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Profinite groups with restricted centralizers of powers

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Abstract
A group G is said to have restricted centralizers if for every $x \in G$ the centralizer $C_G(x)$ either is finite or has finite index in G. Shalev showed that a profinite group with restricted centralizers is virtually abelian. Here we take interest in profinite groups G for which there is an integer n such that $C_G(x^n)$ is either finite or open whenever $x \in G$. It is shown that such a group G has an open normal subgroup T with the property that $G/Z(T)$ has finite exponent.

Keywords Profinite groups · Centralizers · FC-elements

Mathematics Subject Classification 20E18 · 20F24

1 Introduction

A group G is said to have restricted centralizers if for every $g \in G$ the centralizer $C_G(g)$ either is finite or has finite index in G. This notion was introduced by Shalev in [13] where he showed that a profinite group with restricted centralizers is virtually abelian. We say that a profinite group has a property virtually if it has an open subgroup with that property. The article [4] handles profinite groups with restricted centralizers of w-values for a multilinear commutator word w. The main result of [4] says that the verbal subgroup $w(G)$ is virtually

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abelian. More recently, it was shown in [1] that if π is a set of primes and G is a profinite group with restricted centralizers of π-elements, then G has an open subgroup of the form $P \times Q$, where P is an abelian pro-π subgroup and Q is a pro-π' subgroup.

Here we take interest in profinite groups G for which there is an integer n such that $C_G(x^n)$ is either finite or open whenever $x \in G$. This case is really quite different from the aforementioned ones.

Theorem 1.1 *Let n be a positive integer and G a profinite group in which the centralizer of* x^n *is either finite or open for every* $x \in G$. *Then G has an open normal subgroup T such that G/Z(T) has finite exponent.*

Recall that a group G is said to have finite exponent if there is a positive number e such that $x^e = 1$ for every $x \in G$.

The proof of the above theorem is short but it uses a number of highly nontrivial tools. In particular, it uses Zelmanov's theorem that a compact torsion group is locally finite [15], which in turn depends on Wilson's reduction theorem [14] as well as on the classification of finite simple groups. An important role is also played by the theorem of Khukhro that a locally finite group admitting an automorphism of prime order with finite centralizer is virtually nilpotent [8, 9]. Our proof of Theorem 1.1 also depends on some recent results of probabilistic nature (see [2, 3]).

Observe that Theorem 1.1 provides a rather detailed information on the structure of profinite groups with restricted centralizers of *n*th powers.

In particular, it shows that such a group G has an abelian normal subgroup N for which *G*/*N* has finite exponent. Moreover, using a theorem of Mann [10], it is easy to deduce that the commutator subgroup T' has finite exponent and therefore G is (finite exponent)-by-abelian-by-finite.

2 Proof of Theorem 1.1

Throughout, by a subgroup of a topological group we mean closed subgroup. Whenever X is a subset of a topological group, $\langle X \rangle$ stands for the subgroup generated by *X*. We write μ_G to denote the normalized Haar measure on a compact group G (see [6, Chapter 4] or [12, Chapter II]). If *K* is a subgroup of a finite (or compact) group *G*, then the probability that a random element of *G* commutes with a random element of *K* is denoted by *Pr*(*K*, *G*). In [2] the following theorem was established.

Theorem 2.1 *Let K be a subgroup of a compact group G. Then* $Pr(\langle x \rangle, G) > 0$ *for any* $x \in K$ *if and only if G has an open normal subgroup T such that* $K/C_K(T)$ *is torsion.*

Recall that the set of the elements of a group G having centralizer of finite index is denoted by *FC*(*G*). The previous result of probabilistic nature is relevant to our study of profinite groups with restricted centralizers of *n*th powers of elements in view of the following observation.

Lemma 2.2 *Let G be a compact group and* $x \in G$. *Then* $Pr(\langle x \rangle, G) > 0$ *if and only if there is a positive integer n such that* $C_G(x^n)$ *is open in G.*

Proof Suppose that for some positive integer *n* the centralizer $C_G(x^n)$ is open in *G*, and let *k* be the index $[G : C_G(x^n)]$. We have

$$\begin{aligned} Pr(\langle x\rangle, G) &= \int_G \mu_{\langle x\rangle}(C_{\langle x\rangle}(g))d\mu_G(g) = \int_G \mu_{\langle x\rangle}(C_{\langle x\rangle}(g)\langle x^n\rangle)\mu_{\langle x^n\rangle}(C_{\langle x^n\rangle}(g))d\mu_G(g) \\ &\geqslant \int_G \frac{1}{n}\mu_{\langle x^n\rangle}(C_{\langle x^n\rangle}(g))d\mu_G(g) = \frac{1}{n}Pr(\langle x^n\rangle, G) = \frac{1}{nk} > 0. \end{aligned}$$

For the other implication, assume that $Pr(\langle x\rangle, G) > 0$. By [3, Proposition 1.1] there are open subgroups $T \leqslant G$ and $B \leqslant \langle x\rangle$ such that [*B*, *T*] is finite. Since *B* is open in $\langle x\rangle$, there is a positive integer *n* such that x^n is a generator of *B*. It follows from the finiteness of [*B*, *T*] that $C_T(x^n)$ has finite index in *T*. Combining this with the fact that *T* is open in *G*, we deduce that the centralizer $C_G(x^n)$ has finite index in *G* and so is open, as desired. □

It is a longstanding open problem, raised in 1970 by Hewitt and Ross [7], whether every compact torsion group has finite exponent. In some cases the answer is known to be positive. We will need the following lemma.

Lemma 2.3 *Let n be a positive integer and G a profinite torsion group in which* $x^n \in FC(G)$ *for any* $x \in G$. *Then G has finite exponent.*

Proof Let $K = G^n$ be the subgroup generated by the *n*th powers of elements of *G*. Since *G*/*K* has finite exponent, it is sufficient to show that *K* has finite exponent. For every positive integer *s*, let X_s denote the set of all elements $g \in K$ such that $g^s = 1$. Obviously the sets X_s are closed in *G* and the union of X_s is exactly *K*. Therefore, by Baire Category Theorem [11, Theorem 34], at least one of the sets X_s contains non-empty interior. Hence, there is an integer *i*, an element $g \in K$ and an open normal subgroup *N* of *G* such that $g(K \cap N)$ is contained in X_i and so $(gx)^i = 1$ whenever $x \in K \cap N$. Set $N_0 = K \cap N$.

Thus, N_0 is a normal subgroup of *G* having finite index in *K*. Every element of *K* can be written as a product of (possibly infinitely many) *n*th powers of elements of *G*. Since N_0 has finite index in *K*, every element of *K* can be written as a product of finitely many *n*th powers of elements of *G* and an element of N_0. It follows that the coset gN_0 contains an element that is a a product of finitely many *n*th powers of elements of *G*. Without loss of generality we can assume that *g* is such an element. Since all *n*th powers are contained in *FC*(*G*), we deduce that $g \in FC(G)$.

In particular, $C_{N_0}(g)$ has finite index in N_0. Since $(gx)^i = 1$ whenever $x \in N_0$, it follows that every element of $C_{N_0}(g)$ has order dividing *i*. In other words, the exponent of $C_{N_0}(g)$ divides *i*. Taking into account that N_0 has finite index in *K* and $C_{N_0}(g)$ has finite index in N_0, we conclude that *K* has finite exponent. This completes the proof. □

We are now ready to prove Theorem 1.1.

Proof Recall that G is a profinite group in which the centralizer of x^n is either finite or open for every $x \in G$. We need to show that G has an open normal subgroup T such that $G/Z(T)$ has finite exponent.

Suppose first that $x^n \in FC(G)$ for all $x \in G$. In view of Lemma 2.2 this means that $Pr(\langle x\rangle, G) > 0$ for any $x \in G$. It follows from Theorem 2.1 that G has an open normal subgroup T such that $\bar{G} = G/C_G(T)$ is torsion. In view of Lemma 2.3 the quotient $\bar{G}$ has finite exponent. Note that $Z(T)$ is open in $C_G(T)$ since $Z(T) = T \cap C_G(T)$. Therefore $G/Z(T)$ has finite exponent, as desired.

Thus, without loss of generality we assume that G contains at least one element x such that $C_G(x^n)$ is finite. If G has an element g of infinite order, set $D = C_G(g^n)$. Observe that D contains $\langle g^n\rangle$. Therefore D is infinite and hence open in G. Moreover, for any $y \in D$ the centralizer $C_D(y)$ is infinite because it contains $\langle g^n\rangle$. It follows that $C_D(y^n)$ is open in G and so $y^n \in FC(G)$ for any $y \in D$. In view of the above we conclude that D has an open normal subgroup K such that $D/Z(K)$ has finite exponent. Of course, K can be chosen normal in G and so $G/Z(K)$ has finite exponent, as desired.

Therefore we assume now that G does not contain elements of infinite order, that is, G is torsion. Moreover, by Zelmanov's theorem, G is locally finite. Choose an nth power $a = b^n \in G$ of minimal possible order such that $C_G(a)$ is finite. For some prime p the order of a^p is smaller than that of a and so by the minimality of $|a|$ the centralizer $C_G(a^p)$ is open in G. Note that a induces an automorphism of order p on $C_G(a^p)$ and the centralizer $C_{C_G(a^p)}(a)$ is finite. Khukhro's theorem [9, 5.4.1 Corollary] now tells us that $C_G(a^p)$ is virtually nilpotent. Thus, G is virtually nilpotent. Without loss of generality we can assume that G is nilpotent and infinite. Since the centre of an infinite nilpotent profinite group is always infinite (see for instance [5, Lemma 2.1]), we deduce that $C_G(x)$ is infinite for any $x \in G$. Thus $[G : C_G(x^n)]$ is finite for any $x \in G$. Since $C_G(a)$ is finite, it follows that G is finite, a contradiction. The proof is now complete. □

Remark that the proof shows that when G is a virtually nilpotent profinite group containing a subset X such that $C_G(x)$ is either finite or open for every $x \in X$, then $X \subseteq FC(G)$.

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