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Enhancement of the Zakharov–Glassey's method for Blow-up in nonlinear Schrödinger equations

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Abstract
In this paper we give a sharper sufficient condition for blow-up of the solution to a nonlinear Schrödinger equation with free/Stark/quadratic potential by improving the well known Zakharov–Glassey's method.

Keywords: nonlinear Schrödinger equation, blow-up solutions, Zakharov–Glassey's method, Ehrenfest's theorem

1. Introduction

We consider, in dimension one, the nonlinear Schrödinger equation (hereafter NLS)

$$\left\{ \begin{array}{l} i\hbar \frac{\partial \psi_t}{\partial t} = H\psi_t + \nu |\psi_t|^{2\mu}\psi_t \\ \psi_t(x)|_{t=t_0} = \psi_0(x)\,, \|\psi_0\|_{L^2} = 1\,, \end{array} \right. \quad , \psi_t \in L^2(\mathbb{R}, \mathrm{d}x)\,, \tag{1}$$

where $H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)$ is the linear Schrödigner operator with real-valued potential $V(x)$; $\nu \in \mathbb{R}$ represents the strength of the nonlinear perturbation and $\mu > 0$ is the nonlinearity power. Hereafter, for sake of simplicity, we fix the units such that $\hbar = 1$ and $m = 1$, we further assume that $t_0 = 0$. The restriction to dimension one is just to simplify the discussion, but extension of the ideas developed in this paper to higher dimensions could be possible; however, we do not dwell on this problem here.

The first fundamental question that arises when dealing with a nonlinear Schrödinger equation (1) is the existence of a solution locally in time in some functional space. Thus, for ψ_0

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in such a space and under some assumptions on the potential $V(x)$ (e.g. $V(x)$ is a smooth, real-valued and at most quadratic function [9]), there exists $0 < t_+^\star \leqslant +\infty$ such that $\psi_t \in C([0,t_+^\star))$; furthermore, conservation of the norm

$$\mathcal{N}(\psi_t) = \mathcal{N}(\psi_0) \text{ where } \mathcal{N}(\psi) := \|\psi\|_{L^2}, \tag{2}$$

and of the energy

$$\mathcal{E}(\psi_t) = \mathcal{E}(\psi_0) \text{ where } \mathcal{E}(\psi) := \langle \psi, H\psi \rangle_{L^2} + \frac{\nu}{\mu+1}\|\psi\|_{L^{2\mu+2}}^{2\mu+2}, \tag{3}$$

are satisfied. Concerning global existence in the future three possibilities may occur:

- $t_+^\star = +\infty$ and $\limsup_{t\to+\infty} \|\psi_t\|_{H^1} < +\infty$, that is the solution is global in time and bounded;
- $t_+^\star = +\infty$ and $\limsup_{t\to+\infty} \|\psi_t\|_{H^1} = +\infty$, that is the solution blows up in infinite time;
- $0 < t_+^\star < +\infty$ and $\|\psi_t\|_{H^1} \to +\infty$ as $t \to t_+^\star - 0$, that is the solution blows up in finite time.

A similar analysis can be considered in the past for $t \leqslant 0$.

Our purpose in this paper is to give a blow-up sufficient condition by improving the Zakharov(–Shabat)–Glassey's method.

The method introduced by Zakarhov and Shabat [25] and by Glassey [12] (see also the papers [15, 17, 19]) is quite simple in the case where the virial identity takes a simple form. Let

$$\mathcal{I}(t) = \langle \psi_t, x^2 \psi_t \rangle_{L^2}$$

be the *moment of inertia*. If it can be shown that $\mathcal{I}(T_+^{\mathcal{I}}) = 0$ (resp. $\mathcal{I}(T_-^{\mathcal{I}}) = 0$) for some $\pm T_\pm^{\mathcal{I}} > 0$ then blow-up occurs in the future at some $t_+^\star \in (0, T_+^{\mathcal{I}}]$ (resp. in the past at some $t_-^\star \in [T_-^{\mathcal{I}}, 0)$).

To prove that $\mathcal{I}(t)$ can take a null value at some instant t one usually uses the virial identity, which in the one-dimensional free model where $V \equiv 0$ takes the form

$$\frac{\mathrm{d}^2\mathcal{I}}{\mathrm{d}t^2} = 4\mathcal{E}(\psi_0) + 2\nu\frac{\mu-2}{\mu+1}\|\psi_t\|_{L^{2\mu+2}}^{2\mu+2}. \tag{4}$$

If, for example, $\mu = 2$ and ψ_0 is such that $\mathcal{E}(\psi_0) < 0$, then by the virial identity (4) and by the conservation of the energy, the positive quantity $\mathcal{I}(t)$ is an inverted parabola that must then become negative at finite times $T_\pm^{\mathcal{I}}$, $-\infty < T_-^{\mathcal{I}} < 0 < T_+^{\mathcal{I}} < +\infty$, and thus the solution cannot exist for all time and it blows up at finite time in both the future and the past [18]. This argument is very powerful because of its simplicity; in fact, it is based on a pure Hamiltonian information $\mathcal{E}(\psi_0) < 0$, and it also applies to the super-critical case $\mu > 2$.

We have to point out that this method strongly depends on the fact that the virial identity (4) leads to an equation that can be explicitly solved with respect to $\mathcal{I}(t)$ and thus it cannot simply be applied when an external potential $V(x)$ is present because the associated virial identity is not in general associated to an explicitly solved equation. However, in a sequence of seminal papers by Carles [5–8] this method has been applied to the case where $V(x)$ is a quadratic or Stark potential in any dimension.

Our proposal of enhancement of the Zakarov–Glassey's method is based on a quite simple idea. Let

$$\langle \hat{x} \rangle^t := \langle \psi_t, x\psi_t \rangle_{L^2} \tag{5}$$

be the expectation value of the position observable x, where $\hat{x}$ is the associated operator. Let

$$\mathcal{V}(t) = \left\langle \psi_t, (\hat{x} - \langle \hat{x} \rangle^t)^2 \psi_t \right\rangle = \mathcal{I}(t) - [\langle \hat{x} \rangle^t]^2 \tag{6}$$

be the *variance*. If it can be shown that $\mathcal{V}(T_+^{\mathcal{V}}) = 0$ (resp. $\mathcal{V}(T_-^{\mathcal{V}}) = 0$) at some $\pm T_\pm^{\mathcal{V}} > 0$ then blow-up occurs in the future for some $t_+^\star \in (0, T_+^{\mathcal{V}}]$ (resp. in the past for some $t_-^\star \in [T_-^{\mathcal{V}}, 0)$). Since $\mathcal{V}(t) \leqslant \mathcal{I}(t)$ then we expect to give a sharper sufficient condition for the occurrence of the blow-up; the price to pay is to give an expression of the expectation value $\langle \hat{x} \rangle^t$, but this problem can be easily overcome using the (generalized) Ehrenfest's theorem where $\langle \hat{x} \rangle^t$ is nothing but the solution of the 'classical mechanics equation'. In fact, explicit solution to the 'classical mechanics equation' coming from the (generalized) Ehrenfest's theorem can be easily given only when $V(x)$ is a free, Stark or quadratic potential as discussed in remark 1; indeed, for generic (both regular or singular) potentials the 'classical mechanics equation' does not has an explicit solution (see remark 2).

Finally, we must also emphasize the fact that the enhanced Zakharov–Glassey's method not only gives a sharper sufficient condition for the occurrence of blow-up but it also allows us to give a better estimate of the instants $t_\pm^\star$ at which the solution becomes singular because $|T_\pm^{\mathcal{V}}| \leqslant |T_\pm^{\mathcal{I}}|$.

The paper is organized as follows. In section 2 we recall the Ehrenfest's generalized theorem; in section 3 we review the standard blow-up conditions in the free model where $V(x) \equiv 0$ and we show that these conditions can be easily improved by applying the virial equation for the variance $\mathcal{V}(t)$; in section 4 we consider the case where $V(x) = \alpha x$, $\alpha \in \mathbb{R}$, is a Stark potential; in section 5 we review the blow-up conditions in the case where $V(x) = \alpha x^2$, $\alpha \in \mathbb{R}$, is a quadratic potential and we show that again these conditions can be easily improved by applying the virial equation for the variance $\mathcal{V}(t)$. Finally, appendix A is about some functional inequalities, appendix B is about a comparison result for ordinary differential equations and appendix C is about the formal derivation of the virial identity; some results in appendices B and C are due to the papers [5–8], I collect these results in two short appendices for reader's benefit.

Hereafter, for the sake of simplicity, we omit the dependence on the variable t when this fact does not cause misunderstandings, e.g. ψ instead of ψ_t, $\langle \hat{x} \rangle$ instead of $\langle \hat{x} \rangle^t$, $\langle \hat{p} \rangle$ instead of $\langle \hat{p} \rangle^t$, $\mathcal{I}$ instead of $\mathcal{I}(t)$, $\mathcal{V}$ instead of $\mathcal{V}(t)$, and so on.

By $f' = \frac{\mathrm{d}f}{\mathrm{d}x}$ we denote the derivative with respect to x, by $\langle f, g \rangle_{L^2}$ we denote the scalar product $\int_{\mathbb{R}} \bar{f}(x) g(x) \mathrm{d}x$, and it is sometimes denoted simply by $\langle f, g \rangle$; also $\|f\|$ sometimes simply denotes $\|f\|_{L^2}$.

2. Ehrenfest's generalized theorem for NLS

The extension of the Ehrenfest's theorem to the nonlinear Schrödinger equation (1) has already been considered by [4, 14]. In fact, by means of a straightforward calculation it follows that

Proposition 1. *Let A be a time-independent quantum mechanical operator, and let*

$$\langle A \rangle = \langle \psi_t, A\psi_t \rangle_{L^2} \tag{7}$$

be its expectation value depending on time. Then

$$\frac{\mathrm{d}\langle A \rangle}{\mathrm{d}t} = i \langle \psi_t, [H, A] \psi_t \rangle_{L^2} + i\nu \left\langle \psi_t, \left[|\psi_t|^{2\mu}, A \right] \psi_t \right\rangle_{L^2}, \tag{8}$$

where $[H, A] = HA - AH$ is the commutator operator between the operators H and A, and where $\left[|\psi|^{2\mu}, A \right] \psi = |\psi|^{2\mu} A(\psi) - A(|\psi|^{2\mu} \psi)$. Equation (8) is usually called 'Ehrenfest's generalized theorem'.

As a consequence it follows that

Corollary 1. *Let x be the position observable and let $\hat{x} = x$ be the associated multiplication operator, then*

$$\frac{\mathrm{d}\langle \hat{x} \rangle^t}{\mathrm{d}t} = \langle \hat{p} \rangle^t \tag{9}$$

where $\hat{p} = -i\frac{\partial}{\partial x}$ is the associated operator to the momentum observable p.

Proof. Corollary 1 immediately follows from (8) since $[|\psi|^{2\mu}, \hat{x}] = 0$; hence

$$\frac{\mathrm{d}\langle \hat{x} \rangle}{\mathrm{d}t} = i\langle \psi, [H, \hat{x}]\psi \rangle = i\left\langle \psi, \left[\frac{\hat{p}^2}{2}, \hat{x}\right]\psi \right\rangle = \langle \hat{p} \rangle .$$

□

Similarly

Corollary 2. *Let p be the momentum observable with associated operator $\hat{p} = -i\frac{\partial}{\partial x}$, then*

$$\frac{d\langle \hat{p} \rangle^t}{dt} = -\left\langle \frac{\mathrm{d}V}{\mathrm{d}x} \right\rangle^t, \textit{where } \left\langle \frac{\mathrm{d}V}{\mathrm{d}x} \right\rangle^t = \left\langle \psi_t, \frac{\mathrm{d}V}{\mathrm{d}x}\psi_t \right\rangle_{L^2}. \tag{10}$$

Proof. Corollary 2 follows from (8) if we prove that $\left\langle \psi, \left[|\psi|^{2\mu}, \hat{p}\right]\psi \right\rangle = 0$; indeed

$$\begin{aligned}\left\langle \psi, \left[|\psi|^{2\mu}, \hat{p}\right]\psi \right\rangle &= -i\int_{\mathbb{R}} \bar{\psi}\left[|\psi|^{2\mu}\frac{\partial \psi}{\partial x} - \frac{\partial\left(|\psi|^{2\mu}\psi\right)}{\partial x}\right]\mathrm{d}x \\ &= -i\int_{\mathbb{R}} |\psi|^{2\mu}\left[\bar{\psi}\frac{\partial \psi}{\partial x} + \psi\frac{\partial \bar{\psi}}{\partial x}\right]\mathrm{d}x = -i\int_{\mathbb{R}} \rho^{\mu}\frac{\partial \rho}{\partial x}\mathrm{d}x = 0\end{aligned}$$

where $\rho = |\psi|^2$. Hence

$$\frac{\mathrm{d}\langle \hat{p} \rangle}{\mathrm{d}t} = i\langle \psi, [H, \hat{p}]\psi \rangle = i\langle \psi, [V, \hat{p}]\psi \rangle = -\left\langle \frac{\mathrm{d}V}{\mathrm{d}x} \right\rangle .$$

□

Remark 1. Let ψ_t be the solution to the hereafter NLS (1); then the expectation values $\langle \hat{x} \rangle$ of the position observable and $\langle \hat{p} \rangle$ of the momentum observable satisfy to the 'classical canonical equation of motion' (9) and (10). In the case where $V(x) = \frac{1}{2}ax^2 + bx$, for some $a, b \in \mathbb{R}$, then the system (9) and (10) takes the form

$$\begin{cases} \frac{\mathrm{d}\langle \hat{x} \rangle}{\mathrm{d}t} = \langle \hat{p} \rangle \\ \frac{\mathrm{d}\langle \hat{p} \rangle}{\mathrm{d}t} = -a\langle \hat{x} \rangle - b \end{cases}$$

and it has an explicit solution that does not depend on the nonlinearity parameter ν.

Remark 2. We have to point out that the system (9) and (10) has not an explicit solution for generic potentials, both regular like the double-well one or singular like point defect interactions. Indeed, if $V(x) = (x^2 - a^2)^2$, $a > 0$, is a double-well potential then the system (9) and (10) takes the form

$$\begin{cases} \frac{\mathrm{d}\langle \hat{x} \rangle}{\mathrm{d}t} = \langle \hat{p} \rangle \\ \frac{\mathrm{d}\langle \hat{p} \rangle}{\mathrm{d}t} = -4\langle \hat{x}^3 \rangle - 4a^2\langle \hat{x} \rangle \end{cases}$$

that does not has an explicit solution. Similarly, if $V(x)$ is a singular potential like a Dirac's δ at $x=0$ (one can consider several kinds of point defects in nonlinear Schrödinger equations, see, e.g. [1, 2]) then the system (9) and (10) takes the form

$$\begin{cases} \frac{\mathrm{d}\langle\hat{x}\rangle}{\mathrm{d}t} = \langle\hat{p}\rangle \\ \frac{\mathrm{d}\langle\hat{p}\rangle}{\mathrm{d}t} = 2\Re\left[\psi_t(0)\bar{\psi}_t'(0)\right] \end{cases}$$

and it does not has an explicit solution, too.

Remark 3. Ehrenfest's generalized theorem was also proved by [3] for nonlinear Schrödinger equations with a two or three-dimensional confining harmonic potential and under the effect of a rotating force. In such a framework it has also been proved that, under some circumstances (see proposition 4.3 in [3]), the solution is such that $\langle\hat{x}\rangle^t$ and $\langle\hat{p}\rangle^t$ go to $+\infty$ when t goes to $\pm\infty$.

Remark 4. We should point out that the Ehrenfest's generalized theorem (8) for nonlinear Schrödinger does not give the same result of the usual one for linear Schrödinger equations if the quantum operator is the Hamiltonian H; indeed, in such a case

$$\frac{\mathrm{d}\langle H\rangle}{\mathrm{d}t} = i\nu\left\langle\psi,\left[|\psi|^{2\mu},H\right]\psi\right\rangle = -\nu\Im\left\langle\psi,|\psi|^{2\mu}\hat{p}^2\psi\right\rangle$$

is not generically zero. In fact, $\langle H\rangle$ is an integral of motion only when $\nu=0$; otherwise the integral of motion is the energy $\mathcal{E}(\psi)$ defined by (3).

3. Blow-up for the free NLS

We consider now the case where the external potential is zero: $V(x)\equiv 0$. We assume that

$$\psi_0 \in \Sigma := H^1(\mathbb{R})\cap\mathcal{D}(\hat{x}), \tag{11}$$

where $\mathcal{D}(\hat{x})$ is the domain of the operator $\hat{x}$. Then the solution $\psi(x,t)$ to (1) locally exists and it belongs to $C((t_-^\star,t_+^\star),\Sigma)$, for some $t_-^\star<0<t_*^\star$, and the conservation of the norm $\|\psi\|_{L^2}$ and of the energy $\mathcal{E}$ hold true (see, e.g. theorem 3.10 in [22]). If $t_\pm^\star=\pm\infty$ then the solution globally exists; if not, i.e. $t_+^\star<+\infty$ (resp. $t_-^\star>-\infty$) then

$$\lim_{t\to t_\pm^\star\mp 0}\|\psi\|_{H^1}=\infty$$

and thus blow-up occurs in the future (resp. in the past). We observe that blow-up cannot occur when $\nu\geqslant 0$ because of the conservation of the energy (3). Furthermore, we can also point out that when blow-up occurs for $\nu<0$ then we also have that

$$\lim_{t\to t_\pm^\star\mp 0}\|\psi\|_{L^{2\mu+2}}^{2\mu+2}=\infty$$

because conservation of the energy.

3.1. Criterion for blow-up by means of the Zakharov–Glassey method

Estimates of the moment of inertia $\mathcal{I}(t)$ can be obtained by means of the one-dimensional virial identity (4) with initial conditions

$$\mathcal{I}_0 := \mathcal{I}(0) = \|x\psi_0\|_{L^2}^2 \tag{12}$$

and

$$\dot{\mathcal{I}}_0 := \frac{d\mathcal{I}(0)}{dt} = 2\Im\left[\int_{\mathbb{R}} x\bar{\psi}_0(x)\frac{\partial\psi_0(x)}{\partial x}\mathrm{d}x\right] = 2\Re\langle\hat{x}\psi_0,\hat{p}\psi_0\rangle\ . \tag{13}$$

Theorem 5.1 in [22] gives a sufficient condition for blow-up in the future (and similarly in the past). Specifically, when $\nu<0$ and $\mu\geqslant 2$ then there exists a $t_+^\star\in(0,+\infty)$ such that

$$\lim_{t\to t_+^\star-0}\|\psi\|_{H^1}=\infty$$

if any of the following conditions is satisfied:

(a) $\mathcal{E}(\psi_0)<0$;
(b) $\mathcal{E}(\psi_0)=0$ and $\dot{\mathcal{I}}_0<0$;
(c) $\mathcal{E}(\psi_0)>0$ and $\dot{\mathcal{I}}_0\leqslant-\sqrt{8\mathcal{E}(\psi_0)\mathcal{I}_0}$.

The proof of theorem 5.1 in [22] is quite simple: if $\mu\geqslant 2$ and $\nu\leqslant 0$ then (4) implies that

$$\frac{\mathrm{d}^2\mathcal{I}}{\mathrm{d}t^2}\leqslant 4\mathcal{E}(\psi_0)$$

and thus

$$\mathcal{I}(t)\leqslant M(t):=2\mathcal{E}(\psi_0)t^2+\dot{\mathcal{I}}_0 t+\mathcal{I}_0. \tag{14}$$

If any of the three conditions (a)–(c) is satisfied then there exists $\tilde{T}_+^{\mathcal{I}}>0$ such that $M(\tilde{T}_+^{\mathcal{I}})=0$ and thus there exists a $0<T_+^{\mathcal{I}}<\tilde{T}_+^{\mathcal{I}}$ such that $\mathcal{I}(T_+^{\mathcal{I}})=0$. From this fact and from (31) the occurrence of blow-up in the future follows at some $t_+^\star<T_+^{\mathcal{I}}$.

3.2. *Criterion for blow-up by means of the enhanced Zakharov–Glassey method*

We improve now the previous criterion by applying the same argument to the analysis of the variance $\mathcal{V}(t)$ and making use of the Ehrenfest's generalized theorem. Indeed, if the potential $V(x)$ is exactly zero then (9) and (10) imply that

$$\langle\hat{p}\rangle\equiv\hat{p}_0\text{and }\langle\hat{x}\rangle=\hat{p}_0t+\hat{x}_0\,,\text{where }\hat{x}_0:=\langle\hat{x}\rangle^t|_{t=0}\text{ and }\hat{p}_0:=\langle\hat{p}\rangle^t|_{t=0}\,. \tag{15}$$

Remark 5. We point out that in the free NLS problem the conservation of the momentum $\langle\hat{p}\rangle$ and the fact that the *center of mass* of the wavepacket $\langle\hat{x}\rangle$ moves at constant speed can be also derived by making use of arguments of invariance of space translation (see, e.g. section 2.3 in [22]).

Since (15) we have that

$$\mathcal{V}(t)=\mathcal{I}(t)-\langle\hat{x}\rangle^2=\mathcal{I}(t)-[\hat{p}_0t+\hat{x}_0]^2\leqslant N(t)$$

where

$$\begin{aligned}N(t)&:=M(t)-[\hat{p}_0t+\hat{x}_0]^2\\&=\left[2\mathcal{E}(\psi_0)-\hat{p}_0^2\right]t^2+\left[\dot{\mathcal{I}}_0-2\hat{p}_0\hat{x}_0\right]t+\left[\mathcal{I}_0-\hat{x}_0^2\right]\,.\end{aligned} \tag{16}$$

Thus we have the following improvement of theorem 5.1 in [22].

Theorem 1. *Let $\nu < 0$ and $\mu \geqslant 2$, let $\psi_0 \in \Sigma$; then we have blow-up in the future if any of the following conditions is satisfied:*

(a) $2\mathcal{E}(\psi_0) < \hat{p}_0^2$;
(b) $2\mathcal{E}(\psi_0) = \hat{p}_0^2$ *and* $\dot{\mathcal{I}}_0 < 2\hat{p}_0\hat{x}_0$;
(c) $2\mathcal{E}(\psi_0) > \hat{p}_0^2$ *and*

$$\left[\dot{\mathcal{I}}_0 - 2\hat{p}_0\hat{x}_0\right] \leqslant -2\sqrt{\left[2\mathcal{E}(\psi_0) - \hat{p}_0^2\right]\left[\mathcal{I}_0 - \hat{x}_0^2\right]}\,.$$

Proof. Indeed, if any of the three conditions (a)–(c) is satisfied then there exists $\tilde{T}_+^{\mathcal{V}} > 0$ such that $N(\tilde{T}_+^{\mathcal{V}}) = 0$ and thus there exists $0 < T_+^{\mathcal{V}} \leqslant \tilde{T}_+^{\mathcal{V}}$ such that $\mathcal{V}(T_+^{\mathcal{V}}) = 0$. From this fact and from (32) the occurrence of blow-up follows for some $t_+^\star \leqslant T_+^{\mathcal{V}}$. □

Remark 6. In fact, under condition (a) we have blow-up in the future and in the past, too; under conditions (b) and (c) we have blow-up in the future only.

Remark 7. We remark that condition (a) for blow-up is not new and it has been already proved under some circumstances, see e.g. corollary 1.2 in [11] and theorem 7 in [19].

4. Blow-up for the NLS with Stark potential

Let the potential $V(x) = \alpha x$ be a Stark potential, where $\alpha \in \mathbb{R} \setminus \{0\}$, the occurrence of blow-up in such a case has been considered by [7, 16, 20] . Again we assume (11).

4.1. Criterion for blow-up by means of the Zakharov–Glassey method

In the case of Stark potentials it has been proved that the solutions to the NLS (1) with a Stark potential can be derived from the ones of the free NLS, see theorem 2.1 in [7]. Then one can make use of the results obtained in section 3.1; in particular, corollary 3.3 in [7] states that blow-up occurs in the past and in future when

$$\frac{1}{2}\|\psi_0'\|_{L^2} + \frac{\nu}{\mu+1}\|\psi_0\|_{L^{2\mu+2}}^{2\mu+2} < 0\,. \tag{17}$$

4.2. Criterion for blow-up by means of the enhanced Zakharov–Glassey method

If the potential $V(x) = \alpha x$ is a Stark potential, where $\alpha \in \mathbb{R} \setminus \{0\}$ then (9) and (10) imply that

$$\langle \hat{p} \rangle^t = -\alpha t + \hat{p}_0 \text{ and } \langle \hat{x} \rangle^t = -\frac{1}{2}\alpha t^2 + \hat{p}_0 t + \hat{x}_0 \tag{18}$$

where

$$\hat{x}_0 = \langle \hat{x} \rangle^t|_{t=0} \text{ and } \hat{p}_0 = \langle \hat{p} \rangle^t|_{t=0}\,.$$

Estimates of the moment of inertia $\mathcal{I}(t)$ can be obtained by means of the one-dimensional virial identity (38) with initial conditions (12) and (13).

If $\nu(\mu-2) \leqslant 0$ then (18) and (38) imply that

$$\frac{d^2\mathcal{I}}{dt^2} \leqslant 4\mathcal{E} - 6\alpha\left(-\frac{1}{2}\alpha t^2 + \hat{p}_0 t + \hat{x}_0\right)$$

where

$$\mathcal{E} = \frac{1}{2}\|\psi'\|_{L^2}^2 + \alpha\langle\hat{x}\rangle + \frac{\nu}{\mu+1}\|\psi\|_{L^{2\mu+2}}^{2\mu+2},$$

and thus

$$\mathcal{I}(t) \leqslant \frac{1}{4}\alpha^2 t^4 - \alpha\hat{p}_0 t^3 + [2\mathcal{E} - 3\alpha\hat{x}_0]\, t^2 + \dot{\mathcal{I}}_0 t + \mathcal{I}_0 .$$

$\mathcal{I}_0$ and $\dot{\mathcal{I}}_0$ are given by (12) and (13). Therefore,

$$\mathcal{V}(t) = \mathcal{I}(t) - [\langle\hat{x}\rangle^t]^2 \leqslant \left[\|\psi_0'\|^2 + \frac{2\nu}{\mu+1}\|\psi\|_{L^{2\mu+2}}^{2\mu+2} - \hat{p}_0^2\right] t^2 + 2\left[\Re\langle\hat{x}\psi_0, \hat{p}\psi_0\rangle - \hat{p}_0\hat{x}_0\right] t + \mathcal{V}(0).$$

Thus, we can conclude that

Theorem 2. *If*

(a) $\|\psi_0'\|^2 + \frac{2\nu}{\mu+1}\|\psi_0\|_{L^{2\mu+2}}^{2\mu+2} < \hat{p}_0^2$ *then we have blow-up in the past and in the future;*
(b) $\|\psi_0'\|^2 + \frac{2\nu}{\mu+1}\|\psi_0\|_{L^{2\mu+2}}^{2\mu+2} = \hat{p}_0^2$ *and* $\Re\langle\hat{x}\psi_0, \hat{p}\psi_0\rangle - \hat{p}_0\hat{x}_0 \neq 0$ *we have blow-up in the past or in the future;*
(c) $\|\psi_0'\|^2 + \frac{2\nu}{\mu+1}\|\psi_0\|_{L^{2\mu+2}}^{2\mu+2} > \hat{p}_0^2$ *and*

$$\left[\Re\langle\hat{x}\psi_0, \hat{p}\psi_0\rangle - \hat{p}_0\hat{x}_0\right]^2 > \left[\|\psi_0'\|^2 + \frac{2\nu}{\mu+1}\|\psi_0\|_{L^{2\mu+2}}^{2\mu+2} - \hat{p}_0^2\right]\mathcal{V}(0)$$

we have blow-up in the past or in the future.

Remark 8. Since $\|\psi_0'\|^2 = \|\hat{p}\psi_0\|^2$ and $|\hat{p}_0| = |\langle\psi_0, \hat{p}\psi_0\rangle| \leqslant \|\hat{p}\psi_0\|$ then conditions (a) and (b) hold true only when $\nu < 0$.

Remark 9. We remark that the blow-up condition (17) given by [7] agrees with theorem 2, indeed (17) implies (a).

5. Blow-up for NLS with harmonic/inverted oscillator potential

In this section we consider the cases of *harmonic oscillator* potential $V(x) = \alpha x^2$, where $\alpha > 0$, and *inverted oscillator* potential, where $\alpha < 0$. The occurrence of blow-up in these cases has been considered by several authors under different assumptions [5, 6, 8, 10, 13, 21, 23, 24, 26].

In this section we consider the blow-up conditions obtained by means of the enhanced Zakharov–Glassey's method and then we compare these results with the previous ones obtained by Carles [5, 6, 8].

We require now some preliminary results.

Also in this case we assume (11), then local in time existence of the solution to (1) in Σ and conservation of the norm and of the energy $\mathcal{E}$ follows (see, e.g. [8]).

Corollary 2 implies that $\frac{\mathrm{d}\langle\hat{p}\rangle^t}{\mathrm{d}t} = -2\alpha\langle\hat{x}\rangle^t$; hence the expectation value of the position observable coincides with the classical solution. More precisely, let

$$\lambda^2 = 2|\alpha|, \hat{x}_0 = \langle\hat{x}\rangle^t|_{t=0} \text{ and } \hat{p}_0 = \langle\hat{p}\rangle^t|_{t=0};$$

then the Ehrenfest's generalized theorem implies that:

- In the case of the *harmonic oscillator* where $\alpha > 0$, then

$$\begin{cases} \langle \hat{x} \rangle^t = \hat{x}_0 \cos(\lambda t) + \frac{\hat{p}_0}{\lambda} \sin(\lambda t) \\ \langle \hat{p} \rangle^t = -\lambda \hat{x}_0 \sin(\lambda t) + \hat{p}_0 \cos(\lambda t) \end{cases} . \tag{19}$$

- In the case of the *inverted oscillator* where $\alpha < 0$, then

$$\begin{cases} \langle \hat{x} \rangle^t = \hat{x}_0 \cosh(\lambda t) + \frac{\hat{p}_0}{\lambda} \sinh(\lambda t) \\ \langle \hat{p} \rangle^t = \lambda \hat{x}_0 \sinh(\lambda t) + \hat{p}_0 \cosh(\lambda t) \end{cases} . \tag{20}$$

In a previous paper [6] devoted to the analysis of the occurrence of blow-up it has been found that, in the case of harmonic/inverted potential, the moment of inertia $\mathcal{I}$ satisfies to the following equation

$$\frac{\mathrm{d}^2 \mathcal{I}}{\mathrm{d}t^2} + 8\alpha \mathcal{I} = 4\mathcal{E}(\psi_0) + 2\nu \frac{\mu - 2}{\mu + 1} \|\psi\|_{L^{2\mu+2}}^{2\mu+2} . \tag{21}$$

As in the free case we consider now the equation for the variance $\mathcal{V}$.

Lemma 1. *The variance $\mathcal{V}$ satisfies to the following equation*

$$\frac{\mathrm{d}^2 \mathcal{V}}{\mathrm{d}t^2} + 8\alpha \mathcal{V} = C_{\mathcal{V}} + 2\nu \frac{\mu - 2}{\mu + 1} \|\psi\|_{L^{2\mu+2}}^{2\mu+2} , \tag{22}$$

where

$$C_{\mathcal{V}} = -2\hat{p}_0^2 - 4\alpha \hat{x}_0^2 + 4\mathcal{E}(\psi_0) . \tag{23}$$

Proof. Indeed, form (21) it turns out that the variance is a solution to the equation

$$\frac{\mathrm{d}^2 \mathcal{V}}{\mathrm{d}t^2} + 8\alpha \mathcal{V} = -\frac{\mathrm{d}^2 \langle \hat{x} \rangle^2}{\mathrm{d}t^2} - 8\alpha \langle \hat{x} \rangle^2 + 4\mathcal{E}(\psi_0) + 2\nu \frac{\mu - 2}{\mu + 1} \|\psi\|_{L^{2\mu+2}}^{2\mu+2} ,$$

where $\langle \hat{x} \rangle$ simply denotes $\langle \hat{x} \rangle^t$ and it is given by (19) (resp. (20)) when $\alpha > 0$ (resp. $\alpha < 0$). We may remark that the term

$$C = -\frac{\mathrm{d}^2 \langle \hat{x} \rangle^2}{\mathrm{d}t^2} - 8\alpha \langle \hat{x} \rangle^2$$

is constant. Indeed,

$$C = -2 \left(\frac{\mathrm{d} \langle \hat{x} \rangle}{\mathrm{d}t} \right)^2 - 2 \langle x \rangle \frac{\mathrm{d}^2 \langle \hat{x} \rangle}{\mathrm{d}t^2} - 8\alpha \langle x \rangle^2 = -2 \left(\frac{\mathrm{d} \langle \hat{x} \rangle}{\mathrm{d}t} \right)^2 - 4\alpha \langle x \rangle^2$$

since $\frac{\mathrm{d}^2 \langle \hat{x} \rangle}{\mathrm{d}t^2} = -2\alpha \langle \hat{x} \rangle$ and thus

$$\frac{\mathrm{d}C}{\mathrm{d}t} = -4 \frac{\mathrm{d} \langle \hat{x} \rangle}{\mathrm{d}t} \frac{\mathrm{d}^2 \langle \hat{x} \rangle}{\mathrm{d}t^2} - 8\alpha \langle \hat{x} \rangle \frac{\mathrm{d} \langle \hat{x} \rangle}{\mathrm{d}t} = 0 .$$

Hence,

$$C = -2 \left(\frac{\mathrm{d} \langle \hat{x} \rangle}{\mathrm{d}t} \right)^2_{t=0} - 4\alpha \hat{x}_0^2 = -2\hat{p}_0^2 - 4\alpha \hat{x}_0^2$$

and (22) follows. □

We recall that the initial condition associated to (21) and (22) are

$$\mathcal{V}_0 := \mathcal{V}(0) = \mathcal{I}_0 - \hat{x}_0^2 = \|\hat{x}\psi_0\|^2 - \hat{x}_0^2 \tag{24}$$

and

$$\dot{\mathcal{V}}_0 := \frac{d\mathcal{V}(0)}{dt} = 2\left[\Re\langle \hat{x}\psi_0, \hat{p}\psi_0\rangle - \hat{x}_0\hat{p}_0\right] . \tag{25}$$

Let us consider now the differential equation (22) for $\mu \geqslant 2$ and $\nu < 0$. From lemma 3 in appendix B we have that $0 \leqslant \mathcal{V}(t) \leqslant \zeta(t)$ where $\zeta(t)$ is the solution to

$$\begin{cases} \frac{d^2\zeta}{dt^2} + 8\alpha\zeta = C_{\mathcal{V}} \\ \zeta(0) = \mathcal{V}_0 \text{and } \frac{d\zeta(0)}{dt} = \dot{\mathcal{V}}_0 \end{cases} .$$

If we set $\Omega = 2\lambda = \sqrt{8|\alpha|}$ then the solution $\zeta(t)$ is given by

$$\zeta(t) = \begin{cases} \zeta_H(t) := \frac{\dot{\mathcal{V}}_0}{\Omega}\sin(\Omega t) + \mathcal{V}_0\cos(\Omega t) + \frac{1}{\Omega^2}C_{\mathcal{V}}\left[1-\cos(\Omega t)\right] & ,\text{if } \alpha > 0 \\ \\ \zeta_I(t) := \frac{\dot{\mathcal{V}}_0}{\Omega}\sinh(\Omega t) + \mathcal{V}_0\cosh(\Omega t) - \frac{1}{\Omega^2}C_{\mathcal{V}}\left[1-\cosh(\Omega t)\right] & ,\text{if } \alpha < 0 \end{cases} .$$

Now, we are ready to apply the enhanced Zakharov–Glassey method.

5.1. Harmonic oscillator—criterion for blow-up

In the case of the harmonic oscillator potential, where $\alpha > 0$, from lemma 3 in appendix B it follows that the variance $\mathcal{V}(t)$ is bounded from above by the function

$$\zeta_H(t) = \sqrt{a^2+b^2}\sin(\Omega t + \varphi) + c$$

for any t such that $\Omega|t| \leqslant \pi$, where φ is a phase term such that

$$\frac{a}{\sqrt{a^2+b^2}} = \cos\varphi\,, \frac{b}{\sqrt{a^2+b^2}} = \sin\varphi\,, a := \frac{\dot{\mathcal{V}}_0}{\Omega}\,, b := \mathcal{V}_0 - \frac{C_{\mathcal{V}}}{\Omega^2}\,, c := \frac{C_{\mathcal{V}}}{\Omega^2}\,.$$

Since $\zeta_H(\pm\pi/\Omega) = \frac{2}{\Omega^2}C_{\mathcal{V}} - \mathcal{V}_0$ then we have blow-up in the future and in the past if

$$2\frac{C_{\mathcal{V}}}{\Omega^2} - \mathcal{V}_0 \leqslant 0\,. \tag{26}$$

If not, since by means of a straightforward calculation it follows that

$$\min_{t\in[-\pi/\Omega,+\pi/\Omega]} \mathcal{V}(t) \leqslant \min_{t\in[-\pi/\Omega,+\pi/\Omega]} \zeta_H(t) = \frac{C_{\mathcal{V}}}{\Omega^2} - \sqrt{\frac{\dot{\mathcal{V}}_0^2}{\Omega^2} + \left(\mathcal{V}_0 - \frac{C_{\mathcal{V}}}{\Omega^2}\right)^2}$$

and then there exists blow-up in the past or in the future if

$$\dot{\mathcal{V}}_0^2 + \mathcal{V}_0^2\Omega^2 - 2\mathcal{V}_0 C_{\mathcal{V}} \geqslant 0\,. \tag{27}$$

Thus, we have proved the following results.

Theorem 3. *Let $\psi_0 \in \Sigma$ be the normalized initial wavefunction; let $\mu \geqslant 2$, $\alpha > 0$ and $\Omega = \sqrt{8\alpha}$; let $C_{\mathcal{V}}$, $\mathcal{V}_0$ and $\dot{\mathcal{V}}_0$ defined as in (23)–(25). Then, in the focusing nonlinearity case such that $\nu < 0$ blow-up occurs in the past and in the future at some instants $\tilde{T}_- \leqslant t_-^\star < 0 < t_+^\star \leqslant \tilde{T}_+$ if (26) is satisfied; where $\tilde{T}_\pm$ are the solutions to the equation $\zeta_H(t) = 0$ in the interval $[-\pi/\Omega, \pi/\Omega]$. If (26) is not satisfied, but (27) holds true then blow-up occurs in the past or in the future in the interval $[-\pi/\Omega, +\pi/\Omega]$.*

Remark 10. We compare now the results above with the ones given by proposition 3.2 [5] where occurrence of blow-up in the past and in the future was proved in the case of harmonic potential, where $\alpha > 0$, focusing nonlinearity, where $\nu < 0$, and under the conditions $\mu \geqslant 2$ and

$$\frac{1}{2}\|\nabla\psi_0\|_{L^2}^2 + \frac{\nu}{\mu+1}\|\psi_0\|_{L^{2\mu+2}}^{2\mu+2} \leqslant 0\,. \tag{28}$$

In fact, condition (28) implies that (since $\mathcal{V}_0 \geqslant 0$)

$$\Omega^2\mathcal{I}_0 \geqslant 8\mathcal{E} \Leftrightarrow \Omega^2\mathcal{V}_0 - 2C_{\mathcal{V}} \geqslant 4\hat{p}_0^2\,.$$

That is, if (28) occurs then (26) is satisfied (but not vice versa).

5.2. *Inverted oscillator—criterion for blow-up*

In the case of the inverted oscillator potential where $\alpha < 0$ a similar argument proves that the variance $\mathcal{V}(t)$ is bounded from above by the function $\zeta_I(t)$ for any $t \in \mathbb{R}$. Then, it follows that

Theorem 4. *Let $\psi_0 \in \Sigma$ be the normalized initial wavefunction; let $\mu \geqslant 2$, $\alpha < 0$ and $\Omega = \sqrt{8|\alpha|}$; let $C_{\mathcal{V}}$, $\mathcal{V}_0$ and $\dot{\mathcal{V}}_0$ defined as in (23)–(25). Let it now*

$$a := \frac{\dot{\mathcal{V}}_0}{\Omega}\,, b := \mathcal{V}_0 + \frac{C_{\mathcal{V}}}{\Omega^2} \text{and } c := -\frac{C_{\mathcal{V}}}{\Omega^2}\,.$$

Then, in the focusing nonlinearity case such that $\nu < 0$ blow-up occurs if

(a) $b < -|a|$; in that case blow-up occurs in both the past and in the future.
(b) $|a| < b$ and $\sqrt{b^2 - a^2} + c \leqslant 0$; in that case blow-up occurs only in the future (if $a < 0$) or only in the past (if $a > 0$).
(c) $|a| > |b|$; in that case we have blow-up in the past if $a > 0$ or in the future if $a < 0$.
(d) $|a| = |b|$; in that case we have blow-up if $bc < 0$, in particular we have blow-up in the past if $a > 0$ or in the future if $a < 0$.

Proof. Let us introduce the function $\zeta(\tau) = \zeta_I(t)$ where $\tau = \Omega t$, then

$$\zeta(\tau) := a\sinh(\tau) + b\cosh(\tau) + c\,, \zeta(0) = \mathcal{V}(0) > 0\,,$$

and where a, b and c are defined above. If

(a) $|a| < |b|$ then $\frac{d\zeta(\tau_1)}{d\tau} = 0$ where $\tau_1 = \text{arctanh}\left(-\frac{a}{b}\right)$. In particular, if:
  1. $b < 0$ then $\lim_{\tau\to\pm\infty}\zeta(\tau) = -\infty$ and thus there exists $\mathcal{T}_- < 0 < \mathcal{T}_+$ such that $\zeta(\mathcal{T}_\pm) = 0$. In such a case we have blow-up in the past and in the future.
  2. $0 < b$ then $\lim_{\tau\to\pm\infty}\zeta(\tau) = +\infty$. We compute now
  $$\zeta(\tau_1) = \sqrt{b^2 - a^2} + c\,.$$
  Thus, if
  $$\sqrt{b^2 - a^2} + c \leqslant 0$$
  then we have blow-up in the future if $a < 0$ *or* in the past if $a > 0$.

(b) $|a| > |b|$ then $\zeta(\tau)$ is a monotone increasing (resp. decreasing) function if $a > 0$ (resp. $a < 0$) such that $\lim_{\tau\to\pm\infty}\zeta(\tau) = \pm\infty$ (resp. $\mp\infty$); therefore there exists $\mathcal{T}_- < 0$ (resp. $0 < \mathcal{T}_+$) such that $\zeta(\mathcal{T}_-) = 0$ (resp. $\zeta(\mathcal{T}_+) = 0$), and thus we have blow-up in the past (resp. in the future).

(c) $a=b$ then $\frac{d\zeta(\tau)}{d\tau}\neq 0$ for any τ. Hence, if:
   1. $a>0$ then $\frac{d\zeta(0)}{d\tau}>0$ and then $\frac{d\zeta(\tau)}{d\tau}>0$ for any τ; furthermore, $\lim_{\tau\to-\infty}\zeta(\tau)=c$ and $\lim_{\tau\to+\infty}\zeta(\tau)=+\infty$. Thus, if $c<0$ then there exists $\mathcal{T}_-<0$ such that $\zeta(\mathcal{T}_-)=0$ and so we have blow-up in the past.
   2. $a<0$ then $\frac{d\zeta(0)}{d\tau}<0$ and then $\frac{d\zeta(\tau)}{d\tau}<0$ for any τ; furthermore, $\lim_{\tau\to-\infty}\zeta(\tau)=c$ and $\lim_{\tau\to+\infty}\zeta(\tau)=-\infty$. Thus, if $c>0$ then there exists $0<\mathcal{T}_+$ such that $\zeta(\mathcal{T}_+)=0$ and so we have blow-up in the future.

(d) $a=-b$ then $\frac{d\zeta(\tau)}{d\tau}\neq 0$ for any τ. Hence, if:
   1. $a>0$ then $\frac{d\zeta(0)}{d\tau}>0$ and then $\frac{d\zeta(\tau)}{d\tau}>0$ for any τ; furthermore, $\lim_{\tau\to-\infty}\zeta(\tau)=-\infty$ and $\lim_{\tau\to+\infty}\zeta(\tau)=c$. Thus, if $c>0$ then there exists $\mathcal{T}_-<0$ such that $\zeta(\mathcal{T}_-)=0$ and so we have blow-up in the past.
   2. $a<0$ then $\frac{d\zeta(0)}{d\tau}<0$ and then $\frac{d\zeta(\tau)}{d\tau}<0$ for any τ; furthermore, $\lim_{\tau\to-\infty}\zeta(\tau)=+\infty$ and $\lim_{\tau\to+\infty}\zeta(\tau)=c$. Thus, if $c<0$ then there exists $\mathcal{T}_-<0$ such that $\zeta(\mathcal{T}_-)=0$ and so we have blow-up in the past.

Collecting all these results then theorem 4 follows. □

Remark 11. We compare now the results above with those given by theorem 1.1 [8]. For example, [8] proved that in the case of inverted potential, where $\alpha<0$, and focusing nonlinearity, where $\nu<0$, under the condition $\mu\geqslant 2$ and

$$\frac{1}{2}\|\nabla\psi_0\|^2_{L^2}+\frac{\nu}{\mu+1}\|\psi_0\|^{2\mu+2}_{L^{2\mu+2}}<-|\alpha|\|x\psi_0\|^2_{L^2}-\sqrt{2|\alpha|}\,|\Re\langle\hat{x}\psi_0,\hat{p}\psi_0\rangle| \tag{29}$$

then blow-up occurs in the future and in the past at some instant. By means of a straightforward calculation it can be proved that if condition (29) is satisfied, then condition (a) of theorem 4 holds true, but not vice versa.

Data availability statement

No new data were created or analyzed in this study.

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Appendix A. Functional inequalities

Lemma 2. *The following inequality holds true: let $y\in\mathbb{R}$ and let*

$$\Gamma:=\Gamma(y)=\langle f,(x-y)^2f\rangle_{L^2}$$

for any test function $f\in L^2(\mathbb{R},dx)$ such that $xf\in L^2(\mathbb{R},dx)$. Then, for any $q\geqslant 0$:

$$\|f\|^{2q+2}_{L^{2q+2}}\leqslant C\sqrt{\Gamma}\|f\|^q_{L^2}\|f'\|^{q+1}_{L^2}, \tag{30}$$

for some positive constant C, where $f'=\frac{df}{dx}$.

Proof. Indeed:

$$\|f\|_{L^{2q+2}}^{2q+2} = \int_{\mathbb{R}} \frac{\partial(x-y)}{\partial x} f^{q+1}\bar{f}^{q+1}\mathrm{d}x = -(q+1)\int_{\mathbb{R}} (x-y)|f|^{2q}\left[f'\bar{f}+f\bar{f}'\right]\mathrm{d}x .$$

Hence

$$\|f\|_{L^{2q+2}}^{2q+2} \leqslant 2(q+1)\|(x-y)f\|_{L^2}\|f\|_{L^\infty}^{2q}\|f'\|_{L^2} .$$

Now, recalling that from the Gagliardo–Nirenberg inequality one has that

$$\|f\|_{L^\infty} \leqslant \sqrt{2}\|f'\|_{L^2}^{1/2}\|f\|_{L^2}^{1/2}$$

then it follows that

$$\|f\|_{L^{2q+2}}^{2q+2} \leqslant C\sqrt{\Gamma}\|f\|_{L^2}^{q}\|f'\|_{L^2}^{q+1} ,$$

for some positive constant C. □

Corollary 3. *In particular, if $y=0$ and $q=0$ then we have that for some positive constant C:*

$$\|f\|_{L^2}^2 \leqslant C\sqrt{\mathcal{I}}\|f'\|_{L^2} , \tag{31}$$

where $\Gamma(0)=\|xf\|_{L^2}^2=\mathcal{I}$ is the moment of inertia; if $y=\langle\hat{x}\rangle=\langle f,xf\rangle_{L^2}$ and $q=0$ then we have that:

$$\|f\|_{L^2}^2 \leqslant C\sqrt{\mathcal{V}}\|f'\|_{L^2} , \tag{32}$$

where $\Gamma(\langle\hat{x}\rangle)=\|(x-\langle\hat{x}\rangle)f\|_{L^2}^2=\mathcal{V}$ is the variance.

Appendix B. Comparison between solutions of the harmonic/inverted oscillator

Let $\mathcal{V}_\pm(t)$ be the solution to the differential equation

$$\begin{cases} \frac{\mathrm{d}^2\mathcal{V}_\pm}{\mathrm{d}t^2} \pm \Omega^2\mathcal{V}_\pm = C + f(t) \\ \mathcal{V}_\pm(0)=\mathcal{V}_{\pm,0} \text{ and } \frac{\mathrm{d}\mathcal{V}_\pm(0)}{\mathrm{d}t}=\dot{\mathcal{V}}_{\pm,0} \end{cases} , \tag{33}$$

where C is a constant factor and $f(t)\leqslant 0$ for any t; and let $\zeta_\pm(t)$ be the solution to the differential equation

$$\begin{cases} \frac{\mathrm{d}^2\zeta_\pm}{\mathrm{d}t^2} \pm \Omega^2\zeta_\pm = C \\ \zeta_\pm(0)=\mathcal{V}_{\pm,0} \text{ and } \frac{\mathrm{d}\zeta_\pm(0)}{\mathrm{d}t}=\dot{\mathcal{V}}_{\pm,0} \end{cases} . \tag{34}$$

Then, the difference $\mathcal{Z}_\pm(t)=\mathcal{V}_\pm(t)-\zeta_\pm(t)$ solves the differential equation

$$\begin{cases} \frac{\mathrm{d}^2\mathcal{Z}_\pm}{\mathrm{d}t^2} \pm \Omega^2\mathcal{Z}_\pm = f(t) \\ \mathcal{Z}_\pm(0)=0 \text{ and } \frac{\mathrm{d}\mathcal{Z}_\pm(0)}{\mathrm{d}t}=0 \end{cases} .$$

Hence, we have that

$$\mathcal{Z}_+(t)=\frac{1}{\Omega}\int_0^t \sin[\Omega(t-s)]f(s)\mathrm{d}s \leqslant 0 \text{ if } \Omega|t|\leqslant\pi$$

and

$$\mathcal{Z}_-(t)=\frac{1}{\Omega}\int_0^t \sinh[\Omega(t-s)]f(s)\mathrm{d}s \leqslant 0 , \forall t\in\mathbb{R} .$$

In conclusion,

Lemma 3. *Let $\mathcal{V}_\pm$ be the solution to (33), and let*

$$\zeta_+(t) = \frac{\dot{\mathcal{V}}_{+,0}}{\Omega}\sin(\Omega t) + \mathcal{V}_{+,0}\cos(\Omega t) + \frac{1}{\Omega^2}C\left[1-\cos(\Omega t)\right]$$

and

$$\zeta_-(t) = \frac{\dot{\mathcal{V}}_{-,0}}{\Omega}\sinh(\Omega t) + \mathcal{V}_{-,0}\cosh(\Omega t) - \frac{1}{\Omega^2}C\left[1-\cosh(\Omega t)\right]$$

be the solution to (34). Then

$$\mathcal{V}_+(t) \leqslant \zeta_+(t)\,, \forall t \in \left[-\frac{\pi}{\Omega}, +\frac{\pi}{\Omega}\right]$$

and

$$\mathcal{V}_-(t) \leqslant \zeta_-(t)\,, \forall t \in \mathbb{R}\,.$$

Appendix C. A formal touch—the virial identity

Here we formally derive the virial identity for any real-valued potential $V(x)$.

Hereafter, we denote ψ_t by ψ and $\psi' = \frac{\partial \psi}{\partial x}$, $\psi'' = \frac{\partial^2 \psi}{\partial x^2}$, $\dot{\psi} = \frac{\partial \psi}{\partial t}$, $\dot{\mathcal{I}} = \frac{\mathrm{d}\mathcal{I}}{\mathrm{d}t}$, $\ddot{\mathcal{I}} = \frac{\mathrm{d}^2\mathcal{I}}{\mathrm{d}t^2}$, and so on.

Let (3) be the energy integral of motion (here we make no assumptions about the values of the mass m and of the Planck constant $\hbar$):

$$\mathcal{E}(\psi) := \frac{\hbar^2}{2m}\langle \psi', \psi' \rangle + \langle \psi, V\psi \rangle + \frac{\nu}{\mu+1}\|\psi\|_{L^{2\mu+2}}^{2\mu+2}\,.$$

Let

$$\mathcal{I}(t) = \langle \hat{x}^2 \rangle^t = \langle \psi_t, x^2 \psi_t \rangle_{L^2}$$

be the moment of inertia. It satisfies to the following *virial identity*:

$$\ddot{\mathcal{I}} = \frac{4}{m}\mathcal{E} - \frac{2}{m}\left[\langle \psi, xV'\psi \rangle + 2\langle \psi, V\psi \rangle\right] + \frac{2\nu(\mu-2)}{m(\mu+1)}\|\psi\|_{L^{2\mu+2}}^{2\mu+2}. \tag{35}$$

In order to compute the derivatives of $\mathcal{I}(t)$ from (8) it follows that

$$\dot{\mathcal{I}} = \frac{i}{\hbar}\langle \psi, [H, \hat{x}^2]\psi \rangle$$

since $[|\psi|^{2\mu}, \hat{x}^2] = 0$. From this fact and since

$$[H, \hat{x}^2]\psi = -\frac{\hbar^2}{2m}(2\psi + 4x\psi')$$

then

$$\dot{\mathcal{I}} = -i\frac{\hbar}{m}\|\psi\|^2 - 2i\frac{\hbar}{m}\langle x\psi, \psi' \rangle\,. \tag{36}$$

From equation (36) and since the norm $\|\psi\|$ is a constant function with respect to the time then

$$\ddot{\mathcal{I}} = -2i\frac{\hbar}{m}\langle x\dot{\psi}, \psi' \rangle - 2i\frac{\hbar}{m}\langle x\psi, \dot{\psi}' \rangle = 2i\frac{\hbar}{m}\langle \psi, \dot{\psi} \rangle + 4\frac{\hbar}{m}\Im\left[\langle x\dot{\psi}, \psi' \rangle\right]$$

where

$$\langle \psi, \dot{\psi} \rangle = \frac{i}{\hbar}\langle \psi, H\psi + \nu|\psi|^{2\mu}\psi \rangle = -\frac{i}{\hbar}\mathcal{E} - \frac{i}{\hbar}\frac{\nu\mu}{\mu+1}\|\psi\|_{L^{2\mu+2}}^{2\mu+2}$$

because $\dot\psi = -\frac{i}{\hbar}H\psi - i\frac{\nu}{\hbar}|\psi|^{2\mu}\psi$, and

$$\langle x\dot\psi, \psi'\rangle = -\frac{i}{\hbar}\langle H\psi + \nu|\psi|^{2\mu}\psi, x\psi'\rangle = \frac{i}{\hbar}B + \frac{i\nu}{\hbar}A\,,$$

where

$$B = \langle H\psi, x\psi'\rangle \text{ and } A = \langle |\psi|^{2\mu}\psi, x\psi'\rangle\,.$$

By integrating by parts then

$$\begin{aligned} A &= \int_{\mathbb{R}} x\bar\psi^{\mu+1}\psi^{\mu}\psi'\mathrm{d}x \\ &= -\int_{\mathbb{R}} \psi^{\mu+1}\bar\psi^{\mu+1}\mathrm{d}x - (\mu+1)\int_{\mathbb{R}} x\bar\psi^{\mu}\psi^{\mu+1}\bar\psi'\mathrm{d}x - \mu\int_{\mathbb{R}} x\bar\psi^{\mu+1}\psi^{\mu}\psi'\mathrm{d}x \\ &= -\|\psi\|^{2\mu+2}_{L^{2\mu+2}} - (\mu+1)\bar A - \mu A \end{aligned}$$

from which it follows that

$$(A+\bar A) = -\frac{1}{\mu+1}\|\psi\|^{2\mu+2}_{L^{2\mu+2}}\,.$$

Now, let

$$B = B_1 + B_2 \text{ where } B_1 = -\frac{\hbar^2}{2m}\langle\psi'', x\psi'\rangle \text{ and } B_2 = \langle V\psi, x\psi'\rangle\,.$$

A straightforward calculation yields

$$B_2 = -\langle V\psi', x\psi\rangle - \langle (xV)'\psi, \psi\rangle = -\bar B_2 - \langle (xV)'\psi, \psi\rangle\,,$$

hence

$$(B_2 + \bar B_2) = -\langle (xV)'\psi, \psi\rangle\,.$$

Similarly

$$\begin{aligned} B_1 &= -\frac{\hbar^2}{2m}\langle\psi'', x\psi'\rangle = \frac{\hbar^2}{2m}\langle\psi', \hat x\psi''\rangle + \frac{\hbar^2}{2m}\langle\psi', \psi'\rangle \\ &= -\bar B_1 + \mathcal{E} - \langle\psi, V\psi\rangle - \frac{\nu}{\mu+1}\|\psi\|^{2\mu+2}_{L^{2\mu+2}} \end{aligned}$$

from which follows that

$$(B_1 + \bar B_1) = \mathcal{E} - \langle\psi, V\psi\rangle - \frac{\nu}{\mu+1}\|\psi\|^{2\mu+2}_{L^{2\mu+2}}\,.$$

In conclusion:

$$\begin{aligned} \ddot{\mathcal{I}} &= 2i\frac{\hbar}{m}\left[-\frac{i}{\hbar}\mathcal{E} - \frac{i}{\hbar}\frac{\nu\mu}{\mu+1}\|\psi\|^{2\mu+2}_{L^{2\mu+2}}\right] + 4\frac{\hbar}{m}\Im\left[\frac{i}{\hbar}A + \frac{i\nu}{\hbar}B\right] \\ &= \frac{2}{m}\mathcal{E} + \frac{2\nu\mu}{m(\mu+1)}\|\psi\|^{2\mu+2}_{L^{2\mu+2}} + \frac{4}{m}\Re[B + \nu A] \\ &= \frac{4}{m}\mathcal{E} - \frac{2}{m}[\langle\psi, xV'\psi\rangle + 2\langle\psi, V\psi\rangle] + \frac{2\nu(\mu-2)}{m(\mu+1)}\|\psi\|^{2\mu+2}_{L^{2\mu+2}}\,. \end{aligned}$$

Thus (35) follows.

Remark 12. We remark that the virial identity (35) in the particular cases $V(x)\equiv 0$, $V(x) = \alpha x$ and $V(x) = \alpha x^2$, for $\alpha\in\mathbb{R}$, respectively becomes

$$\ddot{\mathcal{I}} = \frac{4}{m}\mathcal{E} + \frac{2\nu(\mu-2)}{m(\mu+1)}\|\psi\|^{2\mu+2}_{L^{2\mu+2}}\,, \text{if } V(x)\equiv 0\,, \qquad (37)$$

$$\ddot{\mathcal{I}} = \frac{4}{m}\mathcal{E} - \frac{6\alpha}{m}\langle\hat{x}\rangle^t + \frac{2\nu(\mu-2)}{m(\mu+1)}\|\psi\|_{L^{2\mu+2}}^{2\mu+2}, \text{if } V(x) = \alpha x, \tag{38}$$

and

$$\begin{aligned}\ddot{\mathcal{I}} &= \frac{4}{m}\mathcal{E} - \frac{8\alpha}{m}\langle\hat{x}^2\rangle^t + \frac{2\nu(\mu-2)}{m(\mu+1)}\|\psi\|_{L^{2\mu+2}}^{2\mu+2} \\ &= \frac{4}{m}\mathcal{E} - \frac{8\alpha}{m}\mathcal{I} + \frac{2\nu(\mu-2)}{m(\mu+1)}\|\psi\|_{L^{2\mu+2}}^{2\mu+2}, \text{if } V(x) = \alpha x^2. \end{aligned} \tag{39}$$

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