbar,\nu}_\lambda u(x) = \int K^{\hbar,\nu}_\lambda(x,\tau,y)u(y)\,d\tau\,dy, \tag{3.3}$$

where $K^{\hbar,\nu}_\lambda(x,\tau,y) = \chi^\nu_\lambda(\tau)\cdot K^\hbar_\lambda(x,\tau,y)$.

Let $\mathcal{R}^\nu_{y,\lambda^{-1}} = \{x \mid |\langle x' - G(x'', \tau^\nu_\lambda, y), \tau^\nu_\lambda\rangle| \le \lambda^{-1}, |x' - G(x'', \tau^\nu_\lambda, y)| \le \lambda^{-\frac{1}{2}}\}$, with $\left|\mathcal{R}^\nu_{y,\lambda^{-1}}\right| \le \operatorname{const}\lambda^{-1}\cdot\lambda^{-\frac{m-1}{2}}$. We set $\bar\chi_{\mathcal{R}^\nu_{y,\lambda^{-1}}}(x)$ to be the characteristic function of $\mathcal{R}^\nu_{y,\lambda^{-1}}$.

Let a_{Q} be an atom supported in a box Q with sidelength r:

$$|\mathrm{Q}| = r^n, \qquad \|a_{\mathrm{Q}}\|_{L^\infty} \le R^{-n}, \qquad \|a_{\mathrm{Q}}\|_{L^1} \le 1, \qquad \int a_{\mathrm{Q}} = 0.$$

We consider

$$\begin{aligned}\mathfrak{F}^\hbar_\lambda a_{\mathrm{Q}}(x) &= \sum_\nu \int \bar\chi_{\mathcal{R}^\nu_{y,\lambda^{-1}}}(x) K^{\hbar,\nu}_\lambda(x,\tau,y) a_{\mathrm{Q}}(y)\,d\tau\,dy \\ &\quad + \sum_\nu \int (1-\bar\chi_{\mathcal{R}^\nu_{y,\lambda^{-1}}}(x)) K^{\hbar,\nu}_\lambda(x,\tau,y) a_{\mathrm{Q}}(y)\,d\tau\,dy.\end{aligned} \tag{3.4}$$

We need to know the L^1-norm of this expression.

(i) The L^1-norm of first term in the right-hand side is bounded by

$$\begin{aligned}&\sum_\nu \int \bar\chi_{\mathcal{R}^\nu_{y,\lambda^{-1}}}(x)\left|K^{\hbar,\nu}_\lambda(x,\tau,y)a_{\mathrm{Q}}(y)\right|\,dx\,d\tau\,dy \\ &\le C\lambda^{-\frac{m-1}{2}}\sum_\nu \int \bar\chi_{\mathcal{R}^\nu_{y,\lambda^{-1}}}(x)\chi^\nu_\lambda(\tau)\beta(\frac{|\tau|}{\lambda})\beta(\hbar^{-1}h(x'',\tau,y))|a_{\mathrm{Q}}(y)|\,dx\,d\tau\,dy.\end{aligned} \tag{3.5}$$

∘ In (3.5), we have already applied the bound $C\lambda^{-\frac{m-1}{2}}$ on the symbol $b(x,\tau,y)$ at $|\tau| \sim \lambda$.

∘ Due to the support properties of $\bar\chi_{\mathcal{R}^\nu_{y,\lambda^{-1}}}(x)$, the integration in x' (with x'' fixed) contributes $\lambda^{-\frac{m+1}{2}}$.

∘ Summation in ν converges, since $\sum_\nu \chi^\nu_\lambda(\tau) = 1$.

∘ If the projection π_R is of type $w = 1$ relative to $\ker d(\pi_Y \circ \pi_R)$, then the integration in τ and x'' contributes $\hbar\cdot\lambda^m$, where $\hbar$ appears due to the support properties of $\beta(\hbar^{-1}h)$ (recall that, according to (1.10), w is equal to 1 if $d_{x'',\tau}h \ne 0$).

More generally, assume that at a point $p \in \mathbf{C}$ the projection π_R is of type at most $w \in \mathbb{N}$ relative to $\ker d(\pi_Y \circ \pi_R)$. We define $\mathcal{T} = \frac{\tau}{\lambda} \in \mathbb{R}^m$, so that the region of integration in x'' and $\mathcal{T}$ is bounded uniformly in λ. Note that $dx''\,d\tau = \lambda^m dx''\,d\mathcal{T}$. According to Lemma 0.2 and to (1.10), we may choose new coordinates, $z = (z_1, \dots, z_n)$, $z = z(x'', \mathcal{T})$, such that $\partial^w_{z_n} h \ne 0$ in an open neighborhood of p. The expression $\partial^w_{z_n} h$ is homogeneous of degree zero in λ, so that $|\partial^w_{z_n} h| \ge \operatorname{const} > 0$ uniformly in λ, $\hbar$. Therefore (see Lemma 3.1 at the end of this section),

$$\int_{\mathbb{R}} dz_n\,\beta(\hbar^{-1}h) \le \operatorname{const}\hbar^{\frac{1}{w}}. \tag{3.6}$$

The integration in $z_1, \dots, z_{n-1}$ converges since the support of (3.5) in $(x'', \mathcal{T} = \frac{\tau}{\lambda})$ (and hence in $\{z_1, \dots, z_n\}$) is bounded uniformly in λ, $\hbar$. We conclude that the integration in x'' and τ contributes $\operatorname{const}\lambda^m\hbar^{\frac{1}{w}}$.

∘ Finally, we integrate in y (using the bound $\int |a_{\mathrm{Q}}(y)|\,dy \le \operatorname{const}$).

Taking the product of all of the above factors, we obtain $\operatorname{const}\hbar^{\frac{1}{w}}$, proving (2.9).

(ii) For the L^1-norm of the second term in the right-hand side of (3.4) we have:

∘ In each ν-term, we can integrate by parts as in [SeSoSt91] (we need the assumption $\hbar \ge \lambda^{-\frac{1}{2}}$ to obtain an analogue of the inequalities (3.19) in [SeSoSt91]; the argument is the same as theirs), getting the factor

$$\left(1+\lambda^2|\langle x'-G(x'',\tau_\lambda^\nu,y),\tau_\lambda^\nu\rangle|^2+\lambda|x'-G(x'',\tau_\lambda^\nu,y)|^2\right)^{-N}. \tag{3.7}$$

The integration of this expression with respect to x' contributes the same factor $\lambda^{-\frac{m+1}{2}}$ as above. The rest of the analysis is the same as for the first term in the right-hand side of (3.4), and we obtain the same bound (2.9).

We conclude that, for any atom a_{Q}, $\|\mathfrak{F}_\lambda^\hbar a_{\mathrm{Q}}\|_{L^1}\le \mathrm{const}\,\hbar^{\frac{1}{w}}$. This completes the proof of Proposition 2.2.

Let us recall how from $|\partial_{z_n}^w h|\ge \mathrm{const}>0$ one may obtain the bound (3.6):

Lemma 3.1. *Let $h\in C^w(\mathbb{R})$ be a function such that $|h^{(w)}(t)|\ge\varkappa>0$ for t in some interval $I\subset\mathbb{R}$. Then the set $I^\hbar=\{t\in I\mid |h(t)|<\hbar\}$ consists of at most 2^{w-1} intervals $I_\sigma^\hbar$, possibly with joint ends, of total measure $|I_\sigma^\hbar|\le 2^{w-1}\,(2w!/\varkappa)^{\frac{1}{w}}\,\hbar^{\frac{1}{w}}$.*

This lemma is well-known; see, e.g., [Ch95]. Instead of giving a proof, let us simply note that the mentioned intervals are those where the derivatives h', h'', ..., $h^{(n-1)}$ do not change signs.

4. Stronger asymptotics for $\mathsf{H}^1\to L^1$ estimates

In this section, we will prove Proposition 2.5: Given an atom a_{Q} supported in a cube Q with the side r, $|\mathrm{Q}|=r^n$, with $\lambda^{-1}\le r\le 1$, we want to prove (2.15):

$$\left\|\mathfrak{F}_\lambda^\hbar a_{\mathrm{Q}}\right\|_{L^1}\le \mathrm{const}\,\hbar^{-\frac12}(\lambda r)^{-\frac{n-1}{2}}.$$

We introduce unit vectors τ_r^ρ, $1\le\rho\le N(r^{\frac12})\approx r^{-\frac{m-1}{2}}$, equidistributed over the unit sphere in the τ-space $\mathbb{R}^m$:

$$|\tau_r^\rho-\tau_r^{\rho'}|\ge \mathrm{const}\,r^{1/2}\quad \text{if}\quad \rho\ne\rho'.$$

We introduce a corresponding partition of unity, as in (3.1):

$$1=\sum_{\rho=1}^{N(r^{1/2})}\chi_r^\rho(\tau),$$

where $\chi_r^\rho(\tau)$ is homogeneous of degree 0 in τ and supported in the spherical angle Ω_r^ρ with the span $\sim r^{1/2}$, centered at τ_r^ρ.

The set of the exceptional values of x which correspond to a particular value of y and to a particular direction τ_r^ρ in τ-space is given by

$$\mathcal{R}_{y,r}^\rho=\{x\mid\ |\langle x'-G(x'',\tau_r^\rho,y),\tau_r^\rho\rangle|\le r,\ \ |x'-G(x'',\tau_r^\rho,y)|\le r^{1/2}\ \}, \tag{4.1}$$

so that $|\mathcal{R}_{y,r}^\rho|\le\mathrm{const}\,r^{\frac{m+1}{2}}$. We then define

$$\mathcal{N}_{\mathrm{Q}}^\rho=\bigcup_{y\in\mathrm{Q}}\mathcal{R}_{y,r}^\rho,\qquad \text{with}\quad |\mathcal{N}_{\mathrm{Q}}^\rho|\le\mathrm{const}\,r^{\frac{m+1}{2}}. \tag{4.2}$$

The bound on $|\mathcal{N}_{\mathrm{Q}}^\rho|$ is valid since the range of change of y is bounded by r; hence $\mathcal{N}_{\mathrm{Q}}^\rho$ is not much different from any individual $\mathcal{R}_{y,r}^\rho$. Note that in the terminology of [SeSoSt91] the exceptional set associated with the atom a_{Q} is given by

$$\mathcal{N}_{\mathrm{Q}}=\bigcup_\rho\mathcal{N}_{\mathrm{Q}}^\rho.$$

We set $\bar{\chi}_{\mathcal{N}^\rho_{\mathrm{Q}}}(x)$ to be the characteristic function of the set $\mathcal{N}^\rho_{\mathrm{Q}}$, and split

$$\begin{aligned}\mathfrak{F}^\hbar_\lambda a_{\mathrm{Q}}(x) &= \sum_\rho \int \bar{\chi}_{\mathcal{N}^\rho_{\mathrm{Q}}}(x)\chi^\rho_r(\tau)K^\hbar_\lambda(x,\tau,y)a_{\mathrm{Q}}(y)\,d\tau\,dy \\ &+ \sum_\rho\sum_\nu \int (1-\bar{\chi}_{\mathcal{N}^\rho_{\mathrm{Q}}}(x))\chi^\rho_r(\tau)K^{\hbar,\nu}_\lambda(x,\tau,y)a_{\mathrm{Q}}(y)\,d\tau\,dy.\end{aligned} \tag{4.3}$$

(i) Let us start with the L^1-norm of the second part of the right-hand side:
∘ In each ν,ρ-term, we integrate by parts, getting the factor similar to (3.7). Since we are off the exceptional set $\mathcal{N}^\rho_{\mathrm{Q}}$, this factor is bounded by

$$\left(1+\lambda^2|\langle x'-G(x'',\tau^\rho_r,y),\tau^\nu_\lambda\rangle|^2+\lambda|x'-G(x'',\tau^\rho_r,y)|^2\right)^{-N} \le \mathrm{const}\,(\lambda r)^{-N}. \tag{4.4}$$

∘ Due to the factor (4.4), integration in x' would contribute $\lambda^{-\frac{m+1}{2}}$.
∘ The symbol $b(x,\tau,y)$ is bounded by $\mathrm{const}\,\lambda^{-\frac{m-1}{2}}$.
∘ Due to the support properties, summations in ρ and ν converge uniformly in λ, $\hbar$.
∘ As in Section 3, the integration in τ and x'' contributes $\hbar^{\frac{1}{w}}\cdot\lambda^m$.
∘ The integration in y converges, since for atoms we have $\int|a_{\mathrm{Q}}(y)|\,dy\le 1$.

The product of all the above terms yields the bound

$$\left\|\sum_{\rho,\nu}\int(1-\bar{\chi}_{\mathcal{N}^\rho_{\mathrm{Q}}}(x))\chi^\rho_r(\tau)K^{\hbar,\nu}_\lambda(x,\tau,y)a_{\mathrm{Q}}(y)\,d\tau\,dy\right\|_{L^1} \le \mathrm{const}\,\hbar^{\frac{1}{w}}(\lambda r)^{-N'}.$$

(ii) Now we need to bound the L^1-norm of the first term in the right-hand side of (4.3). First, we apply the Cauchy-Schwarz inequality to the summation in ρ:

$$\begin{aligned}&\left\|\sum_\rho\int\bar{\chi}_{\mathcal{N}^\rho_{\mathrm{Q}}}(x)\chi^\rho_r(\tau)K^\hbar_\lambda(x,\tau,y)a_{\mathrm{Q}}(y)\,d\tau\,dy\right\|_{L^1_x} \\ &\le \left\|\left(\sum_\rho\bar{\chi}_{\mathcal{N}^\rho_{\mathrm{Q}}}(x)\right)^{\frac12}\left(\sum_\rho\left|\int\bar{\chi}_{\mathcal{N}^\rho_{\mathrm{Q}}}(x)\chi^\rho_r(\tau)K^\hbar_\lambda(x,\tau,y)a_{\mathrm{Q}}(y)\,d\tau\,dy\right|^2\right)^{\frac12}\right\|_{L^1_x}\end{aligned}$$

Now we apply the Cauchy-Schwarz inequality to the integration in the x variable, bounding the above by

$$\begin{aligned}&\left[\int dx\sum_\rho\bar{\chi}_{\mathcal{N}^\rho_{\mathrm{Q}}}(x)\right]^{\frac12}\left[\int dx\sum_\rho\left|\int\bar{\chi}_{\mathcal{N}^\rho_{\mathrm{Q}}}(x)\chi^\rho_r(\tau)K^\hbar_\lambda(x,\tau,y)a_{\mathrm{Q}}(y)\,d\tau\,dy\right|^2\right]^{\frac12} \\ &\le\left[\sum_\rho\left|\mathcal{N}^\rho_{\mathrm{Q}}\right|\right]^{\frac12}\left[\sum_\rho\left\|\int\chi^\rho_r(\tau)K^\hbar_\lambda(x,\tau,y)a_{\mathrm{Q}}(y)\,d\tau\,dy\right\|^2_{L^2_x}\right]^{\frac12}\end{aligned} \tag{4.5}$$

Due to (4.2), the first factor is bounded by

$$\left[\sum_\rho|\mathcal{N}^\rho_{\mathrm{Q}}|\right]^{\frac12} \le \mathrm{const}\,[r^{-\frac{m-1}{2}}\cdot r^{\frac{m+1}{2}}]^{\frac12} = \mathrm{const}\,r^{\frac12}. \tag{4.6}$$

We can focus on the second factor:

$$\left[\sum_\rho \left\| \int \chi_r^\rho(\tau) K_\lambda^\hbar(x,\tau,y) a_{\mathrm{Q}}(y)\, d\tau\, dy \right\|_{L_x^2}^2 \right]^{\frac12}. \tag{4.7}$$

Getting rid of x'-dependence of the symbol b, we substitute

$$b(x,\tau,y) = \int_{\mathbb{R}^m} e^{ix'\cdot\xi'} \hat b_{\xi'}(x'',\tau,y) d\xi',$$

where $\hat b$ has infinite rate of decay for large ξ' since b is compactly supported in x. We then rewrite (4.7) as

$$\left[\sum_\rho \left\| \int \chi_r^\rho(\tau) e^{ix'\cdot\xi'} \hat b_{\xi'}(x'',\tau,y) e^{i\langle x' - G(x'',\tau,y),\tau\rangle} \beta(\frac{|\tau|}{\lambda}) \beta(\hbar^{-1}h) a_{\mathrm{Q}}(y)\, d\tau\, dy\, d\xi' \right\|_{L_x^2}^2 \right]^{\frac12}.$$

By the Minkowski inequality this is bounded by

$$\int_{\mathbb{R}^m} d\xi'$$

$$\cdot \left[\sum_\rho \left\| e^{ix'\cdot\xi'} \int \chi_r^\rho(\tau) \hat b_{\xi'}(x'',\tau,y) e^{i\langle x' - G(x'',\tau,y),\tau\rangle} \beta(\frac{|\tau|}{\lambda}) \beta(\hbar^{-1}h) a_{\mathrm{Q}}(y)\, d\tau\, dy \right\|_{L_x^2}^2 \right]^{\frac12}.$$

The integration in ξ' converges, so we may focus on

$$\left[\sum_\rho \left\| \int \chi_r^\rho(\tau) \hat b_{\xi'}(x'',\tau,y) e^{i\langle x' - G(x'',\tau,y),\tau\rangle} \beta(\frac{|\tau|}{\lambda}) \beta(\hbar^{-1}h) a_{\mathrm{Q}}(y)\, d\tau\, dy \right\|_{L_x^2}^2 \right]^{\frac12}.$$

By Plancherel's theorem this is the same as

$$\left[\sum_\rho \left\| \int \chi_r^\rho(\tau) \hat b_{\xi'}(x'',\tau,y) e^{-i\langle G(x'',\tau,y),\tau\rangle} \beta(\frac{|\tau|}{\lambda}) \beta(\hbar^{-1}h) a_{\mathrm{Q}}(y)\, dy \right\|_{L_{x'',\tau}^2}^2 \right]^{\frac12}$$

$$= \left[\sum_\rho \int dx''\, d\tau \left| \int \chi_r^\rho(\tau) \hat b_{\xi'}(x'',\tau,y) e^{-i\langle G(x'',\tau,y),\tau\rangle} \beta(\frac{|\tau|}{\lambda}) \beta(\hbar^{-1}h) a_{\mathrm{Q}}(y)\, dy \right|^2 \right]^{\frac12}.$$

We interchange integration (in x'', τ) and summation (in ρ) and bound the above with

$$\left[\int dx''\, d\tau \sum_\rho (\chi_r^\rho(\tau))^2 \left| \int \hat b_{\xi'}(x'',\tau,y) e^{-i\langle G(x'',\tau,y),\tau\rangle} \beta(\frac{|\tau|}{\lambda}) \beta(\hbar^{-1}h) a_{\mathrm{Q}}(y)\, dy \right|^2 \right]^{\frac12}$$

$$\le C \left[\int dx''\, d\tau \left| \int \hat b_{\xi'}(x'',\tau,y) e^{-i\langle G(x'',\tau,y),\tau\rangle} \beta(\frac{|\tau|}{\lambda}) \beta(\hbar^{-1}h) a_{\mathrm{Q}}(y)\, dy \right|^2 \right]^{\frac12}.$$

We used the fact that the ρ-partition of the sphere is locally finite, so that for each direction in the τ-space we have $\sum_\rho (\chi_r^\rho(\tau))^2 \le C^2$, with C independent of τ.

Thus, we need to know the L^2-norm of the expression

$$I_{\lambda,\xi'}^\hbar(x'',\tau) = \int \hat b_{\xi'}(x'',\tau,y) e^{-i\langle G(x'',\tau,y),\tau\rangle} \beta(\frac{|\tau|}{\lambda}) \beta(\hbar^{-1}h) a_{\mathrm{Q}}(y)\, dy \tag{4.8}$$

in (x'',τ).

Let us rewrite (4.8) using the rescaling $\mathcal{T} = \tau/\lambda$:

$$\begin{aligned} &I^{\hbar}_{\lambda,\xi'}(x'',\tau) \\ (4.9) &= \int \lambda^{-\frac{m-1}{2}} \left(\lambda^{\frac{m-1}{2}} \hat{b}_{\xi'}(x'',\lambda\mathcal{T},y)\right) e^{-i\lambda\langle G(x'',\mathcal{T},y),\mathcal{T}\rangle} \beta(|\mathcal{T}|)\beta(\hbar^{-1}h) a_{\mathrm{Q}}(y)\,dy \\ &\equiv B^{\hbar}_{\lambda,\xi'} a_{\mathrm{Q}}(x'',\mathcal{T}). \end{aligned}$$

Note that $\lambda^{\frac{m-1}{2}} \hat{b}_{\xi'}(x'',\lambda\mathcal{T},y)$ is bounded uniformly in λ (together with its derivatives in $\mathcal{T}$). There is no λ in $G(x'',\mathcal{T},y)$, since G is homogeneous in τ of degree 0. Therefore, $B^{\hbar}_{\lambda,\xi'} a_{\mathrm{Q}}(x'',\mathcal{T})$ is an oscillatory integral operator (which we denote by $B^{\hbar}_{\lambda,\xi'}$) associated to a one-sided Whitney fold and with a symbol of magnitude $\lambda^{-\frac{m-1}{2}}$, which acts on the atom a_{Q}. The integral kernel of $B^{\hbar}_{\lambda,\xi'}$ is localized by $\beta(\hbar^{-1}h)$ to the variety where the magnitude of the determinants of the Jacobi matrices of the projections from the canonical relation are of magnitude $h \approx \hbar$. Therefore, according to [Co99], the $L^2(y) \to L^2(x'',\mathcal{T})$ norm of $B^{\hbar}_{\lambda,\xi'}$ is bounded by $\lambda^{-\frac{m-1}{2}} \cdot \lambda^{-\frac{n}{2}} \hbar^{-\frac{1}{2}}$; the L^2-norm of a_{Q} is $r^{-\frac{n}{2}}$. This gives

$$\left\| B^{\hbar}_{\lambda,\xi'} a_{\mathrm{Q}} \right\|_{L^2} \le \mathrm{const}\, r^{-\frac{n}{2}} \cdot \lambda^{-\frac{m-1}{2}} \lambda^{-\frac{n}{2}} \hbar^{-\frac{1}{2}}.$$

Finally, the rescaling to $\tau = \lambda\mathcal{T}$ contributes $\lambda^{\frac{m}{2}}$ to the L^2 norm:

$$\left[\int_{\mathbb{R}^{n-m}\times\mathbb{R}^m} dx''\,d\tau \left| I^{\hbar}_{\lambda,\xi'}(x'',\tau)\right|^2\right]^{\frac{1}{2}} = \lambda^{\frac{m}{2}} \left[\int_{\mathbb{R}^{n-m}\times\mathbb{R}^m} dx''\,d\mathcal{T} \left| B^{\hbar}_{\lambda,\xi'} a_{\mathrm{Q}}(x'',\mathcal{T})\right|^2\right]^{\frac{1}{2}}.$$

Therefore, the L^2-norm (4.8) in (x'',τ) is bounded by

$$\left\| B^{\hbar}_{\lambda,\xi'} a_{\mathrm{Q}} \right\|_{L^2} \le \mathrm{const}\, r^{-\frac{n}{2}} \cdot \lambda^{\frac{1}{2}} \lambda^{-\frac{n}{2}} \hbar^{-\frac{1}{2}}. \tag{4.10}$$

The product of two factors, (4.6) and (4.10), gives the bound

$$r^{\frac{1}{2}} \cdot r^{-\frac{n}{2}} \cdot \lambda^{\frac{1}{2}} \lambda^{-\frac{n-m}{2}} \hbar^{-\frac{1}{2}} = (\lambda r)^{-\frac{n-1}{2}} \hbar^{-\frac{1}{2}}$$

on (4.5), which yields the estimate (2.15).

5. Trace regularity of solutions to hyperbolic differential equations

We use Theorem 2 stated in the Introduction to derive a general result on the L^p smoothness of restrictions of solutions to hyperbolic equations onto hypersurfaces, which incorporates the results obtained in [SeSoSt91] (L^p estimates; spacelike hypersurfaces) and [Ta98] (curved timelike hypersurfaces).

We follow Chapter 5 of the book of Duistermaat [Ds95]. Let Ω be a paracompact C^∞ manifold of dimension $n+1$, $n \ge 2$, and let $P(z,D)$ be a properly supported pseudodifferential operator of order 2 on Ω with a principal symbol $p(z,\zeta)$ which is smooth, real-valued, and homogeneous of degree 2 on $T^*\Omega\backslash 0$. The characteristic set of the operator P is a closed conic subset of $T^*\Omega\backslash 0$ defined by

$$\operatorname{Char} P = \{(z,\zeta) \in T^*\Omega\backslash 0 \mid p(z,\zeta) = 0\}. \tag{5.1}$$

We assume that P is hyperbolic, so that on each fiber $T^*_z\Omega$ the principal symbol $p(z,\zeta)$ defines a real quadratic form $\mathcal{Q}_z(\ ,\ )$ of Lorentz signature $(1,n)$:

$$\text{for } \zeta,\, \nu \in T^*_z\Omega, \qquad \mathcal{Q}_z(\zeta,\nu) \equiv \frac{1}{2}(p(z,\zeta+\nu) - p(z,\zeta) - p(z,\nu)). \tag{5.2}$$

A smooth hypersurface $\Sigma \subset \Omega$ is called spacelike with respect to P at a point $z \in \Sigma$ if $p(z, d\phi) > 0$, where $\phi \in C^\infty(\Omega)$ vanishes simply on Σ. Σ is called timelike with respect to P at z if instead $p(z, d\phi) < 0$. If $p(z, d\phi) = 0$, Σ is called characteristic at z. Σ is called spacelike if it is spacelike at each point. Similarly one defines timelike and characteristic hypersurfaces.

The derivative of $f \in C^\infty(\Omega)$ in the normal direction at $z \in \Sigma$ is defined with the aid of the quadratic form $\mathcal{Q}_z(\ ,\ )$ by

$$\partial_{\nu_\Sigma} f = \mathcal{Q}_z(d\phi, df), \tag{5.3}$$

where $\phi \in C^\infty(\Omega)$ vanishes simply on Σ. At non-characteristic points $z \in \Sigma$, where $\mathcal{Q}_z(d\phi, d\phi) \neq 0$, we normalize $\partial_{\nu_\Sigma} f$, dividing it by $|\mathcal{Q}_z(d\phi, d\phi)|^{1/2}$, to make the directional derivative independent of the choice of ϕ (up to a sign).

The Hamiltonian vector field $H_p \in C^\infty(\Gamma(T(T^*\Omega)))$ associated to $p(z, \zeta)$ is defined by the relation $\mathrm{int}_{H_p} \sigma = -dp$, where $\sigma \in \Lambda^2(T^*\Omega)$ is the canonical symplectic form on $T^*\Omega$ and int is the interior multiplication. Given the coordinates z on Ω, in the induced coordinates (z, ζ) on $T^*\Omega$ we have $\sigma = d\zeta \wedge dz$ and

$$H_p = p_\zeta(z, \zeta)\partial_z - p_z(z, \zeta)\partial_\zeta.$$

The null bicharacteristics $\gamma : \mathbb{R} \to T^*\Omega$ of P are defined as the integral curves of the Hamiltonian vector field H_p, corresponding to the value $p = 0$:

$$\dot\gamma(s) = H_p(\gamma(s)), \qquad \gamma(0) \in \mathrm{Char}\, P. \tag{5.4}$$

The projections of the bicharacteristics onto Ω under the natural projection $\pi_\Omega : T^*\Omega \to \Omega$ are called the rays of P.

Due to the assumption that P is hyperbolic, so that $p(z, \zeta)$ defines a quadratic form of signature $(1, n)$, one readily checks that P is strictly hyperbolic with respect to any spacelike hypersurface Σ, which means that all bicharacteristic curves of P are transversal to Σ and for every $(y, \eta) \in T^*\Sigma$ the equations

$$p(y, \zeta) = 0, \qquad \zeta|_{T_y\Sigma} = \eta \tag{5.5}$$

have exactly two distinct roots $\zeta \in T^*_y\Omega$.

We consider the Cauchy problem

$$\begin{cases} Pu = 0, \\ u|_S = f, \\ \partial_{\nu_S} = g, \end{cases} \tag{5.6}$$

where $S \overset{\imath_S}{\hookrightarrow} \Omega$ is a spacelike hypersurface and ∂_{ν_S} is the derivative in the direction normal to S. We make the following assumptions about P and S (see [Ds95]):

(1) Every ray intersects S at most once.

(2) No ray starting on S stays in a compact subset of Ω.

(3) For every pair of compact subsets $K_0 \subset S$, $K \subset \Omega$ there is a compact subset $K' \subset \Omega$ such that if I is an interval on a ray with one end point in K_0 and the other in K, then $I \subset K'$.

(4) For every compact subset $K \subset \Omega$ there is a compact subset $K_0 \subset S$ such that every ray starting in K only hits S in K_0.

One immediately checks that all these assumptions are satisfied for the wave operator $P = -\partial_t^2 + \nabla_x^2$ over $\Omega = \mathbb{R} \times \mathbb{R}^n$ and with the initial data on $S = (0 \times \mathbb{R}^n)$.

We denote by $\mathfrak{p}$ the parametrix $\mathfrak{p} : (f, g) \mapsto u$ of the Cauchy problem (5.6). For convenience we will assume that $u|_S = f = 0$, so that $\mathfrak{p} : \partial_{\nu_S} u \mapsto u$.

The assumptions *(1)-(4)* lead to the following two results:

Lemma 5.1 [Ds95]. *The parametrix* $\mathfrak{p}:\ \partial_{\nu_S}u \mapsto u$ *is a Fourier integral operator from* $I^{-1\frac{1}{4}}(\Omega, S, \mathbf{C}_0)$ *associated to the canonical relation*

$$\mathbf{C}_0 = \{(z,\zeta),\ \imath_S^*(y,\eta)\} \subset T^*\Omega \times T^*S, \tag{5.7}$$

where $z \in \Omega$*,* $y \in S$*, and* (z,ζ)*,* $(y,\eta) \in \operatorname{Char} P$ *lie on the same null bicharacteristic. The map* $\imath_S^*:\ T_S^*\Omega \to T^*S$ *is induced by the inclusion* $\imath_S:\ S \hookrightarrow \Omega$*. The immersed* $\mathbf{C}_0$ *is an embedded closed submanifold of* $(T^*\Omega \times T^*S)\backslash 0$*.*

The restriction $\rho_\Sigma\ :\ C^\infty(\Omega) \to C^\infty(\Sigma)$ is a Fourier integral operator $\rho_\Sigma \in I^{\frac{1}{4}}(\Sigma, \Omega, \mathbf{R}_\Sigma)$ associated to the canonical relation

$$\mathbf{R}_\Sigma = \{\imath_\Sigma^*(z,\zeta),\ (z,\zeta)\} \subset T^*\Sigma \times T_\Sigma^*\Omega, \tag{5.8}$$

where $(z,\zeta) \in T_\Sigma^*\Omega$ and the map $\imath_\Sigma^*\ :\ T_\Sigma^*\Omega \to T^*\Sigma$ is induced by the inclusion $\imath_\Sigma:\ \Sigma \hookrightarrow \Omega$.

One can readily verify that $\mathbf{R}_\Sigma \times \mathbf{C}_0$ and $T^*\Sigma \times \operatorname{diag}(T^*\Omega \times T^*\Omega) \times T^*S$ intersect transversally in $T^*\Sigma \times T^*\Omega \times T^*\Omega \times T^*S$. Therefore, the composition $\rho_\Sigma \circ \mathfrak{p}$ defines a Fourier integral operator:

Lemma 5.2 [Ds95]. *The composition* $\rho_\Sigma \circ \mathfrak{p}:\ \partial_{\nu_S}u \mapsto u|_\Sigma$ *is a Fourier integral operator from* $I^{-1}(\Sigma, S, \mathbf{C})$ *associated to the canonical relation*

$$\mathbf{C} = \mathbf{R}_\Sigma \circ \mathbf{C}_0 = \{\imath_\Sigma^*(z,\zeta),\ \imath_S^*(y,\eta)\} \subset T^*\Sigma \times T^*S, \tag{5.9}$$

where $z \in \Sigma$*,* $y \in S$*, and* (z,ζ)*,* $(y,\eta) \in \operatorname{Char} P$ *lie on the same null bicharacteristic.*

Since the rays are transversal to S and, due to the assumption *(1)* after (5.6), the condition $\pi_\Omega(\gamma(s)) \in S$ defines s implicitly as a smooth function of the initial data, it follows that $\gamma(0) = (z,\zeta) \in T_\Sigma^*\Omega$. We conclude that the point $(y,\eta) = \gamma(s) \in T_S^*\Omega$ depends smoothly on $(z,\zeta) \in \operatorname{Char}_\Sigma P$:

$$(z,\zeta) \mapsto (y,\eta) = (\hat{y}(z,\zeta),\ \hat{\eta}(z,\zeta)), \qquad \hat{y} \oplus \hat{\eta} \in C^\infty(\operatorname{Char}_\Sigma P, \operatorname{Char}_S P), \tag{5.10}$$

where $\operatorname{Char}_\Sigma P = \operatorname{Char} P \cap T_\Sigma^*\Omega$ and $\operatorname{Char}_S P = \operatorname{Char} P \cap T_S^*\Omega$. Therefore, we may parameterize $\mathbf{C}$ by $\operatorname{Char}_\Sigma P$. Let $\phi \in C^\infty(\Omega)$ be a smooth function which vanishes simply on Σ. Since $d\phi|_\Sigma \neq 0$ and $d_\zeta p|_{T^*\Omega\backslash 0} \neq 0$ are linearly independent, the set $\operatorname{Char}_\Sigma P = \{(z,\zeta) \in T^*\Omega \mid \phi(z) = 0,\ p(z,\zeta) = 0\}$ is a smooth submanifold of $T^*\Omega$ of codimension 2, which we identify with $\mathbf{C}$:

$$\operatorname{Char}_\Sigma P \xrightarrow{\cong} \mathbf{C}. \tag{5.11}$$

The glancing variety $\mathbf{G} \subset \mathbf{C}$ is defined by

$$\mathbf{G} = \mathbf{C} \cap \mathbf{Z}, \quad \text{where} \quad \mathbf{Z} = \{(z,\zeta) \in T^*\Omega \mid \{p,\phi\} = 0\}, \tag{5.12}$$

with $\{p,\phi\} = H_p\phi$ being the Poisson bracket determined by the canonical symplectic structure on $T^*\Omega$. The glancing variety $\mathbf{G}$ consists of the points $(z,\zeta) \in \operatorname{Char}_\Sigma P$ such that the corresponding bicharacteristics are tangent to $T_\Sigma^*\Omega = \{(z,\zeta) \in T^*\Omega \mid \phi(z) = 0\}$:

$$(H_p\phi)(z,\zeta) = \{p,\phi\}(z,\zeta) = 0.$$

The corresponding ray $\pi_\Omega(\gamma(s))$ in the direction $d(\pi_\Omega)_{(z,\zeta)}(H_p) \in T_z\Omega$ is tangent to $\Sigma \subset \Omega$:

$$((\pi_\Omega)_* H_p)\phi(z) = (H_p\phi)(z,\zeta) = \{p,\phi\}(z,\zeta) = 0.$$

We do not distinguish between $\phi \in C^\infty(\Omega)$ and its pull-back $\pi_\Omega^*\phi$ onto $T^*\Omega$.

Let us consider the projections from $\mathbf{C}$:

$$\begin{cases} \pi_L : \operatorname{Char}_\Sigma P \to T^*\Sigma, & (z,\zeta) \mapsto \imath_\Sigma^*(z,\zeta); \\ \pi_R : \operatorname{Char}_\Sigma P \to T^*S, & (z,\zeta) \mapsto \imath_S^*(\hat{y}(z,\zeta),\hat{\eta}(z,\zeta)). \end{cases} \tag{5.13}$$

Geometrically, the singular part of π_L, represented by $\zeta \to \zeta|_{T_z\Sigma}$, is the projection from the characteristic cone $\operatorname{Char}_z P$ onto the hyper'splane $T_z^*\Sigma$. If Σ is timelike, then $\pi_L|_z$ is a Whitney fold; if Σ is spacelike, then π_L is a diffeomorphism. The singular component of π_R is represented by $z \mapsto \hat{y}(z,\zeta) \in S$, which is the projection from Σ onto S in the direction of the ray from $z \in \Sigma$ which corresponds to $\zeta \in \operatorname{Char}_z P$.

In order to characterize the singularities of the projections π_L and π_R with the aid of Definitions 0.1 and 0.2, we need to compute the determinants of their Jacobi matrices $d\pi_L$, $d\pi_R$.

Lemma 5.3. *Up to nonzero factors (which depend on the choice of local coordinates), both* $\det d\pi_L$ *and* $\det d\pi_R$ *are equal to* $\{p,\phi\}|_{\mathbf{C}}$.

Corollary. *The critical variety of both* π_L *and* π_R *coincides with the glancing variety* $\mathbf{G} \subset \mathbf{C}$.

Proof of the Lemma. According to [Ho71], it suffices to show that the determinant of the Jacobi matrix of $\pi_L : \operatorname{Char}_\Sigma P \ni (z,\zeta) \mapsto \imath_\Sigma^*(z,\zeta) = (z, \zeta|_{T_z\Sigma})$ is equal to $\{p,\phi\}|_{\mathbf{C}}$, up to a nonzero factor.

We first give an informal argument. It suffices to consider the restriction $\pi_L|_z$: $\operatorname{Char}_z P \to T_z^*\Sigma$, $\zeta \mapsto \zeta|_{T_z\Sigma}$. If we identify the tangent and cotangent fibers, we could say that the differential of this map is the orthogonal projection from $T_\zeta(\operatorname{Char}_z P)$ onto $T_\zeta(T_z^*\Sigma) \cong T_z^*\Sigma \cong T_z\Sigma$, and the determinant of the matrix of this projection is the dot product of the normals: $p_\zeta \cdot \phi_z = H_p\phi = \{p,\phi\}$.

Let us give a rigorous proof. The differential of the map $\imath_\Sigma^* : T_\Sigma^*\Omega \to T^*\Sigma$, $(z,\zeta) \mapsto (z,\zeta|_{T_z\Sigma})$, is given by $d\imath_\Sigma^* : (Z,\mathcal{Z}) \mapsto (Z,\mathcal{Z}|_{T_z\Sigma})$, where $Z \in T_z\Sigma$ and $\mathcal{Z} \in T_z^*\Omega$. The kernel of $d\imath_\Sigma^*$ is generated by $(0,d\phi)$, which is the coordinate representation of (the negative of) the Hamiltonian vector field of ϕ:

$$H_\phi = \phi_\zeta\partial_z - \phi_z\partial_\zeta = -\phi_z\partial_\zeta \in \Gamma(T(T^*\Omega)).$$

Since $H_\phi \neq 0$ on Σ and $H_\phi \notin T(T^*\Sigma)$, we can introduce a 1-form $\theta \in \Lambda^1(T_\Sigma^*\Omega)$ such that $\theta(H_\phi) = 1$, $\theta|_{T(T^*\Sigma)} = 0$. Let $\beta \in \Lambda^{2n}(T^*\Sigma)$ be a volume form on $T^*\Sigma$. The wedge product of θ with the lift of β onto $\Lambda^{2n}(T_\Sigma^*\Omega)$ defines the volume form $d\operatorname{vol}_{T_\Sigma^*\Omega}$ on $T_\Sigma^*\Omega$:

$$d\operatorname{vol}_{T_\Sigma^*\Omega} = (\imath_\Sigma^*)^*\beta \wedge \theta \in \Lambda^{2n+1}(T_\Sigma^*\Omega).$$

Alternatively, we can represent $d\operatorname{vol}_{T_\Sigma^*\Omega}$ by $\alpha \wedge dp$, where $\alpha \in \Lambda^{2n}(T_\Sigma^*\Omega)$. Since $dp|_{T\mathbf{C}} = 0$, the restriction of α onto vectors from $T\mathbf{C}$ is nonzero and can be considered as a volume form $d\operatorname{vol}_{\mathbf{C}}$ on $\mathbf{C}$.

Let us evaluate both sides of the identity

$$\alpha \wedge dp = (\imath_\Sigma^*)^*\beta \wedge \theta$$

on $X_1 \wedge \cdots \wedge X_{2n} \wedge H_\phi \in \wedge^{2n+1}(T_{(z,\zeta)}(T_\Sigma^*\Omega))$, where $X_i \in T_{(z,\zeta)}\mathbf{C} \subset T_{(z,\zeta)}(T_\Sigma^*\Omega)$. From $dp(X_i) = 0$ we see that the left-hand side is equal to $\alpha(X_1 \wedge \cdots \wedge X_{2n})dp(H_\phi)$.

Since $H_\phi \in \ker d\imath_\Sigma^*$, the interior multiplication $\mathrm{int}(H_\phi)((\imath_\Sigma^*)^*\beta) = 0$, and therefore $((\imath_\Sigma^*)^*\beta \wedge \theta)(X_1 \wedge \cdots \wedge X_{2n} \wedge H_p) = (\imath_\Sigma^*)^*\beta(X_1 \wedge \cdots \wedge X_{2n})\theta(H_p)$. We thus have

$$\alpha(X_1 \wedge \cdots \wedge X_{2n}) dp(H_\phi) = \beta(d\imath_\Sigma^*(X_1) \wedge \cdots \wedge d\imath_\Sigma^*(X_{2n})).$$

Therefore, up to a nonzero factor which depends on the choice of local coordinates, $\det d\pi_L$ is equal to

$$\frac{d\,\mathrm{vol}_{T^*\Sigma}(d\pi_L(X_1) \wedge \cdots \wedge d\pi_L(X_{2n}))}{d\,\mathrm{vol}_{\mathbf{C}}(X_1 \wedge \cdots \wedge X_{2n})} = \frac{d\beta(d\imath_\Sigma^*(X_1) \wedge \cdots \wedge d\imath_\Sigma^*(X_{2n}))}{d\alpha(X_1 \wedge \cdots \wedge X_{2n})} = dp(H_\phi).$$

□

Lemma 5.4. *If Σ is spacelike, then both π_L and π_R are (local) diffeomorphisms.*

Proof. If Σ is spacelike at $z \in \Sigma$, then $\mathcal{Q}_z(d\phi, d\phi) > 0$, and hence

$$\{p, \phi\}(z, \zeta) = -H_\phi p(z, \zeta) = (\phi_z \partial_\zeta)\mathcal{Q}_z(\zeta, \zeta) = 2\mathcal{Q}_z(d\phi, \zeta)$$

does not vanish when $\zeta \in \mathrm{Char}_z\, P$. This becomes transparent in the local coordinates where $\mathcal{Q}_z(\,,\,) = \mathrm{diag}\,(1, -1, \ldots, -1)$. □

Therefore, if Σ is spacelike, the Fourier integral operator $\rho_\Sigma \circ \mathfrak{p}$ is associated to a local graph, and the smoothness of $u|_\Sigma = (\rho_\Sigma \circ \mathfrak{p})g$ follows from the L^p estimates for Fourier integral operators associated to local diffeomorphisms [SeSoSt91].

Lemma 5.5. *The map $\partial_{\nu_\Sigma} \circ \mathfrak{p}:\ \partial_{\nu_S} u \mapsto \partial_{\nu_\Sigma} u$ is a Fourier integral operator from $I^0(\Sigma, S, \mathbf{C})$ associated to the same canonical relation as $\mathbf{R}_\Sigma \circ \mathfrak{p}$, with the symbol vanishing (simply) on the critical variety of the projections from $\mathbf{C}$.*

Proof. Let $\Phi \in C^\infty(\Omega \times \mathbb{R}^N \times S)$ be the phase function of the Fourier integral operator $\mathfrak{p}$. Using a usual representation of $\mathfrak{p}$ by the Fourier integral operator, we compute that $\partial_{\nu_\Sigma} \circ \mathfrak{p}$ is a Fourier integral operator associated to the same canonical relation as $\mathbf{R}_\Sigma \circ \mathfrak{p}$, with its leading symbol proportional to $\mathcal{Q}_z(d\phi, d_z\Phi(z, \theta, y))$, $z \in \Sigma$. According to (5.7), if $(z, d_z\Phi(z, \theta, y), y, -d_y\Phi(z, \theta, y))$ is a point on $\mathbf{C}_0$, then $\zeta \equiv d_z\Phi(z, \theta, y) \in \mathrm{Char}_z\, P$. Given $z \in \Sigma$, we have

$$\mathcal{Q}_z(d\phi, d_z\Phi(z, \theta, y)) = \mathcal{Q}_z(d\phi, \zeta) = -H_\phi p(z, \zeta)/2 = \{p, \phi\}(z, \zeta)/2,$$

which vanishes on the glancing set $\mathbf{G}$. □

Lemma 5.6. *If Σ is timelike, then π_L is a Whitney fold.*

Proof. Taking any smooth projection of the vector field H_ϕ onto $T\mathbf{C}$, we obtain a smooth vector field over $\mathbf{C}$ which we denote $K_L \in \Gamma(T\mathbf{C})$. Since $H_\phi p|_{\mathbf{G}} = -\{p, \phi\}|_{\mathbf{G}} = 0$, we know that $H_\phi|_{\mathbf{G}} \in T\mathbf{C}$, and therefore $K_L|_{\mathbf{G}} = H_\phi$ and K_L does not vanish on $\mathbf{G}$.

Since the kernel of $d(\imath_\Sigma^*)$ is generated by $H_\phi = (0, -d\phi) \in T(T^*\Omega)$, the kernel of the differential of $\pi_L = \imath_\Sigma^*|_{\mathbf{C}}$ is nontrivial on $\mathbf{G}$ (where $H_\phi \in T\mathbf{C}$) and is generated by $K_L|_{\mathbf{G}} = H_\phi$. Up to a nonzero factor $\det d\pi_L$ is equal to $\{p, \phi\}(z, \zeta) = 2\mathcal{Q}_z(d\phi, \zeta)$, while $K_L|_{\mathbf{G}} = H_\phi$. We compute at $(z, \zeta) \in \mathbf{G}$:

$$K_L\{p, \phi\}|_{\mathbf{G}}(z, \zeta) = H_\phi\{p, \phi\}(z, \zeta) = 2(-\phi_z \partial_\zeta)\mathcal{Q}_z(d\phi, \zeta) = -2\mathcal{Q}_z(d\phi, d\phi),$$

and, since Σ is timelike, $\mathcal{Q}_z(d\phi, d\phi) < 0$ on Σ. This proves that π_L is of type one and hence is a Whitney fold. □

We use the following straightforward classification of glancing points (a similar definition is in [MeSj78]):

Definition 5.1. We will say that a point $(z,\zeta) \in \mathbf{G}$ is a glancing point of type $k \in \mathbb{N}$ if it is the smallest integer such that

$$H_p^{k+1}\phi(z,\zeta) = \underbrace{\{p,\{p,\dots\{p}_{k+1},\phi\}...\}\}(z,\zeta) \neq 0. \tag{5.14}$$

This definition does not depend on the choice of ϕ. Type 1 corresponds to simple glancing.

Lemma 5.7. *If $(z,\zeta) \in \mathbf{G}$ is a glancing point of type k, then the projection π_R is also of type k at that point.*

Proof. Taking any smooth projection of the vector field $H_p|_{\mathbf{C}}$ onto the fibers of $T\mathbf{C}$, we obtain a smooth vector field $K_R \in \Gamma(T\mathbf{C})$. Since $H_p|_{\mathbf{G}} \in T\mathbf{C}$, we know that $K_R|_{\mathbf{G}} = H_p|_{\mathbf{G}}$ and that K_R does not vanish on $\mathbf{G}$.

Let us show that $K_R|_{\mathbf{G}} \in \ker d\pi_R$. The vector field H_p is tangent to the bicharacteristics, so that $d\hat{y}(H_p) = d\hat{\eta}(H_p) = 0$; therefore,

$$d\pi_R(K_R)|_{\mathbf{G}} = d(\imath_S^* \circ (\hat{y} \oplus \hat{\eta}))(H_p) = 0,$$

where $\hat{y} \oplus \hat{\eta} : \mathrm{Char}_\Sigma P \to \mathrm{Char}_S P$ is the blow-off along the bicharacteristic defined in (5.10).

We need a smooth vector field $\hat{K}_R$ over $T^*\Omega$ such that $\hat{K}_R|_{\mathbf{C}} = K_R \in \Gamma(T\mathbf{C})$ and $\hat{K}_R|_{\mathbf{Z}} = H_p$. To construct it, we extend K_R to a smooth vector field $\tilde{K}_R$ over $T^*\Omega$ and define

$$\hat{K}_R|_{\mathbf{Z}} = H_p, \qquad \hat{K}_R|_{(T^*\Omega)\setminus\mathbf{Z}} = \frac{(p^2+\phi^2)H_p + \{p,\phi\}^2\tilde{K}_R}{p^2+\phi^2+\{p,\phi\}^2}.$$

Since $\hat{K}_R|_{\mathbf{Z}} = H_p$ and since $\{p,\phi\} = H_p\phi$ vanishes simply on $\mathbf{Z} \subset T^*\Omega$, there is a smooth vector field W over $T^*\Omega$ such that

$$\hat{K}_R = H_p + (H_p\phi)W.$$

The type of π_R at a point $(z,\zeta) \in \mathbf{G} \subset \mathbf{C}$ is defined as the smallest integer k such that $K_R^k \det d\pi_R|_{(z,\zeta)} \neq 0$ (see Definition 0.1). The determinant of the Jacobi matrix of π_R is equal to $H_p\phi$, up to a factor $\varphi \in C^\infty(\mathbf{C})$, $\varphi|_{\mathbf{G}} \neq 0$, which depends on the local coordinates. We fix the local coordinates and the corresponding factor φ, which we extend to a smooth function $\varphi \in C^\infty(T^*\Omega)$. Since $\hat{K}_R|_{\mathbf{C}} = K_R \in \Gamma(T\mathbf{C})$, we have $\hat{K}_R^j(\varphi H_p\phi)|_{\mathbf{C}} = K_R^j(\varphi H_p\phi|_{\mathbf{C}}) = K_R^j \det d\pi_R$. Therefore,

$$K_R^j \det d\pi_R = (H_p + (H_p\phi)W)^j\,(\varphi H_p\phi) = \varphi H_p^{j+1}\phi + \sum_{m=1}^{j} a_m H_p^m\phi,$$

where $a_m \in C^\infty(T^*\Omega)$ are certain combinations of H_p and W acting on ϕ and φ. According to Definition 5.1, if $(z,\zeta) \in \mathbf{G}$ is a glancing point of type k, then at that point $H_p^m\phi = 0$ for $1 \le m \le k$, and therefore

$$K_R^j \det d\pi_R|_{(z,\zeta)} = \varphi(z,\zeta)H_p^{j+1}\phi|_{(z,\zeta)}, \qquad 1 \le j \le k.$$

Since $\varphi|_{\mathbf{G}} \neq 0$, we conclude that π_R is of type k at (z,ζ). □

We apply Theorem 2 from the Introduction to the operator $\rho_\Sigma \circ \mathfrak{p}$, and use Lemmas 5.6 and 5.7, which characterize the singularities of the projections from the associated canonical relation $\mathbf{C}$. This gives the following result:

Theorem 5.1. *Let P be a properly supported hyperbolic pseudodifferential operator with real and homogeneous principal symbol, let $S \hookrightarrow \Omega$ be a smooth hypersurface which is spacelike with respect to P, and let the assumptions* (1)-(4) *after* (5.6) *be satisfied. Let $\Sigma \hookrightarrow \Omega$ be a smooth timelike hypersurface and let $\mathcal{O}$ be an open bounded subset of Ω such that $\mathcal{O}\cap\Sigma \neq \emptyset$. Assume that the glancing points $(z,\zeta) \in \mathbf{G}$, $z \in \bar{\mathcal{O}} \cap \Sigma$, are of type at most k.*

If u is a solution to the Cauchy problem (5.6) *with the initial data (f,g) on S, then for $1 < p < \frac{k+2}{k+1}$ and $3 < p < \infty$ the L^p-regularity of the restriction $u|_{\Sigma}$ is the same as if Σ were spacelike* [SeSoSt91]*:*

$$\|u|_{\Sigma}\|_{L^p_{\alpha-\alpha_p}(\bar{\mathcal{O}}\cap\Sigma)} \leq C\left(\|f\|_{L^p_\alpha(S)} + \|g\|_{L^p_{\alpha-1}(S)}\right); \tag{5.15}$$

here $1 < p < \frac{k+2}{k+1}$ or $3 < p < \infty$, and $\alpha_p = (n-1)|\frac{1}{p} - \frac{1}{2}|$.

The estimates on $\|u|_{\Sigma}\|_{L^p_{\alpha-\alpha_p}(\bar{\mathcal{O}}\cap\Sigma)}$ for $\frac{k+2}{k+1} \leq p < 2$ and $2 < p \leq 3$ are obtained by interpolating between (5.15) *and the Sobolev estimates,*

$$\|u|_{\Sigma}\|_{H^{\alpha-1/(4+2k^{-1})}(\bar{\mathcal{O}}\cap\Sigma)} \leq C\left(\|f\|_{H^\alpha(S)} + \|g\|_{H^{\alpha-1}(S)}\right). \tag{5.16}$$

The normal derivative has the optimal regularity for any $p > 1$, $p < \infty$:

$$\|\partial_{\nu_\Sigma} u\|_{L^p_{\alpha-1}(\bar{\mathcal{O}}\cap\Sigma)} \leq C\left(\|f\|_{L^p_\alpha(S)} + \|g\|_{L^p_{\alpha-1}(S)}\right). \tag{5.17}$$

The constants in (5.15)-(5.17) *depend on p, n, the coefficients of $P(z,D)$, and the geometric data: Ω, S, Σ, and $\mathcal{O}$.*

The regularity (5.17) of the normal derivative of u at Σ follows from Lemma 5.5 and the estimates on singular Fourier integral operators with the damping factor [Co98]. Under extra conditions on Σ, the boundary value $p = \frac{k+2}{k+1}$ in (5.15) could be improved (up to $p = \frac{3}{2}$; see Theorem 2 in the Introduction).

6. Regularity of restrictions of solutions onto hypersurfaces with characteristic points

Let the hypersurface $\Sigma \subset \Omega$ be characteristic at certain points: $p(z, d\phi) = 0$. Our methods are applicable if Σ is curved (has simple contact with the rays of P):

$$H_p^2\phi = \{p, \{p, \phi\}\} \neq 0. \tag{6.1}$$

If (6.1) holds, then $\pi_R : \mathbf{C} \to T^*S$ is of type 1 and hence a Whitney fold.

At points z where Σ is characteristic, the map $\pi_L|_{z\text{ fixed}} : \operatorname{Char}_z P \to T^*_z\Sigma$ is the projection from the characteristic cone $\operatorname{Char}_z P$ in the direction of one of the null covectors which form $\operatorname{Char}_z P$. This projection has a singularity of infinite type (the Jacobi matrix vanishes identically in the direction of the kernel of the projection), but we will prove that the second derivative of $\det d\pi_L$ in a direction transversal to the kernel is different from zero. In terms of Definition 0.2, the map π_L is of type $w = 2$ relative to $\ker d(\pi_\Sigma \circ \pi_L)$, where π_Σ is the natural projection $T^*\Sigma \to \Sigma$.

The projection $\pi_S \circ \pi_R : \mathbf{C} \to S$ may fail to be a submersion at the characteristic points (where $p(z, d\phi) = 0$). This is illustrated by the example $p(z,\zeta) = \tau^2 - \xi^2$, $\zeta = (\tau, \xi) \in \mathbb{R} \times \mathbb{R}^n$, $\Sigma = \{(t,x) \in \mathbb{R} \times \mathbb{R}^n \mid t = 1 - |x|^2\}$, when the projections from Σ onto S along the rays of P are not surjective at the points $t = 3/4$, $|x| = 1/2$. At the same time, since $\pi_\Sigma \circ \pi_L : \operatorname{Char}_\Sigma P \to \Sigma$ is a submersion, we can apply the reduction from Section 1 to the adjoint of $\rho_\Sigma \circ \mathfrak{p}$. Then Proposition 2.2 and hence the part of Theorem 2 with $p \le 2$ become applicable to the operator $(\rho_\Sigma \circ \mathfrak{p})^*$. This yields the smoothness of $u|_\Sigma$ in L^p for $p \ge 2$:

Theorem 6.1. *Let P be a properly supported hyperbolic pseudodifferential operator with real and homogeneous principal symbol, let $S \hookrightarrow \Omega$ be a smooth hypersurface which is spacelike with respect to P, and let the assumptions* (1)-(4) *after* (5.6) *be satisfied. Let $\Sigma \hookrightarrow \Omega$ be a smooth curved hypersurface, and let $\mathcal{O}$ be an open bounded subset of Ω such that $\mathcal{O} \cap \Sigma \neq \emptyset$. We allow that Σ be characteristic with respect to P at some points in $\bar{\mathcal{O}} \cap \Sigma$.*

If u is a solution to the Cauchy problem (5.6) *with the initial data (f, g) on S, then for $p > 4$ the L^p-regularity of the restriction $u|_\Sigma$ is the same as if Σ were spacelike* [SeSoSt91]*:*

$$\|u|_\Sigma\|_{L^p_{\alpha-\alpha_p}(\bar{\mathcal{O}}\cap\Sigma)} \le C\left(\|f\|_{L^p_\alpha(S)} + \|g\|_{L^p_{\alpha-1}(S)}\right). \tag{6.2}$$

For $2 < p \le 4$, the regularity (not sharp) is obtained by interpolating (6.2) *with the Sobolev L^2 estimate,*

$$\|u|_\Sigma\|_{H^{\alpha-\frac{1}{4}}(\bar{\mathcal{O}}\cap\Sigma)} \le C\left(\|f\|_{H^\alpha(S)} + \|g\|_{H^{\alpha-1}(S)}\right). \tag{6.3}$$

The normal derivative has the optimal regularity for any $p \ge 2$:

$$\|\partial_{\nu_\Sigma} u\|_{L^p_{\alpha-1}(\bar{\mathcal{O}}\cap\Sigma)} \le C\left(\|f\|_{L^p_\alpha(S)} + \|g\|_{L^p_{\alpha-1}(S)}\right). \tag{6.4}$$

The constants in (6.2)-(6.4) *depend on p, n, the coefficients of $P(z, D)$, and the geometric data: Ω, S, Σ, and $\mathcal{O}$.*

Proof. We already know that π_R is a Whitney fold. We now need to consider the properties of the map π_L.

Lemma 6.1. *If Σ is characteristic at the point $z_c \in \Sigma$, then the critical variety of the projections from $\mathbf{C}$ is given by $N^*_{z_c}\Sigma \subset \mathbf{C}$.*

Proof. Let Σ be characteristic at z_c:

$$p(z_c, d\phi) \equiv \mathcal{Q}_{z_c}(d\phi, d\phi) = 0, \tag{6.5}$$

and let $\zeta_c \in \mathrm{Char}_{z_c} P$ be such that $(z_c, \zeta_c) \in \mathbf{C}$ is a critical point of π_L and π_R:

$$p(z_c, \zeta_c) \equiv \mathcal{Q}_{z_c}(\zeta_c, \zeta_c) = 0, \qquad \{p, \phi\}(z_c, \zeta_c) = 2\mathcal{Q}_{z_c}(d\phi, \zeta_c) = 0. \tag{6.6}$$

From (6.5) and (6.6) we conclude that ζ_c is parallel to $d\phi|_{z_c}$:

$$\zeta_c = c(z_c, \zeta_c)\, d\phi|_{z_c}. \tag{6.7}$$

This is transparent in the local coordinates where $\mathcal{Q}_{z_c}(\ ,\ ) = \mathrm{diag}\,(1, -1, \ldots, -1)$. □

Lemma 6.2. *The type of π_L relative to $\ker d(\pi_\Sigma \circ \pi_L)$ is at most* 2.

Proof. Instead of giving a coordinate-independent argument, let us give an illustrative proof in the local coordinates (z, ζ), $\zeta = (\zeta_0, \boldsymbol{\zeta}) \in \mathbb{R} \times \mathbb{R}^n$, where $p(z_c, \zeta) = \zeta_0^2 - |\boldsymbol{\zeta}|^2$. Let (z_c, ζ_c) be a critical point of π_L. We only need to consider the case when Σ is characteristic at the point z_c (otherwise π_L is at most a Whitney fold). We rotate the axes ζ_i, $i = 1, \ldots, n$, so that $\zeta_c = (\lambda, \lambda, 0, \ldots, 0) \in \mathbb{R} \times \mathbb{R}^n$, where $\lambda = |\zeta_{c0}| = |\boldsymbol{\zeta}_c|$.

Let V be a vector field on $\mathrm{Char}_{z_c} P$ defined by

$$V = \zeta_1 \partial_{\zeta_2} - \zeta_2 \partial_{\zeta_1}, \qquad \zeta \in \mathrm{Char}_{z_c} P. \tag{6.8}$$

One immediately checks that indeed $Vp(z_c,\zeta)=0$. We also have $V\in\ker d(\pi_\Sigma\circ\pi_L)$. We claim that there exists a smooth continuation of V onto an open neighborhood $\mathcal{O}$ of (z_c,ζ_c) in $\mathbf{C}$, which we denote by W, such that

$$W|_{\mathrm{Char}_{z_c}P}=V,\qquad W\in\ker d(\pi_\Sigma\circ\pi_L).$$

Indeed, since $\pi_\Sigma\circ\pi_L:\ \mathrm{Char}_\Sigma P\to\Sigma$ is a submersion, then $\ker d(\pi_\Sigma\circ\pi_L)\to\mathbf{C}$ is a vector bundle (see Lemma 0.2), and $V\in\ker d(\pi_\Sigma\circ\pi_L)$ can be extended from $\mathrm{Char}_{z_c}P\subset\mathbf{C}$ onto an open neighborhood of (z_c,ζ_c) in $\mathbf{C}$.

We want to prove that $W^2\det d\pi_L|_{(z_c,\zeta_c)}\neq 0$. According to Lemma 5.3, $\det d\pi_L$ equals $\{p,\phi\}|_{\mathbf{C}}$, up to a nonzero factor. Therefore, we need to show that at the point (z_c,ζ_c) we have $W\{p,\phi\}=0$, $W^2\{p,\phi\}\neq 0$. According to Lemma 6.1, if Σ is characteristic at z_c and if (z_c,ζ_c) is a critical point, then $d\phi|_{z_c}=c^{-1}\zeta_c=c^{-1}(\lambda,\lambda,0,\dots,0)$, and we obtain

$$\{p,\phi\}|_{z_c}=2\mathfrak{Q}_{z_c}(d\phi,\zeta)=2c^{-1}\mathfrak{Q}_{z_c}(\zeta_c,\zeta)=2c^{-1}\lambda(\zeta_0-\zeta_1). \tag{6.9}$$

Let us notice that the critical variety at z_c is given by the line $\zeta_0=\zeta_1$. The first derivative of $\{p,\phi\}$ by the vector field W is equal to $W\{p,\phi\}=V\{p,\phi\}=2c^{-1}\lambda\zeta_2$, which vanishes at $\zeta=\zeta_c=(\lambda,\lambda,0,\dots,0)$. The second derivative of $\{p,\phi\}$ by W is

$$W^2\{p,\phi\}|_{z_c}=V(2c^{-1}\lambda\zeta_2)=2c^{-1}\lambda\zeta_1, \tag{6.10}$$

which does not vanish at $\zeta=\zeta_c$, and hence at (z_c,ζ_c) the map π_L is of type $w=2$ relative to $\ker d(\pi_\Sigma\circ\pi_L)$. $\square$

We need to overcome one more hindrance before Theorem 6.1 is proved: If Σ is characteristic at z_c, then π_L is of infinite type at critical points (z_c,ζ_c). Indeed, both $H_\phi p(z,\zeta)=-2\mathfrak{Q}_z(d\phi,\zeta)$ and $H_\phi^2 p(z,\zeta)=-2\mathfrak{Q}_z(d\phi,d\phi)$ vanish at (z_c,ζ_c), while $H_\phi^l p(z,\zeta)\equiv 0$ for $l>2$. To be able to apply the analogue of Theorem 2 with $k=\infty$ to the operator $\rho_\Sigma\circ\mathfrak{p}$, we need the following analogue of the estimates (2.6), (2.7) in Proposition 2.1:

Lemma 6.3. *Let $(\rho_\Sigma\circ\mathfrak{p})_\lambda^\hbar$, $\overline{(\rho_\Sigma\circ\mathfrak{p})}_\lambda^\hbar$ be constructed from $\rho_\Sigma\circ\mathfrak{p}\in I^{-1}(\Sigma,S,\mathbf{C})$ according to* (2.1), (2.2). *Then*

$$\|\,(\rho_\Sigma\circ\mathfrak{p})_\lambda^\hbar\,\|_{L^2\to L^2}\le \mathrm{const}\,\lambda^{-1}\cdot\hbar^{-\frac12},$$
$$\|\overline{(\rho_\Sigma\circ\mathfrak{p})}_\lambda^\hbar\|_{L^2\to L^2}\le \mathrm{const}\,\lambda^{-1}\cdot\lambda^{\frac12}\hbar^{\frac12}.$$

This result follows from [Co99], except that since π_L is of infinite type at some points, we need to check directly that π_L satisfies the "convexity assumption" from [Co99]:

$$|\pi_L(z,\zeta_1)-\pi_L(z,\zeta_2)|\ge c\hbar|\zeta_1-\zeta_2|, \tag{6.11}$$

where $\hbar$ is between $\det d\pi|_{(z,\zeta_1)}$ and $\det d\pi|_{(z,\zeta_2)}$, while the constant $c>0$ depends on the choice of local coordinates where $\det d\pi_L$ is evaluated. The inequality (6.11) is important for the van der Corput type estimates (stationary phase estimates for oscillatory integral operators with singular phase functions).

Let us show that $\pi_L:\ \mathrm{Char}_\Sigma P\to T^*\Sigma$ satisfies (6.11). For a given point $z\in\Sigma$ we choose some local coordinates so that $\mathrm{Char}_z P$ is the union of standard cones $\mathbf{C}_\pm=\{(\pm|\boldsymbol{\zeta}|,\boldsymbol{\zeta})\mid\boldsymbol{\zeta}\in\mathbb{R}^n\}$, and then $\pi_L|_z:\ \mathrm{Char}_z P\to T_z^*\Sigma$ is the orthogonal projection from $\mathbf{C}_-\cup\mathbf{C}_+$. We may restrict our attention to $\mathbf{C}_+$. Let s_1 and s_2 be the unit normals to $\mathbf{C}_+$ at ζ_1 and ζ_2, and let ν be the unit normal to Σ at z. Since

$\mathbf{C}_+$ is convex, the orthogonal projection Π from $\mathbf{C}_+$ in the direction ν satisfies the inequality

$$|\,\Pi(\zeta_1) - \Pi(\zeta_2)\,| \geq h|\zeta_1 - \zeta_2|, \tag{6.12}$$

where h is a number between $\nu \cdot s_1$ and $\nu \cdot s_2$. The dot product $\nu \cdot s_i$ is proportional to the determinant of the Jacobi matrix of Π at ζ_i, and hence (6.12) is equivalent to the "convexity" condition (6.11). □

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Department of Mathematics, University of North Carolina, Chapel Hill, North Carolina 27599

Department of Mathematics, University of Virginia, Charlottesville, Virginia 22903