TRANSFER MATRICES OF RATIONAL SPIN CHAINS VIA NOVEL BGG-TYPE RESOLUTIONS

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Abstract. We obtain BGG-type formulas for transfer matrices of irreducible finite-dimensional representations of the classical Lie algebras $\mathfrak{g}$, whose highest weight is a multiple of a fundamental one and which can be lifted to the representations over the Yangian $Y(\mathfrak{g})$. These transfer matrices are expressed in terms of transfer matrices of certain infinite-dimensional highest weight representations (such as parabolic Verma modules and their generalizations) in the auxiliary space. We further factorise the corresponding infinite-dimensional transfer matrices into the products of two Baxter Q-operators, arising from our previous study [FPT,FT] of the degenerate Lax matrices. Our approach is crucially based on the new BGG-type resolutions of the finite-dimensional $\mathfrak{g}$-modules, which naturally arise geometrically as the restricted duals of the Cousin complexes of relative local cohomology groups of ample line bundles on the partial flag variety G/P stratified by B_--orbits.

Contents

1. Introduction

1.1. Summary.

The main results of the present paper are:

- The construction of new BGG-type resolutions of finite-dimensional $\mathfrak{g}$-modules by infinite-dimensional highest weight modules such as parabolic Verma and their "W-translations". These resolutions admit an elegant geometric interpretation via the Cousin complexes of relative local cohomology, similar to the parabolic BGG resolutions of [L]. The crucial difference from the latter setup, however, is that our modules are also defined when the "dominant integral" condition is lifted, and become generically irreducible in this setting.
- The explicit expression of the finite-dimensional transfer matrices $T_{i,t}(x)$ (with i as in (1.22) and $t \in \mathbb{N}$) via the infinite-dimensional transfer matrices. This allows to analytically continue $T_{i,t}(x)$ from $t \in \mathbb{N}$ to the entire complex plane $t \in \mathbb{C}$ and study its t-symmetries.
- The extra symmetry of the rational $ABCD$-type R-matrices in the first fundamental representations gives rise to explicit realizations of the aforementioned infinite-dimensional irreducible highest weight $\mathfrak{g}$-representations in the Fock spaces of oscillator algebras. This also immediately extends the action of $\mathfrak{g}$ on these modules to that of the Yangian $Y(\mathfrak{g})$.
- The factorisation of the above infinite-dimensional transfer matrices into two commuting Baxter Q-operators, which arise by realizing the corresponding non-degenerate Lax matrices as fusion of two *degenerate* Lax matrices. While the latter can be recovered as *renormalized limits* of the former, they also can be viewed as "W-translations" of the single one arising from the antidominantly shifted Yangians $Y_{-\omega_i}(\mathfrak{g})$ following our earlier work [FPT, FT].

1.2. Overview of type A.

For $\mathfrak{gl}_n$-type rational spin chains, both T- and Q-operators can be built within the framework of the quantum inverse scattering method [Fad] from appropriate solutions of the *RTT relation*

$$R_{12}(z-w)L_1(z)L_2(w) = L_2(w)L_1(z)R_{12}(z-w) \tag{1.1}$$

involving the rational R-matrix

$$R(z) = z\mathrm{I}_n + \mathrm{P}\,, \qquad \mathrm{I}_n = \sum_{i,j=1}^{n} e_{ii} \otimes e_{jj}\,, \qquad \mathrm{P} = \sum_{i,j=1}^{n} e_{ij} \otimes e_{ji}\,, \tag{1.2}$$

where $(e_{ij})_{k\ell} = \delta_i^k \delta_j^\ell$ is the standard basis of $\mathfrak{gl}_n$, satisfying the *quantum Yang-Baxter equation*:

$$R_{12}(z-w)R_{13}(z)R_{23}(w) = R_{23}(w)R_{13}(z)R_{12}(z-w)\,. \tag{1.3}$$

More precisely, the operators $\mathsf{T}(z), \mathsf{Q}(z) \in \mathrm{End}(\mathbb{C}^n)^{\otimes N}$ (with N being the length of the spin chain) are then defined as traces in the auxiliary space of the *monodromy matrices* (followed by a twist)

$$M(z) = \underbrace{L(z) \otimes \cdots \otimes L(z)}_{N} \tag{1.4}$$

built from the *Lax matrices* $L(z)$, i.e. solutions of (1.1), with the product $\otimes$ denoting the tensor product in the quantum space and the usual multiplication in the auxiliary space. The key difference between $\mathsf{T}(z)$ and $\mathsf{Q}(z)$, however, is that the former correspond to *non-degenerate* Lax matrices (with a non-degenerate coefficient of the leading z-power) while the latter ones are built from the *degenerate* Lax matrices.

Using the obvious *invariance* of the R-matrix (1.2) under the action of GL_n:

$$[R(z), G\otimes G] = 0 \qquad \forall\, G\in GL_n\,, \tag{1.5}$$

one can multiply the non-degenerate Lax matrix $L(z)$ by a z-independent $G\in GL_n$ to bring the coefficient of its leading z-power to the identity matrix I_n. Such Lax matrices are governed by the *RTT Yangian* $Y^{\mathrm{rtt}}(\mathfrak{gl}_n)$ (the explicit identification of which with the Drinfeld Yangian $Y(\mathfrak{gl}_n)$ was carried out in [BK]). Here, $Y^{\mathrm{rtt}}(\mathfrak{gl}_n)$ is the associative algebra generated by $\{t_{ij}^{(r)}\}_{1\le i,j\le n}^{r\ge 1}$ subject to the defining relation (1.1), where $L(z)$ has been replaced by $T(z)\in \mathrm{End}(\mathbb{C}^n)\otimes Y^{\mathrm{rtt}}(\mathfrak{gl}_n)[[z^{-1}]]$:

$$T(z) = \sum_{i,j=1}^{n} e_{ij}\otimes t_{ij}(z) \qquad \text{with} \qquad t_{ij}(z) = \delta_i^j + \sum_{r\ge 1} t_{ij}^{(r)} z^{-r}\,. \tag{1.6}$$

In particular, any $Y^{\mathrm{rtt}}(\mathfrak{gl}_n)$-representation V gives rise to the $\mathfrak{gl}_n$-type $\mathrm{End}(V)$-valued Lax matrix.

The special feature of type A is the presence of the *evaluation* homomorphism to the universal enveloping algebra of $\mathfrak{gl}_n$:

$$\mathrm{ev}\colon Y^{\mathrm{rtt}}(\mathfrak{gl}_n)\to U(\mathfrak{gl}_n) \quad \text{given by} \quad t_{ij}^{(r)}\mapsto \delta_r^1 E_{ji}\,. \tag{1.7}$$

In particular, given any $\mathfrak{gl}_n$-module $\sigma\colon \mathfrak{gl}_n\to \mathrm{End}(V)$, the matrix $L(z) = z\mathrm{I}_n + \sum_{i,j=1}^n e_{ij}\sigma(E_{ji})$, with E_{ji} denoting the generators of $\mathfrak{gl}_n$, is an $\mathrm{End}(V)$-valued Lax matrix, i.e. satisfies (1.1). Conversely any such Lax matrix endows V with a $\mathfrak{gl}_n$-action, since ev of (1.7) admits a right inverse:

$$\iota\colon U(\mathfrak{gl}_n)\hookrightarrow Y^{\mathrm{rtt}}(\mathfrak{gl}_n) \quad \text{given by} \quad E_{ij}\mapsto t_{ji}^{(1)}\,. \tag{1.8}$$

In contrast, the Q-operators are known to arise from degenerate Lax matrices. As has been recently realized in [FPT] (cf. [CGY] for an interpretation via the $4d$ Chern-Simons theory), under certain *antidominance* condition, the corresponding Lax matrices are governed by the *antidominantly shifted RTT Yangians* $Y_{-\mu}^{\mathrm{rtt}}(\mathfrak{gl}_n)$. The identification of the latter with the shifted Drinfeld Yangian $Y_{-\mu}(\mathfrak{gl}_n)$ defined as in [BFN, Appendix B] goes through the Gauss decomposition of the generating matrix $T(z)$ as in the non-shifted case [BK], see [FPT, Theorem 2.54]. Using the GL_n-invariance (1.5) one can further obtain other degenerate Lax matrices which do not admit a Gauss decomposition (and thus are no longer directly related to the *shifted Drinfeld Yangians*).

According to [BFLMS], see also [Ts1,KLT], there is a whole family of 2^n Q-operators $\mathsf{Q}_I(z)$, labelled by all subsets $I\subseteq\{1,2,\dots,n\}$. However, all of those can be expressed through n fundamental ones $\{\mathsf{Q}_i(z)\}_{i=1}^n$, due to so-called *QQ-relations*:

$$\Delta_{j,i}\cdot \mathsf{Q}_{I\sqcup i\sqcup j}(z)\mathsf{Q}_I(z) = \mathsf{Q}_{I\sqcup i}(z-\tfrac12)\mathsf{Q}_{I\sqcup j}(z+\tfrac12) - \mathsf{Q}_{I\sqcup j}(z-\tfrac12)\mathsf{Q}_{I\sqcup i}(z+\tfrac12)\,, \tag{1.9}$$

where the scalar factor $\Delta_{j,i} = \frac{\tau_j-\tau_i}{\sqrt{\tau_i\tau_j}}$ depends only on the twist parameters and the brackets have been suppressed for the one element sets. The two key components in the proof of (1.9) are: the fusion of the corresponding Lax matrices and the *Desnanot-Jacobi-Dodgson-Sylvester theorem* from linear algebra. In particular, fusing all n fundamental Q-operators, one obtains the transfer matrix $\mathsf{T}_\lambda^+(z)$ associated with the non-degenerate linear Lax matrix corresponding to a $\mathfrak{gl}_n$ Verma module:

$$\Delta_{\{1,\dots,n\}}\cdot \mathsf{T}_\lambda^+(z) = \mathsf{Q}_1(z+\lambda_1')\mathsf{Q}_2(z+\lambda_2')\cdots \mathsf{Q}_n(z+\lambda_n')\,, \tag{1.10}$$

where $\lambda\in\mathbb{C}^n$ is the highest weight of the Verma module, λ_i' are the components of the shifted weight $\lambda' = (\lambda_1',\dots,\lambda_n') = \lambda+\rho$ with $\rho = \left(\frac{n-1}{2},\frac{n-3}{2},\dots,\frac{1-n}{2}\right)$, and $\Delta_{\{1,\dots,n\}} = \prod_{1\le i<j\le n}\Delta_{i,j}$.

Finally, for a dominant integral $\mathfrak{gl}_n$-weight λ, the finite-dimensional transfer matrix $\mathsf{T}_\lambda(z)$ can be expressed as an alternating sum of the infinite-dimensional $\mathsf{T}_\mu^+(z)$, due to so-called *BGG-relations*:

$$\mathsf{T}_\lambda(z) = \sum_{\sigma\in S_n}(-1)^{l(\sigma)}\mathsf{T}^+_{\sigma(\lambda+\rho)-\rho}(z)\,. \tag{1.11}$$

Combining (1.11) with (1.10), one finally gets the *determinant formula* for type A transfer matrices:

$$\Delta_{\{1,\dots,n\}} \cdot \mathsf{T}_\lambda(z) = \det \Big\| \mathsf{Q}_i(z+\lambda'_j) \Big\|_{1\le i,j\le n} \,. \tag{1.12}$$

1.3. **BGG resolutions.**

Formula (1.11) follows directly from the famous *Bernstein-Gelfand-Gelfand resolution* [BGG] of the finite-dimensional $\mathfrak{g}$-module L_λ by means of the infinite-dimensional Verma modules M_μ:

$$0 \to M_{w_0\cdot\lambda} \to \cdots \to \bigoplus_{w\in W}^{l(w)=2} M_{w\cdot\lambda} \to \bigoplus_{w\in W}^{l(w)=1} M_{w\cdot\lambda} \to M_\lambda \to L_\lambda \to 0\,. \tag{1.13}$$

Here, W is the Weyl group of $\mathfrak{g}$, equipped with the length function $l\colon W \to \mathbb{Z}_{\ge 0}$ as an abstract Coxeter group. Furthermore, we use the **dot action** of W on the space of weights, defined via:

$$w\cdot\lambda = w(\lambda+\rho)-\rho \qquad \forall\, w\in W\,,\ \lambda\in\mathfrak{h}^*\,, \tag{1.14}$$

with the weight $\rho\in\mathfrak{h}^*$ defined in the standard way:

$$\rho = \tfrac{1}{2}\sum_{\alpha\in\Delta^+}\alpha\,, \tag{1.15}$$

where Δ^+ denotes the set of positive roots of $\mathfrak{g}$. The resolution (1.13) involves the total of $|W|$ Verma modules and has a length equal to $|\Delta^+|$, with the leftmost nontrivial term corresponding to

$$w_0 = \text{the longest element of } W\,. \tag{1.16}$$

The BGG resolution (1.13) can be thought of as a *categorification* of the *Weyl character formula* (expressing the character of the finite-dimensional $\mathfrak{g}$-module L_λ via those of Verma modules M_μ):

$$\mathrm{ch}_{L_\lambda} = \sum_{w\in W}(-1)^{l(w)}\frac{e^{w(\lambda+\rho)-\rho}}{\prod_{\alpha\in\Delta^+}(1-e^{-\alpha})} = \sum_{w\in W}(-1)^{l(w)}\,\mathrm{ch}_{M_{w\cdot\lambda}}\,, \tag{1.17}$$

the *character limit* of (1.13). For $\lambda=0$, the formula (1.17) recovers the *Weyl denominator formula*:

$$\sum_{w\in W}(-1)^{l(w)}e^{w(\rho)-\rho} = \prod_{\alpha\in\Delta^+}(1-e^{-\alpha})\,. \tag{1.18}$$

1.4. **Generalization to other classical types.**

Let us now consider the rational spin chains of types B_r, C_r, D_r. The corresponding rational R-matrices $R(z)$ were first discovered in [ZZ] and take the following form (we use the conventions in mathematics literature, related to the original formulas of [ZZ] via similarity transformations):

$$R(z) = z(z+\kappa)\mathrm{I}_{\mathsf{K}} + (z+\kappa)\mathrm{P} - z\mathrm{Q} \tag{1.19}$$

with $\kappa = r+1$ for $\mathfrak{sp}_{2r}$, $\kappa = r-1 = \frac{\mathsf{K}}{2}-1$ for $\mathfrak{so}_{2r}$, $\kappa = r-\frac{1}{2} = \frac{\mathsf{K}}{2}-1$ for $\mathfrak{so}_{2r+1}$, where $\mathsf{K}=2r$ for $\mathfrak{sp}_{2r}$ and $\mathfrak{so}_{2r}$ while $\mathsf{K}=2r+1$ for $\mathfrak{so}_{2r+1}$, and the linear operators $\mathrm{P},\mathrm{Q}\in\mathrm{End}(\mathbb{C}^{\mathsf{K}})$ defined by:

$$\mathrm{P} = \sum_{i,j=1}^{\mathsf{K}} e_{ij}\otimes e_{ji}\,, \qquad \mathrm{Q} = \sum_{i,j=1}^{\mathsf{K}} \epsilon_i\epsilon_j\, e_{ij}\otimes e_{i'j'}\,, \tag{1.20}$$

where

$$i' = \mathsf{K}+1-i \qquad \text{for} \qquad 1\le i\le \mathsf{K} \tag{1.21}$$

and $\epsilon_1=\cdots=\epsilon_{\mathsf{K}}=1$ for $\mathfrak{so}_{\mathsf{K}}$ while $\epsilon_1=\cdots=\epsilon_r=1,\ \epsilon_{r+1}=\cdots=\epsilon_{2r}=-1$ for $\mathfrak{sp}_{2r}$.

Similarly to type A, the non-degenerate Lax matrices $L(z)$ of types BCD with the coefficient of the leading z-power equal to I_{K} are governed by the corresponding *extended RTT Yangians* $X^{\mathrm{rtt}}(\mathfrak{g})$ (whose explicit relation to the Drinfeld Yangian $Y(\mathfrak{g})$ was obtained quite recently in [JLM, GRW]).

However, the key difference from type A is in the absence of the evaluation homomorphism (1.7). In particular, the action of $\mathfrak{g}$ on L_λ in general cannot be extended to that of $Y(\mathfrak{g})$, cf. (1.8). Nonetheless, extending the earlier works [SW,Re], a certain family of linear and quadratic oscillator-type non-degenerate Lax matrices $\mathcal{L}(z)$ is known [Fr, FT, KK]. These $\mathcal{L}(z)$ depend on a parameter

$t \in \mathbb{C}$ and give rise to an action of $X^{\mathrm{rtt}}(\mathfrak{g})$ on the parabolic Verma $\mathfrak{g}$-modules. That way we also obtain an action of $Y(\mathfrak{g})$ on $L_{t\omega_i}$ for the fundamental weights ω_i classified in (1.22) and all $t \in \mathbb{N}$.

Given a simple Lie algebra $\mathfrak{g}$ of rank r, one may ask for which indices $i \in \{1, \ldots, r\}$ do the finite-dimensional irreducible $\mathfrak{g}$-modules $L_{t\omega_i}$ (where ω_i denotes the i-th fundamental weight of $\mathfrak{g}$) admit a compatible (through the embedding $U(\mathfrak{g}) \hookrightarrow Y(\mathfrak{g})$, cf. (1.8)) action of $Y(\mathfrak{g})$ for all $t \in \mathbb{N}$. In the classical $ABCD$ types, the answer to this question has been provided long ago in [KR, §2]:

$$\Big\{1 \leq i \leq r \,\Big|\, \mathfrak{g} \curvearrowright L_{t\omega_i} \text{ extends to } Y(\mathfrak{g}) \curvearrowright L_{t\omega_i} \ \forall\, t \in \mathbb{N}\Big\} = \begin{cases} i = 1, \ldots, r & \text{for } A_r \\ i = 1 & \text{for } B_r \\ i = r & \text{for } C_r \\ i = 1, r-1, r & \text{for } D_r \end{cases} . \tag{1.22}$$

Let us emphasize that the above Lax matrix approach provides a <u>constructive existence proof</u> for all these cases. Furthermore, we note that the corresponding values of the index i can be characterized by either of the following two equivalent conditions:

- the label of the vertex i in the Dynkin diagram of $\mathfrak{g}$ is equal to 1
- the i-th fundamental coweight $\omega_i^\vee$ is minuscule (minuscule weight of the Langlands dual $\mathfrak{g}^L$)

It has been recently shown in [CGY, Appendix B] by cohomological arguments that these conditions indeed guarantee the positive answer to the above question in all types. This also adds Lax matrices in the exceptional types E_6, E_7, but we presently do not have explicit formulas for those.

Yet another obstacle to generalize the results of Section 1.2 to other types is that even if L_λ extends to a module over $Y(\mathfrak{g})$, it may not be the case for the Verma modules $M_{w\cdot\lambda}$ featuring in the BGG resolution (1.13). However, as explained above, the Lax matrix approach does provide a natural $Y(\mathfrak{g})$-action on the corresponding parabolic Verma modules. We shall now discuss <u>new</u> BGG-type resolutions of L_λ which involve only those parabolic Verma modules and their "W-translations".

1.5. **New BGG-type resolutions.**

To state the main results of this subsection, let us first introduce some more notation. Consider the root decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathbb{C}e_\alpha$. Let $\{\alpha_i\}_{i=1}^r \subset \Delta^+$ be the positive simple roots of $\mathfrak{g}$. Given a subset $S \subseteq \{1, \ldots, r\}$, one defines the standard parabolic Lie algebra $\mathfrak{p}_S \subseteq \mathfrak{g}$ via:

$$\mathfrak{p}_S = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta^+} \mathbb{C}e_\alpha \oplus \bigoplus_{\alpha\in\Delta_S^+} \mathbb{C}e_{-\alpha} \qquad \text{with} \qquad \Delta_S^+ = \Delta^+ \cap \bigoplus_{i\in S} \mathbb{Z}\alpha_i \,. \tag{1.23}$$

Let us note that $\mathfrak{p}_{\{1,\ldots,r\}} = \mathfrak{g}$, while $\mathfrak{p}_\emptyset$ coincides with the Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta^+} \mathbb{C}e_\alpha$. It is well-known that all parabolic subalgebras $\mathfrak{p}$ satisfying $\mathfrak{b} \subseteq \mathfrak{p} \subseteq \mathfrak{g}$ are necessarily of that form. Such $\mathfrak{p}_S$ further decomposes into the semidirect product:

$$\mathfrak{p}_S = \mathfrak{l} \ltimes \mathfrak{u} \qquad \text{with} \qquad \mathfrak{l} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta_S^+} \mathbb{C}e_\alpha \oplus \bigoplus_{\alpha\in\Delta_S^+} \mathbb{C}e_{-\alpha} \qquad \text{and} \qquad \mathfrak{u} = \bigoplus_{\alpha\in\Delta^+\setminus\Delta_S^+} \mathbb{C}e_\alpha \,, \tag{1.24}$$

of the semisimple part $\mathfrak{l}$ (the *Levi subalgebra*) and the nilpotent part $\mathfrak{u}$ (the *nilpotent radical*). Then

$$\Delta_\mathfrak{l} = \Delta_\mathfrak{l}^+ \sqcup (-\Delta_\mathfrak{l}^+) \quad \text{with} \quad \Delta_\mathfrak{l}^+ = \Delta_S^+ \tag{1.25}$$

is the root system of $\mathfrak{l}$, and the subgroup $W_\mathfrak{l}$ of W generated by the simple reflections $\{s_{\alpha_j}\}_{j\in S}$ is the Weyl group of $\mathfrak{l}$. In analogy with (1.15), we also define

$$\rho_\mathfrak{l} = \tfrac{1}{2} \sum_{\alpha\in\Delta_\mathfrak{l}^+} \alpha \,. \tag{1.26}$$

The finite-dimensional irreducible $\mathfrak{l}$-modules are indexed by the dominant integral weights of $\mathfrak{l}$:

$$P_\mathfrak{l}^+ = \Big\{\lambda \in \mathfrak{h}^* \,|\, \lambda(h_{\alpha_j}) \in \mathbb{N} \ \ \forall j \in S\Big\} \,. \tag{1.27}$$

For $\lambda \in P_\mathfrak{l}^+$, let $L_\lambda^\mathfrak{l}$ be the corresponding $\mathfrak{l}$-module. One makes it into a $\mathfrak{p}_S$-module by letting the nilpotent radical $\mathfrak{u} \subset \mathfrak{p}_S$ act trivially. Then, the **parabolic Verma module** $M_\lambda^{\mathfrak{p}_S}$ is defined via:

$$M_\lambda^{\mathfrak{p}_S} = \mathrm{Ind}_{\mathfrak{p}_S}^\mathfrak{g}(L_\lambda^\mathfrak{l}) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_S)} L_\lambda^\mathfrak{l} \,. \tag{1.28}$$

Let $P^+_{\mathfrak{g}}$ be the set of dominant integral weights of $\mathfrak{g}$ (apply (1.27) to $S = \{1, \dots, r\}$), and define

$$P^+_{\mathfrak{g}/\mathfrak{l}} = \left\{ \lambda \in P^+_{\mathfrak{g}} \,|\, \lambda(h_{\alpha_j}) = 0 \;\; \forall j \in S \right\} , \tag{1.29}$$

that is, $P^+_{\mathfrak{g}/\mathfrak{l}}$ is the set of dominant integral weights λ of $\mathfrak{g}$ such that $\dim(L^{\mathfrak{l}}_\lambda) = 1$.

Let us now state our <u>key mathematical construction</u>: for any parabolic $\mathfrak{p} \subset \mathfrak{g}$ and $\lambda \in P^+_{\mathfrak{g}/\mathfrak{l}}$ (1.29), the irreducible finite-dimensional $\mathfrak{g}$-module L_λ admits the following *truncated BGG resolution*:

$$0 \to M'_{{}^{\mathfrak{l},0}w\,\cdot\lambda} \to \cdots \to \bigoplus_{w \in {}^{\mathfrak{l}}W}^{l(w)=2} M'_{w\,\cdot\lambda} \;\to \bigoplus_{w \in {}^{\mathfrak{l}}W}^{l(w)=1} M'_{w\,\cdot\lambda} \to M'_\lambda \to L_\lambda \to 0 \tag{1.30}$$

with each term admitting further a resolution by the usual Verma modules:

$$0 \to M_{ww_{\mathfrak{l},0}\,\cdot\lambda} \to \cdots \to \bigoplus_{v \in W_{\mathfrak{l}}}^{l(v)=2} M_{wv\,\cdot\lambda} \;\to \bigoplus_{v \in W_{\mathfrak{l}}}^{l(v)=1} M_{wv\,\cdot\lambda} \to M_{w\,\cdot\lambda} \to M'_{w\,\cdot\lambda} \to 0\,. \tag{1.31}$$

Here, ${}^{\mathfrak{l}}W$ is defined via:

$${}^{\mathfrak{l}}W = \{ w \in W \,|\, w(\Delta^+_{\mathfrak{l}}) \subseteq \Delta^+ \}\,. \tag{1.32}$$

According to [Kos, §5.13], the set ${}^{\mathfrak{l}}W$ can be equivalently described as:

$${}^{\mathfrak{l}}W = \{\text{shortest representatives of the left cosets } W/W_{\mathfrak{l}}\}\,, \tag{1.33}$$

and each element $w \in W$ admits a unique factorisation:

$$w = {}^{\mathfrak{l}}w\, w_{\mathfrak{l}} \qquad \text{with} \qquad w_{\mathfrak{l}} \in W_{\mathfrak{l}}\,,\; {}^{\mathfrak{l}}w \in {}^{\mathfrak{l}}W\,. \tag{1.34}$$

In particular, the longest elements ${}^{\mathfrak{l},0}w \in {}^{\mathfrak{l}}W$ and $w_{\mathfrak{l},0} \in W_{\mathfrak{l}}$ featuring in (1.30) and (1.31) arise from the decomposition (1.34) applied to the longest element $w_0 \in W$ of (1.16):

$$w_0 = {}^{\mathfrak{l},0}w\, w_{\mathfrak{l},0}\,. \tag{1.35}$$

The above modules $\{M'_{w\,\cdot\lambda}\}_{w \in {}^{\mathfrak{l}}W}$ are defined as explicit quotients of the Verma modules $M_{w\,\cdot\lambda}$:

$$M'_{w\,\cdot\lambda} = M_{w\,\cdot\lambda}/M^{\text{sing}}_{w\,\cdot\lambda} \tag{1.36}$$

with $M^{\text{sing}}_{w\,\cdot\lambda}$ being the $\mathfrak{g}$-submodule of $M_{w\,\cdot\lambda}$ generated by the singular vectors of weights

$$s_{w(\alpha)}\,(w(\lambda+\rho)) - \rho\,, \qquad \alpha \in \Delta^+_{\mathfrak{l}}\,. \tag{1.37}$$

Here, $s_{w(\alpha)}$ denotes a reflection in the positive root $w(\alpha)$, see (1.32), while the existence (note that uniqueness is standard) of such singular vectors is guaranteed by $(w(\lambda+\rho), w(\alpha)) = (\rho, \alpha) \in \mathbb{Z}_{>0}$.

One can think of (1.30, 1.31) as a *categorification* of the following character equality expressing the character of the finite-dimensional $\mathfrak{g}$-module L_λ via those of the modules $\{M'_{w\,\cdot\lambda}\}$, cf. (1.17):

$$\text{ch}_{L_\lambda} = \sum_{w \in {}^{\mathfrak{l}}W} \frac{(-1)^{l(w)} e^{w(\lambda+\rho)-\rho}}{\prod_{\alpha \in \Delta^+ \setminus w(\Delta^+_{\mathfrak{l}})}(1-e^{-\alpha})} = \sum_{w \in {}^{\mathfrak{l}}W} (-1)^{l(w)}\, \text{ch}_{M'_{w\,\cdot\lambda}}\,. \tag{1.38}$$

Main Theorem 1. *For $\lambda \in P^+_{\mathfrak{g}/\mathfrak{l}}$, the irreducible finite-dimensional $\mathfrak{g}$-module L_λ has a finite length resolution* (1.30)*, with each term admitting a finite length resolution* (1.31) *by Verma modules.*

The resolutions (1.30, 1.31) are reminiscent of the well-known Lepowsky's *parabolic BGG resolutions* [L] of any irreducible finite-dimensional $\mathfrak{g}$-module L_λ by parabolic Verma modules $\{M^{\mathfrak{p}}_\mu\}_{\mu \in P^+_{\mathfrak{l}}}$:

$$0 \to \cdots \to \bigoplus_{w \in W^{\mathfrak{l}}}^{l(w)=2} M^{\mathfrak{p}}_{w\,\cdot\lambda} \;\to \bigoplus_{w \in W^{\mathfrak{l}}}^{l(w)=1} M^{\mathfrak{p}}_{w\,\cdot\lambda} \to M^{\mathfrak{p}}_\lambda \to L_\lambda \to 0 \tag{1.39}$$

with each term admitting further a resolution by the usual Verma modules:

$$0 \to M_{w_{\mathfrak{l},0}w\,\cdot\lambda} \to \cdots \to \bigoplus_{v \in W_{\mathfrak{l}}}^{l(v)=2} M_{vw\,\cdot\lambda} \;\to \bigoplus_{v \in W_{\mathfrak{l}}}^{l(v)=1} M_{vw\,\cdot\lambda} \to M_{w\,\cdot\lambda} \to M^{\mathfrak{p}}_{w\,\cdot\lambda} \to 0\,. \tag{1.40}$$

Here, the indexing subset $W^{\mathfrak{l}}$ of W is defined as:

$$W^{\mathfrak{l}} = \{w \in W \,|\, \Delta_{\mathfrak{l}}^+ \subseteq w(\Delta^+)\} = \{\text{shortest representatives of the right cosets } W_{\mathfrak{l}}\backslash W\}\,. \tag{1.41}$$

Note that $W^{\mathfrak{l}} = \{w^{-1} \,|\, w \in {}^{\mathfrak{l}}W\}$ and each element $w \in W$ admits a unique factorisation:

$$w = w_{\mathfrak{l}} w^{\mathfrak{l}} \qquad \text{with} \qquad w_{\mathfrak{l}} \in W_{\mathfrak{l}}\,,\; w^{\mathfrak{l}} \in W^{\mathfrak{l}}\,. \tag{1.42}$$

While the BGG resolution (1.13) and consequently the Lepowsky-BGG resolution (1.39) were originally constructed algebraically, we are presently not aware of the algebraic construction of (1.30): the key difficulty is that non-simple reflections are involved in the definition of $M'_{w\cdot\lambda}$. Instead, we shall construct (1.30) by interpreting its *restricted dual* as a *Cousin complex* of relative local cohomology groups of the corresponding line bundle on the partial flag variety $X = G/P$ stratified by B_--orbits. Here, $B \subset P$ are the Borel and parabolic subgroups of the Lie group G with $\mathrm{Lie}(B) = \mathfrak{b}$, $\mathrm{Lie}(P) = \mathfrak{p}$, $\mathrm{Lie}(G) = \mathfrak{g}$, and B_- is the opposite Borel subgroup of G. A similar geometric interpretation of (1.13) goes back to [Ke,Br], while the analogous treatment of the parabolic BGG resolutions (1.39) was presented in [MR] by considering instead the complete flag variety $Y = G/B$ stratified by P-orbits. However, let us point out that [MR] was not self-contained as it used some algebraic properties established in [L] (and also contained two substantial inaccuracies which we fix in our Remark 2.60).

1.6. **BGG-relations for transfer matrices of classical types.**

Despite the aforementioned similarity between our resolution (1.30) and the Lepowsky-BGG resolution (1.39), they have several major differences. First of all, the latter is constructed for any choice of the parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ independent of $\lambda \in P_{\mathfrak{g}}^+$ and consists only of the parabolic Verma modules $\{M_\mu^{\mathfrak{p}}\}_{\mu \in P_{\mathfrak{l}}^+}$ (though $\dim(L_\mu^{\mathfrak{l}}) > 1$ in general). But an even more striking difference is the fact that our modules $M'_{w\cdot\lambda}$ admit an <u>analytic continuation</u> in λ to the domain

$$P_{\mathfrak{g}/\mathfrak{l}} = \Big\{\lambda \in \mathfrak{h}^* \,|\, \lambda(h_{\alpha_j}) = 0 \;\; \forall j \in S\Big\}\,. \tag{1.43}$$

Indeed, one may apply the construction (1.36, 1.37) for any $\lambda \in P_{\mathfrak{g}/\mathfrak{l}}$ to define $\mathfrak{g}$-modules $M'_{w\cdot\lambda}$ of the highest weight $w\cdot\lambda$, see (1.14). Moreover, the resulting highest-weight $\mathfrak{g}$-modules $\{M'_{w\cdot\lambda}\}_{w\in{}^{\mathfrak{l}}W}$ are <u>generically irreducible</u> for $\lambda \in P_{\mathfrak{g}/\mathfrak{l}}$ (in particular, they are irreducible for $\lambda \in -P_{\mathfrak{g}/\mathfrak{l}}^+ \subset P_{\mathfrak{g}/\mathfrak{l}}$) as follows from the classical description [J,S] of the weights of singular vectors in the Verma modules.

We note that the module $M'_\lambda = M'_{\mathrm{id}\cdot\lambda}$ coincides with the parabolic Verma module $M_\lambda^{\mathfrak{p}}$, and all other modules $\{M'_{w\cdot\lambda}\}_{w\in{}^{\mathfrak{l}}W}$ can be thought of as "W-translations" of $M'_\lambda = M_\lambda^{\mathfrak{p}}$, as follows from (1.38). In the particular case $\lambda = t\omega_i$ with the label of vertex i equal to 1 (equivalent to (1.22) in the classical types, see Section 1.4), the corresponding parabolic Verma $\mathfrak{g}$-modules $M_{t\omega_i}^{\mathfrak{p}_{\{1,\ldots,r\}\backslash\{i\}}}$, $t \in \mathbb{C}$, can be extended to the modules over $Y(\mathfrak{g})$, cf. [CGY, Appendix B], and so should all other $M'_{w\cdot t\omega_i}$. Thus, our resolution (1.30) can be actually regarded as a resolution of $Y(\mathfrak{g})$-modules, giving rise to the desired <u>*BGG-relation*</u> expressing the finite-dimensional transfer matrix $T_{i,t}(z) = T_{L_{t\omega_i}}(z)$ via:

$$T_{i,t}(z) = \sum_{w\in{}^{\mathfrak{l}}W} (-1)^{l(w)} T_{M'_{w\cdot t\omega_i}}(z)\,, \qquad \forall\, t \in \mathbb{N}\,, \tag{1.44}$$

cf. (1.11). The length $N = 0$ case of (1.44) recovers back the character formula (1.38).

To make this even more feasible in the classical types, let us recall that in each case of (1.22), the resulting action of the Yangian $Y(\mathfrak{g})$ on the parabolic Verma module $M_{t\omega_i}^{\mathfrak{p}_{\{1,\ldots,r\}\backslash\{i\}}}$ is given by an explicit oscillator-type Lax matrix, as has been emphasized in Section 1.4. Utilizing further the Weyl group symmetry of the rational R-matrices (1.2, 1.19) combined with the appropriate particle-hole automorphisms of the corresponding oscillator algebra $\mathcal{A}$ (in order for our $\mathfrak{g}$-modules to be in the category $\mathcal{O}$ of [BGG]) one obtains a family of Lax matrices parametrized by the same set ${}^{\mathfrak{l}}W$. This gives rise to $Y(\mathfrak{g})$-modules $\{M_{w\cdot t\omega_i}^+\}_{w\in{}^{\mathfrak{l}}W}$ explicitly realized in the Fock representation F of the algebras $\mathcal{A}$. As $\mathfrak{g}$-modules, they have the same characters as $\{M'_{w\cdot t\omega_i}\}_{w\in{}^{\mathfrak{l}}W}$ and furthermore the Fock vacuum $|0\rangle \in \mathsf{F}$ is a highest weight vector of the highest weight $w\cdot t\omega_i$. Thus, if $M'_{w\cdot t\omega_i}$ is irreducible, then $M_{w\cdot t\omega_i}^+ \simeq M'_{w\cdot t\omega_i}$ and the corresponding transfer matrices $T_{w,t\omega_i}^+(z) = T_{M_{w\cdot t\omega_i}^+}(z)$

and $T_{M'_{w\cdot t\omega_i}}(z)$ coincide. Combining this with the above observation that $M'_{w\cdot t\omega_i}$ are generically irreducible for $t \in \mathbb{C}$ and the fact that both transfer matrices depend continuously on the parameter $t \in \mathbb{C}$, we obtain the uniform equality of the corresponding transfer matrices:

$$T^+_{w,t\omega_i}(z) = T_{M'_{w\cdot t\omega_i}}(z) \qquad \forall\, t \in \mathbb{C}\,,\ w \in {}^{\mathsf{l}}W\,, \tag{1.45}$$

even though for $w \neq \mathrm{id}$ the $\mathfrak{g}$-modules $M^+_{w\cdot t\omega_i}$ and $M'_{w\cdot t\omega_i}$ may be non-isomorphic at some $t \in \mathbb{N}$ (that are exactly the values featuring in (1.30)). The equality (1.45) allows us to recast (1.44) as:

Main Theorem 2. *For a classical rank r Lie algebra $\mathfrak{g}$, $1 \leq i \leq r$ as in* (1.22)*, $t \in \mathbb{N}$, we have:*

$$T_{i,t}(z) = \sum_{w \in {}^{\mathsf{l}}W} (-1)^{l(w)} T^+_{w,t\omega_i}(z)\,, \tag{1.46}$$

expressing the finite-dimensional transfer matrix $T_{i,t}(z)$ as an alternating sum of the infinite-dimensional transfer matrices $T^+_{w,t\omega_i}(z)$ of the $Y(\mathfrak{g})$-modules $M^+_{w\cdot t\omega_i}$ explicitly realized in the Fock $\mathcal{A}$-module F.

We refer the reader to our Theorems 4.37, 5.41, 6.46, 7.43 for a case-by-case treatment presented in the way that highlights the combinatorial description of ${}^{\mathsf{l}}W$ (1.32, 1.33) in each case of (1.22).

In type D, this establishes the key assumption from [FFK] (the joint work of the first-named author with G. Ferrando and V. Kazakov) essential for the study of the entire QQ-system in *loc.cit.*

Let us note that the *BGG-relation* (1.46) allows to analytically continue the transfer matrices $T_{i,t}(z)$ of the finite-dimensional representations $L_{t\omega_i}$, $t \in \mathbb{N}$, to the entire complex plane $t \in \mathbb{C}$. With this in mind, we establish the t-symmetries of the resulting family of endomorphisms $T_{i,t}(z)$, see Propositions 5.59, 6.52, 7.58, by crucially utilizing the QQ-factorisation which we discuss next.

1.7. **Factorisations.**

The infinite-dimensional transfer matrices $T^+_{w,t\omega_i}(z)$ factorise into products of Q-operators. This factorisation can be traced back to the factorisation of oscillator-type Lax matrices used to construct the transfer matrices. In the case of $U_q(\widehat{\mathfrak{sl}}_2)$, such factorisation formula was initially proposed in [BLZ], see also [KT]. Here, the Lax matrix entering the definition of the infinite-dimensional transfer matrix factorises into the product of two degenerate Lax matrices that are used to construct Baxter Q-operators. The degenerate Lax matrices employed in the factorisation are solutions of (1.1) with degenerate coefficients of the leading term, which are no longer related to quantum groups. They have a long history, going back to [IK, KSS]. The relation to Q-operators is discussed for $U_q(\widehat{\mathfrak{sl}}_2)$ and $Y(\mathfrak{sl}_2)$ in [AF, BLZ, RW, Kor, BLMS], and for higher rank cases in [BHK, BT, BGKNR, BFLMS, Ts3], while cases beyond A-type were first found in [Fr, CGY, FT].

By now, the role of the degenerate Lax matrices, arising from the shifted Yangians and viewed as certain *normalized limits* of the non-degenerate ones (as some of the representation labels tend to infinity), is well understood. However the actual factorisation that relates Lax matrices of different kinds remains yet to be understood. For a discussion of A-type we refer the reader to [FP] (which also indicates an intriguing relation to the cluster structures). For completeness of our exposition, we also refer the reader to the slightly different approach [DM1, DM2, DM3] going back to [GP].

Let us summarize the third main result of this paper that factorises the infinite-dimensional transfer matrices $T^+_{w,t\omega_i}(z)$ from the previous subsection into the product of two commuting Q-operators (in type D, this was first observed in [Fr, FFK], while an interpretation of this factorisation in terms of the $4d$ Chern-Simons theory for general types has appeared very recently in [CGY, §16]):

Main Theorem 3. *For a classical rank r Lie algebra $\mathfrak{g}$, an index $1 \leq i \leq r$ as in* (1.22)*, a scalar $t \in \mathbb{C}$, and the element $w \in {}^{\mathsf{l}}W$ as in (1.32, 1.33) for the standard parabolic $\mathfrak{p}_{\{1,\dots,r\}\setminus\{i\}}$, we have:*

$$T^+_{w,t\omega_i}(z) = \mathrm{ch}^+_{w,t\omega_i} \cdot Q^+_{w,t\omega_i}(z) Q^-_{w,t\omega_i}(z)\,, \tag{1.47}$$

with the scalar factor $\mathrm{ch}^+_{w,t\omega_i}$ arising as a trace of the z-independent twist and the Q-operators $Q^\pm_{w,t\omega_i}(z)$ arising in a similar fashion to $T^+_{w,t\omega_i}(z)$ but rather from the degenerate Lax matrices.

We refer the reader to our formulas (8.29, 8.79, 8.98, 9.24) for a case-by-case treatment presented in the way that highlights the aforementioned combinatorial description of the set ${}^{\mathsf{I}}W$ in each case.

Combining our Main Theorems 2 and 3, we thus obtain expressions for the finite-dimensional transfer matrices $T_{i,t}(z)$ (with the index i from (1.22) and $t \in \mathbb{N}$) in terms of the above Q-operators, see Propositions 8.38, 8.81, 8.100, 9.26.

1.8. **Transfer matrices from the universal R-matrix.**

We conclude our Introduction with a more general, but less explicit, construction of the transfer matrices of rational spin chains (trigonometric version of which is much better understood by now). Let $\mathfrak{g}$ be a semisimple Lie algebra with a non-degenerate invariant form $(\cdot,\cdot)$ and $Y_\hbar(\mathfrak{g})$ denote the Drinfeld Yangian, which is a Hopf $\mathbb{C}[\hbar]$-algebra deforming the current algebra $Y_{\hbar=0}(\mathfrak{g}) \simeq U(\mathfrak{g}[u])$. As the specializations $Y_{\hbar=a}(\mathfrak{g}) \simeq Y_{\hbar=b}(\mathfrak{g})$ are canonically isomorphic for $a, b \in \mathbb{C}^\times$, we shall omit $\hbar$-dependence by rather considering $Y(\mathfrak{g}) = Y_{\hbar=1}(\mathfrak{g})$. The latter is a Hopf algebra with a coproduct

$$\Delta\colon Y(\mathfrak{g}) \to Y(\mathfrak{g}) \otimes Y(\mathfrak{g})\,, \tag{1.48}$$

and admits a one-parameter group of Hopf algebra automorphisms $\{\tau_a\}_{a\in\mathbb{C}}$, quantizing the shift automorphisms $\{\bar\tau_a\}_{a\in\mathbb{C}}$ of $U(\mathfrak{g}[u])$ given by $Xu^k \mapsto X(u+a)^k$ for $X \in \mathfrak{g}$ and $k \in \mathbb{N}$, which may be further viewed (upon replacing $a \in \mathbb{C}$ with a formal variable z) as an algebra homomorphism

$$\tau_z\colon Y(\mathfrak{g}) \to Y(\mathfrak{g})[z]\,. \tag{1.49}$$

Let $\Omega_\mathfrak{g} \in \mathfrak{g} \otimes \mathfrak{g}$ be the Casimir tensor corresponding to $(\cdot,\cdot)$, and Δ^{op} be the opposite coproduct.

Theorem ([D, Theorem 3]). *There is a unique formal series*

$$\mathcal{R}(z) = 1 + \sum_{k=1}^{\infty} \mathcal{R}_k z^{-k} \in Y(\mathfrak{g}) \otimes Y(\mathfrak{g})[[z^{-1}]] \tag{1.50}$$

satisfying the following relations:

$$\begin{aligned} \text{intertwining identity:}\quad & (\mathrm{id}\otimes\tau_z)\Delta^{\mathrm{op}}(y) = \mathcal{R}(z)^{-1}\cdot(\mathrm{id}\otimes\tau_z)\Delta(y)\cdot\mathcal{R}(z) \quad \forall\, y \in Y(\mathfrak{g})\,,\\ \text{cabling identity:}\quad & (\mathrm{id}\otimes\Delta)(\mathcal{R}(z)) = \mathcal{R}_{12}(z)\mathcal{R}_{13}(z)\,. \end{aligned} \tag{1.51}$$

It also satisfies the quantum Yang-Baxter equation (1.3) *and is called* ***the universal R-matrix****. Moreover, $\mathcal{R}(z)$ also satisfies the following identities:*

$$\mathcal{R}(z) = 1 + \Omega_\mathfrak{g}\cdot z^{-1} + O(z^{-2})\,, \quad \mathcal{R}(z)^{-1} = \mathcal{R}_{21}(-z)\,, \quad (\tau_a\otimes\tau_b)\mathcal{R}(z) = \mathcal{R}(z+b-a)\,. \tag{1.52}$$

For any two representations $\rho^V\colon Y(\mathfrak{g}) \to \mathrm{End}(V)$ and $\rho^W\colon Y(\mathfrak{g}) \to \mathrm{End}(W)$, consider the evaluation of $\mathcal{R}(z)$:

$$R^{VW}(z) = (\rho^V\otimes\rho^W)(\mathcal{R}(z)) \in \mathrm{End}(V)\otimes\mathrm{End}(W)[[z^{-1}]]\,. \tag{1.53}$$

For $W = V$, $R^{VV}(z)$ is thus a solution of the quantum Yang-Baxter equation (1.3). For irreducible finite-dimensional V and W, $R^{VW}(z)$ is actually a rational function in z, up to an overall (possibly divergent) power series $f(z)$, see [D, Theorem 4] (cf. [GRW, Theorem 3.10] for more details). This way one recovers the rational R-matrices (1.2) and (1.19). Indeed, for $\mathfrak{g} = \mathfrak{sl}_n$ and $V = \mathbb{C}^n$, we have $R^{VV}(z) = f(z)R(z)$ with $R(z)$ as in (1.2), according to [D, Example 1]. Likewise, for $\mathfrak{g} = \mathfrak{so}_\mathsf{K}, \mathfrak{sp}_\mathsf{K}$ and $V = \mathbb{C}^\mathsf{K}$, we have $R^{VV}(z) = f(z)R(z)$ with $R(z)$ as in (1.19), due to [GRW, Proposition 3.13].

For any $Y(\mathfrak{g})$-module W and a group-like element x in an appropriate completion of $Y(\mathfrak{g})$, consider

$$\mathcal{T}_{W,x}(z) = (\mathrm{id}\otimes\mathrm{tr}_W)\left((1\otimes x)\mathcal{R}(z)\right) \tag{1.54}$$

whenever the latter is well-defined. The above properties of the universal R-matrix $\mathcal{R}(z)$ imply:

$$\mathcal{T}_{W_1\oplus W_2,x}(z) = \mathcal{T}_{W_1,x}(z) + \mathcal{T}_{W_2,x}(z)\,, \qquad \mathcal{T}_{W_1\otimes W_2,x}(z) = \mathcal{T}_{W_1,x}(z)\cdot\mathcal{T}_{W_2,x}(z)\,. \tag{1.55}$$

For a $Y(\mathfrak{g})$-module W and $a \in \mathbb{C}$, we set $W(a) = \tau_a^*(W)$. If further $\tau_z(x) = x$, then (1.52) implies:

$$\mathcal{T}_{W(a),x}(z) = \mathcal{T}_{W,x}(z+a) \qquad \forall\, a \in \mathbb{C}\,. \tag{1.56}$$

Combining (1.55, 1.56), we get the commutativity of the resulting **universal transfer matrices**:

$$\mathcal{T}_{W_1,x}(z+a)\cdot\mathcal{T}_{W_2,x}(z+b) = \mathcal{T}_{W_2,x}(z+b)\cdot\mathcal{T}_{W_1,x}(z+a) \qquad \forall\, a, b \in \mathbb{C}\,. \tag{1.57}$$

Thus, for every finite-dimensional representation $\rho^V\colon Y(\mathfrak{g}) \to \mathrm{End}(V)$ we obtain a commuting family of endomorphisms of V, defined by extracting the coefficients of the power series

$$T_W(z) = \rho^V(\mathcal{T}_{W,x}(z)) \in \mathrm{End}(V)[[z^{-1}]]\,, \tag{1.58}$$

as we vary the auxiliary representation W (we suppress x in $T_W(z)$ for simplicity of notation).

The explicit constructions of the present paper should arise as particular examples of this general setup with $V = (\mathbb{C}^n)^{\otimes N}$ for $\mathfrak{g} = \mathfrak{gl}_n$ (resp. $V = (\mathbb{C}^{\mathsf{K}})^{\otimes N}$ for $\mathfrak{g} = \mathfrak{so}_{\mathsf{K}}, \mathfrak{sp}_{\mathsf{K}}$), the $Y(\mathfrak{g})$-modules W being isomorphic to the modules $\{M'_{w\cdot t\omega_i}\}$ as $\mathfrak{g}$-modules with i from (1.22), and finally $x = \exp(h)$ for a general Cartan element $h \in \mathfrak{h} \subset \mathfrak{g}$ (equivalently, $x = \prod_{i=1}^{\mathrm{rk}(\mathfrak{g})} \tau_i^{\epsilon_i^*}$ with ϵ_i^* being a basis of $\mathfrak{h}$).

1.9. **Outline of the paper.**

The structure of the present paper is as follows. In Section 2, we construct (by using a geometric approach) the novel BGG-type resolutions (1.30) and (1.31) on which the functional relations presented in this article are based on. The reader interested only in the functional relations can start in Section 3 where we recall the well studied case of A-type and the standard BGG resolution. In Section 4, we apply the new BGG-type resolutions to type A recovering functional relations that follow from the standard BGG resolution. Section 5, Section 6, and Section 7 are dedicated to the functional relations obtained from the BGG-type resolutions for type BCD. The factorisation of the corresponding infinite-dimensional transfer matrices is then discussed in Sections 8 and 9. Finally, we mention some generalizations (to be presented elsewhere) of our work in Section 10.

1.10. **Acknowledgments.**

R.F. and A.T. are indebted to Vasily Pestun for the inspiring discussions and the collaboration on [FPT], thus bringing them together close to the subject of the present note. R.F. is grateful to Gwenaël Ferrando and Volodya Kazakov for fruitful discussions. I.K. and A.T. are extremely grateful to Boris Feigin and Michael Finkelberg for the enlightening discussions of the BGG-type resolutions. A.T. is deeply grateful to Kevin Costello for a correspondence on the inspiring physics paper [CGY]; to David Hernandez for a correspondence on q-characters; to Sachin Gautam for a discussion of the Yangian's universal R-matrices; to IHES (Bures-sur-Yvette) for the hospitality and great working conditions in July 2021 when the first stages of the present work took place. We are grateful to Zengo Tsuboi and the anonymous referee for useful suggestions and comments.

R.F. received funding of the German research foundation (Deutsche Forschungsgemeinschaft DFG) Research Fellowships Programme 416527151 and support of the GNFM - INdAM. A.T. would like to gratefully acknowledge the support from NSF Grant DMS-2037602. This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (QUASIFT grant agreement 677368).

2. Truncated BGG resolutions as Cousin complexes

In this section, we construct both resolutions (1.30) and (1.31) by interpreting their "restricted dual" as Cousin complexes of relative local cohomology groups, in the spirit of [Ke, Br, MR, Ku].

2.1. **Cohomology with relative support and Cousin complexes.**

Let X be a topological space and $Z \subset X$ be a closed subset. Consider the functor Γ_Z sending a sheaf $\mathcal{F}$ of abelian groups on X to the module $\ker(\Gamma(X,\mathcal{F}) \to \Gamma(X-Z,\mathcal{F}))$. Let $\mathcal{F} \mapsto H^i_Z(X,\mathcal{F})$ denote the i-th right derived functor of $\Gamma_Z(X,-)$, the ***i*-th cohomology group of $\mathcal{F}$ with support on a closed subset Z**. This construction admits the following relative version. Suppose that A and B are two closed subsets of X such that $B \subset A$. Consider the functor $\Gamma_{A/B}$ which sends a sheaf $\mathcal{F}$ of abelian groups on X to the module $\mathrm{coker}(\Gamma_B(X,\mathcal{F}) \to \Gamma_A(X,\mathcal{F}))$. Let $\mathcal{F} \mapsto H^i_{A/B}(X,\mathcal{F})$ denote the i-th right derived functor[1] of $\Gamma_{A/B}(X,-)$, the ***i*-th cohomology group of $\mathcal{F}$ with relative support (A,B)**. We note that $H^i_A(X,\mathcal{F}) = H^i_{A/\emptyset}(X,\mathcal{F})$ and $H^i_X(X,\mathcal{F}) = H^i(X,\mathcal{F})$.

[1]The issues with coker not being left exact are carefully resolved by Kempf in the beginning of [Ke, §7].

Lemma 2.1 ([Ke]). *The functor $H^i_{A/B}(X,-)$ satisfies the following properties:*

(a) There is a long exact sequence

$$\cdots \to H^i_B(X,\mathcal{F}) \to H^i_A(X,\mathcal{F}) \to H^i_{A/B}(X,\mathcal{F}) \to H^{i+1}_B(X,\mathcal{F}) \to \cdots \tag{2.2}$$

(b) Every inclusion of closed subsets $C \subseteq B$ induces a natural morphism

$$H^i_{A/C}(X,\mathcal{F}) \to H^i_{A/B}(X,\mathcal{F}) \tag{2.3}$$

functorial with respect to A and B, and a morphism of the corresponding exact sequences (2.2).

(c) There is an "excision" isomorphism:

$$H^i_{A/B}(X,\mathcal{F}) \xrightarrow{\sim} H^i_{A\setminus B}(X\setminus B,\mathcal{F}) \tag{2.4}$$

functorial with respect to A and B (here, $A\setminus B$ denotes the complement of B in A).

We will also need the following simple corollary of Lemma 2.1 and the above definitions:

Lemma 2.5. *For any two disjoint closed subsets Z_1 and Z_2 of X there exist isomorphisms:*

$$H^i_{Z_1}(X,\mathcal{F}) \xrightarrow{\sim} H^i_{Z_1}(X\setminus Z_2,\mathcal{F}|_{X\setminus Z_2}) \tag{2.6}$$

and

$$H^i_{Z_1\sqcup Z_2}(X,\mathcal{F}) \xrightarrow{\sim} H^i_{Z_1}(X,\mathcal{F}) \oplus H^i_{Z_2}(X,\mathcal{F}) \tag{2.7}$$

functorial with respect to Z_1 and Z_2.

Let now X be a smooth algebraic variety. Let $\mathcal{O}_X$ be the structure sheaf and $\mathcal{D}_X$ be the sheaf of algebraic differential operators of finite order on X. We also define $\mathcal{D} = \Gamma(X,\mathcal{D}_X)$, the **algebra of global differential operators on X**. The crucial observation is that the above constructions remain true in the D-module theoretic setting:

Lemma 2.8 ([Br, §2]). *For a coherent sheaf $\mathcal{F}$ of left $\mathcal{D}_X$-modules, all cohomology groups $H^i_{A/B}(X,\mathcal{F})$ carry the natural $\mathcal{D}$-action. Moreover, all maps in Lemmas 2.1, 2.5 are $\mathcal{D}$-equivariant.*

Suppose now that X is a G-variety. Then, there is an evident Lie algebra homomorphism $\mathfrak{g} \to \mathcal{D}$ (the target endowed with the commutator bracket). It obviously induces a $\mathfrak{g}$-action on $H^\bullet_{A/B}(X,\mathcal{F})$. However, whenever considering $H^\bullet_{A/B}(X,\mathcal{F})$ as a $\mathfrak{g}$-module, we will be interested not literally in this $\mathfrak{g}$-action, but in the one **twisted by the Chevalley involution ϕ of $\mathfrak{g}$** determined by:

$$\phi\colon \quad h \mapsto -h\,, \qquad e_\alpha \mapsto e_{-\alpha}\,, \qquad \forall\, h\in\mathfrak{h}\,,\ \alpha\in\Delta\,. \tag{2.9}$$

Let us now recall the key tool of Cousin complexes (our exposition closely follows that of [MR, §3]). Suppose that X is a topological space equipped with a (<u>not necessarily exhaustive</u>) filtration

$$Z_n \subseteq Z_{n-1} \subseteq \cdots \subseteq Z_0 \subseteq X \tag{2.10}$$

of closed subsets, and let $\mathcal{E}$ be a sheaf of abelian groups on X. Picking a flabby resolution $\mathcal{E}^\bullet$ of $\mathcal{E}$ and considering the mapping cones $C_j = C(\Gamma_{Z_{j+1}}(X,\mathcal{E}^\bullet) \to \Gamma_{Z_j}(X,\mathcal{E}^\bullet))$, whose cohomology are naturally isomorphic to the relative cohomology $H^\bullet_{Z_j/Z_{j+1}}(X,\mathcal{E})$, one can construct a double complex $C_{\bullet,\bullet}$ with exact rows and whose j-th column is $C_j[j]$, the j-th cone C_j shifted by degree j. Then, on the one hand, the exactness of rows implies that the cohomology of the total complex $\mathrm{Tot}(C_{\bullet,\bullet})$ is isomorphic to the cohomology of $\Gamma_{Z_0}(X,\mathcal{E}^\bullet)$, i.e. to $H^\bullet_{Z_0}(X,\mathcal{E})$. On the other hand, the vertical cohomology of this double complex $C_{\bullet,\bullet}$ is $H^{k+j}_{Z_j/Z_{j+1}}(X,\mathcal{E})$, as noted above, and the horizontal differential in the k-th row gives rise to the so-called **k-th Cousin complex**:

$$\mathcal{C}_k\colon \qquad H^k_{Z_0/Z_1}(X,\mathcal{E}) \to H^{k+1}_{Z_1/Z_2}(X,\mathcal{E}) \to H^{k+2}_{Z_2/Z_3}(X,\mathcal{E}) \to \cdots \tag{2.11}$$

Therefore, by applying the vertical spectral sequence of the double complex $C_{\bullet,\bullet}$, we obtain:[2]

[2]While the Cousin complexes were introduced by Grothendieck and were first applied in the above context in [Ke], we choose to follow the exposition of [MR] for its simplicity.

Theorem 2.12 ([MR]). *If all except the k-th Cousin complexes are zero, then we have:*

$$H^\bullet(\mathcal{C}_k) = H^\bullet_{Z_0}(X, \mathcal{E})\,. \tag{2.13}$$

2.2. **The geometry of partial flag varieties.**

In what follows, we shall freely use the notation of Subsection 1.5. In particular, let G be a connected algebraic group with $\mathrm{Lie}(G) = \mathfrak{g}$ and $H \subset B \subset P$ be the Cartan torus, the Borel, and parabolic subgroups of G with the corresponding Lie algebras $\mathrm{Lie}(H) = \mathfrak{h}$, $\mathrm{Lie}(B) = \mathfrak{b}$, $\mathrm{Lie}(P) = \mathfrak{p}$. Consider the corresponding partial flag variety

$$X = G/P\,. \tag{2.14}$$

Let $B_- \subset G$ be the opposite Borel subgroup of B containing H, and $U_- \subset B_-$ be its unipotent radical, so that $\mathrm{Lie}(U_-) = \bigoplus_{\alpha\in\Delta^+} \mathbb{C}e_{-\alpha}$ and $\mathrm{Lie}(B_-) = \mathfrak{h} \oplus \mathrm{Lie}(U_-)$. Then, B_- naturally acts on X, giving rise to the stratification of X by B_--orbits (see e.g. [Ku, §7.21]):

$$X \;=\; \bigsqcup_{w\in {}^{\mathfrak{l}}W} X_w\,, \qquad X_w = B_- wP/P = U_- wP/P\,, \qquad \mathrm{codim}_X(X_w) = l(w)\,. \tag{2.15}$$

Here, the indexing set ${}^{\mathfrak{l}}W$ consists of the shortest length representatives of the left cosets $W/W_{\mathfrak{l}}$, precisely as in (1.33), and X_w is an affine space of dimension equal to $l(w)$, the length of $w \in {}^{\mathfrak{l}}W$.

Following the setup of Subsection 1.5, let $\lambda \in P^+_{\mathfrak{g}/\mathfrak{l}}$ be a dominant integral weight of $\mathfrak{g}$ <u>vanishing</u> on the coroot lattice of the Levi subalgebra $\mathfrak{l} \subset \mathfrak{p}$. Let $\tilde{L}_{-\lambda}$ be the one-dimensional P-representation corresponding to the weight $-\lambda$, and $\tilde{\mathcal{L}}_\lambda$ be the corresponding G-equivariant line bundle on X:

$$\tilde{\mathcal{L}}_\lambda = G \times_P \tilde{L}_{-\lambda}\,. \tag{2.16}$$

For any subset Y of a topological space X, we use $\overline{\mathsf{Y}}$ and $\partial(\mathsf{Y})$ to denote its closure and boundary. Since X_w is locally closed (as an orbit of an algebraic group), we have:

$$X_w = \overline{X}_w \setminus \partial(X_w)\,. \tag{2.17}$$

All $\mathfrak{g}$-modules we consider in this paper do belong to the category $\mathcal{O}$ of [BGG]. In particular, every such module V has the weight space decomposition with all components being finite-dimensional:

$$V = \bigoplus_{\nu\in\mathfrak{h}^*} V[\nu]\,, \qquad V[\nu] = \Big\{ v \in V \mid h(v) = \nu(h)v \;\forall h \in \mathfrak{h} \Big\}\,. \tag{2.18}$$

In this setup, one may define **the restricted dual module** $V^\vee \subseteq V^*$: as a vector space

$$V^\vee = \bigoplus_{\nu\in\mathfrak{h}^*} V[\nu]^*\,, \tag{2.19}$$

while the $\mathfrak{g}$-action is the restriction of the natural one on V^* **twisted** by the Chevalley involution ϕ of (2.9). This defines an involutive antiautoequivalence Φ of the category $\mathcal{O}$:

$$\Phi\colon V \mapsto V^\vee\,. \tag{2.20}$$

For the finite-dimensional $\mathfrak{g}$-modules, we have:

$$L_\lambda^\vee \simeq L_\lambda\,, \qquad \forall\, \lambda \in P^+_{\mathfrak{g}}\,. \tag{2.21}$$

Theorem 2.22. *There exists a finite length exact sequence of $\mathfrak{g}$-modules of the form:*

$$0 \to M_{ww_{\mathfrak{l},0}\cdot\lambda} \to \cdots \to \bigoplus_{v\in W_{\mathfrak{l}}}^{l(v)=2} M_{wv\cdot\lambda} \to \bigoplus_{v\in W_{\mathfrak{l}}}^{l(v)=1} M_{wv\cdot\lambda} \to M_{w\cdot\lambda} \to H^{l(w)}_{\overline{X}_w/\partial(X_w)}(X, \tilde{\mathcal{L}}_\lambda)^\vee \to 0\,. \tag{2.23}$$

Proof. Using Φ of (2.20) it suffices to prove that there exists an exact sequence of $\mathfrak{g}$-modules of the form:

$$0 \to H^{l(w)}_{\overline{X}_w/\partial(X_w)}(X, \tilde{\mathcal{L}}_\lambda) \to M^\vee_{w\cdot\lambda} \to \bigoplus_{v\in W_{\mathfrak{l}}}^{l(v)=1} M^\vee_{wv\cdot\lambda} \to \bigoplus_{v\in W_{\mathfrak{l}}}^{l(v)=2} M^\vee_{wv\cdot\lambda} \to \cdots \to M^\vee_{ww_{\mathfrak{l},0}\cdot\lambda} \to 0\,. \tag{2.24}$$

To this end, consider the complete flag variety

$$Y = G/B\,, \tag{2.25}$$

and let $\pi\colon Y \to X$ denote the natural projection:

$$\pi\colon G/B \to G/P\,. \tag{2.26}$$

Note that Y admits a natural Bruhat decomposition by B_--orbits, cf. (2.15):

$$Y = \bigsqcup_{u\in W} Y_u\,, \qquad Y_u = B_-uB/B = U_-uB/B\,, \qquad \mathrm{codim}_Y(Y_u) = l(u)\,. \tag{2.27}$$

For any $w \in {}^{\mathfrak{l}}W$, define $Q_w \subseteq Y$ via:

$$Q_w = \pi^{-1}(B_-wP)\,, \tag{2.28}$$

which is naturally stratified by B_--orbits:

$$Q_w = \pi^{-1}(B_-wP) = \bigsqcup_{v\in W_{\mathfrak{l}}} Y_{wv}\,. \tag{2.29}$$

Let us also note the following useful equality:

$$l(wv) = l(w) + l(v) \qquad \text{for any} \qquad w \in {}^{\mathfrak{l}}W\,,\ v \in W_{\mathfrak{l}}\,. \tag{2.30}$$

Let $L_{-\lambda}$ be the one-dimensional B-representation corresponding to the weight $-\lambda$, and $\mathcal{L}_\lambda$ be the corresponding G-equivariant line bundle on Y:

$$\mathcal{L}_\lambda = G \times_B L_{-\lambda}\,. \tag{2.31}$$

Similarly to [MR], for any $w \in {}^{\mathfrak{l}}W$ consider

$$U_w = Y \setminus \partial(Q_w) = Y \setminus \pi^{-1}\left(\partial(X_w)\right) \tag{2.32}$$

(the second equality is due to π being proper), so that Q_w is closed in U_w. Note that U_w is naturally stratified by B_--orbits, which gives rise to the following filtration $Z_\bullet$ of U_w by closed subsets:

$$Z_i = Q_w \cap \tilde{Z}_i \qquad \text{with} \qquad \tilde{Z}_i = \bigsqcup_{\substack{u\in W\\ l(u)\geq l(w)+i}} Y_u\,. \tag{2.33}$$

We note that $Z_0 = Q_w$, and according to (2.29, 2.30) we have:

$$\tilde{Z}_i \setminus \tilde{Z}_{i+1} = \bigsqcup_{u\in W\colon l(u)=l(w)+i} Y_u\,, \qquad Z_i \setminus Z_{i+1} = \bigsqcup_{v\in W_{\mathfrak{l}}\colon l(v)=i} Y_{wv}\,. \tag{2.34}$$

We shall now apply the results of Subsection 2.1 to U_w equipped with the filtration (2.33) and the sheaf $\mathcal{F} = \mathcal{L}_\lambda|_{U_w}$. We claim that Theorem 2.12 applies in this setting, and furthermore the corresponding Cousin complex, which calculates $H^\bullet_{Z_0}(U_w, \mathcal{F})$, provides the desired exact sequence (2.24).

Indeed, according to [Ku, Proposition 9.3.7], for any $u \in W$ we have:

$$H^k_{\overline{Y_u}/\partial(Y_u)}(G/B, \mathcal{L}_\lambda) = \begin{cases} M^\vee_{u\cdot\lambda} & \text{if } k = \mathrm{codim}_Y(Y_u) = l(u) \\ 0 & \text{otherwise} \end{cases}\,. \tag{2.35}$$

Combining this with (2.34) and Lemma 2.5, we obtain:

$$H^{i+l(w)}_{Z_i/Z_{i+1}}(U_w, \mathcal{F}) = H^{i+l(w)}_{Z_i\setminus Z_{i+1}}(U_w \setminus Z_{i+1}, \mathcal{L}_\lambda) \simeq \bigoplus_{v\in W_{\mathfrak{l}}}^{l(v)=i} M^\vee_{wv\cdot\lambda} \tag{2.36}$$

and

$$H^k_{Z_i/Z_{i+1}}(U_w, \mathcal{F}) = 0 \qquad \text{for} \qquad k \neq i + l(w)\,. \tag{2.37}$$

Remark 2.38. We note that Lemma 2.5 does apply, due to:

(1) $Z_i \setminus Z_{i+1} = \bigsqcup_{v\in W_{\mathfrak{l}}\colon l(v)=i} Y_{wv}$;

(2) for every cell $Y_u \subset Z_i \setminus Z_{i+1}$, we have $\partial(Y_u) \subseteq \partial(Q_w) \cup Z_{i+1}$, so that

$$U_w \setminus Z_{i+1} = Y \setminus (\partial(Q_w) \cup Z_{i+1}) = V_u \setminus ((V_u \cap \partial(Q_w)) \cup (V_u \cap Z_{i+1}))\,,$$

where $V_u = Y \setminus \partial(Y_u)$, and $V_u \cap \partial(Q_w), V_u \cap Z_{i+1}$ are closed subsets of V_u disjoint from Y_u.

Thus, all Cousin complexes $\mathcal{C}_k$ of (2.11) vanish for $k \neq l(w)$, while the terms of $\mathcal{C}_{l(w)}$ precisely coincide with the terms of the exact sequence (2.24). Applying Theorem 2.12, we therefore get $H^i(\mathcal{C}_{l(w)}) = H^{i+l(w)}_{Z_0}(U_w, \mathcal{F})$.

Hence, it remains to prove that:

(I) $H^0(\mathcal{C}_{l(w)}) = H^{l(w)}_{Z_0}(U_w, \mathcal{F})$ coincides with the $\mathfrak{g}$-module

$$N_w(\lambda) = H^{l(w)}_{\overline{X}_w/\partial(X_w)}(X, \tilde{\mathcal{L}}_\lambda) \tag{2.39}$$

(II) $H^i(\mathcal{C}_{l(w)}) = H^{i+l(w)}_{Z_0}(U_w, \mathcal{F})$ vanishes for $i \neq 0$.

Both results follow immediately from a B-version of Theorem 2.51 below, the excision isomorphism

$$H^k_{\overline{X}_w/\partial(X_w)}(X, \tilde{\mathcal{L}}_\lambda) \simeq H^k_{X_w}((G/P) \setminus \partial(X_w), \tilde{\mathcal{L}}_\lambda)\,,$$

cf. (2.17), and the following two lemmas:

Lemma 2.40 ([GS, p. 286])**.** *$R^0\pi_*(\mathcal{L}_\lambda) = \tilde{\mathcal{L}}_\lambda$ and $R^{>0}\pi_*(\mathcal{L}_\lambda) = 0$.*

Lemma 2.41 ([GR, Exposé 5, Lemme 3.2])**.** *Let $f\colon X \to X'$ be an arbitrary morphism, $S \subset X'$ be a closed subset, and $\mathcal{F}$ be a sheaf of abelian groups on X. Then, there is a spectral sequence:*

$$H^i_S\left(X', R^j f_* \mathcal{F}\right) \Rightarrow H^{i+j}_{f^{-1}(S)}(X, \mathcal{F})\,.$$

Therefore, the $l(w)$-th Cousin complex $\mathcal{C}_{l(w)}$ realizes the exact sequence (2.24) of $\mathfrak{g}$-modules, which produces the exact sequence (2.23) upon a further application of the antiautoequivalence Φ of (2.20). □

Let us now recall the highest weight $\mathfrak{g}$-modules $M'_{w\cdot\lambda} = M_{w\cdot\lambda}/M^{\mathrm{sing}}_{w\cdot\lambda}$ for $w \in {}^{\mathfrak{l}}W$, defined by the formulas (1.36, 1.37) in the Introduction. They admit the following geometric interpretation:

Lemma 2.42. *For any $w \in {}^{\mathfrak{l}}W$, we have the isomorphism of $\mathfrak{g}$-modules:*

$$M'_{w\cdot\lambda} \simeq N_w(\lambda)^\vee\,. \tag{2.43}$$

Proof. This immediately follows from the following fragment of (2.23) using the notation (2.39):

$$\bigoplus_{v \in W_{\mathfrak{l}}}^{l(v)=1} M_{wv\cdot\lambda} \to M_{w\cdot\lambda} \to N_w(\lambda)^\vee \to 0\,.$$

□

Combining Theorem 2.22 and Lemma 2.42, we immediately obtain:

Corollary 2.44. *All $\mathfrak{g}$-modules $\{M'_{w\cdot\lambda}\}_{w\in{}^{\mathfrak{l}}W}$ admit resolutions* (1.31) *by Verma modules.*

As a direct corollary of (1.31), we obtain the character formula for the $\mathfrak{g}$-modules $M'_{w\cdot\lambda}$, cf. (1.38):

Lemma 2.45. *For $w \in {}^{\mathfrak{l}}W$, we have:*

$$\mathrm{ch}_{M'_{w\cdot\lambda}} = \frac{e^{w(\lambda+\rho)-\rho}}{\prod_{\alpha\in\Delta^+\setminus w(\Delta^+_{\mathfrak{l}})}(1-e^{-\alpha})}\,. \tag{2.46}$$

Proof. The existence of the resolution (1.31) implies the following equality of characters:

$$\mathrm{ch}_{M'_{w\cdot\lambda}} = \sum_{v\in W_{\mathfrak{l}}} (-1)^{l(v)}\,\mathrm{ch}_{M_{wv\cdot\lambda}} = \frac{\sum_{v\in W_{\mathfrak{l}}}(-1)^{l(v)}e^{wv(\lambda+\rho)-\rho}}{\prod_{\alpha\in\Delta^+}(1-e^{-\alpha})} =$$

$$e^{w(\lambda+\rho)-\rho}\cdot\frac{\sum_{v\in W_{\mathfrak{l}}}(-1)^{l(v)}e^{w(v(\rho)-\rho)}}{\prod_{\alpha\in\Delta^+}(1-e^{-\alpha})} = e^{w(\lambda+\rho)-\rho}\cdot\frac{\sum_{v\in W_{\mathfrak{l}}}(-1)^{l(v)}e^{w(v(\rho_{\mathfrak{l}})-\rho_{\mathfrak{l}})}}{\prod_{\alpha\in\Delta^+}(1-e^{-\alpha})}\,, \tag{2.47}$$

where ρ is defined in (1.15), $\rho_{\mathfrak{l}}$ is defined in (1.26), and the last two equalities follow from:

$$v(\lambda)=\lambda\,,\qquad v(\rho)-\rho = v(\rho_{\mathfrak{l}})-\rho_{\mathfrak{l}}\,,\qquad \forall\ v\in W_{\mathfrak{l}}\,,\ \lambda\in P^+_{\mathfrak{g}/\mathfrak{l}}\,. \tag{2.48}$$

Applying the Weyl denominator formula (1.18) for $\mathfrak{l}$:

$$\sum_{v\in W_{\mathfrak{l}}}(-1)^{l(v)}e^{v(\rho_{\mathfrak{l}})-\rho_{\mathfrak{l}}} = \prod_{\alpha\in\Delta^+_{\mathfrak{l}}}(1-e^{-\alpha}) \tag{2.49}$$

and noting $w(\Delta^+_{\mathfrak{l}})\subseteq\Delta^+$, due to (1.32), we get precisely the character formula (2.46). □

Remark 2.50. The corresponding formula for $\mathrm{ch}_{N_w(\lambda)}$ goes back to [Ke, Lemma 12.8].

The next result is the key point of the further discussion:

Theorem 2.51. *For any $w\in{}^{\mathfrak{l}}W$, we have:*

$$H^i_{\overline{X}_w/\partial(X_w)}(X,\tilde{\mathcal{L}}_\lambda)=0 \quad \text{for any} \quad i\neq l(w)\,. \tag{2.52}$$

Proof. It is an immediate consequence of the local purity theorems (see [AL, Proposition 4.1]). □

2.3. **Derivation of the truncated BGG resolutions.**

Let $\lambda\in P^+_{\mathfrak{g}/\mathfrak{l}}$ be a dominant integral weight of $\mathfrak{g}$ vanishing on the coroot lattice of $\mathfrak{l}$, see (1.29). Now we are ready to derive both resolutions (1.30, 1.31) of the Introduction, cf. Main Theorem 1.

Theorem 2.53. *For $\lambda\in P^+_{\mathfrak{g}/\mathfrak{l}}$, the irreducible finite-dimensional $\mathfrak{g}$-module L_λ has a finite length resolution* (1.30)*, with each term admitting a finite length resolution* (1.31) *by Verma modules.*

Proof. We will prove the dualized version of the desired statement, just as in our proof of Theorem 2.22. Let us consider the sheaf $\mathcal{F}=\tilde{\mathcal{L}}_\lambda$ and the following filtration of X by closed subsets:

$$Z_i = \bigsqcup_{w\in{}^{\mathfrak{l}}W:\,l(w)\geq i} X_w\,. \tag{2.54}$$

We claim that Theorem 2.12 applies in this setup and gives rise to the following exact sequence of $\mathfrak{g}$-modules:

$$0\to L^\vee_\lambda\to(M'_\lambda)^\vee\to\bigoplus_{w\in{}^{\mathfrak{l}}W}^{l(w)=1}(M'_{w\cdot\lambda})^\vee\to\bigoplus_{w\in{}^{\mathfrak{l}}W}^{l(w)=2}(M'_{w\cdot\lambda})^\vee\to\cdots\to(M'_{\mathfrak{l},0_w\cdot\lambda})^\vee\to 0\,. \tag{2.55}$$

Indeed, all Cousin complexes $\mathcal{C}_k$ of (2.11) vanish for $k\neq 0$, due to Theorem 2.51 and Lemma 2.5:

$$H^{k+j}_{Z_j/Z_{j+1}}(X,\mathcal{F}) = H^{k+j}_{Z_j\setminus Z_{j+1}}(X\setminus Z_{j+1},\mathcal{F}) = \bigoplus_{w\in{}^{\mathfrak{l}}W}^{l(w)=j} H^{k+j}_{\overline{X}_w/\partial(X_w)}(X,\tilde{\mathcal{L}}_\lambda)=0\,. \tag{2.56}$$

Therefore, Theorem 2.12 applies, and we get:

$$H^i(\mathcal{C}_0)=H^i_X(X,\mathcal{F})=H^i(X,\mathcal{F})\,. \tag{2.57}$$

According to the parabolic version of the Borel-Weil-Bott theorem [Kos, Theorem 6.4], combined with our conventions of all geometric $\mathfrak{g}$-actions being twisted by the Chevalley involution ϕ of (2.9) that also enters our definition (2.19, 2.20) of the restricted dual $\mathfrak{g}$-module, we have (cf. (2.21)):

$$H^i(X,\mathcal{F}) = H^i(X,\tilde{\mathcal{L}}_\lambda) \simeq \begin{cases} L_\lambda^\vee \simeq L_\lambda & \text{for } i=0 \\ 0 & \text{for } i\neq 0 \end{cases} . \tag{2.58}$$

On the other hand, the j-th term of $\mathcal{C}_0$ is computed using Lemmas 2.5 and 2.42 similarly to (2.56):

$$H^j_{Z_j/Z_{j+1}}(X,\mathcal{F}) \;=\; \bigoplus_{w\in {}^{\mathfrak{l}}W}^{l(w)=j} H^{l(w)}_{\overline{X}_w/\partial(X_w)}(X,\tilde{\mathcal{L}}_\lambda) \;=\; \bigoplus_{w\in {}^{\mathfrak{l}}W}^{l(w)=j} (M'_{w\cdot\lambda})^\vee . \tag{2.59}$$

Combining (2.57, 2.58, 2.59), we see that the Cousin complex $\mathcal{C}_0$ realizes the exact sequence (2.55).

Applying the antiautoequivalence Φ of (2.20) to (2.55) produces the resolution (1.30), while the resolutions (1.31) were constructed in Corollary 2.44. This completes our proof of the theorem. □

Remark 2.60. Our argument above has been strongly influenced by [MR], where the Lepowsky parabolic BGG resolution (1.39) was interpreted via the Cousin complex on the complete flag variety $Y = G/B$ stratified by P-orbits. Nevertheless, there are two subtle points in [MR]:

(1) For $u \in W$, let Z_u denote the B-orbit $BuB/B \subseteq Y$. It is stated in [MR, after (3.2)] that:

$$H^k_{\overline{Z}_u/\partial(Z_u)}(G/B,\mathcal{L}_\lambda) = \begin{cases} M^\vee_{u\cdot\lambda} & \text{if } k=\operatorname{codim}_Y(Z_u) \\ 0 & \text{otherwise} \end{cases} . \tag{2.61}$$

This (as well as $\lambda\neq 0$ case at [Br, p. 55]) is wrong, as we rather have:

Lemma 2.62.

$$H^k_u(\lambda) := H^k_{\overline{Z}_u/\partial(Z_u)}(G/B,\mathcal{L}_\lambda) = \begin{cases} M^\vee_{uw_0\cdot\lambda} & \textit{if } k=\operatorname{codim}_Y(Z_u) \\ 0 & \textit{otherwise} \end{cases} . \tag{2.63}$$

Proof. When $u = w_0$, this statement is well-known (cf. [Be, Claim 2.4.2]). One can show that the general case holds along the lines of [Br, Proposition 7]. Indeed, let $w_0u^{-1} = s_{\alpha_1}\cdot\ldots\cdot s_{\alpha_{N-\ell(u)}}$ be a reduced decomposition, where $N = l(w_0) = |\Delta^+|$. Then, similarly to the argument in [Br, Lemma 4], it is easy to see that we still have a sequence of the following $\mathfrak{g}$-module epimorphisms:

$$H^0_{w_0}(\lambda) = H^0_{s_{\alpha_1}s_{\alpha_2}\cdots s_{\alpha_{N-l(u)}}u}(\lambda) \twoheadrightarrow H^1_{s_{\alpha_2}\cdots s_{\alpha_{N-l(u)}}u}(\lambda) \twoheadrightarrow \cdots \twoheadrightarrow H^{c(u)-1}_{s_{\alpha_{N-l(u)}}u}(\lambda) \twoheadrightarrow H^{c(u)}_u(\lambda)\,,$$

where $c(u) = \operatorname{codim}_Y(Z_u) = N - l(u)$. Thus, the argument of [Br] still applies and we get:

$$H^{c(u)}_u(\lambda) = M^\vee_{(s_{\alpha_{N-l(u)}}\cdots s_{\alpha_1})\cdot\lambda} = M^\vee_{uw_0\cdot\lambda}\,.$$

On the other hand, the vanishing result in (2.63) is just the B-case of Theorem 2.51. □

(2) The use of $H^\bullet_\bullet(-,\mathcal{L}_\lambda\otimes\mathcal{K})$ in [MR] is wrong.

However, both of the above inaccuracies can be easily fixed by replacing $\mathcal{L}_\lambda\otimes\mathcal{K}$ with $\mathcal{L}_\lambda$ and considering the stratification of Y by P_--orbits, where $P_-\subset G$ is the opposite parabolic subgroup.

Remark 2.64. (a) It is instructive to point out that the results of [MR] provide the answer to [Ku, Open Problem 9.3.19]. The only difference is that *loc.cit.* treats the case of an arbitrary Kac-Moody algebra. Nonetheless, the results of [MR], as well as ours, admit natural generalizations to such infinite-dimensional setup through the usual stratification by Schubert varieties.

(b) We also note that the other possible way to generalize our results is by considering an arbitrary dominant integral weight $\lambda\in P^+_{\mathfrak{g}}$, not necessarily vanishing on the coroot lattice of $\mathfrak{l}$. In this case, one obtains (exactly as above) the resolutions of the form (1.30, 1.31) with $\tilde{\mathcal{L}}_\lambda$ being replaced

by $R^0\pi_*(\mathcal{L}_\lambda)$ (note that $R^{>0}\pi_*(\mathcal{L}_\lambda)=0$, according to [GS]). However, the corresponding infinite-dimensional $\mathfrak{g}$-modules (realized as $H^{l(w)}_{\overline{X}_w/\partial(X_w)}(X, R^0\pi_*(\mathcal{L}_\lambda))^\vee$) are not defined for $\lambda \notin P^+_{\mathfrak{g}}$ in this case, in contrast to such a key feature of our modules $M'_{w\cdot\lambda}$ of (1.36) as discussed in Subsection 1.6.

3. Standard BGG

In this section, we recall the standard relation between the transfer matrices of A-type spin chains corresponding to finite-dimensional and infinite-dimensional (dual Verma) $\mathfrak{gl}_n$-modules provided via oscillator Lax matrices, as summarized in the Introduction. This exposition is mostly to motivate the key constructions and results of the upcoming sections. We also provide an overview of the factorisation and the determinant formulas in this setup, as mentioned in the Introduction.

3.1. **Oscillator realization in type A (Verma).**

For any $n \in \mathbb{Z}_{\geq 2}$, let $\mathcal{A}$ denote the oscillator algebra generated by $\frac{n(n-1)}{2}$ pairs of oscillators $\{(\mathbf{a}_{j,i}, \bar{\mathbf{a}}_{i,j})\}_{1\leq i<j\leq n}$ subject to the standard defining relations:

$$[\mathbf{a}_{j,i}, \bar{\mathbf{a}}_{k,\ell}] = \delta_i^k \delta_j^\ell\,, \qquad [\mathbf{a}_{j,i}, \mathbf{a}_{\ell,k}] = 0\,, \qquad [\bar{\mathbf{a}}_{i,j}, \bar{\mathbf{a}}_{k,\ell}] = 0\,, \tag{3.1}$$

so that

$$\mathcal{A} = \mathbb{C}\Big\langle \mathbf{a}_{j,i}\,,\, \bar{\mathbf{a}}_{i,j}\Big\rangle_{1\leq i<j\leq n} \Big/ (3.1)\,. \tag{3.2}$$

Fix $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$. Following [DM1, §2] (going back to [GN]), let us consider the $\mathfrak{gl}_n$-type $\mathcal{A}[x]$-valued Lax matrix (i.e. a solution of the RTT relation (1.1) with the R-matrix of (1.2)):

$$\mathcal{L}_\lambda(x) = U^{-1}(x + D_\lambda)U \tag{3.3}$$

defined through the matrices:

$$U = \begin{pmatrix} 1 & -\bar{\mathbf{a}}_{1,2} & \cdots & -\bar{\mathbf{a}}_{1,n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & -\bar{\mathbf{a}}_{n-1,n} \\ 0 & \cdots & 0 & 1 \end{pmatrix}, \qquad D_\lambda = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ \tilde{\mathbf{a}}_{2,1} & \lambda_2 - 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \tilde{\mathbf{a}}_{n,1} & \cdots & \tilde{\mathbf{a}}_{n,n-1} & \lambda_n - n + 1 \end{pmatrix}, \tag{3.4}$$

with

$$\tilde{\mathbf{a}}_{j,i} = -\mathbf{a}_{j,i} + \sum_{k=j+1}^{n} \bar{\mathbf{a}}_{j,k}\mathbf{a}_{k,i}\,. \tag{3.5}$$

Writing (3.3) in the form

$$\mathcal{L}_\lambda(x) = x\mathrm{I}_n + \sum_{i,j=1}^{n} e_{ij}\mathcal{E}_{ji}\,, \tag{3.6}$$

we note that the RTT relation (1.1) implies that $\{\mathcal{E}_{ij}\}_{i,j=1}^n$ satisfy the $\mathfrak{gl}_n$ commutation relations:

$$[\mathcal{E}_{ij}, \mathcal{E}_{k\ell}] = \delta_j^k \mathcal{E}_{i\ell} - \delta_\ell^i \mathcal{E}_{kj}\,. \tag{3.7}$$

Let us consider the standard Fock module F of $\mathcal{A}$, generated by the Fock vacuum $|0\rangle \in \mathsf{F}$ satisfying

$$\mathbf{a}_{j,i}|0\rangle = 0\,, \qquad 1 \leq i < j \leq n\,. \tag{3.8}$$

Thus, F has a basis obtained by the action of the pairwise commuting *creation* operators on $|0\rangle$:

$$|\vec{m}\,\rangle = \prod_{1\leq i<j\leq n} \bar{\mathbf{a}}_{i,j}^{m_{i,j}}\,|0\rangle\,, \qquad \forall\, \vec{m} = (m_{i,j})_{1\leq i<j\leq n} \in \mathbb{N}^{\frac{n(n-1)}{2}}\,. \tag{3.9}$$

We shall use $\langle\vec{m}|$ to denote the dual basis of F^*, so that

$$\langle\vec{k}|X|\vec{m}\rangle = \langle\vec{k}|\Big(X|\vec{m}\rangle\Big) \tag{3.10}$$

denotes the $(|\vec{k}\rangle, |\vec{m}\rangle)$-matrix coefficient of any linear operator X acting on the Fock space F.

By straightforward computation, we find:

$$\mathcal{E}_{ii} = \lambda_i + \sum_{k<i} \bar{\mathbf{a}}_{k,i}\mathbf{a}_{i,k} - \sum_{k>i} \bar{\mathbf{a}}_{i,k}\mathbf{a}_{k,i}\,, \qquad 1 \le i \le n\,, \tag{3.11}$$

$$\mathcal{E}_{ij} = -\mathbf{a}_{j,i} + \sum_{k<i} \bar{\mathbf{a}}_{k,i}\mathbf{a}_{j,k}\,, \qquad 1 \le i < j \le n\,. \tag{3.12}$$

Hence, the Fock vacuum $|0\rangle$ is a highest weight state of the resulting $\mathfrak{gl}_n$-action:

$$\mathcal{E}_{ij}|0\rangle = 0\,, \qquad 1 \le i < j \le n\,, \tag{3.13}$$

with the highest weight λ, that is:

$$\mathcal{E}_{ii}|0\rangle = \lambda_i|0\rangle\,, \qquad 1 \le i \le n\,. \tag{3.14}$$

We can now identify the resulting $\mathfrak{gl}_n$-modules F with those featuring in the Introduction:

Lemma 3.15. *There is a $\mathfrak{gl}_n$-module isomorphism:*

$$\mathsf{F} \simeq M_\lambda^\vee\,, \tag{3.16}$$

identifying F with the restricted dual (2.19) *of the highest weight Verma module M_λ.*

Proof. Since the restricted dual $\mathsf{F}^\vee$ has the same highest weight and the character as M_λ, it suffices to prove that it is also a highest weight $\mathfrak{gl}_n$-module, i.e. generated by its highest weight vector $\langle 0|$. To this end, we note that the formula (3.12) implies that $\langle \vec{m}|$ is in fact a non-zero multiple of

$$(\mathcal{E}_{12}^*)^{m_{1,2}} \cdots (\mathcal{E}_{1n}^*)^{m_{1,n}} (\mathcal{E}_{23}^*)^{m_{2,3}} \cdots (\mathcal{E}_{2n}^*)^{m_{2,n}} \cdots (\mathcal{E}_{n-1,n}^*)^{m_{n-1,n}} \langle 0|$$

for any $\vec{m} = (m_{i,j})_{1\le i<j\le n} \in \mathbb{N}^{\frac{n(n-1)}{2}}$, where $\mathcal{E}_{ij}^* \in \mathrm{End}(\mathsf{F}^*)$ is the dual of the $\mathcal{E}_{ij}$-action on F. □

Combining this with the determinant formula of [J] and the isomorphism (2.21), we obtain:

Corollary 3.17. *(a) The Fock space F is irreducible as a $\mathfrak{gl}_n$-module if and only if*

$$\lambda_i - \lambda_j \notin i - j + \mathbb{Z}_{>0}\,, \qquad \forall\, 1 \le i < j \le n\,. \tag{3.18}$$

(b) The Fock vacuum $|0\rangle$ generates an irreducible finite-dimensional $\mathfrak{gl}_n$-module L_λ if and only if

$$\lambda \in P^+ = \Big\{\mu \in \mathbb{C}^n \,|\, \mu_i - \mu_{i+1} \in \mathbb{N} \;\; \forall\, 1 \le i < n\Big\}\,. \tag{3.19}$$

3.2. **Transfer matrices.**

Recall the notion of transfer matrices $\{T_W(x)\}_{W\in \mathrm{Rep}\, Y(\mathfrak{gl}_n)}$, as discussed in Subsection 1.8. In particular, we shall consider the following explicit infinite-dimensional transfer matrices:

$$T_\lambda^+(x) = \mathrm{tr} \prod_{i=1}^n \tau_i^{\mathcal{E}_{ii}} \underbrace{\mathcal{L}_\lambda(x) \otimes \cdots \otimes \mathcal{L}_\lambda(x)}_{N}\,. \tag{3.20}$$

Here, we use the N-fold tensor product and the trace is taken over the entire Fock space (3.9, 3.10):

$$\mathrm{tr}(X) = \sum_{\vec{m}} \langle \vec{m}|X|\vec{m}\rangle\,. \tag{3.21}$$

Remark 3.22. The twist parameters $\tau_i \in \mathbb{C}$ lift the degeneracies in the spectrum, i.e. break the $\mathfrak{sl}_n$ invariance of the transfer matrix, and regularize the infinite-dimensional trace.

For a dominant integral $\lambda \in P^+$, see (3.19), we also consider the finite-dimensional transfer matrices $T_\lambda(x)$ corresponding to the modules L_λ in the auxiliary space: those are defined similarly to (3.20), but with the trace taken over the finite-dimensional submodule L_λ of F, see Corollary 3.17(b).

Recall the dot action (1.14) of S_n on $\mathbb{C}^n$, cf. (1.11):

$$\sigma \cdot \lambda = \sigma(\lambda + \rho) - \rho\,, \qquad \rho = \Big(\tfrac{n-1}{2}, \tfrac{n-3}{2}, \ldots, \tfrac{1-n}{2}\Big) \in \mathbb{C}^n\,. \tag{3.23}$$

Then, according to [BFLMS, DM3], while for low ranks it goes back to [BLZ, BHK], we have:

Theorem 3.24. *For $\lambda \in P^+$, we have:*

$$T_\lambda(x) = \sum_{\sigma \in S_n} (-1)^{l(\sigma)} T^+_{\sigma\cdot\lambda}(x)\,. \tag{3.25}$$

As recalled in Subsection 1.3, the formula (3.25) is a consequence of the BGG resolution of the finite-dimensional $\mathfrak{gl}_n$-module L_λ by means of the infinite-dimensional dual Verma $\mathfrak{gl}_n$-modules:

$$0 \to L_\lambda \to M^\vee_\lambda \to \bigoplus_{\sigma\in S_n}^{l(\sigma)=1} M^\vee_{\sigma\cdot\lambda} \to \bigoplus_{\sigma\in S_n}^{l(\sigma)=2} M^\vee_{\sigma\cdot\lambda} \to \cdots \to M^\vee_{(\lambda_n+1-n,\lambda_{n-1}+3-n,\ldots,\lambda_1+n-1)} \to 0\,, \tag{3.26}$$

cf. (2.19, 2.55). We note that the character limit of (3.25), corresponding to its special case with $N = 0$ (zero length of the spin chain), recovers the classical Weyl character formula:

$$\mathrm{ch}_{L_\lambda} = \sum_{\sigma\in S_n} (-1)^{l(\sigma)} \frac{e^{\sigma(\lambda+\rho)-\rho}}{\prod_{\alpha\in\Delta^+}(1-e^{-\alpha})} = \sum_{\sigma\in S_n} (-1)^{l(\sigma)}\, \mathrm{ch}_{M^\vee_{\sigma\cdot\lambda}}\,, \tag{3.27}$$

where Δ^+ denotes the set of positive roots of $\mathfrak{g} = \mathfrak{sl}_n$. In particular, identifying simple roots with $\alpha_i = \epsilon_i - \epsilon_{i+1}$ in the standard way, we get $\Delta^+ = \{\epsilon_i - \epsilon_j\}_{1\le i<j\le n}$ and (3.23) agrees with (1.15).

Remark 3.28. For the physics' reader, let us explain how (3.27) fits into the rest of our notation. Consider a general Cartan element $h = \sum_{i=1}^n x_i \epsilon_i^* \in \mathfrak{h} \subset \mathfrak{gl}_n$ (here, one can think of $x_i \in \mathbb{C}$ or as a formal parameter), and let $\tau_i = e^{x_i}$. Then, e^h and $\prod_{i=1}^n \tau_i^{\mathcal{E}_{ii}}$ agree and the formula (3.27) reads:

$$\mathrm{ch}_{L_\lambda}(e^h) = \mathrm{tr}_{L_\lambda} \prod_{1\le i\le n} \tau_i^{\mathcal{E}_{ii}} = \sum_{\sigma\in S_n} (-1)^{l(\sigma)} \frac{e^{(\sigma(\lambda+\rho)-\rho,h)}}{\prod_{i<j}(1-\frac{\tau_j}{\tau_i})} = \sum_{\sigma\in S_n} (-1)^{l(\sigma)}\, \mathrm{ch}_{M^\vee_{\sigma\cdot\lambda}}(e^h) \tag{3.29}$$

with

$$\mathrm{ch}_{M^\vee_\lambda}(e^h) = \mathrm{tr}_{M^\vee_\lambda} \prod_{1\le i\le n} \tau_i^{\mathcal{E}_{ii}} = \prod_{1\le i\le n} \tau_i^{\lambda_i} \prod_{1\le i<j\le n} \frac{1}{1-\frac{\tau_j}{\tau_i}}\,, \tag{3.30}$$

where the left-hand sides denote the traces of e^h (invertible diagonal element of GL_n) on $L_\lambda, M^\vee_\lambda$.

3.3. **Factorisation.**

The factorisation formula for the transfer matrices $T^+_\lambda(x)$ of the restricted dual of Verma modules, cf. (2.19) and Lemma 3.15, was proven in [BFLMS]. It was further combined with the BGG-relation (3.25) to derive the determinant expression for the finite-dimensional transfer matrices $T_\lambda(x)$. We review these constructions in the present subsection.

Following [BFLMS, (1.16)], let us consider the following $\mathfrak{gl}_n$-type $\mathcal{A}[x]$-valued Lax matrices:

$$L_i(x) = \begin{pmatrix} 1 & & & & \bar{\mathbf{a}}_{i,1} & & & \\ & \ddots & & & \vdots & & & \\ & & 1 & & \bar{\mathbf{a}}_{i,i-1} & & & \\ \mathbf{a}_{1,i} & \cdots & \mathbf{a}_{i-1,i} & x+\sum_{j=1}^{i-1}\mathbf{a}_{j,i}\bar{\mathbf{a}}_{i,j} - \sum_{j=i+1}^n \bar{\mathbf{a}}_{i,j}\mathbf{a}_{j,i} & \bar{\mathbf{a}}_{i,i+1} & \cdots & \bar{\mathbf{a}}_{i,n} \\ & & & -\mathbf{a}_{i+1,i} & 1 & & \\ & & & \vdots & & \ddots & \\ & & & -\mathbf{a}_{n,i} & & & 1 \end{pmatrix} \tag{3.31}$$

with $1 \le i \le n$ and the oscillators $\{\mathbf{a}_{j,i}, \bar{\mathbf{a}}_{i,j}\}_{j\ne i}$ subject to the standard commutation relations (3.1).

The following factorisation formula has been shown in [BFLMS, Appendix B]:

$$L_1(x+\ell_1)L_2(x+\ell_2)\cdots L_n(x+\ell_n) = \mathbf{S}\mathcal{L}_\lambda(x)\mathrm{G}\mathbf{S}^{-1}\,, \tag{3.32}$$

where $\mathcal{L}_\lambda(x)$ is the Lax matrix of (3.3) and the shifted weights $\{\ell_i\}_{i=1}^n$ are defined via:

$$\ell_i = \lambda_i - i + 1\,, \qquad 1 \le i \le n\,. \tag{3.33}$$

Here, the similarity transformation $\mathbf{S} = \mathbf{S}_n \cdots \mathbf{S}_1$ is defined via:

$$\mathbf{S}_i = \exp\left[\sum_{j=1}^{i-1}\left(\bar{\mathbf{a}}_{ji} - \sum_{k=j+1}^{i-1}\bar{\mathbf{a}}_{ki}\bar{\mathbf{a}}_{kj}\right)\mathbf{a}_{ji}\right], \tag{3.34}$$

while the matrix G reads:

$$\mathrm{G} = \begin{pmatrix} 1 & -\bar{\mathbf{a}}_{2,1} & \cdots & -\bar{\mathbf{a}}_{n,1} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & -\bar{\mathbf{a}}_{n,n-1} \\ 0 & \cdots & 0 & 1 \end{pmatrix}^{-1}. \tag{3.35}$$

As explained below, the factorisation formula (3.32) can be lifted to the level of transfer matrices. To this end, let us first define single-index Q-operators $\{Q_i(x)\}_{i=1}^n \subset \mathrm{End}(\mathbb{C}^n)^{\otimes N}$ via:

$$Q_i(x) = \widehat{\mathrm{tr}}_{D_i}\Big(\underbrace{L_i(x)\otimes\cdots\otimes L_i(x)}_{N}\Big), \tag{3.36}$$

where we use the *normalized trace* $\widehat{\mathrm{tr}}_{D_i}$ defined through:

$$\widehat{\mathrm{tr}}_{D_i}(X) = \frac{\mathrm{tr}(D_i X)}{\mathrm{tr}(D_i)}, \tag{3.37}$$

cf. (3.21). The twist D_i in (3.36) acts only on the Fock space and is defined via:

$$D_i = \prod_{1\le j<i}\left(\frac{\tau_i}{\tau_j}\right)^{\mathbf{a}_{ji}\bar{\mathbf{a}}_{ij}} \prod_{i<j\le n}\left(\frac{\tau_j}{\tau_i}\right)^{\bar{\mathbf{a}}_{ij}\mathbf{a}_{ji}}. \tag{3.38}$$

We note that the action of this twist on the Fock module is uniquely determined (up to a scalar function) by the following condition:

$$DL_i(x)D^{-1} = D_i^{-1}L_i(x)D_i, \tag{3.39}$$

with

$$D = \mathrm{diag}(\tau_1,\ldots,\tau_n). \tag{3.40}$$

The relation (3.39) ensures the commutativity of $Q_i(x)$ defined via (3.36) and the transfer matrix $T_{(1,0,\ldots,0)}(y)$ of the defining fundamental representation in the auxiliary space, see Remark 3.67(b).

Next, we consider the N-fold tensor product on the matrix space of the factorisation formula (3.32). Taking the normalized traces of each of the resulting monodromies $L_i(x+\ell_i)\otimes\cdots\otimes L_i(x+\ell_i)$ on the left-hand side of the corresponding relation yields a product of the Q-operators. On the right-hand side, we recover the transfer matrix $T_\lambda^+(x)$ multiplied by the inverse of the character

$$\mathrm{ch}_\lambda^+ = \mathrm{tr}\prod_{1\le i\le n}\tau_i^{\mathcal{E}_{ii}} = \prod_{1\le i\le n}\tau_i^{\lambda_i}\prod_{1\le i<j\le n}\frac{\tau_i}{\tau_i-\tau_j} = \prod_{1\le i\le n}\tau_i^{\ell_i}\prod_{1\le i<j\le n}\frac{1}{\tau_j^{-1}-\tau_i^{-1}}. \tag{3.41}$$

Remark 3.42. The above computation requires a few steps. First, let us note the following relation among the twists of the Q-operators and the transfer matrices:

$$\prod_{i=1}^n D_i = \prod_{1\le i\le n}\tau_i^{\mathcal{E}_{ii}-\lambda_i}\prod_{1\le j<i\le n}\left(\frac{\tau_j}{\tau_i}\right)^{-\mathbf{a}_{ji}\bar{\mathbf{a}}_{ij}}. \tag{3.43}$$

Second, we note that apart from its diagonal the matrix G in (3.35) only contains creation operators which do not contribute to the trace. Finally, we need the commutativity (3.59) established below.

Thus, we finally obtain the factorisation formula for $T^+_\lambda(x)$, cf (1.10):

$$T^+_\lambda(x) = \mathrm{ch}^+_\lambda \cdot Q_1(x+\ell_1)Q_2(x+\ell_2)\cdots Q_n(x+\ell_n)\,. \tag{3.44}$$

Combining (3.44) with the BGG-relation (3.25) and evoking the Vandermonde determinant:

$$\prod_{1\le i<j\le n}\left(\tau_j^{-1}-\tau_i^{-1}\right) = \det\left\|\tau_i^{-j+1}\right\|_{1\le i,j\le n}\,, \tag{3.45}$$

we get the determinant formula for $T_\lambda(x)$ in type A, cf. (1.12):

Theorem 3.46. *For $\lambda\in P^+$, we have:*

$$T_\lambda(x) = \frac{\det\left\|\tau_i^{\ell_j}Q_i(x+\ell_j)\right\|_{1\le i,j\le n}}{\det\left\|\tau_i^{-j+1}\right\|_{1\le i,j\le n}}\,. \tag{3.47}$$

Remark 3.48. The *partonic* Lax matrices $\mathsf{L}_i(x)$ of [BFLMS, (1.16)] are related to ours (3.31) via:

$$\mathsf{L}_i(x) = L_i\left(x-\tfrac{n-1}{2}\right) \tag{3.49}$$

upon the following identification of the oscillators $\{\mathbf{b}^\dagger_{j,i},\mathbf{b}_{i,j}\}_{1\le i\ne j\le n}$ of *loc.cit.* with ours:

$$\mathbf{b}^\dagger_{i,j} = \begin{cases}\bar{\mathbf{a}}_{i,j} & \text{for}\quad i<j\\ \mathbf{a}_{j,i} & \text{for}\quad i>j\end{cases}\,,\qquad \mathbf{b}_{i,j} = \begin{cases}-\bar{\mathbf{a}}_{j,i} & \text{for}\quad i<j\\ \mathbf{a}_{i,j} & \text{for}\quad i>j\end{cases}\,. \tag{3.50}$$

To continue this comparison, we note that the Q-operators $\{\mathsf{Q}_i(x)\}_{i=1}^n$ of [BFLMS] slightly differ from ours. Explicitly, to simplify the functional relations an extra factor τ_i^x has been introduced in [BFLMS, (4.23)], so that the Q-operators of [BFLMS] are related to ours (3.36) via:

$$\mathsf{Q}_i(x) = \tau_i^x\cdot Q_i\left(x-\tfrac{n-1}{2}\right)\,. \tag{3.51}$$

Here, the exponential twist parameters of [BFLMS] are related to our conventions simply through

$$\tau_a = e^{\mathrm{i}\Phi_a}\,,\qquad 1\le a\le n\,. \tag{3.52}$$

Likewise, the T-operators $\mathsf{T}^+_\lambda(x)$ and $\mathsf{T}_\lambda(x)$ of [BFLMS, (4.15, 4.16)] are related to ours via:

$$\mathsf{T}^+_\lambda(x) = \prod_{i=1}^n \tau_i^x\cdot T^+_\lambda(x)\,,\qquad \mathsf{T}_\lambda(x) = \prod_{i=1}^n \tau_i^x\cdot T_\lambda(x)\,, \tag{3.53}$$

thus differing by an overall factor $\prod_{i=1}^n\tau_i^x$. However, we note that the constraint $\sum_{a=1}^n\Phi_a=0$ was imposed in [BFLMS, (4.4)], thus translating into $\prod_{i=1}^n\tau_i=1$, see (3.52). Finally, our determinant formula (3.47) is equivalent to the determinant formula of [BFLMS, (5.10)], see (1.12):

$$\Delta_{\{1,\dots,n\}}\cdot\mathsf{T}_\lambda(x) = \det\left\|\mathsf{Q}_i(x+\lambda'_j)\right\|_{1\le i,j\le n}\,, \tag{3.54}$$

where

$$\Delta_{\{1,\dots,n\}} = \det\left\|\tau_i^{-j+1}\right\|_{1\le i,j\le n} = \det\left\|\tau_i^{-j+1}\right\|_{1\le i,j\le n}\cdot\prod_{i=1}^n\tau_i^{\frac{n-1}{2}} = \prod_{1\le i<j\le n}\frac{\tau_i-\tau_j}{\sqrt{\tau_i\tau_j}} \tag{3.55}$$

and

$$\lambda'_i = \ell_i+\frac{n-1}{2} = \lambda_i+\frac{n+1-2i}{2}\,,\qquad 1\le i\le n\,, \tag{3.56}$$

cf. [BFLMS, (3.19, 5.3)]. Indeed, identifying our twist parameters τ_a with the Φ_a of [BFLMS] via (3.52), we get precisely the formulas of [BFLMS], due to:

$$\sqrt{\frac{\tau_a}{\tau_b}}-\sqrt{\frac{\tau_b}{\tau_a}} = 2\mathrm{i}\sin\left(\frac{\Phi_a-\Phi_b}{2}\right)\,. \tag{3.57}$$

Remark 3.58. An essential step used in the derivation of (3.44) is the following commutativity:

$$\Big[\mathbf{S}, \prod_{i=1}^{n} D_i\Big] = 0\,. \tag{3.59}$$

As the proof of this result was missing in [BFLMS], we provide the corresponding argument below. To this end, it is more convenient to switch to the oscillators $(\mathbf{b}_{i,j}, \mathbf{b}^{\dagger}_{j,i})_{i\neq j}$ of [BFLMS], related to our $(\mathbf{a}_{i,j}, \bar{\mathbf{a}}_{j,i})_{i\neq j}$ via (3.50). With this choice of conventions, the twist D_i of (3.38) reads:

$$D_i = \prod_{j\neq i} \Big(\frac{\tau_j}{\tau_i}\Big)^{\mathbf{b}^{\dagger}_{ij}\mathbf{b}_{ji}}\,, \tag{3.60}$$

so that the product of twists in (3.59) simplifies to:

$$\prod_{i=1}^{n} D_i = \prod_{i=1}^{n} \tau_i^{\mathbf{N}_i}\,, \tag{3.61}$$

where $\mathbf{N}_i$ are defined via:

$$\mathbf{N}_i = \sum_{j\neq i} \Big(\mathbf{b}^{\dagger}_{ji}\mathbf{b}_{ij} - \mathbf{b}^{\dagger}_{ij}\mathbf{b}_{ji}\Big)\,. \tag{3.62}$$

Likewise, the similarity transformation $\mathbf{S}_i$ of (3.34) reads:

$$\mathbf{S}_i = \exp\left[\sum_{1\leq j<i} \mathbf{b}^{\dagger}_{ji}\mathbf{b}^{\dagger}_{ij} + \sum_{1\leq j<k<i} \mathbf{b}^{\dagger}_{ki}\mathbf{b}_{jk}\mathbf{b}^{\dagger}_{ij}\right]\,, \tag{3.63}$$

which can be further simplified to:

$$\mathbf{S}_i = \prod_{1\leq j<i} \exp\Big[\mathbf{b}^{\dagger}_{ij}\mathbf{b}^{\dagger}_{ji}\Big] \prod_{1\leq j<k<i} \exp\Big[\mathbf{b}^{\dagger}_{ij}\mathbf{b}_{jk}\mathbf{b}^{\dagger}_{ki}\Big]\,, \tag{3.64}$$

since the oscillators $\mathbf{b}^{\dagger}_{\bullet,\bullet}$ in (3.63) always have one of $\bullet$ equal to i, while the $\mathbf{b}_{\bullet,\bullet}$'s never have. Therefore, to prove (3.59) it suffices to verify that:

$$[\mathbf{N}_i, \mathbf{b}^{\dagger}_{k\ell}\mathbf{b}^{\dagger}_{\ell k}] = 0\,, \qquad [\mathbf{N}_i, \mathbf{b}^{\dagger}_{k\ell}\mathbf{b}_{\ell m}\mathbf{b}^{\dagger}_{mk}] = 0 \tag{3.65}$$

for any $1 \leq i \leq n$ and $1 \leq \ell < m < k \leq n$. These equalities follow immediately from the fact that $[\mathbf{N}_i, -]$ acts on a given state by counting the number of creation and annihilation operators via:

$$\sum_{j\neq i} \Big[(\#\mathbf{b}^{\dagger}_{ji} - \#\mathbf{b}_{ij}) - (\#\mathbf{b}^{\dagger}_{ij} - \#\mathbf{b}_{ji})\Big]\,. \tag{3.66}$$

Remark 3.67. Let us conclude with the commutativity of $\{T^{+}_{\lambda}(x)\}_{\lambda\in\mathbb{C}^n}$, $\{T_{\mu}(x)\}_{\mu\in P^{+}}$, $\{Q_i(x)\}_{i=1}^{n}$.

(a) The commutativity

$$[T^{+}_{\lambda}(x), T^{+}_{\mu}(y)] = 0\,, \qquad \lambda, \mu \in \mathbb{C}^n\,, \tag{3.68}$$

is a direct consequence of their realization through the universal R-matrix as outlined in Section 1.8. To this end, we also would like to point out that an explicit form of the R-matrix intertwining the monodromies of these infinite-dimensional transfer matrices was obtained in [DM2, (2.28)]. By a direct application of Theorem 3.24 (or, alternatively, the construction of Section 1.8), we also get:

$$[T^{+}_{\lambda}(x), T_{\mu}(y)] = 0\,, \qquad [T_{\nu}(x), T_{\mu}(y)] = 0\,, \qquad \lambda \in \mathbb{C}^n\,,\ \mu, \nu \in P^{+}\,. \tag{3.69}$$

(b) The commutativity of the above transfer matrices with all single-index Q-operators:

$$[T^{+}_{\lambda}(x), Q_i(y)] = 0\,, \qquad [T_{\mu}(x), Q_i(y)] = 0\,, \qquad 1 \leq i \leq n\,,\ \lambda \in \mathbb{C}^n\,,\ \mu \in P^{+}\,, \tag{3.70}$$

follows from the R-matrix of [FLMS, (2.15)] intertwining the monodromies of all non-degenerate Lax matrices and the degenerate one of (3.31). Nonetheless, let us present a self-contained proof of

$$[Q_i(x), T_{(1,0,\ldots,0)}(y)] = 0\,, \qquad 1 \leq i \leq n\,, \tag{3.71}$$

in order to emphasize the role of the relation (3.39), as promised after (3.40). To this end, combining the RTT relation (1.1) for $L_i(x)$ of (3.31) with the identity $R(x)R(-x) = (1-x^2)\mathrm{I}_n$, we obtain:

$$L_i(x-y) \otimes L_i(x)R(y) = R(y)L_i(x) \otimes L_i(x-y)\,. \tag{3.72}$$

Building further the monodromy matrices

$$M_i(x) = \underbrace{L_i(x) \otimes \cdots \otimes L_i(x)}_{N}\,, \qquad M_a(y) = R_{a1}(y)\cdots R_{aN}(y)\,, \tag{3.73}$$

we end up with the following equation:

$$L_i(x-y)M_i(x)M_a(y) = M_a(y)M_i(x)L_i(x-y)\,, \tag{3.74}$$

where $L_i(x-y)$ acts nontrivially on the auxiliary spaces of these two monodromies: the monodromy M_i built from the oscillators and the monodromy M_a of the fundamental transfer matrix with an n-dimensional auxiliary space denoted by a. Multiplying (3.74) by the twists D_i and D on the left and taking the trace, we end up precisely with (3.71) (we should note that to move both twists past $L_i(x)$ in the left-hand side of (3.74), we use the commutativity $[D_iD, L_i(x)] = 0$ of (3.39)).

(c) Finally, to establish the commutativity among the Q-operators $Q_i(x)$, it is convenient to realize them as the *normalized limits* of the transfer matrices $T^+_\lambda(x)$ in which some of the representation labels tend to infinity. On the level of the corresponding Lax matrices, the relevant limit is:

$$\lim_{t\to\infty}\left\{\mathrm{diag}(\underbrace{t^{-1},\dots,t^{-1}}_{n-1},-1)\mathcal{L}_{(\underbrace{t,\dots,t}_{n-1},0)}(x)\,\mathrm{diag}(\underbrace{1,\dots,1}_{n-1},-1)\right\} =$$

$$\left(\begin{array}{c|c} U_{n-1}^{-1} & 0 \\ \hline \mathbf{A}_{n-1} & -x+n-1 \end{array}\right)\left(\begin{array}{c|c} U_{n-1} & -\bar{\mathbf{A}}_{n-1} \\ \hline 0 & 1 \end{array}\right)\left(\begin{array}{c|c} \mathrm{I}_{n-1} & 0 \\ \hline 0 & -1 \end{array}\right) =$$

$$\left(\begin{array}{c|c} \mathrm{I}_{n-1} & U_{n-1}^{-1}\bar{\mathbf{A}}_{n-1} \\ \hline \mathbf{A}_{n-1}U_{n-1} & x-n+1+\mathbf{A}_{n-1}\bar{\mathbf{A}}_{n-1} \end{array}\right) = \mathbf{S}L'_n(x-n+1)\mathbf{S}^{-1}\,,$$

where $\mathbf{A}_{n-1} = (\mathbf{a}_{n,1},\dots,\mathbf{a}_{n,n-1})$, $\bar{\mathbf{A}}_{n-1} = (\bar{\mathbf{a}}_{1,n},\dots,\bar{\mathbf{a}}_{n-1,n})^T$, U_i denotes the upper-left $i\times i$ block of the matrix $U = U_n$ from (3.4), and $L'_n(x)$ is the $\mathfrak{gl}_n$-type Lax matrix obtained from $L_n(x)$ of (3.31) by relabelling $\mathbf{a}_{k,n} \mapsto \mathbf{a}_{n,k}$ and $\bar{\mathbf{a}}_{n,k} \mapsto \bar{\mathbf{a}}_{k,n}$ for $1 \le k < n$. Here, the similarity transformation $\mathbf{S}$ is defined via:

$$\mathbf{S} = \mathbf{S}_{n-1}\cdots\mathbf{S}_1\,, \qquad \mathbf{S}_k = \exp\left[\bar{\mathbf{a}}_{kn}\sum_{i=1}^{k-1}\mathbf{a}_{ni}\bar{\mathbf{a}}_{ik}\right]\,. \tag{3.75}$$

This implies $[Q_n(x), Q_n(y)] = 0$ and combining this with the action of the Weyl group, we obtain:

$$[Q_i(x), Q_i(y)] = 0\,, \qquad 1 \le i \le n\,. \tag{3.76}$$

Finally, the proof of

$$[Q_i(x), Q_j(y)] = 0\,, \qquad 1 \le i \ne j \le n\,, \tag{3.77}$$

follows immediately from the factorisation of the X-operators from [BFLMS, §4-5], see (8.46).

4. A-type: rectangular

Let us now consider A-type Lax matrices corresponding to *rectangular* representations, i.e. those whose highest weight is a multiple of a fundamental weight. These play a special role because their transfer matrices are related by the Hirota equation [KNS]. But also, as we will see below, they will be relevant to our approach to the study of transfer matrices in other classical types.

4.1. **Oscillator realization in type A (parabolic Verma).**

For any $n \in \mathbb{Z}_{\geq 2}$ and $1 \leq a \leq n-1$, let $\mathcal{A}$ denote the oscillator algebra generated by $a(n-a)$ pairs of oscillators $\{(\mathbf{a}_{j,i}, \bar{\mathbf{a}}_{i,j})\}_{1\leq i\leq a}^{a<j\leq n}$ subject to the defining relations (3.1):

$$\mathcal{A} = \mathbb{C}\Big\langle \mathbf{a}_{j,i}\,,\, \bar{\mathbf{a}}_{i,j} \Big\rangle_{1\leq i\leq a}^{a<j\leq n} \Big/ (3.1)\,. \tag{4.1}$$

Following [BFLMS], let us consider the $\mathfrak{gl}_n$-type $\mathcal{A}[x]$-valued Lax matrix (depending on $t \in \mathbb{C}$):

$$\mathcal{L}_a(x) = \left(\begin{array}{c|c} (x+t)\mathrm{I}_a - \bar{\mathbf{A}}\mathbf{A} & -\bar{\mathbf{A}}(t+a-\mathbf{A}\bar{\mathbf{A}}) \\ \hline -\mathbf{A} & (x-a)\mathrm{I}_{n-a} + \mathbf{A}\bar{\mathbf{A}} \end{array}\right) \tag{4.2}$$

with the blocks $\bar{\mathbf{A}} \in \mathrm{Mat}_{a\times(n-a)}(\mathcal{A})$ and $\mathbf{A} \in \mathrm{Mat}_{(n-a)\times a}(\mathcal{A})$ encoding all the generators via:

$$\bar{\mathbf{A}} = \begin{pmatrix} \bar{\mathbf{a}}_{1,a+1} & \cdots & \bar{\mathbf{a}}_{1,n} \\ \vdots & \iddots & \vdots \\ \bar{\mathbf{a}}_{a,a+1} & \cdots & \bar{\mathbf{a}}_{a,n} \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} \mathbf{a}_{a+1,1} & \cdots & \mathbf{a}_{a+1,a} \\ \vdots & \iddots & \vdots \\ \mathbf{a}_{n,1} & \cdots & \mathbf{a}_{n,a} \end{pmatrix}. \tag{4.3}$$

Writing (4.2) in the form

$$\mathcal{L}_a(x) = x\mathrm{I}_n + \sum_{i,j=1}^{n} e_{ij}\mathcal{E}_{ji}\,, \tag{4.4}$$

we note that $\{\mathcal{E}_{ij}\}_{i,j=1}^n$ satisfy the $\mathfrak{gl}_n$ commutation relations (3.7), as a consequence of RTT (1.1). As before, let F denote the Fock module of $\mathcal{A}$, generated by the Fock vacuum $|0\rangle \in \mathsf{F}$ satisfying:

$$\mathbf{a}_{j,i}|0\rangle = 0\,, \qquad 1 \leq i \leq a < j \leq n\,. \tag{4.5}$$

Then, the Fock vacuum $|0\rangle$ is a highest weight state of the resulting $\mathfrak{gl}_n$-action:

$$\mathcal{E}_{ij}|0\rangle = 0\,, \qquad 1 \leq i < j \leq n\,, \tag{4.6}$$

with the highest weight $\lambda = t\omega_a = (\underbrace{t,\ldots,t}_{a},\underbrace{0,\ldots,0}_{n-a})$, that is:

$$\mathcal{E}_{kk}|0\rangle = t\delta_{k\leq a}|0\rangle\,, \qquad 1 \leq k \leq n\,. \tag{4.7}$$

We can now identify the resulting $\mathfrak{gl}_n$-modules F with those featuring in the Introduction:

Lemma 4.8. *There is a $\mathfrak{gl}_n$-module isomorphism:*

$$\mathsf{F} \simeq \Big(M_{t\omega_a}^{\mathfrak{p}_{\{1,\ldots,n-1\}\setminus\{a\}}}\Big)^{\vee}, \tag{4.9}$$

identifying F with the restricted dual (2.19) *of the parabolic Verma module* (1.28)*.*

Proof. Since the restricted dual $\mathsf{F}^\vee$ has the same highest weight and the character as $M_{t\omega_a}^{\mathfrak{p}_{\{1,\ldots,n-1\}\setminus\{a\}}}$, it suffices to prove that it is also a highest weight $\mathfrak{gl}_n$-module, i.e. generated by its highest weight vector $\langle 0|$. To this end, we note that $\langle \vec{m}|$ is in fact a non-zero multiple of the image of $\langle 0|$ under the order-independent product $\prod_{1\leq i\leq a}^{a<j\leq n}(\mathcal{E}_{ij}^*)^{m_{i,j}}$ for any $\vec{m} = (m_{i,j})_{1\leq i\leq a<j\leq n} \in \mathbb{N}^{a(n-a)}$, cf. the proof of Lemma 3.15. □

Combining this with the determinant formula of [J] and the isomorphism (2.21), we obtain:

Corollary 4.10. *(a) For $t \notin \mathbb{Z}_{\geq 2-n}$, the $\mathfrak{gl}_n$-module F is irreducible (thus, is generated by $|0\rangle$).*

(b) For $t \in \mathbb{N}$, the Fock vacuum $|0\rangle$ generates an irreducible finite-dimensional $\mathfrak{gl}_n$-module $L_{t\omega_a}$.

4.2. **More oscillator realizations in type A via underlying symmetries.**

Since the R-matrix (1.2) is invariant, cf. (1.5), with respect to the natural action of the symmetric group S_n (via the standard embedding $S_n \hookrightarrow GL_n$), we can generate more solutions to the RTT relation (1.1) from the Lax matrix (4.2) above by simultaneously permuting its rows and columns. We shall further apply the (unique) particle-hole automorphism of $\mathcal{A}$ to insure that the Fock vacuum $|0\rangle \in \mathsf{F}$ remains to be a $\mathfrak{gl}_n$ highest weight state. To this end, consider the indexing set

$$\mathcal{S}_a = \Big\{ I \subseteq \{1,\dots,n\} \,|\, \#I = a \Big\} . \tag{4.11}$$

Then, we construct the following explicit $\mathfrak{gl}_n$-type $\mathcal{A}[x]$-valued Lax matrices:

$$\mathcal{L}_I(x) = B_I \mathcal{L}_a(x) B_I^{-1}\Big|_{p.h.} = x\mathrm{I}_n + \sum_{i,j=1}^{n} e_{ij}\mathcal{E}^I_{ji} , \qquad \forall\, I \in \mathcal{S}_a , \tag{4.12}$$

with the similarity matrix B_I and the particle-hole transformation (denoted $p.h.$) described below. Let $J = \bar{I}$ denote the complement of the subset I:

$$J = \bar{I} = \{1,\dots,n\} \setminus I , \tag{4.13}$$

and let us order the elements of I and J in the increasing order:

$$\begin{aligned} I &= \{i_1, i_2, \dots, i_a\} , \qquad 1 \le i_1 < i_2 < \dots < i_a \le n , \\ J &= \{j_1, j_2, \dots, j_{n-a}\} , \qquad 1 \le j_1 < j_2 < \dots < j_{n-a} \le n . \end{aligned} \tag{4.14}$$

We consider the following permutation σ_I of the set $\{1,\dots,n\}$:

$$\sigma_I(c) = \begin{cases} i_c & \text{for } 1 \le c \le a \\ j_{c-a} & \text{for } a < c \le n \end{cases} . \tag{4.15}$$

Remark 4.16. Consider the natural transitive action of the symmetric group S_n on the set $\mathcal{S}_a$ (4.11). Then, the stabilizer of $\{1,\dots,a\} \in \mathcal{S}_a$ is the subgroup $S_a \times S_{n-a} \subset S_n$ of those permutations that map $\{1,\dots,a\} \mapsto \{1,\dots,a\}$ and $\{a+1,\dots,n\} \mapsto \{a+1,\dots,n\}$. This gives rise to a set bijection

$$\pi \colon S_n/(S_a \times S_{n-a}) \xrightarrow{\sim} \mathcal{S}_a \tag{4.17}$$

satisfying the property $\sigma_I \in \pi^{-1}(I)$ with σ_I defined in (4.15). Furthermore, σ_I can be characterized as the shortest representative of the corresponding left coset $\pi^{-1}(I)$, and its length is given by:

$$l(\sigma_I) = \#\Big\{ (k,\ell) \in I \times \bar{I} \,|\, k > \ell \Big\} . \tag{4.18}$$

Then, we define B_I in (4.12) as the permutation matrix corresponding to $\sigma_I \in S_n$:

$$B_I = \sum_{i=1}^{n} e_{\sigma_I(i),i} . \tag{4.19}$$

We note that its inverse coincides with its transpose: $B_I^{-1} = B_I^T = \sum_{i=1}^n e_{i,\sigma_I(i)}$.

To determine the particle-hole transformation used in (4.12) so as to preserve the $\mathfrak{gl}_n$ highest weight state condition, let us check where the $\mathbf{a}_{\cdot,\cdot}$-generators from the lower-left block of (4.2) are moved to under the conjugation by the matrix B_I of (4.19). Explicitly, the oscillator $-\mathbf{a}_{j,i}$ $(i \le a < j)$ is located above the main diagonal of $B_I \mathcal{L}_a(x) B_I^{-1}$ if and only if $\sigma_I(j) < \sigma_I(i)$, due to the equality

$$\Big(B_I \mathcal{L}_a(x) B_I^{-1} \Big)_{\sigma_I(j),\sigma_I(i)} = (\mathcal{L}_a(x))_{ji} , \qquad 1 \le i,j \le n . \tag{4.20}$$

Therefore, we consider the following particle-hole automorphism of $\mathcal{A}$ in (4.12):

$$\bar{\mathbf{a}}_{i,j} \mapsto -\mathbf{a}_{j,i} , \quad \mathbf{a}_{j,i} \mapsto \bar{\mathbf{a}}_{i,j} \quad \text{for} \quad 1 \le i \le a < j \le n \quad \text{such that} \quad \sigma_I(j) < \sigma_I(i) . \tag{4.21}$$

The resulting matrix elements $\{\mathcal{E}^I_{ij}\}_{i,j=1}^n \subset \mathcal{A}$ of (4.12) still satisfy the $\mathfrak{gl}_n$ commutation relations (3.7), as follows from the RTT relation (1.1). This makes the Fock module F into a $\mathfrak{gl}_n$-module, denoted by $M^+_{I,t}$. The Fock vacuum $|0\rangle \in M^+_{I,t}$ is easily seen to be a $\mathfrak{gl}_n$ highest weight state:

$$\mathcal{E}^I_{ij}|0\rangle = 0 , \qquad 1 \le i < j \le n . \tag{4.22}$$

To compute its highest weight, we note that

$$\begin{aligned}
\Big(B_I\mathcal{L}_a(x)B_I^{-1}\Big)_{\sigma_I(i),\sigma_I(i)} &= x+t-\sum_{a<j\leq n}\bar{\mathbf{a}}_{ij}\mathbf{a}_{ji}\,, \qquad 1\leq i\leq a\,,\\
\Big(B_I\mathcal{L}_a(x)B_I^{-1}\Big)_{\sigma_I(j),\sigma_I(j)} &= x-a+\sum_{1\leq i\leq a}\mathbf{a}_{ji}\bar{\mathbf{a}}_{ij} = x+\sum_{1\leq i\leq a}\bar{\mathbf{a}}_{ij}\mathbf{a}_{ji}\,, \qquad a<j\leq n\,,
\end{aligned} \tag{4.23}$$

which after implementing the particle-hole transformation (4.21) gives:

$$\begin{aligned}
\mathcal{E}^I_{\sigma_I(i),\sigma_I(i)}|0\rangle &= \Big(t+\#\{a<j\leq n\,|\,\sigma_I(j)<\sigma_I(i)\}\Big)|0\rangle\,, \qquad 1\leq i\leq a\,,\\
\mathcal{E}^I_{\sigma_I(j),\sigma_I(j)}|0\rangle &= \Big(-\#\{1\leq i\leq a\,|\,\sigma_I(j)<\sigma_I(i)\}\Big)|0\rangle\,, \qquad a<j\leq n\,.
\end{aligned} \tag{4.24}$$

Evoking (4.15) and the particular ordering (4.14), we find:

$$\begin{aligned}
\#\Big\{a<j\leq n\,|\,\sigma_I(j)<\sigma_I(i)\Big\} &= \sigma_I(i)-i\,, \qquad 1\leq i\leq a\,,\\
-\#\Big\{1\leq i\leq a\,|\,\sigma_I(j)<\sigma_I(i)\Big\} &= \sigma_I(j)-j\,, \qquad a<j\leq n\,,
\end{aligned} \tag{4.25}$$

so that:

$$\mathcal{E}^I_{kk}|0\rangle = \Big(\delta_{\sigma_I^{-1}(k)\leq a}t+k-\sigma_I^{-1}(k)\Big)\,|0\rangle\,, \qquad 1\leq k\leq n\,. \tag{4.26}$$

Hence, the highest weight of the Fock vacuum $|0\rangle\in M^+_{I,t}$ is precisely

$$\sigma_I\cdot t\omega_a\,,$$

see (1.14), or equivalently:

$$\mathcal{E}^I_{kk}|0\rangle = \begin{cases}(t+\#\{j\notin I\,|\,j<k\})\,|0\rangle & \text{for } k\in I\\ (-\#\{i\in I\,|\,i>k\})\,|0\rangle & \text{for } k\notin I\end{cases}\,. \tag{4.27}$$

Thus, the highest weight of $|0\rangle\in M^+_{I,t}$ is the same as the highest weight of our key modules $M'_{\sigma_I\cdot t\omega_a}$, introduced in (1.36), with respect to the standard parabolic subalgebra $\mathfrak{p}_S\subset\mathfrak{gl}_n$ corresponding to $S=\{1,\dots,n-1\}\setminus\{a\}$. Furthermore, the module $M'_{\sigma_I\cdot t\omega_a}$ has the same character as $M^+_{I,t}$ (according to Lemma 2.45) and is irreducible for $t\notin\mathbb{Z}$ (as follows from [J]). Therefore, we obtain:

Proposition 4.28. *For any $I\in\mathcal{S}_a$ and $t\notin\mathbb{Z}$, we have $\mathfrak{gl}_n$-module isomorphisms:*

$$M^+_{I,t}\simeq M'_{\sigma_I\cdot t\omega_a}\simeq\Big(M'_{\sigma_I\cdot t\omega_a}\Big)^\vee\,. \tag{4.29}$$

Remark 4.30. Let us point out the key difference between Proposition 4.28 and Lemma 4.8:

(a) For $I=\{1,\dots,a\}\in\mathcal{S}_a$, we actually have $M^+_{I,t}\simeq(M'_{\sigma_I\cdot t\omega_a})^\vee$ for any $t\in\mathbb{C}$, due to Lemma 4.8.

(b) Likewise, for $I=\{n-a+1,\dots,n\}\in\mathcal{S}_a$, we have $M^+_{I,t}\simeq M'_{\sigma_I\cdot t\omega_a}$ for any $t\in\mathbb{C}$.

(c) For other $I\in\mathcal{S}_a$, $M^+_{I,t}$ is <u>not isomorphic</u> to either of $M'_{\sigma_I\cdot t\omega_a}$ or $(M'_{\sigma_I\cdot t\omega_a})^\vee$ at certain $t\in\mathbb{Z}$ (but is expected to be isomorphic to one of the twisted Verma modules in the sense of [AL]).

Remark 4.31. (a) The Weyl group $W_{\mathfrak{l}}$ of the Levi subalgebra $\mathfrak{l}$ of $\mathfrak{p}_S$ is precisely $S_a\times S_{n-a}\subset S_n$.

(b) We indeed have $\sigma_I\in{}^{\mathfrak{l}}W$ in the notation (1.32, 1.33), due to Remark 4.16.

(c) For any other permutation $\sigma'\in\sigma_I(S_a\times S_{n-a})$, conjugating $\mathcal{L}_a(x)$ with $B'_I=\sum_{i=1}^n e_{\sigma'(i),i}$ and applying the corresponding particle-hole transformation will produce an isomorphic $\mathfrak{gl}_n$-module.

Evoking the above bijection $\mathcal{S}_a\ni I\leftrightarrow\sigma_I\in{}^{\mathfrak{l}}W$, see (4.15) and Remarks 4.16, 4.31(b), we define:

$$M^\vee_{I,t}=\Big(M'_{\sigma_I\cdot t\omega_a}\Big)^\vee\,, \qquad \forall\,t\in\mathbb{C}\,. \tag{4.32}$$

Then, Proposition 4.28 can be recast as the isomorphism of the following $\mathfrak{gl}_n$-modules:

$$M^+_{I,t}\simeq M^\vee_{I,t}\,, \qquad \forall\,t\in\mathbb{C}\setminus\mathbb{Z}\,. \tag{4.33}$$

For $I \in \mathcal{S}_a$, we also define its *length* $l(I)$ as the length of the corresponding $\sigma_I \in S_n$, see (4.18):

$$l(I) = l(\sigma_I) = \#\left\{(k,\ell) \in I \times \bar{I} \,|\, k > \ell\right\} . \tag{4.34}$$

4.3. **Type A transfer matrices.**

Recall the notion of transfer matrices $\{T_W(x)\}_{W \in \mathrm{Rep}\, Y(\mathfrak{gl}_n)}$, as discussed in Subsection 1.8. In particular, we shall consider the following explicit infinite-dimensional transfer matrices:

$$T^+_{I,t}(x) = \mathrm{tr} \prod_{i=1}^n \tau_i^{\mathcal{E}^I_{ii}} \underbrace{\mathcal{L}_I(x) \otimes \cdots \otimes \mathcal{L}_I(x)}_{N} , \qquad \forall\, I \in \mathcal{S}_a \,,\, t \in \mathbb{C} \,, \tag{4.35}$$

corresponding to $M^+_{I,t}$. For $t \in \mathbb{N}$, we also consider the finite-dimensional transfer matrices $T_{a,t}(x)$ corresponding to the modules $L_{t\omega_a}$ in the auxiliary space: those are defined similarly to (4.35), but with the trace taken over the finite-dimensional submodule $L_{t\omega_a}$ of $M^+_{\{1,\ldots,a\},t}$, see Corollary 4.10(b).

Using the notation (4.32, 4.34), let us recast the resolution (2.55), dual to (1.30), as follows:

$$0 \to L_{t\omega_a} \to M^\vee_{\{1,\ldots,a\},t} \to \bigoplus_{I \in \mathcal{S}_a}^{l(I)=1} M^\vee_{I,t} \to \bigoplus_{I \in \mathcal{S}_a}^{l(I)=2} M^\vee_{I,t} \to \cdots \to M^\vee_{\{n-a+1,\ldots,n\},t} \to 0 \tag{4.36}$$

for any $t \in \mathbb{N}$. Combining this with (4.33) and the fact that the transfer matrices (4.35) depend continuously on $t \in \mathbb{C}$ (as so do the Lax matrices $\mathcal{L}_I(x)$), we obtain the key result of this section:

Theorem 4.37. *For $1 \le a < n$ and $t \in \mathbb{N}$, we have:*

$$T_{a,t}(x) \;=\; \sum_{I \in \mathcal{S}_a} (-1)^{l(I)}\, T^+_{I,t}(x) \,. \tag{4.38}$$

The character limit of (4.38) expresses the character of the $\mathfrak{gl}_n$-modules $\{L_{t\omega_a}\}^{t\in\mathbb{N}}_{1\le a<n}$ defined as

$$\mathrm{ch}_{a,t} = \mathrm{ch}_{a,t}(\tau_1,\ldots,\tau_n) := \mathrm{tr}_{L_{t\omega_a}} \prod_{i=1}^n \tau_i^{\mathcal{E}_{ii}} \,, \tag{4.39}$$

that is the length $N=0$ case of $T_{a,t}(x)$, via:

$$\mathrm{ch}_{a,t} \;=\; \sum_{I \in \mathcal{S}_a} (-1)^{l(I)} \frac{\prod_{k\in I} \tau_k^{t+\#\{\ell\notin I|\ell<k\}} \prod_{\ell\notin I} \tau_\ell^{-\#\{k\in I|k>\ell\}}}{\prod^{k>\ell}_{k\in I,\ell\notin I}\left(1-\frac{\tau_k}{\tau_\ell}\right)\prod^{k<\ell}_{k\in I,\ell\notin I}\left(1-\frac{\tau_\ell}{\tau_k}\right)} \tag{4.40}$$

with the I's summand in the right-hand side of (4.40) equal to the character of $M^+_{I,t}$, up to a sign.

Let us note right away that formulas (4.38) and (4.40) allow to analytically continue the transfer matrices $T_{a,t}(x)$ and their particular length $N=0$ case $\mathrm{ch}_{a,t}$ of (4.39) from the discrete set $t \in \mathbb{N}$ to the entire complex plane $t \in \mathbb{C}$.

Remark 4.41. For the physics' reader who skipped Section 2, let us present a concise proof of (4.40). We shall identify the set Δ^+ of positive roots of $\mathfrak{g} = \mathfrak{gl}_n$ with $\Delta^+ = \{\epsilon_i - \epsilon_j | 1 \le i < j \le n\}$, so that $\rho = (\frac{n-1}{2}, \frac{n-3}{2}, \ldots, \frac{1-n}{2})$ in the basis $\{\epsilon_i\}_{i=1}^n$ and the Weyl group W gets identified with $W \simeq S_n$ (acting by permutations on the basis $\{\epsilon_i\}_{i=1}^n$). According to the Weyl character formula, we have:

$$\mathrm{ch}_{L_{t\omega_a}} \;=\; \sum_{\sigma \in S_n} (-1)^{l(\sigma)} \frac{e^{\sigma(t\omega_a+\rho)-\rho}}{\prod_{1\le i<j\le n}(1-e^{\epsilon_j-\epsilon_i})} \,. \tag{4.42}$$

Following Remark 4.31, let us consider the parabolic subalgebra $\mathfrak{p}_{\{1,\ldots,n-1\}\setminus\{a\}} \subset \mathfrak{g}$ with the Levi subalgebra $\mathfrak{l} \simeq \mathfrak{gl}_a \oplus \mathfrak{gl}_{n-a}$ and the Weyl group $W_\mathfrak{l} \simeq S_a \times S_{n-a} \subset S_n$. We can rewrite (4.42) as:

$$\mathrm{ch}_{L_{t\omega_a}} \;=\; \sum_{\bar\sigma \in W/W_\mathfrak{l}} \sum_{\tau \in W_\mathfrak{l}} (-1)^{l(\sigma\tau)} \frac{e^{\sigma\tau(t\omega_a+\rho)-\rho}}{\prod_{i<j}(1-e^{\epsilon_j-\epsilon_i})} \,, \tag{4.43}$$

where $\sigma \in W$ is a representative of $\bar{\sigma} \in W/W_{\mathfrak{l}}$ (the inner sum is independent of the choice of σ). The key step is to simplify the inner sum of (4.43) using the Weyl denominator formula for $\mathfrak{l}$:

$$\sum_{\tau \in W_{\mathfrak{l}}} (-1)^{l(\tau)} e^{\tau(\rho_{\mathfrak{l}})-\rho_{\mathfrak{l}}} = \prod_{\alpha \in \Delta_{\mathfrak{l}}^{+}} (1-e^{-\alpha}), \tag{4.44}$$

where $\Delta_{\mathfrak{l}}^{+} = \{\epsilon_i - \epsilon_j\}_{1\le i<j\le a} \cup \{\epsilon_i - \epsilon_j\}_{a<i<j\le n} \subset \Delta^{+}$ denotes the set of positive roots of $\mathfrak{l}$ and $\rho_{\mathfrak{l}} = \frac{1}{2}\sum_{\alpha\in\Delta_{\mathfrak{l}}^{+}} \alpha = (\frac{a-1}{2}, \dots, \frac{1-a}{2}, \frac{n-a-1}{2}, \dots, \frac{1+a-n}{2})$. As $\tau(\rho)-\rho = \tau(\rho_{\mathfrak{l}})-\rho_{\mathfrak{l}}$ for $\tau \in W_{\mathfrak{l}}$, we get:

$$\sum_{\tau \in W_{\mathfrak{l}}} (-1)^{l(\tau)} \frac{e^{\tau(t\omega_a+\rho)-\rho}}{\prod_{1\le i<j\le n}(1-e^{\epsilon_j-\epsilon_i})} = \frac{e^{t\omega_a}}{\prod_{1\le i\le a}^{a<j\le n}(1-e^{\epsilon_j-\epsilon_i})}. \tag{4.45}$$

Therefore, the inner sum of (4.43) corresponding to the trivial left coset $W_{\mathfrak{l}} \in W/W_{\mathfrak{l}}$ gives rise to the $I = \{1, \dots, a\}$'s term of (4.40). Likewise, we claim that any I's term of (4.40) precisely arises from the inner sum of (4.43) corresponding to the left coset $\sigma_I W_{\mathfrak{l}}$ with $\sigma_I \in S_n$ of (4.14, 4.15), which amounts to the proof of (4.47) below. To this end, let us apply σ_I to both sides of (4.44):

$$\sum_{\tau \in W_{\mathfrak{l}}} (-1)^{l(\sigma_I\tau)} e^{\sigma_I\tau(\rho)-\rho} = (-1)^{l(\sigma_I)} e^{\sigma_I(\rho)-\rho} \prod_{i,j\in I}^{i<j} (1-e^{\epsilon_j-\epsilon_i}) \prod_{i,j\notin I}^{i<j} (1-e^{\epsilon_j-\epsilon_i}). \tag{4.46}$$

Combining this with the straightforward formula

$$\sigma_I(\rho)-\rho = \sum_{k\in I} \#\{\ell \notin I \,|\, \ell < k\}\epsilon_k - \sum_{\ell\notin I} \#\{k \in I \,|\, k > \ell\}\epsilon_\ell,$$

we obtain the desired equality:

$$\sum_{\tau \in W_{\mathfrak{l}}} (-1)^{l(\sigma_I\tau)} \frac{e^{\sigma_I\tau(t\omega_a+\rho)-\rho}}{\prod_{i<j}(1-e^{\epsilon_j-\epsilon_i})} = (-1)^{l(I)} \frac{\prod_{k\in I} e^{(t+\#\{\ell\notin I|\ell<k\})\epsilon_k} \prod_{\ell\notin I} e^{-\#\{k\in I|k>\ell\}\epsilon_\ell}}{\prod_{k\in I,\ell\notin I}^{k>\ell}(1-e^{\epsilon_k-\epsilon_\ell}) \prod_{k\in I,\ell\notin I}^{k<\ell}(1-e^{\epsilon_\ell-\epsilon_k})}. \tag{4.47}$$

This completes our direct proof of (4.40), due to the bijection π of (4.17), see Remark 4.16 (cf. Remark 3.28 for more details as for perceiving $\mathrm{ch}_{a,t}$ of (4.39) as a specialization of $\mathrm{ch}_{L_{t\omega_a}}$).

5. Resolutions for transfer matrices of C-type

In this section, we generalize the key constructions and results of Section 4 to C-type.

5.1. Oscillator realization in type C (parabolic Verma).

Let $\mathcal{A}$ denote the oscillator algebra generated by $\frac{r(r+1)}{2}$ pairs of oscillators $\{(\mathbf{a}_{j',i}, \bar{\mathbf{a}}_{i,j'})\}_{1\le i\le j\le r}$ with $r \in \mathbb{Z}_{\ge 1}$, cf. notation (1.21), subject to the standard defining relations:

$$[\mathbf{a}_{j',i}, \bar{\mathbf{a}}_{k,\ell'}] = \delta_i^k \delta_j^\ell, \qquad [\mathbf{a}_{j',i}, \mathbf{a}_{\ell',k}] = 0, \qquad [\bar{\mathbf{a}}_{i,j'}, \bar{\mathbf{a}}_{k,\ell'}] = 0, \tag{5.1}$$

so that

$$\mathcal{A} = \mathbb{C}\Big\langle \mathbf{a}_{j',i}\,,\, \bar{\mathbf{a}}_{i,j'} \Big\rangle_{1\le i\le j\le r} \Big/ (5.1). \tag{5.2}$$

Following [FT, p. 593], see also [KK, §6.2], let us consider the C_r-type $\mathcal{A}[x]$-valued Lax matrix:

$$\mathcal{L}(x) = \left(\begin{array}{c|c} (x+t)\mathrm{I}_r - \bar{\mathbf{A}}\mathbf{A} & -\bar{\mathbf{A}}(2t+r+1-\mathbf{A}\bar{\mathbf{A}}) \\ \hline -\mathbf{A} & (x-t-r-1)\mathrm{I}_r + \mathbf{A}\bar{\mathbf{A}} \end{array}\right), \tag{5.3}$$

depending on $t \in \mathbb{C}$, with the blocks $\bar{\mathbf{A}}, \mathbf{A} \in \mathrm{Mat}_{r\times r}(\mathcal{A})$ encoding all the generators via:

$$\bar{\mathbf{A}} = \begin{pmatrix} \bar{\mathbf{a}}_{1,r'} & \cdots & \bar{\mathbf{a}}_{1,2'} & \bar{\mathbf{a}}_1 \\ \vdots & \iddots & \bar{\mathbf{a}}_2 & \bar{\mathbf{a}}_{1,2'} \\ \bar{\mathbf{a}}_{r-1,r'} & \bar{\mathbf{a}}_{r-1} & \iddots & \vdots \\ \bar{\mathbf{a}}_r & \bar{\mathbf{a}}_{r-1,r'} & \cdots & \bar{\mathbf{a}}_{1,r'} \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} \mathbf{a}_{r',1} & \cdots & \mathbf{a}_{r',r-1} & \mathbf{a}_r \\ \vdots & \iddots & \mathbf{a}_{r-1} & \mathbf{a}_{r',r-1} \\ \mathbf{a}_{2',1} & \mathbf{a}_2 & \iddots & \vdots \\ \mathbf{a}_1 & \mathbf{a}_{2',1} & \cdots & \mathbf{a}_{r',1} \end{pmatrix}, \tag{5.4}$$

where the anti-diagonal terms $\{\bar{\mathbf{a}}_i, \mathbf{a}_i\}_{i=1}^r$ are defined via:

$$\bar{\mathbf{a}}_i = 2\bar{\mathbf{a}}_{i,i'}\,, \qquad \mathbf{a}_i = \mathbf{a}_{i',i}\,. \tag{5.5}$$

Remark 5.6. The Lax matrix (5.3) is the specialization of [FT, (3.51)] at $x_1 = t,\ x_2 = -t-r-1$.

Writing (5.3) in the form

$$\mathcal{L}(x) = x\mathrm{I}_{2r} + \sum_{i,j=1}^{2r} e_{ij}\mathcal{F}_{ji}\,, \tag{5.7}$$

we note that the RTT relation (1.1) implies that $\{\mathcal{F}_{ij}\}_{i,j=1}^{2r}$ satisfy the $\mathfrak{sp}_{2r}$ commutation relations:

$$[\mathcal{F}_{ij}, \mathcal{F}_{k\ell}] = \delta_k^j \mathcal{F}_{i\ell} - \delta_i^\ell \mathcal{F}_{kj} - \epsilon_i\epsilon_j(\delta_k^{i'}\mathcal{F}_{j'\ell} - \delta_\ell^{j'}\mathcal{F}_{ki'})\,, \qquad \mathcal{F}_{ij} = -\epsilon_i\epsilon_j\mathcal{F}_{j'i'}\,, \tag{5.8}$$

with $\{\epsilon_i\}_{i=1}^{2r}$ defined as in the Introduction:

$$\epsilon_1 = \cdots = \epsilon_r = 1 \qquad \text{and} \qquad \epsilon_{r+1} = \cdots = \epsilon_{2r} = -1\,. \tag{5.9}$$

As before, let F denote the Fock module of $\mathcal{A}$, generated by the Fock vacuum $|0\rangle \in \mathsf{F}$ satisfying:

$$\mathbf{a}_{j',i}|0\rangle = 0\,, \qquad 1 \le i \le j \le r\,. \tag{5.10}$$

Then, the Fock vacuum $|0\rangle$ is a highest weight state of the resulting $\mathfrak{sp}_{2r}$-action:

$$\mathcal{F}_{ij}|0\rangle = 0\,, \qquad 1 \le i < j \le 2r\,, \tag{5.11}$$

with the highest weight $\lambda = t\omega_r = (\underbrace{t,\ldots,t}_{r})$, that is:

$$\mathcal{F}_{ii}|0\rangle = t|0\rangle\,, \qquad 1 \le i \le r\,. \tag{5.12}$$

The latter is a consequence of the following explicit formulas for any $1 \le i \le r$:

$$\begin{aligned} \mathcal{F}_{ii} &= t - 2\bar{\mathbf{a}}_{ii'}\mathbf{a}_{i'i} - \sum_{k=i+1}^{r} \bar{\mathbf{a}}_{ik'}\mathbf{a}_{k'i} - \sum_{k=1}^{i-1} \bar{\mathbf{a}}_{ki'}\mathbf{a}_{i'k}\,, \\ \mathcal{F}_{i'i'} &= -t-r-1 + 2\mathbf{a}_{i'i}\bar{\mathbf{a}}_{ii'} + \sum_{k=1}^{i-1} \mathbf{a}_{i'k}\bar{\mathbf{a}}_{ki'} + \sum_{k=i+1}^{r} \mathbf{a}_{k'i}\bar{\mathbf{a}}_{ik'} = -\mathcal{F}_{ii}\,. \end{aligned} \tag{5.13}$$

Similarly to Lemma 4.8, we can identify the resulting $\mathfrak{sp}_{2r}$-modules F as follows:

Lemma 5.14. *There is an $\mathfrak{sp}_{2r}$-module isomorphism:*

$$\mathsf{F} \simeq \left(M_{t\omega_r}^{\mathfrak{p}_{\{1,\ldots,r-1\}}}\right)^\vee\,, \tag{5.15}$$

identifying F with the restricted dual (2.19) *of the parabolic Verma module* (1.28).

Combining this with the determinant formula of [J] and the isomorphism (2.21), we obtain:

Corollary 5.16. *(a) For $t \notin \frac{1}{2}\mathbb{Z}_{\ge 2-2r}$, the $\mathfrak{sp}_{2r}$-module F is irreducible (thus, is generated by $|0\rangle$).*

(b) For $t \in \mathbb{N}$, the Fock vacuum $|0\rangle$ generates an irreducible finite-dimensional $\mathfrak{sp}_{2r}$-module $L_{t\omega_r}$.

5.2. **More oscillator realizations in type C via underlying symmetries.**
Consider the following endomorphisms of $\mathbb{C}^{2r}$:

$$B_i = e_{ii'} - e_{i'i} + \sum_{1\le j\le r}^{j\neq i} \left(e_{jj} + e_{j'j'}\right) , \qquad 1 \le i \le r , \tag{5.17}$$

along with their order-independent products:

$$B_{\vec{\mu}} = \prod_{1\le j\le r}^{\mu_j=-1} B_j , \qquad \vec{\mu} = (\mu_1, \dots, \mu_r) \in \{\pm 1\}^r . \tag{5.18}$$

Remark 5.19. For $1 \le i \le r$, we have: $B_{\vec{\mu}}(e_i) = \begin{cases} e_i & \text{if } \mu_i = 1 \\ -e_{i'} & \text{if } \mu_i = -1 \end{cases}$, $B_{\vec{\mu}}(e_{i'}) = \begin{cases} e_{i'} & \text{if } \mu_i = 1 \\ e_i & \text{if } \mu_i = -1 \end{cases}$.

Since the R-matrix (1.19) is invariant under such transformations, cf. (1.5):

$$[R(x), B_{\vec{\mu}} \otimes B_{\vec{\mu}}] = 0 , \qquad \forall \vec{\mu} \in \{\pm 1\}^r , \tag{5.20}$$

we can generate more solutions to the RTT relation (1.1) from the Lax matrix (5.3) via:

$$\hat{\mathcal{L}}_{\vec{\mu}}(x) = B_{\vec{\mu}} \mathcal{L}(x) B_{\vec{\mu}}^{-1} = x \mathrm{I}_{2r} + \sum_{i,j=1}^{2r} e_{ij} \hat{\mathcal{F}}_{ji}^{\vec{\mu}} , \qquad \forall \vec{\mu} \in \{\pm 1\}^r . \tag{5.21}$$

We shall further apply the following particle-hole automorphism of $\mathcal{A}$ (denoted *p.h.*):

$$\bar{\mathbf{a}}_{i,j'} \mapsto -\mathbf{a}_{j',i} , \quad \mathbf{a}_{j',i} \mapsto \bar{\mathbf{a}}_{i,j'} \quad \text{for} \quad 1 \le i \le j \le r \quad \text{such that} \quad \mu_i = -1 . \tag{5.22}$$

Thus, we obtain the following explicit C_r-type $\mathcal{A}[x]$-valued Lax matrices:

$$\mathcal{L}_{\vec{\mu}}(x) = \hat{\mathcal{L}}_{\vec{\mu}}(x)\Big|_{p.h.} = B_{\vec{\mu}} \mathcal{L}(x) B_{\vec{\mu}}^{-1}\Big|_{p.h.} = x \mathrm{I}_{2r} + \sum_{i,j=1}^{2r} e_{ij} \mathcal{F}_{ji}^{\vec{\mu}} , \qquad \forall \vec{\mu} \in \{\pm 1\}^r . \tag{5.23}$$

The resulting matrix elements $\{\mathcal{F}_{ij}^{\vec{\mu}}\}_{i,j=1}^{2r}$ of $\mathcal{A}$ satisfy the $\mathfrak{sp}_{2r}$ commutation relations (5.8), due to the RTT relation (1.1). This makes the Fock module F into an $\mathfrak{sp}_{2r}$-module, denoted by $M_{\vec{\mu},t}^{+}$. We furthermore note that the particular choice of the particle-hole transformation (5.22) is uniquely made to insure that the Fock vacuum $|0\rangle \in M_{\vec{\mu},t}^{+}$ remains to be an $\mathfrak{sp}_{2r}$ highest weight state:

$$\mathcal{F}_{ij}^{\vec{\mu}} |0\rangle = 0 , \qquad 1 \le i < j \le 2r . \tag{5.24}$$

To compute its highest weight, we note that:

$$\mathrm{diag}\left(\hat{\mathcal{F}}^{\vec{\mu}}\right) = \mathrm{diag}\left(B_{\vec{\mu}} \mathcal{F} B_{\vec{\mu}}^{-1}\right) = (\mu_1 \mathcal{F}_{11}, \dots, \mu_r \mathcal{F}_{rr}, -\mu_r \mathcal{F}_{rr}, \dots, -\mu_1 \mathcal{F}_{11}) , \tag{5.25}$$

due to (5.13), which after implementing the particle-hole transformation (5.22) gives:

$$\mathcal{F}_{ii}^{\vec{\mu}} |0\rangle = \mu_i \left(t + (r - i + 1)\delta_{\mu_i}^{-} + \sum_{k=1}^{i} \delta_{\mu_k}^{-} \right) |0\rangle , \qquad 1 \le i \le r . \tag{5.26}$$

Thus, the Fock vacuum $|0\rangle \in M_{\vec{\mu},t}^{+}$ is an $\mathfrak{sp}_{2r}$ highest weight state whose weight is given by (5.26).

We shall now compare the above modules $M_{\vec{\mu},t}^{+}$'s with those from the Introduction. To this end, let us consider the parabolic $\mathfrak{p}_S \subset \mathfrak{sp}_{2r}$ corresponding to $S = \{1, \dots, r-1\}$, see Section 1.5. The Weyl group of $\mathfrak{sp}_{2r}$ can be identified with $W \simeq (\mathbb{Z}/2\mathbb{Z})^r \rtimes S_r \simeq \{\pm 1\}^r \rtimes S_r$, so that elements of W are indexed by pairs $(\vec{\mu}, \sigma)$ with $\vec{\mu} \in \{\pm 1\}^r$ and $\sigma \in S_r$. The Weyl group of the Levi subalgebra $\mathfrak{l} \simeq \mathfrak{gl}_r$ is $W_{\mathfrak{l}} \simeq S_r$ consisting of the elements $((+1, \dots, +1), \sigma)_{\sigma \in S_r} \subset W$. Thus, we have a set bijection

$$\pi \colon W/W_{\mathfrak{l}} \xrightarrow{\sim} \{\pm 1\}^r , \tag{5.27}$$

cf. (4.17). Given any $\vec{\mu} \in \{\pm 1\}^r$, we define the permutation $\sigma_{\vec{\mu}} \in S_r$ via:

$$\sigma_{\vec{\mu}}^{-1}(i) = \begin{cases} \#\{1 \le k \le i \,|\, \mu_k = 1\} & \text{if } \mu_i = 1 \\ r + 1 - \#\{1 \le k \le i \,|\, \mu_k = -1\} & \text{if } \mu_i = -1 \end{cases} , \tag{5.28}$$

and further consider $w_{\vec{\mu}} \in W \simeq \{\pm 1\}^r \rtimes S_r$ defined via:

$$w_{\vec{\mu}} = (\vec{\mu}, \sigma_{\vec{\mu}}) \,. \tag{5.29}$$

It is clear that $w_{\vec{\mu}} \in \pi^{-1}(\vec{\mu})$, see (5.27). Furthermore, it can be characterized as follows:

Lemma 5.30. *$w_{\vec{\mu}}$ is the shortest representative of the left coset $\pi^{-1}(\vec{\mu})$, for any $\vec{\mu} \in \{\pm 1\}^r$.*

Proof. This follows from the standard combinatorial description of the length function on the Weyl group of any Lie algebra $\mathfrak{g}$:

$$l(w) = \#\Big\{\alpha \in \Delta^+ \,|\, w(\alpha) \in -\Delta^+\Big\} \tag{5.31}$$

for any $w \in W$, where Δ^+ denotes the set of positive roots of $\mathfrak{g}$ (cf. Remark 5.45). □

Corollary 5.32. ${}^{\mathsf{t}}W = \{w_{\vec{\mu}}\}_{\vec{\mu} \in \{\pm 1\}^r}$.

Combining now the formula (5.26) with the definition of $w_{\vec{\mu}} \in W$, see (5.28, 5.29), we conclude that the highest weight of the Fock vacuum $|0\rangle \in M^+_{\vec{\mu},t}$ coincides with the highest weight of our key modules $M'_{w_{\vec{\mu}} \cdot t\omega_r}$ introduced in (1.36), see Corollary 5.32. Furthermore, $M'_{w_{\vec{\mu}} \cdot t\omega_r}$ has the same character as $M^+_{\vec{\mu},t}$ (according to Lemma 2.45) and is irreducible for $t \notin \frac{1}{2}\mathbb{Z}$ (as follows from [J]). Therefore, similarly to Proposition 4.28, we obtain:

Proposition 5.33. *For any $\vec{\mu} \in \{\pm 1\}^r$ and $t \notin \frac{1}{2}\mathbb{Z}$, we have $\mathfrak{sp}_{2r}$-module isomorphisms:*

$$M^+_{\vec{\mu},t} \simeq M'_{w_{\vec{\mu}} \cdot t\omega_r} \simeq \Big(M'_{w_{\vec{\mu}} \cdot t\omega_r}\Big)^\vee \,. \tag{5.34}$$

Remark 5.35. Let us point out the key difference between Proposition 5.33 and Lemma 5.14:

(a) For $\vec{\mu} = (+1, \ldots, +1)$, we actually have $M^+_{\vec{\mu},t} \simeq (M'_{w_{\vec{\mu}} \cdot t\omega_r})^\vee$ for any $t \in \mathbb{C}$, due to Lemma 5.14.

(b) Likewise, for $\vec{\mu} = (-1, \ldots, -1)$, we have $M^+_{\vec{\mu},t} \simeq M'_{w_{\vec{\mu}} \cdot t\omega_r}$ for any $t \in \mathbb{C}$.

(c) For other $\vec{\mu} \in \{\pm 1\}^r$, $M^+_{\vec{\mu},t}$ is <u>not isomorphic</u> to either of $M'_{w_{\vec{\mu}} \cdot t\omega_r}$ or $(M'_{w_{\vec{\mu}} \cdot t\omega_r})^\vee$ at certain $t \in \mathbb{Z}$ (but is expected to be isomorphic to one of the twisted Verma modules in the sense of [AL]).

Evoking the above bijection $\{\pm 1\}^r \ni \vec{\mu} \leftrightarrow w_{\vec{\mu}} \in {}^{\mathsf{t}}W$ of Corollary 5.32, let us define:

$$M^\vee_{\vec{\mu},t} = \Big(M'_{w_{\vec{\mu}} \cdot t\omega_r}\Big)^\vee \,, \qquad \forall\, t \in \mathbb{C} \,. \tag{5.36}$$

Then, Proposition 5.33 can be recast as the isomorphism of the following $\mathfrak{sp}_{2r}$-modules:

$$M^+_{\vec{\mu},t} \simeq M^\vee_{\vec{\mu},t} \,, \qquad \forall\, t \in \mathbb{C} \setminus \tfrac{1}{2}\mathbb{Z} \,. \tag{5.37}$$

For $\vec{\mu} \in \{\pm 1\}^r$, we also define its *length* $\mathsf{l}(\vec{\mu})$ as the length of the corresponding element $w_{\vec{\mu}} \in W$. Using formula (5.31) and the explicit description of the set Δ^+ of positive roots of $\mathfrak{sp}_{2r}$, we find:

$$\mathsf{l}(\vec{\mu}) = l(w_{\vec{\mu}}) = \sum_{i=1}^{r} (r - i + 1)\delta^-_{\mu_i} \,. \tag{5.38}$$

Our choice of notation is due to $\mathsf{l}(\vec{\mu}) \neq l(\vec{\mu})$, the latter being used for the length of $(\vec{\mu}, \mathrm{id}) \in W$.

5.3. **Type C transfer matrices.**

Recall the notion of transfer matrices $\{T_W(x)\}_{W\in \mathrm{Rep}\, Y(\mathfrak{sp}_{2r})}$, as discussed in Subsection 1.8. In particular, we shall consider the following explicit infinite-dimensional transfer matrices:

$$T^+_{\vec{\mu},t}(x) = \mathrm{tr} \prod_{i=1}^r \tau_i^{\mathcal{F}^{\vec{\mu}}_{ii}} \underbrace{\mathcal{L}_{\vec{\mu}}(x) \otimes \cdots \otimes \mathcal{L}_{\vec{\mu}}(x)}_{N}\,, \tag{5.39}$$

corresponding to $M^+_{\vec{\mu},t}$. For $t\in\mathbb{N}$, the finite-dimensional transfer matrices $T_{r,t}(x)$ corresponding to the modules $L_{t\omega_r}$ in the auxiliary space are defined similarly to (5.39), but with the trace taken over the finite-dimensional submodule $L_{t\omega_r}$ of $M^+_{(+1,\dots,+1),t}$, see Corollary 5.16(b).

Using the notation (5.36, 5.38), let us recast the resolution (2.55), dual to (1.30), as follows:

$$0 \to L_{t\omega_r} \to M^\vee_{(+1,\dots,+1),t} \to \bigoplus_{\vec{\mu}\in\{\pm1\}^r}^{\mathsf{l}(\vec{\mu})=1} M^\vee_{\vec{\mu},t} \to \bigoplus_{\vec{\mu}\in\{\pm1\}^r}^{\mathsf{l}(\vec{\mu})=2} M^\vee_{\vec{\mu},t} \to \cdots \to M^\vee_{(-1,\dots,-1),t} \to 0 \tag{5.40}$$

for any $t\in\mathbb{N}$. Combining this with (5.37) and the fact that the transfer matrices (5.39) depend continuously on $t\in\mathbb{C}$ (as so do the Lax matrices $\mathcal{L}_{\vec{\mu}}(x)$), we obtain the key result of this section:

Theorem 5.41. *For $t\in\mathbb{N}$, we have:*

$$T_{r,t}(x) \;=\; \sum_{\vec{\mu}\in\{\pm1\}^r} (-1)^{\mathsf{l}(\vec{\mu})}\, T^+_{\vec{\mu},t}(x)\,. \tag{5.42}$$

The character limit of (5.42) expresses the character of the $\mathfrak{sp}_{2r}$-modules $\{L_{t\omega_r}\}_{t\in\mathbb{N}}$ defined as

$$\mathrm{ch}_{r,t} = \mathrm{ch}_{r,t}(\tau_1,\dots,\tau_r) := \mathrm{tr}_{L_{t\omega_r}} \prod_{i=1}^r \tau_i^{\mathcal{F}_{ii}}\,, \tag{5.43}$$

that is the length $N=0$ case of $T_{r,t}(x)$, via:

$$\mathrm{ch}_{r,t} \;=\; \sum_{\vec{\mu}=(\mu_1,\dots,\mu_r)\in\{\pm1\}^r} (-1)^{\mathsf{l}(\vec{\mu})} \frac{\prod_{i=1}^r \tau_i^{\mu_i\left(t+(r-i+1)\delta^-_{\mu_i}+\sum_{k=1}^i \delta^-_{\mu_k}\right)}}{\prod_{1\le i\le j\le r}\left(1-\frac{1}{\tau_i\tau_j^{\mu_i\mu_j}}\right)} \tag{5.44}$$

with the $\vec{\mu}$'s summand in the right-hand side of (5.44) equal to the character of $M^+_{\vec{\mu},t}$, up to a sign.

Remark 5.45. For the physics' reader who skipped Section 2, let us present a concise proof of (5.44). Let us identify the set Δ^+ of positive roots of $\mathfrak{g}=\mathfrak{sp}_{2r}$ with $\Delta^+=\{\epsilon_i-\epsilon_j\}_{1\le i<j\le r}\cup\{\epsilon_i+\epsilon_j\}_{1\le i\le j\le r}$, so that $\rho=(r,r-1,\dots,2,1)$ in the basis $\{\epsilon_i\}_{i=1}^r$ and the Weyl group W gets identified with $W\simeq(\mathbb{Z}/2\mathbb{Z})^r\rtimes S_r\simeq\{\pm1\}^r\rtimes S_r$. According to the Weyl character formula, we have:

$$\mathrm{ch}_{L_{t\omega_r}} \;=\; \sum_{(\vec{\mu},\sigma)\in\{\pm1\}^r\rtimes S_r} (-1)^{l(\vec{\mu},\sigma)} \frac{e^{(\vec{\mu},\sigma)(t\omega_r+\rho)-\rho}}{\prod_{1\le i<j\le r}(1-e^{\epsilon_j-\epsilon_i})\prod_{1\le i\le j\le r}(1-e^{-\epsilon_j-\epsilon_i})}\,. \tag{5.46}$$

Following §5.2, let us consider the parabolic subalgebra $\mathfrak{p}_{\{1,\dots,r-1\}}\subset\mathfrak{g}$ whose Levi subalgebra is $\mathfrak{l}\simeq\mathfrak{gl}_r$ and the Weyl group is $W_{\mathfrak{l}}\simeq S_r=\{((+1,\dots,+1),\sigma)\}_{\sigma\in S_r}\subset W$. We can rewrite (5.46) as:

$$\mathrm{ch}_{L_{t\omega_r}} \;=\; \sum_{\vec{\mu}\in\{\pm1\}^r}\sum_{\sigma\in S_r} (-1)^{l(\vec{\mu},\sigma)} \frac{e^{(\vec{\mu},\sigma)(t\omega_r+\rho)-\rho}}{\prod_{1\le i<j\le r}(1-e^{\epsilon_j-\epsilon_i})\prod_{1\le i\le j\le r}(1-e^{-\epsilon_j-\epsilon_i})}\,. \tag{5.47}$$

The key step is to simplify the inner sum of (5.47) using the Weyl denominator formula for $\mathfrak{l}$:

$$\sum_{\sigma\in S_r}(-1)^{l(\sigma)} e^{\sigma(\rho_{\mathfrak{l}})-\rho_{\mathfrak{l}}} \;=\; \prod_{\alpha\in\Delta^+_{\mathfrak{l}}}(1-e^{-\alpha})\,, \tag{5.48}$$

where $\Delta^+_{\mathfrak{l}}=\{\epsilon_i-\epsilon_j\}_{1\le i<j\le r}\subset\Delta^+$ consists of positive roots of $\mathfrak{l}$, $\rho_{\mathfrak{l}}=\frac12\sum_{\alpha\in\Delta^+_{\mathfrak{l}}}\alpha=(\frac{r-1}{2},\dots,\frac{1-r}{2})$. As $\sigma(\rho)-\rho=\sigma(\rho_{\mathfrak{l}})-\rho_{\mathfrak{l}}$ and $\sigma(\omega_r)=\omega_r$ for any $\sigma\in S_r$, we get:

$$\sum_{\sigma\in S_r}(-1)^{l(\sigma)} \frac{e^{\sigma(t\omega_r+\rho)-\rho}}{\prod_{1\le i<j\le r}(1-e^{\epsilon_j-\epsilon_i})\prod_{1\le i\le j\le r}(1-e^{-\epsilon_j-\epsilon_i})} = \frac{\prod_{i=1}^r e^{t\epsilon_i}}{\prod_{1\le i\le j\le r}(1-e^{-\epsilon_j-\epsilon_i})}\,.$$

Thus, the inner sum of (5.47) indexed by $\vec{\mu} = (+1, \dots, +1)$ gives rise to the corresponding summand of (5.44). We claim that the same holds for any $\vec{\mu} \in \{\pm 1\}^r$, which amounts to the proof of (5.50) below. To this end, let us apply $\vec{\mu} = (\vec{\mu}, 1) \in W$ to both sides of the equality (5.48):

$$\sum_{\sigma \in S_r} (-1)^{l(\vec{\mu},\sigma)} e^{(\vec{\mu},\sigma)(\rho)-\rho} \;=\; (-1)^{l(\vec{\mu})} e^{\vec{\mu}(\rho)-\rho} \prod_{1 \le i < j \le r} (1 - e^{\mu_j \epsilon_j - \mu_i \epsilon_i}) \,. \tag{5.49}$$

We note that

$$1 - e^{\mu_j \epsilon_j - \mu_i \epsilon_i} = \begin{cases} 1 - e^{\epsilon_j - \epsilon_i} & \text{if } \mu_i = 1,\ \mu_j = 1 \\ 1 - e^{-\epsilon_j - \epsilon_i} & \text{if } \mu_i = 1,\ \mu_j = -1 \\ -e^{\epsilon_i - \epsilon_j}(1 - e^{\epsilon_j - \epsilon_i}) & \text{if } \mu_i = -1,\ \mu_j = -1 \\ -e^{\epsilon_i + \epsilon_j}(1 - e^{-\epsilon_j - \epsilon_i}) & \text{if } \mu_i = -1,\ \mu_j = 1 \end{cases}$$

as well as

$$\vec{\mu}(\rho) - \rho = \sum_{i=1}^{r} (\mu_i - 1)(r + 1 - i)\epsilon_i \,, \qquad (-1)^{l(\vec{\mu})} = (-1)^{\sum_{i=1}^{r} \delta^-_{\mu_i}} \,, \qquad \vec{\mu}(t\omega_r) = (t\mu_1, \dots, t\mu_r) \,.$$

Thus, we obtain the desired equality:

$$\sum_{\sigma \in S_r} (-1)^{l(\vec{\mu},\sigma)} \frac{e^{(\vec{\mu},\sigma)(t\omega_r+\rho)-\rho}}{\prod_{1 \le i < j \le r}(1 - e^{\epsilon_j - \epsilon_i}) \prod_{1 \le i \le j \le r}(1 - e^{-\epsilon_j - \epsilon_i})} = \\ (-1)^{\mathsf{l}(\vec{\mu})} \frac{\prod_{i=1}^{r} e^{\mu_i \left(t + (r-i+1)\delta^-_{\mu_i} + \sum_{k=1}^{i} \delta^-_{\mu_k}\right)\epsilon_i}}{\prod_{1 \le i \le j \le r} \left(1 - e^{-\mu_i \mu_j \epsilon_j - \epsilon_i}\right)} \,. \tag{5.50}$$

This completes our direct proof of the character formula (5.44) (see Remark 3.28 for more details in regards to perceiving $\mathrm{ch}_{r,t}$ of (5.43) as a specialization of $\mathrm{ch}_{L_{t\omega_r}}$).

We note that the formula (5.44) allows to analytically continue the character $\mathrm{ch}_{r,t}$ of (5.43) from the discrete set $t \in \mathbb{N}$ to the entire complex plane $t \in \mathbb{C}$. With this convention in mind, we have:

Lemma 5.51. *(a)* $\mathrm{ch}_{r,t} = (-1)^{\frac{r(r+1)}{2}} \mathrm{ch}_{r,-r-1-t}$ *for any* $t \in \mathbb{C}$.

(b) $\mathrm{ch}_{r,t} = 0$ *for* $t \in \left\{ -\frac{2}{2}, -\frac{3}{2}, \dots, -\frac{2r-1}{2}, -\frac{2r}{2} \right\}$.

Proof. (a) For any $\vec{\mu} = (\mu_1, \dots, \mu_r) \in \{\pm 1\}^r$ and $t \in \mathbb{C}$, define $\vec{\bar{\mu}} \in \{\pm 1\}^r$ and $\bar{t} \in \mathbb{C}$ via:

$$\vec{\bar{\mu}} = (\bar{\mu}_1, \dots, \bar{\mu}_r) := -\vec{\mu} \,, \qquad \bar{t} = -r - 1 - t \,. \tag{5.52}$$

Then, the obvious equality $\sum_{i=1}^{r} (r - i + 1)(\delta^-_{\mu_i} + \delta^-_{\bar{\mu}_i}) = \frac{r(r+1)}{2}$ implies:

$$\mathsf{l}(\vec{\mu}) + \mathsf{l}(\vec{\bar{\mu}}) = \tfrac{r(r+1)}{2} \,, \qquad \forall\, \vec{\mu} \in \{\pm 1\}^r \,. \tag{5.53}$$

Let $\mathrm{ch}^+_{\vec{\mu},t}$ denote the $\vec{\mu}$-th summand in the right-hand side of (5.44) without a sign, see (8.80):

$$\mathrm{ch}^+_{\vec{\mu},t} = \frac{\prod_{i=1}^{r} \tau_i^{\mu_i \left(t + (r-i+1)\delta^-_{\mu_i} + \sum_{k=1}^{i} \delta^-_{\mu_k}\right)}}{\prod_{1 \le i \le j \le r} \left(1 - \tau_i^{-1} \tau_j^{-\mu_i \mu_j}\right)} \,. \tag{5.54}$$

Then, we have the symmetry of (5.54) with respect to (5.52):

$$\mathrm{ch}^+_{\vec{\mu},t} = \mathrm{ch}^+_{\vec{\bar{\mu}},\bar{t}} \tag{5.55}$$

as follows from the equality $\mu_i \Big(t + (r - i + 1)\delta^-_{\mu_i} + \sum_{k=1}^{i} \delta^-_{\mu_k}\Big) = \bar{\mu}_i \Big(\bar{t} + (r - i + 1)\delta^-_{\bar{\mu}_i} + \sum_{k=1}^{i} \delta^-_{\bar{\mu}_k}\Big)$. This implies the desired equality:

$$\mathrm{ch}_{r,t} = \sum_{\vec{\mu} \in \{\pm 1\}^r} (-1)^{\mathsf{l}(\vec{\mu})} \mathrm{ch}^+_{\vec{\mu},t} = (-1)^{\frac{r(r+1)}{2}} \sum_{\vec{\mu} \in \{\pm 1\}^r} (-1)^{\mathsf{l}(\vec{\bar{\mu}})} \mathrm{ch}^+_{\vec{\bar{\mu}},\bar{t}} = (-1)^{\frac{r(r+1)}{2}} \mathrm{ch}_{r,\bar{t}} \,, \tag{5.56}$$

by matching the $\vec{\mu}$-th summand in $\mathrm{ch}_{r,t}$ with the $\vec{\bar{\mu}}$-th summand in $\mathrm{ch}_{r,\bar{t}}$ for every $\vec{\mu} \in \{\pm 1\}^r$.

(b) To prove (b), we shall rather use the Weyl character formula (5.46), which implies that $\mathrm{ch}_{r,t}=0$ if and only if one can split elements of the Weyl group $W\simeq\{\pm1\}^r\rtimes S_r$ into pairs (w,w') so that:

$$(-1)^{l(w)}=-(-1)^{l(w')}\qquad\text{and}\qquad w(t\omega_r+\rho)=w'(t\omega_r+\rho)\tag{5.57}$$

with $\rho=(r,\dots,1)$ and $\omega_r=(1,\dots,1)$. Let us indicate such splittings for the desired values of t:

- set $w'=w\Big(\big(\underbrace{+1,\dots,+1}_{k-1},-1,\underbrace{+1\dots,+1}_{r-k}\big),\mathrm{id}\Big)$ if $t=-(r+1-k)$ with $1\le k\le r$;
- set $w'=w\Big(\big(\underbrace{+1,\dots,+1}_{k-1},-1,\underbrace{+1,\dots,+1}_{m-k-1},-1,\underbrace{+1\dots,+1}_{r-m}\big),(k\ m)\Big)$ with $(k\ m)\in S_r$ denoting the transposition exchanging k and m, if $t=-(r+1-\frac{k+m}{2})$ with $1\le k<m\le r$.

By Remark 5.45, note that (5.44) and (5.46) provide the same analytic continuations of (5.43). □

Remark 5.58. Following the above proof, we get $\mathrm{ch}_{r,t}=0$ only for $t\in\Big\{-\frac{2}{2},-\frac{3}{2},\dots,-\frac{2r-1}{2},-\frac{2r}{2}\Big\}$.

In a completely similar way, the formula (5.42) allows to analytically continue the transfer matrices $T_{r,t}(x)$ of the finite-dimensional representations $L_{t\omega_r}$, $t\in\mathbb{N}$, to the entire complex plane $t\in\mathbb{C}$. With this convention in mind, we have the following generalization of Lemma 5.51(a):

Proposition 5.59. $T_{r,t}(x)=(-1)^{\frac{r(r+1)}{2}}T_{r,-r-1-t}(x)$ *for any* $t\in\mathbb{C}$.

Proof. This follows from the factorisation (8.79) of each infinite-dimensional transfer matrix $T^+_{\vec\mu,t}(x)$ into the product of Q-operators that allows to recast Theorem 5.41 in the form of Proposition 8.81:

$$T_{r,t}(x)\;=\;\sum_{\vec\mu\in\{\pm1\}^r}(-1)^{\mathrm{l}(\vec\mu)}\;\mathrm{ch}^+_{\vec\mu,t}\cdot Q_{\vec\mu}(x+t)Q_{\bar{\vec\mu}}(x+\bar t)\,,\tag{5.60}$$

with $\bar{\vec\mu},\bar t$ as in (5.52) and $\mathrm{ch}^+_{\vec\mu,t}$ as in (5.54), see (8.80). Similarly to our proof of Lemma 5.51(a), we claim that any $\vec\mu$-th summand in the right-hand side of (5.60) for $T_{r,t}(x)$ coincides with the $\bar{\vec\mu}$-th summand in the corresponding expression for $T_{r,\bar t}(x)$, up to an overall sign $(-1)^{\frac{r(r+1)}{2}}$:

$$(-1)^{\mathrm{l}(\vec\mu)}\;\mathrm{ch}^+_{\vec\mu,t}\cdot Q_{\vec\mu}(x+t)Q_{\bar{\vec\mu}}(x+\bar t)=(-1)^{\mathrm{l}(\bar{\vec\mu})+\frac{r(r+1)}{2}}\;\mathrm{ch}^+_{\bar{\vec\mu},\bar t}\cdot Q_{\bar{\vec\mu}}(x+\bar t)Q_{\vec\mu}(x+t)\,.\tag{5.61}$$

The latter is a consequence of (5.53, 5.55) combined with the essential property of the Q-operators:

$$[Q_{\vec\mu}(x),Q_{\bar{\vec\mu}}(y)]=0\,,\tag{5.62}$$

that follows from the natural commutativity of the transfer matrices, $[T^+_{\vec\mu}(x),T^+_{\vec\mu}(y)]=0$, combined with the realization of the Q-operators as *renormalized limits* of the transfer matrices, cf. (8.59). □

We also expect the generalization of Lemma 5.51(b) to hold: $T_{r,t}(x)=0$ for $t\in\Big\{-\frac{2}{2},-\frac{3}{2},\dots,-\frac{2r}{2}\Big\}$.

6. Resolutions for transfer matrices of D-type: spinorial representations

In this section, we present a natural counterpart of the results from Section 5 for D-type.

6.1. Oscillator realization in type D (parabolic Verma).

Let $\mathcal{A}$ denote the oscillator algebra generated by $\frac{r(r-1)}{2}$ pairs of oscillators $\{(\mathbf{a}_{j',i},\bar{\mathbf{a}}_{i,j'})\}_{1\le i<j\le r}$ with $r\ge2$, cf. notation (1.21), subject to the defining relations (5.1):

$$\mathcal{A}=\mathbb{C}\Big\langle\mathbf{a}_{j',i}\,,\,\bar{\mathbf{a}}_{i,j'}\Big\rangle_{1\le i<j\le r}\Big/\,(5.1)\,.\tag{6.1}$$

Following [Fr, (5.4)] and similarly to (5.3), let us consider the D_r-type $\mathcal{A}[x]$-valued Lax matrix:

$$\mathcal{L}(x)=\left(\begin{array}{c|c}(x+t)\mathrm{I}_r-\bar{\mathbf{A}}\mathbf{A} & -\bar{\mathbf{A}}(2t+r-1-\mathbf{A}\bar{\mathbf{A}})\\ \hline -\mathbf{A} & (x-t-r+1)\mathrm{I}_r+\mathbf{A}\bar{\mathbf{A}}\end{array}\right),\tag{6.2}$$

depending on $t \in \mathbb{C}$, with the blocks $\bar{\mathbf{A}}, \mathbf{A} \in \mathrm{Mat}_{r\times r}(\mathcal{A})$ encoding all the generators via:

$$\bar{\mathbf{A}} = \begin{pmatrix} \bar{\mathbf{a}}_{1,r'} & \cdots & \bar{\mathbf{a}}_{1,2'} & 0 \\ \vdots & \iddots & 0 & -\bar{\mathbf{a}}_{1,2'} \\ \bar{\mathbf{a}}_{r-1,r'} & 0 & \iddots & \vdots \\ 0 & -\bar{\mathbf{a}}_{r-1,r'} & \cdots & -\bar{\mathbf{a}}_{1,r'} \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} \mathbf{a}_{r',1} & \cdots & \mathbf{a}_{r',r-1} & 0 \\ \vdots & \iddots & 0 & -\mathbf{a}_{r',r-1} \\ \mathbf{a}_{2',1} & 0 & \iddots & \vdots \\ 0 & -\mathbf{a}_{2',1} & \cdots & -\mathbf{a}_{r',1} \end{pmatrix}. \tag{6.3}$$

Writing (6.2) in the form

$$\mathcal{L}(x) = x\mathrm{I}_{2r} + \sum_{i,j=1}^{2r} e_{ij}\mathcal{F}_{ji}\,, \tag{6.4}$$

we note that the RTT relation (1.1) implies that $\{\mathcal{F}_{ij}\}_{i,j=1}^{2r}$ satisfy the $\mathfrak{so}_{2r}$ commutation relations:

$$[\mathcal{F}_{ij}, \mathcal{F}_{k\ell}] = \delta_k^j \mathcal{F}_{i\ell} - \delta_i^\ell \mathcal{F}_{kj} - \delta_k^{i'} \mathcal{F}_{j'\ell} + \delta_\ell^{j'} \mathcal{F}_{ki'}\,, \qquad \mathcal{F}_{ij} = -\mathcal{F}_{j'i'}\,. \tag{6.5}$$

As before, let F denote the Fock module of $\mathcal{A}$, generated by the Fock vacuum $|0\rangle \in \mathsf{F}$ satisfying:

$$\mathbf{a}_{j',i}|0\rangle = 0\,, \qquad 1 \le i < j \le r\,. \tag{6.6}$$

Then, the Fock vacuum $|0\rangle$ is a highest weight state of the resulting $\mathfrak{so}_{2r}$-action:

$$\mathcal{F}_{ij}|0\rangle = 0\,, \qquad 1 \le i < j \le 2r\,, \tag{6.7}$$

with the highest weight $\lambda = 2t\omega_r = (\underbrace{t,\ldots,t}_{r})$, that is:

$$\mathcal{F}_{ii}|0\rangle = t|0\rangle\,, \qquad 1 \le i \le r\,. \tag{6.8}$$

The latter is a consequence of the following explicit formulas for any $1 \le i \le r$:

$$\begin{aligned} \mathcal{F}_{ii} &= t - \sum_{k=i+1}^{r} \bar{\mathbf{a}}_{ik'}\mathbf{a}_{k'i} - \sum_{k=1}^{i-1} \bar{\mathbf{a}}_{ki'}\mathbf{a}_{i'k}\,, \\ \mathcal{F}_{i'i'} &= -t - r + 1 + \sum_{k=1}^{i-1} \mathbf{a}_{i'k}\bar{\mathbf{a}}_{ki'} + \sum_{k=i+1}^{r} \mathbf{a}_{k'i}\bar{\mathbf{a}}_{ik'} = -\mathcal{F}_{ii}\,. \end{aligned} \tag{6.9}$$

Similarly to Lemmas 4.8 and 5.14, we can identify the resulting $\mathfrak{so}_{2r}$-modules F as follows:

Lemma 6.10. *There is an $\mathfrak{so}_{2r}$-module isomorphism:*

$$\mathsf{F} \simeq \left(M_{2t\omega_r}^{\mathsf{p}_{\{1,\ldots,r-1\}}}\right)^\vee\,. \tag{6.11}$$

Combining this with the determinant formula of [J] and the isomorphism (2.21), we obtain:

Corollary 6.12. *(a) For $t \notin \frac{1}{2}\mathbb{Z}_{\ge 4-2r}$, the $\mathfrak{so}_{2r}$-module F is irreducible (thus, is generated by $|0\rangle$).*
(b) For $t \in \frac{1}{2}\mathbb{N}$, the Fock vacuum $|0\rangle$ generates an irreducible finite-dimensional $\mathfrak{so}_{2r}$-module $L_{2t\omega_r}$.

6.2. **More oscillator realizations in type D via underlying symmetries.**
Similarly to the C-type case, let us consider the following endomorphisms of $\mathbb{C}^{2r}$:

$$B_i = e_{ii'} + e_{i'i} + \sum_{1\le j\le r}^{j\ne i} \left(e_{jj} + e_{j'j'}\right)\,, \qquad 1 \le i \le r\,, \tag{6.13}$$

along with their order-independent products:

$$B_{\vec{\mu}} = \prod_{1\le j\le r}^{\mu_j=-1} B_j\,, \qquad \vec{\mu} = (\mu_1,\ldots,\mu_r) \in \{\pm 1\}^r\,. \tag{6.14}$$

Remark 6.15. For $1 \leq i \leq r$, we have: $B_{\vec{\mu}}(e_i) = \begin{cases} e_i & \text{if } \mu_i = 1 \\ e_{i'} & \text{if } \mu_i = -1 \end{cases}$, $B_{\vec{\mu}}(e_{i'}) = \begin{cases} e_{i'} & \text{if } \mu_i = 1 \\ e_i & \text{if } \mu_i = -1 \end{cases}$.

Since the R-matrix (1.19) is invariant under such transformations, cf. (1.5):

$$[R(x), B_{\vec{\mu}} \otimes B_{\vec{\mu}}] = 0\,, \qquad \forall \vec{\mu} \in \{\pm 1\}^r\,, \tag{6.16}$$

we can generate more solutions to the RTT relation (1.1) from the Lax matrix (6.2) via:

$$\mathcal{L}_{\vec{\mu}}(x) = B_{\vec{\mu}} \mathcal{L}(x) B_{\vec{\mu}}^{-1}\Big|_{p.h.} = x\mathrm{I}_{2r} + \sum_{i,j=1}^{2r} e_{ij} \mathcal{F}_{ji}^{\vec{\mu}}\,, \qquad \forall \vec{\mu} \in \{\pm 1\}^r\,. \tag{6.17}$$

Here, we apply the following particle-hole automorphism of $\mathcal{A}$ (denoted *p.h.*), cf. (5.22):

$$\bar{\mathbf{a}}_{i,j'} \mapsto -\mathbf{a}_{j',i}\,, \quad \mathbf{a}_{j',i} \mapsto \bar{\mathbf{a}}_{i,j'} \quad \text{for} \quad 1 \leq i < j \leq r \quad \text{such that} \quad \mu_i = -1\,, \tag{6.18}$$

uniquely chosen to insure that the Fock vacuum $|0\rangle$ remains to be an $\mathfrak{so}_{2r}$ highest weight state:

$$\mathcal{F}_{ij}^{\vec{\mu}} |0\rangle = 0\,, \qquad 1 \leq i < j \leq 2r\,. \tag{6.19}$$

The resulting matrix elements $\{\mathcal{F}_{ij}^{\vec{\mu}}\}_{i,j=1}^{2r}$ of $\mathcal{A}$ satisfy the $\mathfrak{so}_{2r}$ commutation relations (6.5), due to the RTT relation (1.1). This makes the Fock module F into an $\mathfrak{so}_{2r}$-module, denoted by $M_{\vec{\mu},t}^{+}$. The corresponding highest weight of $|0\rangle$ is computed similarly to C-type, see (5.26):

$$\mathcal{F}_{ii}^{\vec{\mu}} |0\rangle = \mu_i \left(t + (r-i-1)\delta_{\mu_i}^{-} + \sum_{k=1}^{i} \delta_{\mu_k}^{-} \right) |0\rangle\,, \qquad 1 \leq i \leq r\,. \tag{6.20}$$

Let us note that for the particular choice $\vec{\mu} = (+1, \ldots, +1, -1)$, the particle-hole transformation (6.18) is the identity, and the resulting $\mathfrak{so}_{2r}$-module $M_{\vec{\mu},t}^{+}$ can be read off the Lax matrix

$$\mathcal{L}_{-}(x) = \mathcal{L}_{(\underbrace{+1, \ldots, +1}_{r-1}, -1)}(x)\,, \tag{6.21}$$

which is obtained from the Lax matrix $\mathcal{L}_{+}(x) = \mathcal{L}(x)$ of (6.2) by permuting its r-th and $(r+1)$-st rows and columns. We also have the following counterparts of Lemma 6.10 and Corollary 6.12:

Lemma 6.22. *There is an $\mathfrak{so}_{2r}$-module isomorphism:*

$$M_{(\underbrace{+1, \ldots, +1}_{r-1}, -1), t}^{+} \simeq \left(M_{2t\omega_{r-1}}^{\mathfrak{p}_{\{1,\ldots,r-2,r\}}} \right)^{\vee}\,. \tag{6.23}$$

Corollary 6.24. *(a) For $t \notin \frac{1}{2}\mathbb{Z}_{\geq 4-2r}$, the $\mathfrak{so}_{2r}$-module $M_{(+1,\ldots,+1,-1),t}^{+}$ is irreducible.*

(b) For $t \in \frac{1}{2}\mathbb{N}$, the Fock vacuum $|0\rangle$ generates an irreducible finite-dimensional $\mathfrak{so}_{2r}$-module $L_{2t\omega_{r-1}}$.

For $\vec{\mu} = (\mu_1, \ldots, \mu_r) \in \{\pm 1\}^r$, we define its *sign* $|\vec{\mu}| \in \{\pm 1\}$ via:

$$|\vec{\mu}| = \mu_1 \cdot \ldots \cdot \mu_r\,. \tag{6.25}$$

We call $\vec{\mu} \in \{\pm 1\}^r$ *even* (resp. *odd*) if $|\vec{\mu}| = 1$ (resp. $|\vec{\mu}| = -1$), and denote the sets of such by

$$\{\pm 1\}_{+}^{r} = \Big\{ \text{even } \vec{\mu} \in \{\pm 1\}^r \Big\}\,, \qquad \{\pm 1\}_{-}^{r} = \Big\{ \text{odd } \vec{\mu} \in \{\pm 1\}^r \Big\}\,. \tag{6.26}$$

For $t \in \frac{1}{2}\mathbb{N}$, let $L_t^{\pm}$ denote the following irreducible finite-dimensional $\mathfrak{so}_{2r}$-representations:

$$L_t^{+} = L_{2t\omega_r}\,, \qquad L_t^{-} = L_{2t\omega_{r-1}}\,, \tag{6.27}$$

which can be uniformly written as $L_t^{\pm} = L_{2t\omega^{\pm}}$ with the weights $\omega^{\pm}$ defined via:

$$\omega^{+} = \omega_r\,, \qquad \omega^{-} = \omega_{r-1}\,. \tag{6.28}$$

Let us now generalize Lemmas 6.10, 6.22 by comparing the above modules $\{M_{\vec{\mu},t}^{+}\}_{\vec{\mu} \in \{\pm 1\}^r}$ to those from the Introduction. To this end, we shall consider two parabolic subalgebras $\mathfrak{p}_{S^{\pm}} \subset \mathfrak{so}_{2r}$ with

$$S^{+} = \{1, \ldots, r-2, r-1\}\,, \qquad S^{-} = \{1, \ldots, r-2, r\}\,. \tag{6.29}$$

The Weyl group of $\mathfrak{so}_{2r}$ can be identified with $W \simeq (\mathbb{Z}/2\mathbb{Z})^{r-1} \rtimes S_r \simeq \{\pm 1\}^r_+ \rtimes S_r$, cf. (6.26), so that elements of W are indexed by pairs $(\vec{\mu}, \sigma)$ with $\sigma \in S_r$ and $\vec{\mu} \in \{\pm 1\}^r$ that are even ($|\vec{\mu}| = 1$).

The Weyl group of the Levi subalgebra $\mathfrak{l}^+ \simeq \mathfrak{gl}_r$ of $\mathfrak{p}_{S^+}$ is $W_{\mathfrak{l}^+} \simeq S_r$, which consists of the elements $((+1, \ldots, +1), \sigma)_{\sigma \in S_r} \subset W$. Equivalently, $W_{\mathfrak{l}^+}$ is the stabilizer of $(+1, \ldots, +1)$ for the natural transitive action of W on the set $\{\pm 1\}^r_+$. Thus, we have a set bijection

$$\pi_+ \colon W/W_{\mathfrak{l}^+} \xrightarrow{\sim} \{\pm 1\}^r_+ \,, \tag{6.30}$$

cf. (5.27). For any $\vec{\mu} \in \{\pm 1\}^r_+$, we define $w_{\vec{\mu}} \in W \simeq \{\pm 1\}^r_+ \rtimes S_r$ via:

$$w_{\vec{\mu}} = (\vec{\mu}, \sigma_{\vec{\mu}}) \,, \tag{6.31}$$

with $\sigma_{\vec{\mu}} \in S_r$ as in (5.28). The element $w_{\vec{\mu}} \in \pi_+^{-1}(\vec{\mu})$ can be characterized similarly to Lemma 5.30:

Lemma 6.32. *$w_{\vec{\mu}}$ is the shortest representative of the left coset $\pi_+^{-1}(\vec{\mu})$, for any $\vec{\mu} \in \{\pm 1\}^r_+$.*

Corollary 6.33. ${}^{\mathfrak{l}^+}W = \{w_{\vec{\mu}}\}_{\vec{\mu} \in \{\pm 1\}^r_+}$.

Likewise, the Weyl group of the Levi subalgebra $\mathfrak{l}^- \simeq \mathfrak{gl}_r$ of $\mathfrak{p}_{S^-}$ is $W_{\mathfrak{l}^-} \simeq S_r$, which consists of the elements $((+1, \ldots, +1, -1, +1, \ldots, +1, -1), \sigma)_{\sigma \in S_r} \subset W$ with -1's at the r-th and $\sigma(r)$-th spots (and there are no -1's at all if $r = \sigma(r)$). Equivalently, $W_{\mathfrak{l}^-}$ is the stabilizer of $(+1, \ldots, +1, -1)$ for the natural transitive action of W on the set $\{\pm 1\}^r_-$. Thus, we have a set bijection

$$\pi_- \colon W/W_{\mathfrak{l}^-} \xrightarrow{\sim} \{\pm 1\}^r_- \,, \tag{6.34}$$

cf. (6.30). For any $\vec{\mu} \in \{\pm 1\}^r_-$, we define $w_{\vec{\mu}} \in W \simeq \{\pm 1\}^r_+ \rtimes S_r$ via:

$$w_{\vec{\mu}} = (\vec{\mu}', \sigma_{\vec{\mu}}) \,, \tag{6.35}$$

with $\sigma_{\vec{\mu}} \in S_r$ as in (5.28) and $\vec{\mu}' \in \{\pm 1\}^r_+$ obtained from $\vec{\mu}$ by replacing the first $\mu_i = -1$ (with minimal i) by $+1$. The above element $w_{\vec{\mu}} \in \pi_-^{-1}(\vec{\mu})$ can be characterized similarly to Lemma 6.32:

Lemma 6.36. *$w_{\vec{\mu}}$ is the shortest representative of the left coset $\pi_-^{-1}(\vec{\mu})$, for any $\vec{\mu} \in \{\pm 1\}^r_-$.*

Corollary 6.37. ${}^{\mathfrak{l}^-}W = \{w_{\vec{\mu}}\}_{\vec{\mu} \in \{\pm 1\}^r_-}$.

Combining now the formula (6.20) with the definition of $w_{\vec{\mu}} \in W$, see (6.31, 6.35), we conclude that the highest weight of the Fock vacuum $|0\rangle \in M^+_{\vec{\mu},t}$ coincides with the highest weight of our key modules $M'_{w_{\vec{\mu}} \cdot t\omega^{|\vec{\mu}|}}$ introduced in (1.36), see Corollaries 6.33, 6.37 (we set $\omega^{\pm 1} = \omega^{\pm}$). Furthermore, $M'_{w_{\vec{\mu}} \cdot t\omega^{|\vec{\mu}|}}$ has the same character as $M^+_{\vec{\mu},t}$ (according to Lemma 2.45) and is irreducible for $t \notin \frac{1}{2}\mathbb{Z}$ (as follows from [J]). Therefore, similarly to Propositions 4.28 and 5.33, we obtain:

Proposition 6.38. *For any $\vec{\mu} \in \{\pm 1\}^r$ and $t \notin \frac{1}{2}\mathbb{Z}$, we have $\mathfrak{so}_{2r}$-module isomorphisms:*

$$M^+_{\vec{\mu},t} \simeq M'_{w_{\vec{\mu}} \cdot t\omega^{|\vec{\mu}|}} \simeq \left(M'_{w_{\vec{\mu}} \cdot t\omega^{|\vec{\mu}|}}\right)^{\vee} . \tag{6.39}$$

Remark 6.40. Let us point out the key difference between Proposition 6.38 and Lemmas 6.10, 6.22:

(a) For $\vec{\mu} = (+1, \ldots, +1, \pm 1)$, we actually have $M^+_{\vec{\mu},t} \simeq (M'_{w_{\vec{\mu}} \cdot t\omega^{\pm}})^{\vee}$ for any $t \in \mathbb{C}$, due to Lemmas 6.10, 6.22.

(b) Likewise, for $\vec{\mu} = (-1, \ldots, -1, \mp 1)$, we have $M^+_{\vec{\mu},t} \simeq M'_{w_{\vec{\mu}} \cdot t\omega^{|\vec{\mu}|}}$ for any $t \in \mathbb{C}$.

(c) For other $\vec{\mu} \in \{\pm 1\}^r$, $M^+_{\vec{\mu},t}$ is <u>not isomorphic</u> to either of $M'_{w_{\vec{\mu}} \cdot t\omega^{|\vec{\mu}|}}$ or $(M'_{w_{\vec{\mu}} \cdot t\omega^{|\vec{\mu}|}})^{\vee}$ at some $t \in \frac{1}{2}\mathbb{Z}$ (but we expect it to be isomorphic to a twisted Verma module in the sense of [AL]).

Evoking the above bijections $\{\pm 1\}^r_\pm \ni \vec{\mu} \leftrightarrow w_{\vec{\mu}} \in {}^{\mathsf{I}\pm}W$ of Corollaries 6.33 and 6.37, let us define:

$$M^\vee_{\vec{\mu},t} = \left(M'_{w_{\vec{\mu}}\cdot t\omega^{|\vec{\mu}|}}\right)^\vee, \qquad \forall\, t \in \mathbb{C}\,. \tag{6.41}$$

Then, Proposition 6.38 can be recast as the isomorphism of the following $\mathfrak{so}_{2r}$-modules:

$$M^+_{\vec{\mu},t} \simeq M^\vee_{\vec{\mu},t}\,, \qquad \forall\, t \in \mathbb{C} \setminus \tfrac{1}{2}\mathbb{Z}\,. \tag{6.42}$$

For $\vec{\mu} \in \{\pm 1\}^r$, we also define its *length* $\mathsf{l}(\vec{\mu})$ as the length of the corresponding element $w_{\vec{\mu}} \in W$. Using formula (5.31) and the explicit description of the set Δ^+ of positive roots of $\mathfrak{so}_{2r}$, we find:

$$\mathsf{l}(\vec{\mu}) = l(w_{\vec{\mu}}) = \sum_{i=1}^r (r-i)\delta^-_{\mu_i}\,, \tag{6.43}$$

cf. (5.38). We note that $\mathsf{l}(\vec{\mu})$ differs from the lengths of $(\vec{\mu}, \mathrm{id}), (\vec{\mu}', \mathrm{id}) \in W$ denoted by $l(\vec{\mu}), l(\vec{\mu}')$.

6.3. **Type D transfer matrices.**

Recall the notion of transfer matrices $\{T_W(x)\}_{W \in \mathrm{Rep}\, Y(\mathfrak{so}_{2r})}$, as discussed in Subsection 1.8. In particular, we shall consider the following explicit infinite-dimensional transfer matrices:

$$T^+_{\vec{\mu},t}(x) = \mathrm{tr} \prod_{i=1}^r \tau_i^{\mathcal{F}^{\vec{\mu}}_{ii}} \underbrace{\mathcal{L}_{\vec{\mu}}(x) \otimes \cdots \otimes \mathcal{L}_{\vec{\mu}}(x)}_{N}\,, \tag{6.44}$$

corresponding to $M^+_{\vec{\mu},t}$. For $t \in \frac{1}{2}\mathbb{N}$, we also consider the finite-dimensional transfer matrices $T^\pm_t(x)$ corresponding to the modules $L^\pm_t$ (6.27) in the auxiliary space: those are defined similarly to (6.44), but with the trace taken over the finite-dimensional submodules $L^\pm_t$ of $M^+_{(+1,\ldots,+1,\pm 1),t}$, see Corollaries 6.12(b), 6.24(b).

Using the notation (6.41, 6.43), let us recast the resolution (2.55), dual to (1.30), as follows:

$$0 \to L^\pm_t \to M^\vee_{(1,\ldots,1,\pm 1),t} \to \bigoplus_{\vec{\mu}\in\{\pm 1\}^r_\pm}^{\mathsf{l}(\vec{\mu})=1} M^\vee_{\vec{\mu},t} \to \bigoplus_{\vec{\mu}\in\{\pm 1\}^r_\pm}^{\mathsf{l}(\vec{\mu})=2} M^\vee_{\vec{\mu},t} \to \cdots \to 0 \tag{6.45}$$

for any $t \in \frac{1}{2}\mathbb{N}$. Combining this with (6.42) and the fact that the transfer matrices (6.44) depend continuously on $t \in \mathbb{C}$ (as so do the Lax matrices $\mathcal{L}_{\vec{\mu}}(x)$), we obtain the key result of this section:

Theorem 6.46. *For $t \in \frac{1}{2}\mathbb{N}$, we have:*

$$T^\pm_t(x) \;=\; \sum_{\vec{\mu}\in\{\pm 1\}^r_\pm} (-1)^{\mathsf{l}(\vec{\mu})}\, T^+_{\vec{\mu},t}(x)\,. \tag{6.47}$$

The character limit of (6.47) expresses the character of the $\mathfrak{so}_{2r}$-modules $\{L^\pm_t\}_{t\in\frac{1}{2}\mathbb{N}}$ defined as

$$\mathrm{ch}^\pm_t = \mathrm{ch}^\pm_t(\tau_1, \ldots, \tau_r) := \mathrm{tr}_{L^\pm_t} \prod_{i=1}^r \tau_i^{\mathcal{F}^{(+1,\ldots,+1,\pm 1)}_{ii}}\,, \tag{6.48}$$

that is the length $N = 0$ case of $T^\pm_t(x)$, via:

$$\mathrm{ch}^\pm_t = \sum_{\vec{\mu}=(\mu_1,\ldots,\mu_r)\in\{\pm 1\}^r_\pm} (-1)^{\mathsf{l}(\vec{\mu})} \frac{\prod_{i=1}^r \tau_i^{\mu_i\left(t+(r-i-1)\delta^-_{\mu_i}+\sum_{k=1}^i \delta^-_{\mu_k}\right)}}{\prod_{1\le i<j\le r}\left(1-\frac{1}{\tau_i\tau_j^{\mu_i\mu_j}}\right)} \tag{6.49}$$

with the $\vec{\mu}$'s summand in the right-hand side of (6.49) equal to the character of $M^+_{\vec{\mu},t}$, up to a sign.

Remark 6.50. The character formula (6.49) can be derived directly from the Weyl character and Weyl denominator formulas, as in Remark 5.45; we leave details to the interested reader.

We note that the formula (6.49) allows to analytically continue the characters $\mathrm{ch}^\pm_t$ of (6.48) from the discrete set $t \in \frac{1}{2}\mathbb{N}$ to the entire complex plane $t \in \mathbb{C}$. With this convention in mind, we obtain:

Lemma 6.51. *(a)* $\mathrm{ch}_t^{\pm} = (-1)^{\frac{r(r-1)}{2}}\,\mathrm{ch}_{-r+1-t}^{\pm(-1)^r}$ *for any* $t \in \mathbb{C}$.

(b) $\mathrm{ch}_t^{\pm} = 0$ *for* $t \in \Big\{-\frac{1}{2}, -\frac{2}{2}, \dots, -\frac{2r-4}{2}, -\frac{2r-3}{2}\Big\}$.

Proof. The proof is completely analogous to that of Lemma 5.51 with the following two changes. In part (a), we should rather use $\bar{t} = -r+1-t$ instead of (5.52), and note that for $\vec{\mu} \in \{\pm 1\}^r_{\pm}$ of (6.26) we have $\vec{\bar{\mu}} = -\vec{\mu} \in \{\pm 1\}^r_{\pm(-1)^r}$. In part (b) for ch_t^{+}, the splitting of elements of W into the pairs satisfying (5.57) is performed following only the second rule in our proof of Lemma 5.51(b). □

In a completely similar way, the formula (6.47) allows to analytically continue the transfer matrices $T_t^{\pm}(x)$ of the finite-dimensional representations $L_t^{\pm}, t \in \frac{1}{2}\mathbb{N}$, to the entire complex plane $t \in \mathbb{C}$. With this convention in mind, we have the following generalization of Lemma 6.51(a):

Proposition 6.52. $T_t^{\pm}(x) = (-1)^{\frac{r(r-1)}{2}} T_{-r+1-t}^{\pm(-1)^r}(x)$ *for any* $t \in \mathbb{C}$.

Proof. The proof is completely analogous to that of Proposition 5.59 and follows from the proof of Lemma 6.51(a) combined with the factorisation (8.98) of the transfer matrices $T_{\vec{\mu},t}^{+}(x)$ into the product of two commuting Q-operators, cf. Proposition 8.100. □

We expect the generalization of Lemma 6.51(b) to hold: $T_t^{\pm}(x) = 0$ for $t \in \Big\{-\frac{1}{2}, -\frac{2}{2}, \dots, -\frac{2r-3}{2}\Big\}$. This was first observed in [FFK] for small length and rank.

7. Resolutions for transfer matrices of BD-types: first fundamental representations

In this section, we apply similar ideas to treat the remaining case of (1.22): $i = 1$ for BD-types.

7.1. Oscillator realization in types BD (parabolic Verma).

For $\mathsf{K} \geq 5$, let $\mathcal{A}$ denote the oscillator algebra generated by $\mathsf{K}-2$ pairs of oscillators $\{(\mathbf{a}_i, \bar{\mathbf{a}}_i)\}_{1<i<1'}$, cf. notation (1.21) so that $1' = \mathsf{K}$, subject to the standard defining relations:

$$[\mathbf{a}_i, \bar{\mathbf{a}}_j] = \delta_i^j\,, \qquad [\mathbf{a}_i, \mathbf{a}_j] = 0\,, \qquad [\bar{\mathbf{a}}_i, \bar{\mathbf{a}}_j] = 0\,, \tag{7.1}$$

so that

$$\mathcal{A} = \mathbb{C}\Big\langle \mathbf{a}_i\,,\, \bar{\mathbf{a}}_i \Big\rangle_{1<i<1'} \Big/ (7.1)\,. \tag{7.2}$$

Following [Fr, (5.36, 5.38)] and [FT, (2.243), §4.3], let us consider the quadratic non-degenerate $\mathcal{A}[x]$-valued Lax matrices of $\mathfrak{so}_{\mathsf{K}}$-type (i.e. of types D_r or B_r with $r = \lfloor \mathsf{K}/2 \rfloor$ for K even or odd, respectively), depending on $x_1, x_2 \in \mathbb{C}$:

$$\mathfrak{L}_{x_1,x_2}(x) = \left(\begin{array}{c|c|c} 1 & \bar{\mathbf{w}} & -\frac{1}{2}\bar{\mathbf{w}}\mathrm{J}_{\mathsf{K}-2}\bar{\mathbf{w}}^T \\ \hline 0 & \mathrm{I}_{\mathsf{K}-2} & -\mathrm{J}_{\mathsf{K}-2}\bar{\mathbf{w}}^T \\ \hline 0 & 0 & 1 \end{array}\right) \cdot D_{x_1,x_2}(x) \cdot \left(\begin{array}{c|c|c} 1 & -\bar{\mathbf{w}} & -\frac{1}{2}\bar{\mathbf{w}}\mathrm{J}_{\mathsf{K}-2}\bar{\mathbf{w}}^T \\ \hline 0 & \mathrm{I}_{\mathsf{K}-2} & \mathrm{J}_{\mathsf{K}-2}\bar{\mathbf{w}}^T \\ \hline 0 & 0 & 1 \end{array}\right) \tag{7.3}$$

with J_k denoting the anti-diagonal $k \times k$-matrix and the middle factor explicitly given by:

$$D_{x_1,x_2}(x) = \left(\begin{array}{c|c|c} (x-x_1)(x-x_1-\frac{\mathsf{K}}{2}+2) & 0 & 0 \\ \hline -\mathbf{w}(x-x_1) & (x-x_1)(x-x_2)\mathrm{I}_{\mathsf{K}-2} & 0 \\ \hline -\frac{1}{2}\mathbf{w}^T\mathrm{J}_{\mathsf{K}-2}\mathbf{w} & \mathbf{w}^T\mathrm{J}_{\mathsf{K}-2}(x-x_2) & (x-x_2)(x-x_2-\frac{\mathsf{K}}{2}+2) \end{array}\right),$$

while the row-vector $\bar{\mathbf{w}} \in \mathrm{Mat}_{1\times(\mathsf{K}-2)}(\mathcal{A})$ and the column-vector $\mathbf{w} \in \mathrm{Mat}_{(\mathsf{K}-2)\times 1}(\mathcal{A})$ encode all the generators via:

$$D_r\colon \quad \bar{\mathbf{w}} = (\bar{\mathbf{a}}_2, \dots, \bar{\mathbf{a}}_r, \bar{\mathbf{a}}_{r'}, \dots, \bar{\mathbf{a}}_{2'})\,, \quad \mathbf{w} = (\mathbf{a}_2, \dots, \mathbf{a}_r, \mathbf{a}_{r'}, \dots, \mathbf{a}_{2'})^T\,, \tag{7.4}$$

$$B_r\colon \quad \bar{\mathbf{w}} = (\bar{\mathbf{a}}_2, \dots, \bar{\mathbf{a}}_r, \bar{\mathbf{a}}_{r+1}, \bar{\mathbf{a}}_{r'}, \dots, \bar{\mathbf{a}}_{2'}), \quad \mathbf{w} = (\mathbf{a}_2, \dots, \mathbf{a}_r, \mathbf{a}_{r+1}, \mathbf{a}_{r'}, \dots, \mathbf{a}_{2'})^T. \tag{7.5}$$

Following [FT, Remark 4.37], we also consider

$$L_{x_{12}}(x) = \mathfrak{L}_{x_1,x_2}(x+c) = x^2 \mathrm{I}_{\mathsf{K}} + x M_{x_{12}} + G_{x_{12}} \tag{7.6}$$

with the shift c of the spectral parameter given by:

$$c = \frac{x_1 + x_2 - 1}{2}, \tag{7.7}$$

and $x_{12} = x_1 - x_2$ (note that the right-hand side of (7.6) depends only on the difference of x_1, x_2). It is straightforward to see that the linear term $M_{x_{12}}$ in (7.6) is given by:

$$M_{x_{12}} = \left(\begin{array}{c|c|c} -x_{12} - \frac{\mathsf{K}}{2} + 1 - \bar{\mathbf{w}}\mathbf{w} & M_{[12]} & 0 \\ \hline -\mathbf{w} & \mathbf{w}\bar{\mathbf{w}} - \mathrm{J}_{\mathsf{K}-2}\bar{\mathbf{w}}^T\mathbf{w}^T\mathrm{J}_{\mathsf{K}-2} - \mathrm{I}_{\mathsf{K}-2} & M_{[23]} \\ \hline 0 & \mathbf{w}^T\mathrm{J}_{\mathsf{K}-2} & x_{12} + \frac{\mathsf{K}}{2} - 1 + \bar{\mathbf{w}}\mathbf{w} \end{array}\right), \tag{7.8}$$

with

$$M_{[12]} = \left(x_{12} + \tfrac{\mathsf{K}}{2} - 2 + \bar{\mathbf{w}}\mathbf{w}\right)\bar{\mathbf{w}} - \tfrac{1}{2}\bar{\mathbf{w}}\mathrm{J}_{\mathsf{K}-2}\bar{\mathbf{w}}^T\mathbf{w}^T\mathrm{J}_{\mathsf{K}-2}, \tag{7.9}$$

$$M_{[23]} = -\left(x_{12} + \tfrac{\mathsf{K}}{2} - 2 + \bar{\mathbf{w}}\mathbf{w}\right)\mathrm{J}_{\mathsf{K}-2}\bar{\mathbf{w}}^T + \tfrac{1}{2}\bar{\mathbf{w}}\mathrm{J}_{\mathsf{K}-2}\bar{\mathbf{w}}^T \cdot \mathbf{w}, \tag{7.10}$$

while the free term $G_{x_{12}}$ in (7.6) is expressed via the linear term $M_{x_{12}}$ as follows:

$$G_{x_{12}} = \tfrac{1}{2}M_{x_{12}}^2 + \tfrac{1}{4}(\mathsf{K}-2)M_{x_{12}} + \tfrac{1}{4}(\mathsf{K} - 3 - x_{12}^2)\mathrm{I}_{\mathsf{K}}. \tag{7.11}$$

As a direct consequence of the RTT relation (1.1), we can identify the generators of $\mathfrak{so}_{\mathsf{K}}$ through:

$$\mathcal{F}_{ij} = \left(M_{1-t-\frac{\mathsf{K}}{2}}\right)_{ji}. \tag{7.12}$$

In particular, we have:

$$\mathcal{F}_{11} = t - \sum_{k=2}^{\mathsf{K}-1} \bar{\mathbf{a}}_k \mathbf{a}_k, \qquad \mathcal{F}_{ii} = \bar{\mathbf{a}}_i\mathbf{a}_i - \bar{\mathbf{a}}_{i'}\mathbf{a}_{i'} \quad \text{for} \quad 1 < i \le r. \tag{7.13}$$

As before, let F denote the Fock module of $\mathcal{A}$, generated by the Fock vacuum $|0\rangle \in \mathsf{F}$ satisfying:

$$\mathbf{a}_i|0\rangle = 0, \qquad 1 < i < 1'. \tag{7.14}$$

Then, the Fock vacuum $|0\rangle$ is obviously a highest weight state of the resulting $\mathfrak{so}_{\mathsf{K}}$-action:

$$\mathcal{F}_{ij}|0\rangle = 0, \qquad 1 \le i < j \le \mathsf{K}, \tag{7.15}$$

with the highest weight $\lambda = t\omega_1 = (t, \underbrace{0, \dots, 0}_{r-1})$, that is:

$$\mathcal{F}_{ii}|0\rangle = t\delta_i^1|0\rangle, \qquad 1 \le i \le r. \tag{7.16}$$

Completely similarly to Lemmas 4.8, 5.14, 6.10, 6.22, we have:

Lemma 7.17. *There is an $\mathfrak{so}_{\mathsf{K}}$-module isomorphism:*

$$\mathsf{F} \simeq \left(M_{t\omega_1}^{\mathfrak{p}_{\{2,\dots,r\}}}\right)^{\vee}. \tag{7.18}$$

Completely similarly to Corollaries 4.10, 5.16, 6.12, 6.24, we thus get:

Corollary 7.19. *(a) For $t \notin 4 - \mathsf{K} + \frac{1}{2}\mathbb{N}$, the $\mathfrak{so}_{\mathsf{K}}$-module F is irreducible.*

(b) For $t \in \mathbb{N}$, the Fock vacuum $|0\rangle$ generates an irreducible finite-dimensional $\mathfrak{so}_{\mathsf{K}}$-module $L_{t\omega_1}$.

7.2. **More oscillator realizations in types BD via underlying symmetries.**
Consider the following $2r$ endomorphisms of $\mathbb{C}^{\mathsf{K}}$:

$$\hat{B}_k = \begin{cases} \sum_{j=1}^{\mathsf{K}} e_{jj} & \text{for } k=1 \\ e_{1k}+e_{k1}+e_{1'k'}+e_{k'1'}+\sum_{1<j<1'}^{j\neq k,k'} e_{jj} & \text{for } 1<k\leq r \\ e_{1k}+e_{k1}+e_{1'k'}+e_{k'1'}+\sum_{1<j<1'}^{j\neq k,k'} e_{jj'} & \text{for } r'\leq k<1' \\ \sum_{j=1}^{\mathsf{K}} e_{jj'} & \text{for } k=1' \end{cases} . \tag{7.20}$$

Since the R-matrix (1.19) is invariant under such transformations, cf. (1.5):

$$\Big[R(x), \hat{B}_k\otimes\hat{B}_k\Big] = 0\,, \qquad \forall\, k\in\{1,\dots,r\}\cup\{r',\dots,1'\}\,, \tag{7.21}$$

we can generate more solutions to the RTT relation (1.1) from the Lax matrix (7.6) via:

$$\mathcal{L}_k(x) = \hat{B}_k L_{x_{12}=1-t-\frac{\mathsf{K}}{2}}(x)\hat{B}_k^{-1}\Big|_{p.h.}\,, \qquad \forall\, k\in\{1,\dots,r\}\cup\{r',\dots,1'\}\,. \tag{7.22}$$

Here, we apply the following particle-hole automorphism of $\mathcal{A}$ (denoted $p.h.$):

$$\begin{aligned} &\bar{\mathbf{a}}_j\mapsto -\mathbf{a}_j\,, \quad \mathbf{a}_j\mapsto\bar{\mathbf{a}}_j \quad \text{for} \quad 1<j\leq k \qquad \text{if} \quad 1\leq k\leq r\,, \\ &\bar{\mathbf{a}}_j\mapsto -\mathbf{a}_j\,, \quad \mathbf{a}_j\mapsto\bar{\mathbf{a}}_j \quad \text{for} \quad k'<j<1' \quad \text{if} \quad r'\leq k\leq 1'\,, \end{aligned} \tag{7.23}$$

uniquely chosen to insure that the Fock vacuum $|0\rangle$ remains to be an $\mathfrak{so}_{\mathsf{K}}$ highest weight state. The resulting $\mathfrak{so}_{\mathsf{K}}$ generators are read off the linear term of (7.22) in the spectral parameter:

$$\mathcal{F}_{ij}^k = \Big(\hat{B}_k M_{1-t-\frac{\mathsf{K}}{2}}\hat{B}_k^{-1}\Big|_{p.h.}\Big)_{ji}\,, \qquad k\in\{1,\dots,r\}\cup\{r',\dots,1'\}\,. \tag{7.24}$$

This makes the Fock module F into an $\mathfrak{so}_{\mathsf{K}}$-module, denoted by $M_{k,t}^+$. For $1\leq i\leq r$, we get:

$$\mathcal{F}_{ii}^k|0\rangle = \begin{cases} \Big((t+k-1)\delta_i^k-\delta_{i<k}\Big)|0\rangle & \text{for } 1\leq k\leq r \\ \Big((-t-k+2)\delta_i^{k'}-\delta_{i<k'}\Big)|0\rangle & \text{for } r'\leq k\leq 1' \end{cases}, \tag{7.25}$$

thus the corresponding highest weight of $|0\rangle\in M_{k,t}^+$ is:

$$\begin{aligned} &(\underbrace{-1,\dots,-1}_{k-1}, t+k-1, \underbrace{0,\dots,0}_{r-k}) \qquad \text{for} \quad 1\leq k\leq r\,, \\ &(\underbrace{-1,\dots,-1}_{k'-1}, -t-k+2, \underbrace{0,\dots,0}_{r-k}) \qquad \text{for} \quad r'\leq k\leq 1'\,. \end{aligned} \tag{7.26}$$

We shall now compare the above modules $M_{k,t}^+$'s with those from the Introduction. To this end, let us consider the parabolic $\mathfrak{p}_S\subset\mathfrak{so}_{\mathsf{K}}$ corresponding to $S=\{2,\dots,r\}$ with $r=\lfloor\frac{\mathsf{K}}{2}\rfloor$, see §1.5. The Weyl group of $\mathfrak{so}_{\mathsf{K}}$ can be identified with $W\simeq\{\pm1\}^r\rtimes S_r$ for $\mathsf{K}=2r+1$ or $W\simeq\{\pm1\}_+^r\rtimes S_r$ for $\mathsf{K}=2r$, cf. (6.26). The Weyl group of the Levi subalgebra $\mathfrak{l}\simeq\mathfrak{so}_{\mathsf{K}-2}\oplus\mathfrak{gl}_1$ is $W_{\mathfrak{l}}\simeq\{\pm1\}^{r-1}\rtimes S_{r-1}$ for $\mathsf{K}=2r+1$ or $W\simeq\{\pm1\}_+^{r-1}\rtimes S_{r-1}$ for $\mathsf{K}=2r$, consisting of those $(\vec{\mu},\sigma)\in W$ such that $\mu_1=+1$ and $\sigma(1)=1$. Equivalently, $W_{\mathfrak{l}}$ is the stabilizer of 1 for the natural transitive action of the Weyl group W on the set $\{1\dots,r\}\cup\{r',\dots,1'\}$. Thus, we have a set bijection

$$\pi\colon W/W_{\mathfrak{l}}\xrightarrow{\sim}\{1,\dots,r\}\cup\{r',\dots,1'\}\,, \tag{7.27}$$

cf. (4.17, 5.27, 6.30, 6.34). For any $1\leq k\leq r$, we define the permutation $\sigma_k\in S_r$ as $\sigma_{\{k\}}$ of (4.15):

$$\sigma_k(1)=k\,,\ \sigma_k(2)=1\,,\ \dots\,,\ \sigma_k(k)=k-1\,,\ \sigma_k(k+1)=k+1\,,\ \dots\,,\ \sigma_k(r)=r\,, \tag{7.28}$$

and further consider $w_k\in W$ given by:

$$w_k = \Big((+1,\dots,+1),\sigma_k\Big)\,, \qquad 1\leq k\leq r\,. \tag{7.29}$$

The element $w_k\in\pi^{-1}(k)$ can be characterized similarly to Lemmas 5.30, 6.32, 6.36:

Lemma 7.30. *w_k is the shortest representative of the left coset $\pi^{-1}(k)$, for any $1\leq k\leq r$.*

Likewise, for $1 \leq k \leq r$, we also define $w_{k'} \in W$ via:

$$w_{k'} = (\mu(k), \sigma_k) \ , \qquad 1 \leq k \leq r \ , \tag{7.31}$$

with $\sigma_k \in S_r$ as in (7.28) and $\mu(k) \in \{\pm 1\}^r$ having -1 components only at the:

(1) k-th spot, if $\mathsf{K} = 2r+1$;
(2) k-th and r-th spots, if $\mathsf{K} = 2r$ and $k < r$;
(3) $(r-1)$-th and r-th spots, if $\mathsf{K} = 2r$ and $k = r$.

Then, similarly to Lemma 7.30, we have the following characterization of such elements:

Lemma 7.32. *w_k is the shortest representative of the left coset $\pi^{-1}(k)$, for any $r' \leq k \leq 1'$.*

Combining Lemmas 7.30 and 7.32 with the set bijection (7.27) and (1.33), we get:

Corollary 7.33. ${}^{\mathfrak{l}}W = \Big\{ w_k \,\Big|\, k \in \{1, \ldots, r\} \cup \{r', \ldots, 1'\} \Big\}$.

Combining now the formula (7.26) with the definition of $w_k \in W$, see (7.29, 7.31), we conclude that the highest weight of the Fock vacuum $|0\rangle \in M^+_{k,t}$ coincides with the highest weight of our key modules $M'_{w_k \cdot t\omega_1}$ introduced in (1.36), see Corollary 7.33, for any $k \in \{1, \ldots, r\} \cup \{r', \ldots, 1'\}$. Furthermore, $M'_{w_k \cdot t\omega_1}$ has the same character as $M^+_{k,t}$ (according to Lemma 2.45) and is irreducible for $t \notin \mathbb{Z}$ (as follows from [J]). Therefore, similarly to Propositions 4.28, 5.33, 6.38, we obtain:

Proposition 7.34. *For $k \in \{1, \ldots, r\} \cup \{r', \ldots, 1'\}$ and $t \notin \mathbb{Z}$, we have $\mathfrak{so}_{\mathsf{K}}$-module isomorphisms:*

$$M^+_{k,t} \simeq M'_{w_k \cdot t\omega_1} \simeq \Big(M'_{w_k \cdot t\omega_1} \Big)^{\vee} \ . \tag{7.35}$$

Remark 7.36. Let us point out the key difference between Proposition 7.34 and Lemma 7.17:

(a) For $k = 1$, we actually have $M^+_{k,t} \simeq (M'_{w_k \cdot t\omega_1})^{\vee}$ for any $t \in \mathbb{C}$, due to Lemma 7.17.

(b) Likewise, for $k = 1'$, we have $M^+_{k,t} \simeq M'_{w_k \cdot t\omega_1}$ for any $t \in \mathbb{C}$.

(c) For other values of k, $M^+_{k,t}$ is <u>not isomorphic</u> to either of $M'_{w_k \cdot t\omega_1}$ or $(M'_{w_k \cdot t\omega_1})^{\vee}$ at certain $t \in \mathbb{Z}$ (but is expected to be isomorphic to one of the twisted Verma modules in the sense of [AL]).

Evoking the above bijection $\{1, \ldots, r\} \cup \{r', \ldots, 1'\} \ni k \leftrightarrow w_k \in {}^{\mathfrak{l}}W$ of Corollary 7.33, we define:

$$M^{\vee}_{k,t} = \Big(M'_{w_k \cdot t\omega_1} \Big)^{\vee} \ , \qquad \forall\, t \in \mathbb{C} \, . \tag{7.37}$$

Then, Proposition 7.34 can be recast as the isomorphism of the following $\mathfrak{so}_{\mathsf{K}}$-modules:

$$M^+_{k,t} \simeq M^{\vee}_{k,t} \, , \qquad \forall\, t \in \mathbb{C} \setminus \mathbb{Z} \, . \tag{7.38}$$

For the key results of the following subsection, let us record the lengths of the above elements:

$$l(w_k) = \begin{cases} k-1 & \text{for } 1 \leq k \leq r \\ k-2 & \text{for } r' \leq k \leq 1' \end{cases} \ , \tag{7.39}$$

which follows from (5.31) and the explicit description of the set Δ^+ of positive roots of $\mathfrak{so}_{\mathsf{K}}$.

7.3. **Type BD transfer matrices.**

Recall the notion of transfer matrices $\{T_W(x)\}_{W\in \mathrm{Rep}\, Y(\mathfrak{so}_{\mathsf{K}})}$, as discussed in Subsection 1.8. In particular, we shall consider the following explicit infinite-dimensional transfer matrices:

$$T^{+}_{k,t}(x) = \mathrm{tr} \prod_{i=1}^{r} \tau_i^{\mathcal{F}^k_{ii}} \underbrace{\mathcal{L}_k(x) \otimes \cdots \otimes \mathcal{L}_k(x)}_{N}, \tag{7.40}$$

corresponding to $M^{+}_{k,t}$. For $t \in \mathbb{N}$, we also consider the finite-dimensional transfer matrices $T_{1,t}(x)$ corresponding to the modules $L_{t\omega_1}$ in the auxiliary space: they are defined similarly to (7.40), but with the trace taken over the finite-dimensional submodule $L_{t\omega_1}$ of $M^{+}_{1,t}$, see Corollary 7.19(b).

Using the notation (7.37) and the formula (7.39), let us recast the resolution (2.55), dual to (1.30):

$$D_r\colon\ 0 \to L_{t\omega_1} \to M^{\vee}_{1,t} \to \cdots \to M^{\vee}_{r-1,t} \to M^{\vee}_{r,t} \oplus M^{\vee}_{r',t} \to M^{\vee}_{(r-1)',t} \to \cdots \to M^{\vee}_{1',t} \to 0\,, \tag{7.41}$$

$$B_r\colon\ 0 \to L_{t\omega_1} \to M^{\vee}_{1,t} \to \cdots \to M^{\vee}_{r-1,t} \to M^{\vee}_{r,t} \to M^{\vee}_{r',t} \to M^{\vee}_{(r-1)',t} \to \cdots \to M^{\vee}_{1',t} \to 0\,, \tag{7.42}$$

for any $t \in \mathbb{N}$. Combining them with (7.38) and the fact that the transfer matrices (7.40) depend continuously on $t \in \mathbb{C}$ (as so do the Lax matrices $\mathcal{L}_k(x)$), we obtain the key result of this section:

Theorem 7.43. *(a) For $t \in \mathbb{N}$, we have the following equality of D_r-type transfer matrices:*

$$T_{1,t}(x) \;=\; \sum_{k=1}^{r} (-1)^{k-1} T^{+}_{k,t}(x) \;+\; \sum_{k=1}^{r} (-1)^{k-1} T^{+}_{k',t}(x)\,. \tag{7.44}$$

(b) For $t \in \mathbb{N}$, we have the following equality of B_r-type transfer matrices:

$$T_{1,t}(x) \;=\; \sum_{k=1}^{r} (-1)^{k-1} T^{+}_{k,t}(x) \;+\; \sum_{k=1}^{r} (-1)^{k} T^{+}_{k',t}(x)\,. \tag{7.45}$$

The character limit of (7.44) expresses the character of the $\mathfrak{so}_{2r}$-modules $\{L_{t\omega_1}\}_{t\in\mathbb{N}}$ defined as

$$\mathrm{ch}_{1,t} = \mathrm{ch}_{1,t}(\tau_1, \ldots, \tau_r) := \mathrm{tr}_{L_{t\omega_1}} \prod_{i=1}^{r} \tau_i^{\mathcal{F}_{ii}}\,, \tag{7.46}$$

that is the length $N = 0$ case of $T_{1,t}(x)$, via:

$$\begin{aligned} \mathrm{ch}_{1,t} = &\sum_{k=1}^{r} (-1)^{k-1} \frac{\tau_1^{-1} \cdots \tau_{k-1}^{-1} \tau_k^{t+k-1}}{\prod_{1\le \ell<k}\left(1-\frac{\tau_k}{\tau_\ell}\right)\prod_{k<\ell\le r}\left(1-\frac{\tau_\ell}{\tau_k}\right)\prod_{\ell\ne k}\left(1-\frac{1}{\tau_k\tau_\ell}\right)} + \\ &\sum_{k=1}^{r} (-1)^{k-1} \frac{\tau_1^{-1} \cdots \tau_{k-1}^{-1} \tau_k^{k+1-2r-t}}{\prod_{1\le \ell<k}\left(1-\frac{\tau_k}{\tau_\ell}\right)\prod_{k<\ell\le r}\left(1-\frac{\tau_\ell}{\tau_k}\right)\prod_{\ell\ne k}\left(1-\frac{1}{\tau_k\tau_\ell}\right)}\,. \end{aligned} \tag{7.47}$$

Likewise, the character limit of (7.45) expresses the character of the $\mathfrak{so}_{2r+1}$-modules $\{L_{t\omega_1}\}_{t\in\mathbb{N}}$

$$\mathrm{ch}_{1,t} = \mathrm{ch}_{1,t}(\tau_1, \ldots, \tau_r) := \mathrm{tr}_{L_{t\omega_1}} \prod_{i=1}^{r} \tau_i^{\mathcal{F}_{ii}}\,, \tag{7.48}$$

that is the length $N = 0$ case of $T_{1,t}(x)$, via:

$$\begin{aligned} \mathrm{ch}_{1,t} = &\sum_{k=1}^{r} (-1)^{k-1} \frac{\tau_1^{-1} \cdots \tau_{k-1}^{-1} \tau_k^{t+k-1}}{\left(1-\frac{1}{\tau_k}\right)\prod_{1\le \ell<k}\left(1-\frac{\tau_k}{\tau_\ell}\right)\prod_{k<\ell\le r}\left(1-\frac{\tau_\ell}{\tau_k}\right)\prod_{\ell\ne k}\left(1-\frac{1}{\tau_k\tau_\ell}\right)} + \\ &\sum_{k=1}^{r} (-1)^{k} \frac{\tau_1^{-1} \cdots \tau_{k-1}^{-1} \tau_k^{k-2r-t}}{\left(1-\frac{1}{\tau_k}\right)\prod_{1\le \ell<k}\left(1-\frac{\tau_k}{\tau_\ell}\right)\prod_{k<\ell\le r}\left(1-\frac{\tau_\ell}{\tau_k}\right)\prod_{\ell\ne k}\left(1-\frac{1}{\tau_k\tau_\ell}\right)}\,. \end{aligned} \tag{7.49}$$

Here, the k-th summand in the first (resp. second) sums in the right-hand side of (7.47, 7.49) is equal to the character of $M^{+}_{k,t}$ (resp. $M^{+}_{k',t}$), up to a sign.

Remark 7.50. For the physics' reader who skipped Section 2, let us present a concise proof of (7.49) (the proof of (7.47) is completely analogous). Let us identify the set Δ^+ of positive roots of $\mathfrak{g} = \mathfrak{so}_{2r+1}$ with $\Delta^+ = \{\epsilon_i \pm \epsilon_j\}_{1\le i<j\le r} \cup \{\epsilon_i\}_{i=1}^r$, so that $\rho = (r-\frac{1}{2}, r-\frac{3}{2}, \dots, \frac{1}{2})$ and the Weyl group W is $W \simeq (\mathbb{Z}/2\mathbb{Z})^r \rtimes S_r \simeq \{\pm 1\}^r \rtimes S_r$. According to the Weyl character formula, we have:

$$\mathrm{ch}_{L_{t\omega_1}} = \sum_{(\vec{\mu},\sigma)\in\{\pm1\}^r\rtimes S_r} (-1)^{l(\vec{\mu},\sigma)} \frac{e^{(\vec{\mu},\sigma)(t\omega_1+\rho)-\rho}}{\prod_{1\le i<j\le r}(1-e^{\epsilon_j-\epsilon_i})(1-e^{-\epsilon_j-\epsilon_i})\prod_{i=1}^r(1-e^{-\epsilon_i})}\,. \tag{7.51}$$

Following §7.2, let us consider the parabolic subalgebra $\mathfrak{p}_{\{2,\dots,r\}} \subset \mathfrak{g}$ whose Levi subalgebra is $\mathfrak{l} \simeq \mathfrak{so}_{2r-1} \oplus \mathfrak{gl}_1$ and the Weyl group is $W_{\mathfrak{l}} \simeq \{\pm1\}^{r-1} \rtimes S_{r-1}$ consisting of $(\vec{\mu},\sigma) \in \{\pm1\}^r \rtimes S_r = W$ such that $\mu_1 = 1$ and $\sigma(1) = 1$. The assignment $W \ni (\vec{\mu},\sigma) \mapsto (\mu_1, \sigma(1)) \in \{\pm1\}\times\{1,\dots,r\}$ gives rise to a set bijection $\pi\colon W/W_{\mathfrak{l}} \xrightarrow{\sim} \{\pm1\}\times\{1,\dots,r\}$, cf. (7.27). For any $(\mu,k) \in \{\pm1\}\times\{1,\dots,r\}$, we consider $((\mu,1,\dots,1),\sigma_k) \in \pi^{-1}(\mu,k)$ with $\sigma_k \in S_r$ as in (7.28). We can rewrite (7.51) as:

$$\mathrm{ch}_{L_{t\omega_1}} = \sum_{(\mu,k)\in\{\pm1\}\times\{1,\dots,r\}} \sum_{(\vec{\nu},\tau)\in W_{\mathfrak{l}}} \frac{(-1)^{l((\mu,\vec{\nu}),\sigma_k\tau)} e^{((\mu,\vec{\nu}),\sigma_k\tau)(t\omega_1+\rho)-\rho}}{\prod_{1\le i<j\le r}(1-e^{\epsilon_j-\epsilon_i})(1-e^{-\epsilon_j-\epsilon_i})\prod_{i=1}^r(1-e^{-\epsilon_i})}\,, \tag{7.52}$$

where $(\mu,\vec{\nu}) \in \{\pm1\}^r$ is obtained by attaching $\mu \in \{\pm1\}$ on the left to $\vec{\nu} \in \{\pm1\}^{r-1}$. The key step is to simplify the inner sum of (7.52) using the Weyl denominator formula for $\mathfrak{l}$:

$$\sum_{(\vec{\nu},\tau)\in W_{\mathfrak{l}}} (-1)^{l(\vec{\nu},\tau)} e^{(\vec{\nu},\tau)(\rho_{\mathfrak{l}})-\rho_{\mathfrak{l}}} = \prod_{\alpha\in\Delta_{\mathfrak{l}}^+}(1-e^{-\alpha})\,, \tag{7.53}$$

where $\Delta_{\mathfrak{l}}^+ = \{\epsilon_i \pm \epsilon_j\}_{2\le i<j\le r} \cup \{\epsilon_i\}_{i=2}^r \subset \Delta^+$ consists of positive roots of $\mathfrak{l}$, $\rho_{\mathfrak{l}} = \frac{1}{2}\sum_{\alpha\in\Delta_{\mathfrak{l}}^+}\alpha = (0, r-\frac{3}{2}, \dots, \frac{1}{2})$. As $(\vec{\nu},\tau)(\rho)-\rho = (\vec{\nu},\tau)(\rho_{\mathfrak{l}})-\rho_{\mathfrak{l}}$, $(\vec{\nu},\tau)(\omega_1) = \omega_1$ for any $(\vec{\nu},\tau) \in W_{\mathfrak{l}} \subset W$, we get:

$$\sum_{(\vec{\nu},\tau)\in W_{\mathfrak{l}}} (-1)^{l(\vec{\nu},\tau)} \frac{e^{(\vec{\nu},\tau)(t\omega_1+\rho)-\rho}}{\prod_{1\le i<j\le r}(1-e^{\epsilon_j-\epsilon_i})(1-e^{-\epsilon_j-\epsilon_i})\prod_{i=1}^r(1-e^{-\epsilon_i})} = \frac{e^{t\epsilon_1}}{(1-e^{-\epsilon_1})\prod_{j=2}^r(1-e^{\epsilon_j-\epsilon_1})(1-e^{-\epsilon_j-\epsilon_1})}\,. \tag{7.54}$$

Thus, the inner sum of (7.52) indexed by $(\mu,k) = (1,1)$ gives rise to the $k=1$ summand of the first sum in (7.49). Likewise, we claim that the inner sum of (7.52) indexed by (μ,k) with $\mu = 1$ (resp. $\mu = -1$) precisely recovers the k-th summand of the first (resp. second) sum in (7.49), which amounts to the proof of (7.56) below (and its $\mu = -1$ counterpart). To prove this claim for $\mu = 1$, let us apply $((+1,\dots,+1),\sigma_k) \in W$ to both sides of the equality (7.53):

$$\sum_{(\vec{\nu},\tau)\in W_{\mathfrak{l}}} (-1)^{l((1,\vec{\nu}),\sigma_k\tau)} e^{((1,\vec{\nu}),\sigma_k\tau)(\rho)-\rho} = (-1)^{l(\sigma_k)} e^{\sigma_k(\rho)-\rho} \prod_{1\le i\le r}^{i\ne k}(1-e^{-\epsilon_i}) \prod_{1\le i<j\le r}^{i,j\ne k}(1-e^{\epsilon_j-\epsilon_i})(1-e^{-\epsilon_j-\epsilon_i})\,. \tag{7.55}$$

Combining this with the formulas

$$\sigma_k(\rho)-\rho = (k-1)\epsilon_k - (\epsilon_1+\dots+\epsilon_{k-1})\,, \qquad l(\sigma_k) = k-1\,, \qquad \sigma_k(\omega_1) = \epsilon_k\,,$$

we obtain the desired equality:

$$\sum_{(\vec{\nu},\tau)\in W_{\mathfrak{l}}} (-1)^{l((1,\vec{\nu}),\sigma_k\tau)} \frac{e^{((1,\vec{\nu}),\sigma_k\tau)(t\omega_1+\rho)-\rho}}{\prod_{1\le i<j\le r}(1-e^{\epsilon_j-\epsilon_i})(1-e^{-\epsilon_j-\epsilon_i})\prod_{i=1}^r(1-e^{-\epsilon_i})} = (-1)^{k-1} \frac{e^{-\epsilon_1}\cdots e^{-\epsilon_{k-1}} e^{(t+k-1)\epsilon_k}}{(1-e^{-\epsilon_k})\prod_{\ell=1}^{k-1}(1-e^{\epsilon_k-\epsilon_\ell})\prod_{\ell=k+1}^r(1-e^{\epsilon_\ell-\epsilon_k})\prod_{1\le\ell\le r}^{\ell\ne k}(1-e^{-\epsilon_k-\epsilon_\ell})}\,. \tag{7.56}$$

The proof of the above claim for $\mu = -1$ is completely analogous with the only difference that:

$$((-1,+1,\dots,+1),\sigma_k)(\rho)-\rho = (k-2r)\epsilon_k - (\epsilon_1+\dots+\epsilon_{k-1})\,, \qquad ((-1,+1,\dots,+1),\sigma_k)(\omega_1) = -\epsilon_k\,.$$

This completes our direct proof of the character formula (7.49) (see Remark 3.28 for more details in regards to perceiving $\mathrm{ch}_{1,t}$ of (7.48) as a specialization of $\mathrm{ch}_{L_{t\omega_1}}$).

Let us note that the formulas (7.47) and (7.49) allow to analytically continue the character $\mathrm{ch}_{1,t}$ of (7.46) and (7.48), from the discrete set $t \in \mathbb{N}$ to the entire complex plane $t \in \mathbb{C}$. With these conventions in mind and similarly to Lemmas 5.51, 6.51, we obtain:

Lemma 7.57. *(a)* $\mathrm{ch}_{1,t} = (-1)^{\mathsf{K}}\, \mathrm{ch}_{1,2-\mathsf{K}-t}$ *for any* $t \in \mathbb{C}$.

(b) $\mathrm{ch}_{1,t} = 0$ *for* $t \in \{-1, -2, \dots, 3-\mathsf{K}\}$.

In a completely similar way, the formulas (7.44, 7.45) allow to analytically continue the transfer matrices $T_{1,t}(x)$ of the finite-dimensional representations $L_{t\omega_1}, t \in \mathbb{N}$, to the entire complex plane $t \in \mathbb{C}$. With this convention in mind, we have the following generalization of Lemma 7.57(a):

Proposition 7.58. $T_{1,t}(x) = (-1)^{\mathsf{K}} T_{1,2-\mathsf{K}-t}(x)$ *for any* $t \in \mathbb{C}$.

Proof. The proof is completely analogous to that of Proposition 5.59 and follows from the proof of Lemma 7.57(a) combined with the factorisation (9.24) of the transfer matrices $T^{+}_{k,t}(x), T^{+}_{k',t}(x)$ into the product of two commuting Q-operators, cf. Proposition 9.26. □

We also expect the generalization of Lemma 7.57(b) to hold: $T_{1,t}(x) = 0$ for $t \in \{-1, -2, \dots, 3-\mathsf{K}\}$. For D-type, this was first observed in [FFK] for small length and rank.

8. Factorisation for linear ACD-types

In this section, we demonstrate the factorisation of the infinite-dimensional transfer matrices (4.35, 5.39, 6.44) into the products of two Baxter Q-operators arising from degenerate Lax matrices (which are *renormalized limits* of the former), linear in the spectral parameter. The factorisation formula is universal for all three types ACD, and we present it in full detail for the case of A-type.

8.1. General two-term linear factorisation.

Consider the following two $n \times n$ matrices written in the block form as:

$$L_a(x) = \left(\begin{array}{c|c} x\mathrm{I}_a - \bar{\mathbf{A}}_1\mathbf{A}_1 & \bar{\mathbf{A}}_1 \\ \hline -\mathbf{A}_1 & \mathrm{I}_{n-a} \end{array}\right), \qquad \bar{L}_a(y) = \left(\begin{array}{c|c} \mathrm{I}_a & \bar{\mathbf{A}}_2 \\ \hline \mathbf{A}_2 & y\mathrm{I}_{n-a} + \mathbf{A}_2\bar{\mathbf{A}}_2 \end{array}\right), \tag{8.1}$$

with $a \times (n-a)$ upper-right blocks $\bar{\mathbf{A}}_1, \bar{\mathbf{A}}_2$ and $(n-a) \times a$ lower-left blocks $\mathbf{A}_1, \mathbf{A}_2$. Then, their product can be factorised as follows:

$$L_a(x)\bar{L}_a(y) = \left(\begin{array}{c|c} x\mathrm{I}_a - \bar{\mathbf{A}}'_1\mathbf{A}'_1 & \left(y - x + \bar{\mathbf{A}}'_1\mathbf{A}'_1\right)\bar{\mathbf{A}}'_1 \\ \hline -\mathbf{A}'_1 & y\mathrm{I}_{n-a} + \mathbf{A}'_1\bar{\mathbf{A}}'_1 \end{array}\right) \left(\begin{array}{c|c} \mathrm{I}_a & \bar{\mathbf{A}}'_2 \\ \hline 0 & \mathrm{I}_{n-a} \end{array}\right), \tag{8.2}$$

where

$$\begin{aligned} \mathbf{A}'_1 &= \mathbf{A}_1 - \mathbf{A}_2\,, \qquad & \mathbf{A}'_2 &= \mathbf{A}_2\,, \\ \bar{\mathbf{A}}'_2 &= \bar{\mathbf{A}}_2 + \bar{\mathbf{A}}_1\,, \qquad & \bar{\mathbf{A}}'_1 &= \bar{\mathbf{A}}_1\,. \end{aligned} \tag{8.3}$$

We note that the right-hand side of (8.2) is independent of $\mathbf{A}'_2$.

8.2. **Two-term factorisation in A-type.**

Let $\mathcal{A}$ denote the oscillator algebra generated by $n(n-1)$ pairs of oscillators $\{(\mathbf{a}_{j,i}, \bar{\mathbf{a}}_{i,j})\}_{1\leq i\neq j\leq n}$ subject to (3.1). For any subset $I \in \mathcal{S}_a$, see our notation (4.11), recall the permutation σ_I of the set $\{1,\dots,n\}$ defined in (4.15) and the corresponding permutation matrix B_I of (4.19). We define:

$$L_I(x) = B_I L_{\{1,\dots,a\}}(x) B_I^{-1}\Big|_{p.h.}\,, \tag{8.4}$$

where

$$L_{\{1,\dots,a\}}(x) = \left(\begin{array}{c|c} x\mathrm{I}_a - \bar{\mathbf{A}}\mathbf{A} & \bar{\mathbf{A}} \\ \hline -\mathbf{A} & \mathrm{I}_{n-a} \end{array}\right) = \left(\begin{array}{c|c} \mathrm{I}_a & \bar{\mathbf{A}} \\ \hline 0 & \mathrm{I}_{n-a} \end{array}\right) \left(\begin{array}{c|c} x\mathrm{I}_a & 0 \\ \hline 0 & \mathrm{I}_{n-a} \end{array}\right) \left(\begin{array}{c|c} \mathrm{I}_a & 0 \\ \hline -\mathbf{A} & \mathrm{I}_{n-a} \end{array}\right) \tag{8.5}$$

with

$$\mathbf{A} = \begin{pmatrix} \mathbf{a}_{a+1,1} & \cdots & \mathbf{a}_{a+1,a} \\ \vdots & \iddots & \vdots \\ \mathbf{a}_{n,1} & \cdots & \mathbf{a}_{n,a} \end{pmatrix}, \qquad \bar{\mathbf{A}} = \begin{pmatrix} \bar{\mathbf{a}}_{1,a+1} & \cdots & \bar{\mathbf{a}}_{1,n} \\ \vdots & \iddots & \vdots \\ \bar{\mathbf{a}}_{a,a+1} & \cdots & \bar{\mathbf{a}}_{a,n} \end{pmatrix}, \tag{8.6}$$

as in (4.3), and the particle-hole transformation (denoted *p.h.*) in (8.4) is chosen as follows:

$$\begin{aligned} &\bar{\mathbf{a}}_{i,j} \mapsto \mathbf{a}_{\sigma_I(j),\sigma_I(i)}\,, \quad \mathbf{a}_{j,i} \mapsto -\bar{\mathbf{a}}_{\sigma_I(i),\sigma_I(j)} \quad \text{if} \quad \sigma_I(j) < \sigma_I(i)\,, \\ &\bar{\mathbf{a}}_{i,j} \mapsto \bar{\mathbf{a}}_{\sigma_I(i),\sigma_I(j)}\,, \quad \mathbf{a}_{j,i} \mapsto \mathbf{a}_{\sigma_I(j),\sigma_I(i)} \quad \text{if} \quad \sigma_I(j) > \sigma_I(i)\,. \end{aligned} \tag{8.7}$$

Let us note that the matrix $L_I(x)$ of (8.4) depends only on the oscillators $(\bar{\mathbf{a}}_{i,j}, \mathbf{a}_{j,i}) \in \mathcal{A}$ with $i \in I$ and $j \in \bar{I} = \{1,\dots,n\} \setminus I$, see (4.13), and can be further factorised similarly to (8.5) as:

$$L_I(x) = \left(\mathrm{I}_n + \sum_{i\in I}^{j\in\bar{I}} (\bar{\mathbf{a}}_{ij}\delta_{i<j} + \mathbf{a}_{ji}\delta_{i>j})\, e_{ij}\right) \left(x\sum_{i\in I} e_{ii} + \sum_{j\in\bar{I}} e_{jj}\right) \left(\mathrm{I}_n + \sum_{i\in I}^{j\in\bar{I}} (\bar{\mathbf{a}}_{ij}\delta_{j<i} - \mathbf{a}_{ji}\delta_{j>i})\, e_{ji}\right).$$

Remark 8.8. We note that the particle-hole (8.7) differs from (4.21) in two aspects: (1) a different sign change, (2) relabelling of the oscillator indices to indicate the row and column of their position.

Let us now apply the general factorisation from Subsection 8.1 to the following choice of (8.1):

$$L_a(x) = L_{\{1,\dots,a\}}(x)\,, \qquad \bar{L}_a(y) = L_{\{a+1,\dots,n\}}(y)\,. \tag{8.9}$$

This fixes $\mathbf{A}_1$ and $\bar{\mathbf{A}}_1$ of (8.1) as $\mathbf{A}$ and $\bar{\mathbf{A}}$ of (8.6), while $\mathbf{A}_2$ and $\bar{\mathbf{A}}_2$ are explicitly given by:

$$\mathbf{A}_2 = \begin{pmatrix} \mathbf{a}_{1,a+1} & \cdots & \mathbf{a}_{a,a+1} \\ \vdots & \iddots & \vdots \\ \mathbf{a}_{1,n} & \cdots & \mathbf{a}_{a,n} \end{pmatrix}, \qquad \bar{\mathbf{A}}_2 = \begin{pmatrix} \bar{\mathbf{a}}_{a+1,1} & \cdots & \bar{\mathbf{a}}_{n,1} \\ \vdots & \iddots & \vdots \\ \bar{\mathbf{a}}_{a+1,a} & \cdots & \bar{\mathbf{a}}_{n,a} \end{pmatrix}. \tag{8.10}$$

We note that these two matrices $L_a(x)$ and $\bar{L}_a(y)$ involve non-intersecting sets of oscillators from the algebra $\mathcal{A}$, hence, they mutually commute, while the only nontrivial commutators are:

$$[\mathbf{a}_{j,i}, \bar{\mathbf{a}}_{i,j}] = 1\,. \tag{8.11}$$

Thus, the transformation (8.3) in this case is in fact induced by the similarity transformation:

$$\begin{aligned} \mathbf{A}_1' &= \mathbf{S}\mathbf{A}_1\mathbf{S}^{-1}\,, \qquad \mathbf{A}_2' = \mathbf{S}\mathbf{A}_2\mathbf{S}^{-1}\,, \\ \bar{\mathbf{A}}_2' &= \mathbf{S}\bar{\mathbf{A}}_2\mathbf{S}^{-1}\,, \qquad \bar{\mathbf{A}}_1' = \mathbf{S}\bar{\mathbf{A}}_1\mathbf{S}^{-1}\,, \end{aligned} \tag{8.12}$$

with

$$\mathbf{S} = \exp\left[\sum_{1\leq i\leq a}^{a<j\leq n} \bar{\mathbf{a}}_{ij}\mathbf{a}_{ij}\right], \tag{8.13}$$

where we note that all the summands in (8.13) pairwise commute.

Combining the factorisation formula (8.2) with the similarity transformation (8.12), we obtain:

$$L_{\{1,\dots,a\}}(x+t) L_{\{a+1,\dots,n\}}(x-a) = \mathbf{S}\mathcal{L}_a(x)\mathrm{G}\mathbf{S}^{-1}\,, \tag{8.14}$$

where

$$\mathrm{G} = \left(\begin{array}{c|c} \mathrm{I}_a & \bar{\mathbf{A}}_2 \\ \hline 0 & \mathrm{I}_{n-a} \end{array}\right), \tag{8.15}$$

$\mathbf{S}$ is given by (8.13), and $\mathcal{L}_a(x)$ is precisely the $\mathfrak{gl}_n$-type Lax matrix of (4.2).

Vice versa, the matrices $L_{\{1,\dots,a\}}(x)$ and $L_{\{a+1,\dots,n\}}(x)$ can be obtained from the Lax matrix $\mathcal{L}_a(x)$ of (4.2) via the *renormalized limit* procedures (which clearly preserve the property of being Lax):

$$\begin{aligned} L_{\{1,\dots,a\}}(x) &= \lim_{t\to\infty} \Big\{\mathcal{L}_a(x-t)\cdot \mathrm{diag}\Big(\underbrace{1,\dots,1}_{a};\underbrace{-\tfrac{1}{t},\dots,-\tfrac{1}{t}}_{n-a}\Big)\Big\}, \\ L_{\{a+1,\dots,n\}}(x) &= \lim_{t\to\infty} \Big\{\mathrm{diag}\Big(\underbrace{\tfrac{1}{t},\dots,\tfrac{1}{t}}_{a};\underbrace{1,\dots,1}_{n-a}\Big)\cdot \mathcal{L}_a(x+a)\Big\}\Big|_{\bar{\mathbf{a}}_{ij}\mapsto -\bar{\mathbf{a}}_{ji}\,,\,\mathbf{a}_{ij}\mapsto -\mathbf{a}_{ji}}. \end{aligned} \tag{8.16}$$

Remark 8.17. This implies that all the matrices $\{L_I(x)\}_{I\in\mathcal{S}_a}$ of (8.4) are in fact Lax, that is, they satisfy the RTT relation (1.1), which is crucial for the entire analysis of the present section.

Conjugating the factorisation formula (8.14) by B_I of (4.19), thus utilizing the Weyl group action, and further performing the particle-hole transformations in both sets of oscillators, we obtain:

$$L_I(x+t)L_{\bar{I}}(x-a) = \mathbf{S}_I\mathcal{L}'_I(x)\mathrm{G}_I\mathbf{S}_I^{-1} \tag{8.18}$$

with the similarity transformation $\mathbf{S}_I$ and the matrix G_I specified in (8.23) and (8.22) below, and the $\mathfrak{gl}_n$-type Lax matrix $\mathcal{L}'_I(x)$ obtained from $\mathcal{L}_I(x)$ of (4.12) through the sign change and relabelling of the oscillator indices precisely as in Remark 8.8. The factorisation (8.18) above follows when performing the particle-hole transformation (8.7) for the oscillators contained in $L_{\{1,\dots,a\}}(x+t)$, see (8.4), and further undoing the particle-hole transformation for the *creation/annihilation* oscillators contained in $L_{\{a+1,\dots,n\}}(x-a)$ that get mapped below/above the main diagonal:

$$\begin{aligned} &\bar{\mathbf{a}}_{j,i}\mapsto -\mathbf{a}_{\sigma_I(i),\sigma_I(j)}\,,\quad \mathbf{a}_{i,j}\mapsto \bar{\mathbf{a}}_{\sigma_I(j),\sigma_I(i)} \quad \text{if} \quad \sigma_I(j)<\sigma_I(i)\,, \\ &\bar{\mathbf{a}}_{j,i}\mapsto \bar{\mathbf{a}}_{\sigma_I(j),\sigma_I(i)}\,,\quad \mathbf{a}_{i,j}\mapsto \mathbf{a}_{\sigma_I(i),\sigma_I(j)} \quad \text{if} \quad \sigma_I(j)>\sigma_I(i)\,, \end{aligned} \tag{8.19}$$

with the permutation σ_I of the set $\{1,\dots,n\}$ as in (4.15). More precisely, the latter particle-hole (denoted $\overline{p.h.}$) is chosen so that the following equality holds:

$$L_{\bar{I}}(x) = B_I L_{\{a+1,\dots,n\}}(x) B_I^{-1}\Big|_{\overline{p.h.}}\,, \tag{8.20}$$

where we note the following natural compatibility between B_I and $B_{\bar{I}}$:

$$B_I B_{\{a+1,\dots,n\}} = B_{\bar{I}}\,, \qquad \forall\, I\in\mathcal{S}_a\,. \tag{8.21}$$

The remaining ingredients in (8.18) can be written as:

$$\mathrm{G}_I = B_I\mathrm{G}B_I^{-1}\Big|_{\overline{p.h.}} = \mathrm{I}_n + \sum_{i\in I}^{j\in\bar{I}}\Big(\bar{\mathbf{a}}_{ji}\delta_{i<j} - \mathbf{a}_{ij}\delta_{i>j}\Big)e_{ij}\,, \tag{8.22}$$

and $\mathbf{S}_I$ is obtained from $\mathbf{S}$ of (8.13) by performing both particle-hole transformations $p.h.$, $\overline{p.h.}$:

$$\mathbf{S}_I = \exp\left[\sum_{i\in I}^{j\in\bar{I}}\Big(\bar{\mathbf{a}}_{ij}\mathbf{a}_{ij}\delta_{i<j} - \mathbf{a}_{ji}\bar{\mathbf{a}}_{ji}\delta_{i>j}\Big)\right]. \tag{8.23}$$

Following [BFLMS], let us now define the Q-operators $\{Q_I(x)\}_{I\in\mathcal{S}_a}\subset \mathrm{End}(\mathbb{C}^n)^{\otimes N}$ via:

$$Q_I(x) = \widehat{\mathrm{tr}}_{D_I}\Big(\underbrace{L_I(x)\otimes\cdots\otimes L_I(x)}_{N}\Big), \tag{8.24}$$

that is, as the *normalized trace* $\widehat{\mathrm{tr}}_{D_I}$, defined as in (3.37), of the N-fold tensor product of $L_I(x)$ from (8.4). The twist D_I in (8.24) acts only on the Fock space and is defined via:

$$D_I = \prod_{i\in I,\, j\in \bar{I}}^{i<j} \left(\frac{\tau_j}{\tau_i}\right)^{\bar{\mathbf{a}}_{ij}\mathbf{a}_{ji}} \prod_{i\in I,\, j\in \bar{I}}^{i>j} \left(\frac{\tau_i}{\tau_j}\right)^{\mathbf{a}_{ji}\bar{\mathbf{a}}_{ij}} , \tag{8.25}$$

which can be further expressed via $\{\mathcal{E}^I_{ii}\}_{i=1}^n$ of (4.12) as (cf. formulas (4.23, 4.24)):

$$D_I = \prod_{i\in I} \tau_i^{-t} \cdot \prod_{i=1}^n \tau_i^{\mathcal{E}^I_{ii}} . \tag{8.26}$$

We note that the action of this twist on the Fock module is uniquely determined (up to a scalar function) by the same condition as in (3.39):

$$DL_I(x)D^{-1} = D_I^{-1} L_I(x) D_I , \tag{8.27}$$

with $D = \mathrm{diag}(\tau_1, \dots, \tau_n)$ of (3.40). This ensures the commutativity of $Q_I(x)$ and the transfer matrix $T_{(1,0,\dots,0)}(y)$ of the defining fundamental representation in the auxiliary space, cf. Remark 3.67.

Remark 8.28. For $a = 1$ and $I = \{i\}$, the Lax matrix (8.4) coincides with $L_i(x)$ of (3.31) and the twist (8.25) coincides with (3.38), hence, the above Q-operator (8.24) recovers $Q_i(x)$ of (3.36).

Building the monodromy matrices (by considering the N-fold tensor product on the matrix space) from (8.18), taking the normalized trace, and evoking the relation (8.26), the factorisation formula (8.18) implies the following factorisation formula for the transfer matrices $T^+_{I,t}(x)$ of (4.35):

$$T^+_{I,t}(x) = \mathrm{ch}^+_{I,t} \cdot Q_I(x+t) Q_{\bar{I}}(x-a) \tag{8.29}$$

with

$$\mathrm{ch}^+_{I,t} = \mathrm{tr} \prod_{i=1}^n \tau_i^{\mathcal{E}^I_{ii}} = \prod_{i\in I} \tau_i^t \prod_{i\in I}^{j\in \bar{I}} \frac{(-1)^{\delta_{i>j}} \tau_i}{\tau_i - \tau_j} , \tag{8.30}$$

cf. the factorisation formula (3.44), the character formula (4.40), and the details of Remark 3.42.

Remark 8.31. Let us stress right away that the transfer matrices constructed from $\mathcal{L}'_I(x)$ and $\mathcal{L}_I(x)$ via (4.35) <u>do coincide</u>, as the sign change and the relabelling of oscillators (see Remark 8.8) do not affect the trace.

Remark 8.32. An essential step used in our derivation of (8.29) is the following commutativity:

$$[\mathbf{S}_I, D_I D_{\bar{I}}] = 0 , \tag{8.33}$$

cf. (3.59). Clearly, it suffices to verify (8.33) for $I = \{1, \dots, a\}$. To this end, we note that:

$$D_{\{1,\dots,a\}} D_{\{a+1,\dots,n\}} = \prod_{1\le i\le a}^{a<j\le n} \left(\frac{\tau_j}{\tau_i}\right)^{\bar{\mathbf{a}}_{ij}\mathbf{a}_{ji} + \mathbf{a}_{ij}\bar{\mathbf{a}}_{ji}} = \prod_{1\le i\le a} \tau_i^{\mathbf{N}_i} \prod_{a<j\le n} \tau_j^{\mathbf{N}_j} \tag{8.34}$$

with

$$\mathbf{N}_i = -\sum_{a<j\le n} (\bar{\mathbf{a}}_{ij}\mathbf{a}_{ji} + \mathbf{a}_{ij}\bar{\mathbf{a}}_{ji}) , \qquad \mathbf{N}_j = \sum_{1\le i\le a} (\bar{\mathbf{a}}_{ij}\mathbf{a}_{ji} + \mathbf{a}_{ij}\bar{\mathbf{a}}_{ji}) , \tag{8.35}$$

while the similarity transformation $\mathbf{S}_I$ is given by (8.13):

$$\mathbf{S}_{\{1,\dots,a\}} = \prod_{1\le i\le a}^{a<j\le n} \exp\left[\bar{\mathbf{a}}_{ij}\mathbf{a}_{ij}\right] . \tag{8.36}$$

Thus, the desired commutativity (8.33) follows from the obvious equalities

$$[\mathbf{N}_i, \bar{\mathbf{a}}_{k\ell}\mathbf{a}_{\ell k}] = 0 , \qquad [\mathbf{N}_j, \bar{\mathbf{a}}_{k\ell}\mathbf{a}_{\ell k}] = 0 , \tag{8.37}$$

for any $1 \le i \le a < j \le n$ and $1 \le k \le a < \ell \le n$, cf. (3.65, 3.66).

Combining the factorisation formula (8.29) with Theorem 4.37, we get (see (4.34) and (8.30)):

Proposition 8.38. *For any $1 \leq a < n$ and $t \in \mathbb{N}$, we have:*

$$T_{a,t}(x) = \sum_{I \in \mathcal{S}_a} (-1)^{l(I)} \operatorname{ch}^+_{I,t} \cdot Q_I(x+t) Q_{\bar{I}}(x-a) \,. \tag{8.39}$$

Remark 8.40. (a) Such formula first appeared in [BHK, (5.12)] for the $n = 3$ trigonometric case, while the general rational case goes back to [Ts1, Ts2]. However, our derivation of (8.39) from (4.38) has a benefit of not using the determinant formula (3.47) that is absent in other types.

(b) Let us note that (8.39) is also a consequence of the determinant formula (3.47) and its analogue expressing any Q-operator $Q_I(x)$ of (8.24) in terms of the single-indexed Q-operators (3.36):

$$Q_I(x) = \frac{\det \left\| \tau_{i_k}^{-\ell+1} Q_{i_k}(x-\ell+1) \right\|_{1 \leq k, \ell \leq a}}{\det \left\| \tau_{i_k}^{-\ell+1} \right\|_{1 \leq k, \ell \leq a}} \,, \tag{8.41}$$

where $I = \{i_1, \ldots, i_a\}$ (the right-hand side of (8.41) is clearly independent of the ordering of i_c's). To establish this formula, one has to consider a family of $\mathfrak{gl}_n$-type Lax matrices $\{\mathbb{L}_I(x)\}_{I \subseteq \{1,\ldots,n\}}$, see [BFLMS, (2.20)], generalizing $L_I(x)$ of (8.4) by letting their matrix coefficients to take values in the bigger algebra $\mathcal{A} \otimes U(\mathfrak{gl}_{|I|})[x]$. Explicitly, we set $\mathbb{L}_{\{1,\ldots,a\}}(x) = L_{\{1,\ldots,a\}}(x) + \sum_{i,j=1}^a e_{ij} E_{ji}$ with $\{E_{ji}\}_{i,j=1}^a$ being the generators of $\mathfrak{gl}_a$, while all other $\mathbb{L}_I(x)$ are again obtained through the similarity and particle-hole transformations precisely as in (8.4), thus resulting in:

$$\mathbb{L}_I(x) = L_I(x) + \sum_{i,j \in I} E_{\sigma_I^{-1}(j), \sigma_I^{-1}(i)} e_{ij} \,. \tag{8.42}$$

Generalizing $Q_I(x)$ of (8.24), one defines the X-operators $\{X_I^+(x,\lambda)\}_{I \in \mathcal{S}_a}^{\lambda \in \mathbb{C}^a} \subset \operatorname{End}(\mathbb{C}^n)^{\otimes N}$ via:

$$X_I^+(x,\lambda) = \operatorname{tr}_{M_\lambda^\vee} \left\{ \prod_{i \in I} \tau_i^{E_{ii}} \, \widehat{\operatorname{tr}}_{D_I} \Big(\underbrace{\mathbb{L}_I(x) \otimes \cdots \otimes \mathbb{L}_I(x)}_{N} \Big) \right\} \,, \tag{8.43}$$

cf. [BFLMS, (4.13)]. Thus, $X_I^+(x,\lambda)$ is the *normalized trace* $\widehat{\operatorname{tr}}_{D_I}$ of (3.37) in the Fock module F of $\mathcal{A}$ followed by the standard trace in the dual Verma module $M_\lambda^\vee$ of $\mathfrak{gl}_a$ of the N-fold tensor product of $\mathbb{L}_I(x)$ with the twist D_I defined in (8.25). Likewise, for a dominant integral weight λ of $\mathfrak{gl}_a$, one defines $X_I(x,\lambda)$ with the outer trace taken over the finite-dimensional $\mathfrak{gl}_a$-submodule L_λ of $M_\lambda^\vee$. The latter construction allows to recover back the Q-operators via:

$$Q_I(x) = X_I\Big(x, (\underbrace{0, \ldots, 0}_{a})\Big) \,. \tag{8.44}$$

Then, evoking the BGG resolution of the finite-dimensional $\mathfrak{gl}_a$-module L_λ, we obtain the following counterpart of Theorem 3.24 (cf. [BFLMS, (4.19)]):

$$X_I(x,\lambda) = \sum_{\sigma \in S_a} (-1)^{l(\sigma)} X_I^+(x, \sigma \cdot \lambda) \,. \tag{8.45}$$

On the other hand, arguing precisely as in our proof of (3.44), we obtain the following counterpart of the latter (cf. [BFLMS, (5.7)]):

$$X_I^+(x,\lambda) = \prod_{1 \leq k < \ell \leq a} \frac{1}{\tau_{i_\ell}^{-1} - \tau_{i_k}^{-1}} \prod_{1 \leq k \leq a} \tau_{i_k}^{\lambda_k - k + 1} Q_{i_1}(x+\lambda_1) Q_{i_2}(x+\lambda_2-1) \cdots Q_{i_a}(x+\lambda_a-a+1) \,, \tag{8.46}$$

where $I = \{i_1, \ldots, i_a\} \in \mathcal{S}_a$ and $\lambda = (\lambda_1, \ldots, \lambda_a) \in \mathbb{C}^a$. Combining (8.46) with (8.45) and evoking the Vandermonde determinant, we obtain the following analogue of Theorem 3.46:

$$X_I(x,\lambda) = \frac{\det \left\| \tau_{i_k}^{\lambda_\ell - \ell + 1} Q_{i_k}(x+\lambda_\ell-\ell+1) \right\|_{1 \leq k, \ell \leq a}}{\det \left\| \tau_{i_k}^{-\ell+1} \right\|_{1 \leq k, \ell \leq a}} \,. \tag{8.47}$$

Specializing this formula at $\lambda = (0, \ldots, 0)$ and evoking (8.44), we recover the desired formula (8.41).

(c) Conversely, plugging the formula (8.41) into (8.39), we recover (by expanding the corresponding $n \times n$ determinant with respect to the first a columns) the determinant formula (3.47) in the particular case of $\lambda = t\omega_a$, the multiples of fundamental weights.

For completeness of our exposition, let us conclude with the QQ-relations in the present conventions:

Lemma 8.48. *For any two disjoint subsets I and $\{i,j\}$ of $\{1,\dots,n\}$, we have:*

$$Q_{I\sqcup i\sqcup j}(x+\tfrac{1}{2})Q_I(x-\tfrac{1}{2}) = \frac{\tau_j}{\tau_j-\tau_i}Q_{I\sqcup i}(x-\tfrac{1}{2})Q_{I\sqcup j}(x+\tfrac{1}{2}) - \frac{\tau_i}{\tau_j-\tau_i}Q_{I\sqcup j}(x-\tfrac{1}{2})Q_{I\sqcup i}(x+\tfrac{1}{2})\,.$$

Proof. Let $I=\{i_1,\dots,i_a\}$ and set $i_0=j$, $i_{a+1}=i$. Then, the QQ-relation stated above follows immediately from the Desnanot-Jacobi-Dodgson-Sylvester theorem applied to the $(a+2)\times(a+2)$ matrix $M=\left(\tau_{i_k}^{-\ell+1}Q_{i_k}(x-\ell+\frac{3}{2})\right)_{0\le k\le a+1}^{1\le\ell\le a+2}$ with $Q_i(x)$ defined in (3.36). □

Remark 8.49. Generalizing our earlier Remarks 3.48 and 8.28, we note that the Q-operators $\mathsf{Q}_I(x)$ of [BFLMS, (4.13, 4.20)] are related to ours from (8.24) via:

$$\mathsf{Q}_I(x) = \prod_{i\in I}\tau_i^x\cdot Q_I\left(x-\tfrac{n-|I|}{2}\right)\,, \tag{8.50}$$

where the twist parameters and the oscillators are identified via (3.52) and (3.50), respectively. Here, the shift of the spectral parameter x by $\frac{n-|I|}{2}$ arises when identifying our Lax matrices (8.5) with those of [BFLMS, (2.20)], while the additional factor $\prod_{i\in I}\tau_i^x$ is due to the conventions of [BFLMS]. Thus, our QQ-relations of Lemma 8.48 are equivalent to the QQ-relations of [BFLMS, (5.12)] (though the latter need to be corrected by a sign as already seen from [BFLMS, (5.9)]), see (1.9):

$$\frac{\tau_j-\tau_i}{\sqrt{\tau_i\tau_j}}\cdot\mathsf{Q}_{I\sqcup i\sqcup j}(x)\mathsf{Q}_I(x) = \mathsf{Q}_{I\sqcup i}(x-\tfrac{1}{2})\mathsf{Q}_{I\sqcup j}(x+\tfrac{1}{2}) - \mathsf{Q}_{I\sqcup j}(x-\tfrac{1}{2})\mathsf{Q}_{I\sqcup i}(x+\tfrac{1}{2})\,. \tag{8.51}$$

Remark 8.52. Let us also note that the commutativity of the single-index Q-operators $\{Q_i(x)\}_{i=1}^n$, see Remark 3.67(c), is essential both to the derivation of the determinant formulas (3.47, 8.41) as well as to the above proof of Lemma 8.48. Furthermore, combining Remark 3.67 with the determinant expression (8.41), we conclude that all the Q-operators $\{Q_I(x)|I\subseteq\{1,\dots,n\}\}$ commute among themselves as well as with the transfer matrices $T^+_{I,t}(x)$ and $T_{a,t}(x)$ of Subsection 4.3.

8.3. **Two-term factorisation in C-type.**

Inspired by (5.3), let us consider the particular example of the general factorisation (8.2) applied in the case when $n=2r$, $a=r$, and the $r\times r$ matrices $\mathbf{A}_1,\bar{\mathbf{A}}_1,\mathbf{A}_2,\bar{\mathbf{A}}_2$ are explicitly given by:

$$\mathbf{A}_1=\begin{pmatrix}\mathbf{a}_{r',1} & \cdots & \mathbf{a}_{r',r-1} & \mathbf{a}_{r',r}\\ \vdots & \iddots & \mathbf{a}_{(r-1)',r-1} & \mathbf{a}_{r',r-1}\\ \mathbf{a}_{2',1} & \mathbf{a}_{2',2} & \iddots & \vdots\\ \mathbf{a}_{1',1} & \mathbf{a}_{2',1} & \cdots & \mathbf{a}_{r',1}\end{pmatrix},\quad \bar{\mathbf{A}}_1=\begin{pmatrix}\bar{\mathbf{a}}_{1,r'} & \cdots & \bar{\mathbf{a}}_{1,2'} & 2\bar{\mathbf{a}}_{1,1'}\\ \vdots & \iddots & 2\bar{\mathbf{a}}_{2,2'} & \bar{\mathbf{a}}_{1,2'}\\ \bar{\mathbf{a}}_{r-1,r'} & 2\bar{\mathbf{a}}_{r-1,(r-1)'} & \iddots & \vdots\\ 2\bar{\mathbf{a}}_{r,r'} & \bar{\mathbf{a}}_{r-1,r'} & \cdots & \bar{\mathbf{a}}_{1,r'}\end{pmatrix},$$

$$\mathbf{A}_2=\begin{pmatrix}\mathbf{a}_{1,r'} & \cdots & \mathbf{a}_{r-1,r'} & 2\mathbf{a}_{r,r'}\\ \vdots & \iddots & 2\mathbf{a}_{r-1,(r-1)'} & \mathbf{a}_{r-1,r'}\\ \mathbf{a}_{1,2'} & 2\mathbf{a}_{2,2'} & \iddots & \vdots\\ 2\mathbf{a}_{1,1'} & \mathbf{a}_{1,2'} & \cdots & \mathbf{a}_{1,r'}\end{pmatrix},\quad \bar{\mathbf{A}}_2=\begin{pmatrix}\bar{\mathbf{a}}_{r',1} & \cdots & \bar{\mathbf{a}}_{2',1} & \bar{\mathbf{a}}_{1',1}\\ \vdots & \iddots & \bar{\mathbf{a}}_{2',2} & \bar{\mathbf{a}}_{2',1}\\ \bar{\mathbf{a}}_{r',r-1} & \bar{\mathbf{a}}_{(r-1)',r-1} & \iddots & \vdots\\ \bar{\mathbf{a}}_{r',r} & \bar{\mathbf{a}}_{r',r-1} & \cdots & \bar{\mathbf{a}}_{r',1}\end{pmatrix}, \tag{8.53}$$

cf. (5.4), where the only nontrivial commutators of the above entries are:

$$\big[\mathbf{a}_{j,i}, \bar{\mathbf{a}}_{i,j}\big] = 1\,, \qquad 1 \le i,j \le 2r\,. \tag{8.54}$$

In this setup, both matrices $L_r(x)$ and $\bar{L}_r(y)$ of (8.1) are actually C_r-type Lax matrices. In fact

$$L_{\underbrace{(+,\dots,+)}_{r}}(x) = L_r(x) = \left(\begin{array}{c|c} x\mathrm{I}_r - \bar{\mathbf{A}}_1\mathbf{A}_1 & \bar{\mathbf{A}}_1 \\ \hline -\mathbf{A}_1 & \mathrm{I}_r \end{array}\right) \tag{8.55}$$

appeared in our recent work [FT, (3.50)]. On the other hand, the Lax matrix $\bar{L}_r(y)$ can be obtained from (8.55) via the Weyl group action followed by a particle-hole transformation:

$$L_{\underbrace{(-,\dots,-)}_{r}}(y) = \bar{L}_r(y) = SL_{(+,\dots,+)}(y)S^{-1}\Big|_{P.H.} = \left(\begin{array}{c|c} \mathrm{I}_r & \mathrm{J}_r\mathbf{A}_1\mathrm{J}_r \\ \hline -\mathrm{J}_r\bar{\mathbf{A}}_1\mathrm{J}_r & y\mathrm{I}_r - \mathrm{J}_r\bar{\mathbf{A}}_1\mathbf{A}_1\mathrm{J}_r \end{array}\right)\Bigg|_{P.H.} \tag{8.56}$$

with the $2r \times 2r$ similarity matrix S given by:

$$S = \left(\begin{array}{c|c} 0 & \mathrm{J}_r \\ \hline -\mathrm{J}_r & 0 \end{array}\right) = B_{(-1,\dots,-1)}\,, \tag{8.57}$$

cf. notation (5.18), and the *total particle-hole* transformation (denoted $P.H.$) given by:

$$\bar{\mathbf{a}}_{i,j'} \mapsto -\mathbf{a}_{i,j'}\,, \quad \mathbf{a}_{j',i} \mapsto \bar{\mathbf{a}}_{j',i} \quad \text{for all} \quad 1 \le i \le j \le r\,. \tag{8.58}$$

Let us stress right away that both $L_{(+,\dots,+)}(x)$ and $L_{(-,\dots,-)}(x)$ can be obtained from the Lax matrix $\mathcal{L}(x)$ of (5.3) via the *renormalized limit* procedures (preserving the property of being Lax):

$$\begin{aligned} L_{(+,\dots,+)}(x) &= \lim_{t\to\infty} \Big\{\mathcal{L}(x-t)\cdot \mathrm{diag}\Big(\underbrace{1,\dots,1}_{r};\underbrace{-\tfrac{1}{2t},\dots,-\tfrac{1}{2t}}_{r}\Big)\Big\}\,, \\ L_{(-,\dots,-)}(x) &= \lim_{t\to\infty} \Big\{\mathrm{diag}\Big(\underbrace{\tfrac{1}{2t},\dots,\tfrac{1}{2t}}_{r};\underbrace{1,\dots,1}_{r}\Big)\cdot \mathcal{L}(x+t+r+1)\Big\}\Big|_{\bar{\mathbf{A}}\mapsto-\bar{\mathbf{A}}_2\,,\,\mathbf{A}\mapsto-\mathbf{A}_2}\,. \end{aligned} \tag{8.59}$$

Remark 8.60. We note that the commutation relations (8.54) are invariant under the transformation $\mathbf{a}_{j,i} \mapsto -\mathbf{a}_{i,j}$, $\bar{\mathbf{a}}_{i,j} \mapsto -\bar{\mathbf{a}}_{j,i}$, and furthermore such transformations do not affect the trace.

Then, the transformation (8.3) is again induced by the similarity transformation (8.12) with

$$\mathbf{S} = \exp\left[\sum_{1\le i\le j\le r} \Big(1+\delta_i^j\Big)\,\bar{\mathbf{a}}_{ij'}\mathbf{a}_{ij'}\right]\,, \tag{8.61}$$

where we note that all the summands in (8.61) pairwise commute.

Combining the factorisation formula (8.2) with the similarity transformation (8.12), we obtain:

$$L_{(+,\dots,+)}(x+t)L_{(-,\dots,-)}(x-t-r-1) = \mathbf{S}\mathcal{L}(x)\mathrm{G}\mathbf{S}^{-1}\,, \tag{8.62}$$

where

$$\mathrm{G} = \left(\begin{array}{c|c} \mathrm{I}_r & \bar{\mathbf{A}}_2 \\ \hline 0 & \mathrm{I}_r \end{array}\right)\,, \tag{8.63}$$

$\mathbf{S}$ is given by (8.61), and $\mathcal{L}(x)$ is precisely the C_r-type Lax matrix of (5.3).

Following (8.24), let us now define the Q-operators $Q_{(+,\dots,+)}(x), Q_{(-,\dots,-)}(x) \in \mathrm{End}(\mathbb{C}^{2r})^{\otimes N}$ via:

$$Q_{(+,\dots,+)}(x) = \widehat{\mathrm{tr}}_{D_{(+,\dots,+)}}\Big(\underbrace{L_{(+,\dots,+)}(x)\otimes\cdots\otimes L_{(+,\dots,+)}(x)}_{N}\Big) \tag{8.64}$$

and

$$Q_{(-,\dots,-)}(x) = \widehat{\mathrm{tr}}_{D_{(-,\dots,-)}}\Big(\underbrace{L_{(-,\dots,-)}(x)\otimes\cdots\otimes L_{(-,\dots,-)}(x)}_{N}\Big)\,, \tag{8.65}$$

cf. (3.37), with the twists $D_{(+,\dots,+)}$ and $D_{(-,\dots,-)}$ defined in analogy with (8.25)–(8.27) via:

$$D_{(+,\dots,+)} = \prod_{1\le i\le j\le r} (\tau_i\tau_j)^{-\bar{\mathbf{a}}_{ij'}\mathbf{a}_{j'i}}\,, \qquad D_{(-,\dots,-)} = \prod_{1\le i\le j\le r} (\tau_i\tau_j)^{-\bar{\mathbf{a}}_{j'i}\mathbf{a}_{ij'}}\,. \tag{8.66}$$

We note that the twist $D_{(+,\dots,+)}$ can be further expressed via $\{\mathcal{F}_{ii}\}_{i=1}^r$ of (5.7, 5.13) as:

$$D_{(+,\dots,+)} = \prod_{i=1}^r \tau_i^{\mathcal{F}_{ii}-t}\,. \tag{8.67}$$

Let us stress right away that the actions of the twists $D_{(+,\dots,+)}$ and $D_{(-,\dots,-)}$ on the corresponding Fock modules are uniquely determined (up to scalar functions) by the following conditions:

$$\begin{aligned} DL_{(+,\dots,+)}(x)D^{-1} &= D^{-1}_{(+,\dots,+)}L_{(+,\dots,+)}(x)D_{(+,\dots,+)}\,,\\ DL_{(-,\dots,-)}(x)D^{-1} &= D^{-1}_{(-,\dots,-)}L_{(-,\dots,-)}(x)D_{(-,\dots,-)}\,, \end{aligned} \tag{8.68}$$

with

$$D = \mathrm{diag}\left(\tau_1,\dots,\tau_r,\tau_r^{-1},\dots,\tau_1^{-1}\right), \tag{8.69}$$

cf. (8.27). The relations (8.68) ensure the commutativity of $Q_{(+,\dots,+)}(x)$ and $Q_{(-,\dots,-)}(x)$ defined via (8.64, 8.65) with the transfer matrix $T_{(1,0,\dots,0)}(y)$ of the defining fundamental representation in the auxiliary space, see Remark 3.67(b).

Building the monodromy matrices (by considering the N-fold tensor product on the matrix space) from (8.62), taking the normalized trace, and evoking the relation (8.67), the factorisation formula (8.62) implies the following factorisation for the transfer matrices $T^+_{(+1,\dots,+1),t}(x)$ of (5.39):

$$T^+_{(+1,\dots,+1),t}(x) = \mathrm{ch}^+_{(+1,\dots,+1),t}\cdot Q_{(+,\dots,+)}(x+t)Q_{(-,\dots,-)}(x-t-r-1) \tag{8.70}$$

with

$$\mathrm{ch}^+_{(+1,\dots,+1),t} = \mathrm{tr}\prod_{i=1}^r \tau_i^{\mathcal{F}_{ii}} = \prod_{i=1}^r \tau_i^t \prod_{1\le i\le j\le r}\frac{1}{1-\tau_i^{-1}\tau_j^{-1}}\,. \tag{8.71}$$

In analogy with Subsection 8.2, the factorisation formula (8.2) can be conjugated by $B_{\vec{\mu}}$ of (5.18) to provide an analogue of (8.18) with the index $I\in\mathcal{S}_a$ being replaced by $\vec{\mu}\in\{\pm1\}^r$. To keep our presentation short, we shall generate the remaining Q-operators directly from $Q_{(+,\dots,+)}(x)$ of (8.64) using the action of the Weyl group via:

$$Q_{\vec{\mu}}(x) = \left(B_{\vec{\mu}}\otimes\cdots\otimes B_{\vec{\mu}}\right)Q_{(+,\dots,+)}(x)\left(B_{\vec{\mu}}\otimes\cdots\otimes B_{\vec{\mu}}\right)^{-1}\Big|_{\left\{\tau_i\mapsto\tau_i^{-1}\,|\,\mu_i=-1\right\}}\,. \tag{8.72}$$

To this end, we should stress right away that the action of the Weyl group on the Q-operator $Q_{(-,\dots,-)}(x)$ of (8.65) generates the same Q-operators $Q_{\vec{\mu}}(x)$. More precisely, we have:

$$Q_{\bar{\vec{\mu}}}(x) = \left(B_{\vec{\mu}}\otimes\cdots\otimes B_{\vec{\mu}}\right)Q_{(-,\dots,-)}(x)\left(B_{\vec{\mu}}\otimes\cdots\otimes B_{\vec{\mu}}\right)^{-1}\Big|_{\left\{\tau_i\mapsto\tau_i^{-1}\,|\,\mu_i=-1\right\}}\,, \tag{8.73}$$

where we use the notation of (5.52):

$$\bar{\vec{\mu}} = -\vec{\mu}\,. \tag{8.74}$$

The above claim, that is the operators (8.72) satisfy (8.73), follows from the equality:

$$B_{\vec{\mu}}L_{(-,\dots,-)}(x)\Big|_{\overline{P.H.}}B^{-1}_{\vec{\mu}} = B_{\bar{\vec{\mu}}}L_{(+,\dots,+)}(x)B^{-1}_{\bar{\vec{\mu}}}\Big|_{\substack{\mathbf{a}_{j'i}\mapsto\mu_i\mu_j\mathbf{a}_{j'i}\\ \bar{\mathbf{a}}_{ij'}\mapsto\mu_i\mu_j\bar{\mathbf{a}}_{ij'}}} \tag{8.75}$$

that relates the Lax matrices from our definition of $Q_{(\pm,\dots,\pm)}(x)$. Here, $\overline{P.H.}$ stands for undoing the total particle-hole (8.58). As mentioned in Remark 8.60, since only powers of $\bar{\mathbf{a}}\mathbf{a}$ contribute to the trace, the resulting Q-operators are invariant under the transformations $\mathbf{a}\to-\mathbf{a},\bar{\mathbf{a}}\to-\bar{\mathbf{a}}$, and thus (8.75) indeed implies (8.73). According to (8.56), the equality (8.75) is equivalent to:

$$\Big(B_{\vec{\mu}}B_{(-1,\dots,-1)}\Big)L_{(+,\dots,+)}(x)\Big(B_{\vec{\mu}}B_{(-1,\dots,-1)}\Big)^{-1} = B_{\bar{\vec{\mu}}}L_{(+,\dots,+)}(x)B^{-1}_{\bar{\vec{\mu}}}\Big|_{\substack{\mathbf{a}_{j'i}\mapsto\mu_i\mu_j\mathbf{a}_{j'i}\\ \bar{\mathbf{a}}_{ij'}\mapsto\mu_i\mu_j\bar{\mathbf{a}}_{ij'}}}\,. \tag{8.76}$$

To prove the latter (and thus establish (8.73)), we note that the endomorphisms (5.18) satisfy:

$$B^{-1}_{(-1,\dots,-1)}B^{-1}_{\vec{\mu}}B_{\vec{\mu}} \;=\sum_{i=1}^{r}\mu_i\left(e_{ii}+e_{i'i'}\right)\,, \tag{8.77}$$

cf. Remark 5.19. Thus, conjugating $L_{(+,\dots,+)}(x)$ by $B^{-1}_{(-1,\dots,-1)}B^{-1}_{\vec{\mu}}B_{\vec{\mu}}$ (which is diagonal in the standard basis) and further applying the above change of oscillators leaves $L_{(+,\dots,+)}(x)$ invariant:

$$L_{(+,\dots,+)}(x)=\left(\sum_{i=1}^{r}\mu_i\left(e_{ii}+e_{i'i'}\right)\right)L_{(+,\dots,+)}(x)\left(\sum_{i=1}^{r}\mu_i\left(e_{ii}+e_{i'i'}\right)\right)^{-1}\Bigg|_{\substack{\mathbf{a}_{j'i}\mapsto\mu_i\mu_j\mathbf{a}_{j'i}\\ \bar{\mathbf{a}}_{ij'}\mapsto\mu_i\mu_j\bar{\mathbf{a}}_{ij'}}}\,. \tag{8.78}$$

Evoking the action of $B_{\vec{\mu}}$ on the Lax matrices $\mathcal{L}_{\vec{\mu}}(x)$ and the behaviour of the twists, as discussed in Subsection 5.2, we see that conjugating (8.62) with $B_{\vec{\mu}}$ and subsequently interchanging the twists, we obtain the following generalization of the factorisation (8.70) for any $\vec{\mu}\in\{\pm1\}^r$:

$$T^{+}_{\vec{\mu},t}(x)=\mathrm{ch}^{+}_{\vec{\mu},t}\cdot Q_{\vec{\mu}}(x+t)Q_{\vec{\mu}}(x-t-r-1) \tag{8.79}$$

with

$$\mathrm{ch}^{+}_{\vec{\mu},t}=\mathrm{tr}\prod_{i=1}^{r}\tau_i^{\mathcal{F}^{\vec{\mu}}_{ii}}=\frac{\prod_{i=1}^{r}\tau_i^{\mu_i\left(t+(r-i+1)\delta^{-}_{\mu_i}+\sum_{k=1}^{i}\delta^{-}_{\mu_k}\right)}}{\prod_{1\le i\le j\le r}\left(1-\tau_i^{-1}\tau_j^{-\mu_i\mu_j}\right)}\,, \tag{8.80}$$

cf. (5.26, 5.44, 5.54).

Combining the factorisation formula (8.79) with Theorem 5.41, we get (cf. (5.38) and (8.80)):

Proposition 8.81. *For any $t\in\mathbb{N}$, we have:*

$$T_{r,t}(x)\;=\sum_{\vec{\mu}\in\{\pm1\}^r}(-1)^{\mathsf{l}(\vec{\mu})}\;\mathrm{ch}^{+}_{\vec{\mu},t}\cdot Q_{\vec{\mu}}(x+t)Q_{\vec{\mu}}(x-t-r-1)\,. \tag{8.82}$$

8.4. **Two-term factorisation in D-type.**

Let us now discuss the straightforward D-type version of the results from the previous subsection (going back to [Fr, §5.1]). Consider the particular example of the general factorisation (8.2) applied in the case when $n=2r$, $a=r$, and the $r\times r$ matrices $\mathbf{A}_1,\bar{\mathbf{A}}_1,\mathbf{A}_2,\bar{\mathbf{A}}_2$ are explicitly given by:

$$\mathbf{A}_1=\begin{pmatrix}\mathbf{a}_{r',1} & \cdots & \mathbf{a}_{r',r-1} & 0\\ \vdots & \iddots & 0 & -\mathbf{a}_{r',r-1}\\ \mathbf{a}_{2',1} & 0 & \iddots & \vdots\\ 0 & -\mathbf{a}_{2',1} & \cdots & -\mathbf{a}_{r',1}\end{pmatrix},\quad \bar{\mathbf{A}}_1=\begin{pmatrix}\bar{\mathbf{a}}_{1,r'} & \cdots & \bar{\mathbf{a}}_{1,2'} & 0\\ \vdots & \iddots & 0 & -\bar{\mathbf{a}}_{1,2'}\\ \bar{\mathbf{a}}_{r-1,r'} & 0 & \iddots & \vdots\\ 0 & -\bar{\mathbf{a}}_{r-1,r'} & \cdots & -\bar{\mathbf{a}}_{1,r'}\end{pmatrix},$$

$$\mathbf{A}_2=\begin{pmatrix}\mathbf{a}_{1,r'} & \cdots & \mathbf{a}_{r-1,r'} & 0\\ \vdots & \iddots & 0 & -\mathbf{a}_{r-1,r'}\\ \mathbf{a}_{1,2'} & 0 & \iddots & \vdots\\ 0 & -\mathbf{a}_{1,2'} & \cdots & -\mathbf{a}_{1,r'}\end{pmatrix},\quad \bar{\mathbf{A}}_2=\begin{pmatrix}\bar{\mathbf{a}}_{r',1} & \cdots & \bar{\mathbf{a}}_{2',1} & 0\\ \vdots & \iddots & 0 & -\bar{\mathbf{a}}_{2',1}\\ \bar{\mathbf{a}}_{r',r-1} & 0 & \iddots & \vdots\\ 0 & -\bar{\mathbf{a}}_{r',r-1} & \cdots & -\bar{\mathbf{a}}_{r',1}\end{pmatrix}, \tag{8.83}$$

cf. (6.3), where the only nontrivial commutators of the above entries are given by (8.54).

In this setup, both matrices $L_r(x)$ and $\bar{L}_r(y)$ of (8.1) are actually D_r-type Lax matrices. In fact, the Lax matrix $L_{(+,\dots,+)}(x)=L_r(x)$, see (8.55), appeared first in [Fr, §4.1], cf. [FT, (2.231)]. On the other hand, the Lax matrix $L_{(-,\dots,-)}(y)=\bar{L}_r(y)$ can be obtained from (8.55) via the Weyl group action (8.57) followed by the total particle-hole transformation (8.58), exactly as in (8.56).

Similarly to (8.59), we note that both $L_{(+,\dots,+)}(x)$ and $L_{(-,\dots,-)}(x)$ can be obtained from the Lax matrix $\mathcal{L}(x)$ of (6.2) via the *renormalized limit* procedures (preserving the property of being Lax):

$$\begin{aligned} L_{(+,\dots,+)}(x) &= \lim_{t\to\infty} \Big\{\mathcal{L}(x-t)\cdot \mathrm{diag}\Big(\underbrace{1,\dots,1}_{r};\underbrace{-\tfrac{1}{2t},\dots,-\tfrac{1}{2t}}_{r}\Big)\Big\}\,,\\ L_{(-,\dots,-)}(x) &= \lim_{t\to\infty} \Big\{\mathrm{diag}\Big(\underbrace{\tfrac{1}{2t},\dots,\tfrac{1}{2t}}_{r};\underbrace{1,\dots,1}_{r}\Big)\cdot \mathcal{L}(x+t+r-1)\Big\}\Big|_{\bar{\mathbf{A}}\mapsto-\bar{\mathbf{A}}_2\,,\,\mathbf{A}\mapsto-\mathbf{A}_2}\,. \end{aligned} \tag{8.84}$$

Then, the transformation (8.3) is again induced by the similarity transformation (8.12) with

$$\mathbf{S} = \exp\left[\sum_{1\le i<j\le r} \bar{\mathbf{a}}_{ij'}\mathbf{a}_{ij'}\right]\,, \tag{8.85}$$

where we note that all the summands in (8.85) pairwise commute.

Combining the factorisation formula (8.2) with the similarity transformation (8.12), we obtain:

$$L_{(+,\dots,+)}(x+t)L_{(-,\dots,-)}(x-t-r+1) = \mathbf{S}\mathcal{L}(x)\mathrm{G}\mathbf{S}^{-1}\,, \tag{8.86}$$

where

$$\mathrm{G} = \left(\begin{array}{c|c} \mathrm{I}_r & \bar{\mathbf{A}}_2 \\ \hline 0 & \mathrm{I}_r \end{array}\right)\,, \tag{8.87}$$

$\mathbf{S}$ is given by (8.85), and $\mathcal{L}(x)$ is precisely the D_r-type Lax matrix of (6.2).

Following (8.24, 8.64, 8.65), we define the Q-operators $Q_{(+,\dots,+)}(x), Q_{(-,\dots,-)}(x) \in \mathrm{End}(\mathbb{C}^{2r})^{\otimes N}$ via:

$$Q_{(+,\dots,+)}(x) = \widehat{\mathrm{tr}}_{D_{(+,\dots,+)}}\Big(\underbrace{L_{(+,\dots,+)}(x)\otimes\cdots\otimes L_{(+,\dots,+)}(x)}_{N}\Big) \tag{8.88}$$

and

$$Q_{(-,\dots,-)}(x) = \widehat{\mathrm{tr}}_{D_{(-,\dots,-)}}\Big(\underbrace{L_{(-,\dots,-)}(x)\otimes\cdots\otimes L_{(-,\dots,-)}(x)}_{N}\Big)\,, \tag{8.89}$$

cf. (3.37), with the twists $D_{(+,\dots,+)}$ and $D_{(-,\dots,-)}$ defined in analogy with (8.25)–(8.27) via:

$$D_{(+,\dots,+)} = \prod_{1\le i<j\le r}(\tau_i\tau_j)^{-\bar{\mathbf{a}}_{ij'}\mathbf{a}_{j'i}}\,,\qquad D_{(-,\dots,-)} = \prod_{1\le i<j\le r}(\tau_i\tau_j)^{-\bar{\mathbf{a}}_{j'i}\mathbf{a}_{ij'}}\,, \tag{8.90}$$

cf. (8.66). Similarly to C-type considered in the previous subsection, we note that the actions of these twists on the Fock modules are uniquely determined (up to scalar functions) by the condition (8.68). Crucially, the twist $D_{(+,\dots,+)}$ can be further expressed via $\{\mathcal{F}_{ii}\}_{i=1}^r$ of (6.4, 6.9) as:

$$D_{(+,\dots,+)} = \prod_{i=1}^r \tau_i^{\mathcal{F}_{ii}-t}\,. \tag{8.91}$$

Building the monodromy matrices (by considering the N-fold tensor product on the matrix space) from (8.86), taking the normalized trace, and evoking the relation (8.91), the factorisation formula (8.86) implies the following factorisation for the transfer matrices $T^+_{(+1,\dots,+1),t}(x)$ of (6.44):

$$T^+_{(+1,\dots,+1),t}(x) = \mathrm{ch}^+_{(+1,\dots,+1),t}\cdot Q_{(+,\dots,+)}(x+t)Q_{(-,\dots,-)}(x-t-r+1) \tag{8.92}$$

with

$$\mathrm{ch}^+_{(+1,\dots,+1),t} = \mathrm{tr}\prod_{i=1}^r \tau_i^{\mathcal{F}_{ii}} = \prod_{i=1}^r \tau_i^t \prod_{1\le i<j\le r}\frac{1}{1-\tau_i^{-1}\tau_j^{-1}}\,. \tag{8.93}$$

Similarly to our treatment of C-type in the previous subsection, we shall generate the remaining Q-operators directly from $Q_{(+,\dots,+)}(x)$ of (8.88) via:

$$Q_{\vec{\mu}}(x) = \left(B_{\vec{\mu}}\otimes\cdots\otimes B_{\vec{\mu}}\right)Q_{(+,\dots,+)}(x)\left(B_{\vec{\mu}}\otimes\cdots\otimes B_{\vec{\mu}}\right)^{-1}\Big|_{\left\{\tau_i\mapsto\tau_i^{-1}\,|\,\mu_i=-1\right\}}\,. \tag{8.94}$$

As in type C, let us note the following key compatibility of this construction:

$$Q_{\vec{\mu}}(x) = \left(B_{\vec{\mu}} \otimes \cdots \otimes B_{\vec{\mu}}\right) Q_{(-,\dots,-)}(x) \left(B_{\vec{\mu}} \otimes \cdots \otimes B_{\vec{\mu}}\right)^{-1} \Big|_{\left\{\tau_i \mapsto \tau_i^{-1} \,|\, \mu_i = -1\right\}} . \tag{8.95}$$

The latter is a consequence of the following analogue of (8.75):

$$B_{\vec{\mu}} L_{(-,\dots,-)}(x)\Big|_{\overline{P.H.}} B_{\vec{\mu}}^{-1} = B_{\vec{\mu}} L_{(+,\dots,+)}(x) B_{\vec{\mu}}^{-1} \tag{8.96}$$

(where $\overline{P.H.}$ stands for undoing the total particle-hole (8.58)), which follows from the natural compatibility among the endomorphisms (6.14), cf. (8.77):

$$B_{(-1,\dots,-1)}^{-1} B_{\vec{\mu}}^{-1} B_{\vec{\mu}} = \mathrm{I}_{2r} . \tag{8.97}$$

Thus, similarly to C-type, the factorisation (8.92) admits the generalization for any $\vec{\mu} \in \{\pm 1\}^r$:

$$T^{+}_{\vec{\mu},t}(x) = \mathrm{ch}^{+}_{\vec{\mu},t} \cdot Q_{\vec{\mu}}(x+t) Q_{\vec{\mu}}(x-t-r+1) \tag{8.98}$$

with

$$\mathrm{ch}^{+}_{\vec{\mu},t} = \mathrm{tr} \prod_{i=1}^{r} \tau_i^{\mathcal{F}_{ii}^{\vec{\mu}}} = \frac{\prod_{i=1}^{r} \tau_i^{\mu_i \left(t + (r-i-1)\delta^{-}_{\mu_i} + \sum_{k=1}^{i} \delta^{-}_{\mu_k}\right)}}{\prod_{1 \le i < j \le r} \left(1 - \tau_i^{-1} \tau_j^{-\mu_i \mu_j}\right)} , \tag{8.99}$$

cf. (6.20) and (6.49).

Combining the factorisation formula (8.98) with Theorem 6.46, we get (cf. (6.26, 6.43) and (8.99))):

Proposition 8.100. *For any $t \in \frac{1}{2}\mathbb{N}$, we have:*

$$T^{\pm}_{t}(x) \;=\; \sum_{\vec{\mu} \in \{\pm 1\}^r_{\pm}} (-1)^{\mathrm{l}(\vec{\mu})} \; \mathrm{ch}^{+}_{\vec{\mu},t} \cdot Q_{\vec{\mu}}(x+t) Q_{\vec{\mu}}(x-t-r+1) . \tag{8.101}$$

9. Factorisation for quadratic BD-types

In this section, we factorise the infinite-dimensional quadratic BD-type transfer matrices (7.40) into the products of two Baxter Q-operators arising from degenerate Lax matrices, alike Section 8.

Consider the following two $\mathsf{K} \times \mathsf{K}$ matrices written in the block form as:

$$L_1(x) = \left(\begin{array}{c|c|c} x^2 + x\left(2 - \frac{\mathsf{K}}{2} - \bar{\mathbf{w}}_1 \mathbf{w}_1\right) + \frac{1}{4}\bar{\mathbf{w}}_1 \mathrm{J} \bar{\mathbf{w}}_1^T \mathbf{w}_1^T \mathrm{J} \mathbf{w}_1 & x\bar{\mathbf{w}}_1 - \frac{1}{2}\bar{\mathbf{w}}_1 \mathrm{J} \bar{\mathbf{w}}_1^T \mathbf{w}_1^T \mathrm{J} & -\frac{1}{2}\bar{\mathbf{w}}_1 \mathrm{J} \bar{\mathbf{w}}_1^T \\ \hline -x\mathbf{w}_1 + \frac{1}{2}\mathrm{J}\bar{\mathbf{w}}_1^T \mathbf{w}_1^T \mathrm{J} \mathbf{w}_1 & x\mathrm{I} - \mathrm{J}\bar{\mathbf{w}}_1^T \mathbf{w}_1^T \mathrm{J} & -\mathrm{J}\bar{\mathbf{w}}_1^T \\ \hline -\frac{1}{2}\mathbf{w}_1^T \mathrm{J} \mathbf{w}_1 & \mathbf{w}_1^T \mathrm{J} & 1 \end{array}\right) \tag{9.1}$$

and

$$L_{\mathsf{K}}(y) = \left(\begin{array}{c|c|c} 1 & \bar{\mathbf{w}}_2 & -\frac{1}{2}\bar{\mathbf{w}}_2 \mathrm{J} \bar{\mathbf{w}}_2^T \\ \hline \mathbf{w}_2 & y\mathrm{I} + \mathbf{w}_2 \bar{\mathbf{w}}_2 & -y\mathrm{J}\bar{\mathbf{w}}_2^T - \frac{1}{2}\mathbf{w}_2 \bar{\mathbf{w}}_2 \mathrm{J} \bar{\mathbf{w}}_2^T \\ \hline -\frac{1}{2}\mathbf{w}_2^T \mathrm{J} \mathbf{w}_2 & -y\mathbf{w}_2^T \mathrm{J} - \frac{1}{2}\mathbf{w}_2^T \mathrm{J} \mathbf{w}_2 \bar{\mathbf{w}}_2 & y^2 + y\left(2 - \frac{\mathsf{K}}{2} + \mathbf{w}_2^T \bar{\mathbf{w}}_2^T\right) + \frac{1}{4}\mathbf{w}_2^T \mathrm{J} \mathbf{w}_2 \bar{\mathbf{w}}_2 \mathrm{J} \bar{\mathbf{w}}_2^T \end{array}\right) \tag{9.2}$$

with $\mathrm{I} = \mathrm{I}_{\mathsf{K}-2}$, $\mathrm{J} = \mathrm{J}_{\mathsf{K}-2}$, while the length $\mathsf{K}-2$ rows $\bar{\mathbf{w}}_1, \bar{\mathbf{w}}_2$ and columns $\mathbf{w}_1, \mathbf{w}_2$ are given by:

$$\begin{aligned} \mathbf{w}_1 &= \left(\mathbf{a}_{2,1}, \dots, \mathbf{a}_{\mathsf{K}-1,1}\right)^T , \qquad & \bar{\mathbf{w}}_1 &= \left(\bar{\mathbf{a}}_{1,2}, \dots, \bar{\mathbf{a}}_{1,\mathsf{K}-1}\right) , \\ \mathbf{w}_2 &= \left(\mathbf{a}_{1,2}, \dots, \mathbf{a}_{1,\mathsf{K}-1}\right)^T , \qquad & \bar{\mathbf{w}}_2 &= \left(\bar{\mathbf{a}}_{2,1}, \dots, \bar{\mathbf{a}}_{\mathsf{K}-1,1}\right) , \end{aligned} \tag{9.3}$$

cf. (7.4, 7.5), where the only nontrivial commutators of the above entries are given by (8.54).

Then, their product can be factorised as follows:

$$L_1\left(x - 1 + \tfrac{t}{2} + \tfrac{\mathsf{K}}{4}\right) L_{\mathsf{K}}\left(x - \tfrac{t}{2} - \tfrac{\mathsf{K}}{4}\right) = \mathcal{L}_1'(x) \mathrm{G}' , \tag{9.4}$$

where

$$\mathrm{G}' = \left(\begin{array}{c|c|c} 1 & \bar{\mathbf{w}}_2' & -\frac{1}{2}\bar{\mathbf{w}}_2'\mathrm{J}(\bar{\mathbf{w}}_2')^T \\ \hline 0 & \mathrm{I} & -\mathrm{J}(\bar{\mathbf{w}}_2')^T \\ \hline 0 & 0 & 1 \end{array}\right) \tag{9.5}$$

and $\mathcal{L}_1'(x)$ is the $\mathfrak{so}_{\mathsf{K}}$-type Lax matrix obtained from $\mathcal{L}_1(x)$ of (7.22) by replacing $\bar{\mathbf{w}} \mapsto \bar{\mathbf{w}}_1', \mathbf{w} \mapsto \mathbf{w}_1'$, with the following transformation in place:

$$\begin{aligned} \mathbf{w}_1' &= \mathbf{w}_1 - \mathbf{w}_2\,, \qquad & \mathbf{w}_2' &= \mathbf{w}_2\,, \\ \bar{\mathbf{w}}_2' &= \bar{\mathbf{w}}_2 + \bar{\mathbf{w}}_1\,, \qquad & \bar{\mathbf{w}}_1' &= \bar{\mathbf{w}}_1\,, \end{aligned} \tag{9.6}$$

cf. (8.3). We note that the right-hand side of (9.4) is independent of $\mathbf{w}_2'$.

Actually, both $L_1(x)$, $L_{\mathsf{K}}(y)$ of (9.1, 9.2) are $\mathfrak{so}_{\mathsf{K}}$-type Lax matrices. In fact, the Lax matrix $L_1(x)$ appeared first in [Fr, §4.2], cf. [FT, (2.237) and §4.3]. On the other hand, the Lax matrix $L_{\mathsf{K}}(y)$ can be obtained from (9.1) via the Weyl group action followed by a particle-hole transformation:

$$L_{\mathsf{K}}(x) = \hat{B}_{1'} L_1(x) \hat{B}_{1'}^{-1}\Big|_{P.H.} \tag{9.7}$$

with the similarity matrix $\hat{B}_{1'} = \mathrm{J}_{\mathsf{K}}$, cf. (7.20), and the following particle-hole transformation $P.H.$:

$$\bar{\mathbf{w}}_1 \mapsto -\mathbf{w}_2^T\,, \qquad \mathbf{w}_1 \mapsto \bar{\mathbf{w}}_2^T\,. \tag{9.8}$$

We also note that the transformation (9.6) is in fact induced by the similarity transformation:

$$\begin{aligned} \mathbf{w}_1' &= \mathbf{S}\mathbf{w}_1\mathbf{S}^{-1}\,, \qquad & \mathbf{w}_2' &= \mathbf{S}\mathbf{w}_2\mathbf{S}^{-1}\,, \\ \bar{\mathbf{w}}_2' &= \mathbf{S}\bar{\mathbf{w}}_2\mathbf{S}^{-1}\,, \qquad & \bar{\mathbf{w}}_1' &= \mathbf{S}\bar{\mathbf{w}}_1\mathbf{S}^{-1}\,, \end{aligned} \tag{9.9}$$

with

$$\mathbf{S} = \exp\left[\sum_{\ell=2}^{\mathsf{K}-1} \bar{\mathbf{a}}_{1\ell}\mathbf{a}_{1\ell}\right]\,, \tag{9.10}$$

where all the summands in the right-hand side of (9.10) pairwise commute, cf. (8.12, 8.13).

Thus, combining the factorisation formula (9.4) with the similarity transformation (9.9), we get:

$$L_1\left(x - 1 + \tfrac{t}{2} + \tfrac{\mathsf{K}}{4}\right) L_{\mathsf{K}}\left(x - \tfrac{t}{2} - \tfrac{\mathsf{K}}{4}\right) = \mathbf{S}\mathcal{L}_1(x)\mathrm{G}\mathbf{S}^{-1}\,, \tag{9.11}$$

where $\mathcal{L}_1(x)$ is precisely the $\mathfrak{so}_{\mathsf{K}}$-type Lax matrix of (7.22), the matrix G is obtained from (9.5) by replacing $\bar{\mathbf{w}}_2', \mathbf{w}_2'$ with $\bar{\mathbf{w}}_2, \mathbf{w}_2$, and $\mathbf{S}$ is given by (9.10).

Remark 9.12. In type D, the factorisation formula (9.11) first appeared in [Fr, (5.30)].

Let us stress right away that both $L_1(x)$ and $L_{\mathsf{K}}(x)$ can be obtained from the Lax matrix $\mathcal{L}_1(x)$ of (7.22) via the *renormalized limit* procedures (which clearly preserve the property of being Lax):

$$\begin{aligned} L_1(x) &= \lim_{t\to\infty} \left\{\mathcal{L}_1\left(x + 1 - \tfrac{t}{2} - \tfrac{\mathsf{K}}{4}\right) \cdot \mathrm{diag}\Big(1; \underbrace{-\tfrac{1}{t}, \dots, -\tfrac{1}{t}}_{\mathsf{K}-2}; \tfrac{1}{t^2}\Big)\right\}\Big|_{\bar{\mathbf{w}}\mapsto\bar{\mathbf{w}}_1\,,\,\mathbf{w}\mapsto\mathbf{w}_1}\,, \\ L_{\mathsf{K}}(x) &= \lim_{t\to\infty} \left\{\mathrm{diag}\Big(\tfrac{1}{t^2}; \underbrace{\tfrac{1}{t}, \dots, \tfrac{1}{t}}_{\mathsf{K}-2}; 1\Big) \cdot \mathcal{L}_1\left(x + \tfrac{t}{2} + \tfrac{\mathsf{K}}{4}\right)\right\}\Big|_{\bar{\mathbf{w}}\mapsto-\bar{\mathbf{w}}_2\,,\,\mathbf{w}\mapsto-\mathbf{w}_2}\,. \end{aligned} \tag{9.13}$$

Following (8.24, 8.64, 8.65, 8.88, 8.89), we define the Q-operators $Q_1(x), Q_{\mathsf{K}}(x) \in \mathrm{End}(\mathbb{C}^{\mathsf{K}})^{\otimes N}$ via:

$$Q_1(x) = \widehat{\mathrm{tr}}_{D_1}\Big(\underbrace{L_1(x) \otimes \cdots \otimes L_1(x)}_{N}\Big) \tag{9.14}$$

and

$$Q_{\mathsf{K}}(x) = \widehat{\mathrm{tr}}_{D_{\mathsf{K}}}\Big(\underbrace{L_{\mathsf{K}}(x) \otimes \cdots \otimes L_{\mathsf{K}}(x)}_{N}\Big)\,, \tag{9.15}$$

cf. (3.37), with the twists D_1 and D_{K} defined in analogy with (8.25)–(8.27) via:

$$D_1 = \tau_1^{-\sum_{\ell=2}^{\mathsf{K}-1} \bar{\mathbf{a}}_{1\ell}\mathbf{a}_{\ell 1}} \prod_{i=2}^{r} \tau_i^{\bar{\mathbf{a}}_{1i}\mathbf{a}_{i1} - \bar{\mathbf{a}}_{1i'}\mathbf{a}_{i'1}} , \qquad D_{\mathsf{K}} = \tau_1^{-\sum_{\ell=2}^{\mathsf{K}-1} \bar{\mathbf{a}}_{\ell 1}\mathbf{a}_{1\ell}} \prod_{i=2}^{r} \tau_i^{\bar{\mathbf{a}}_{i1}\mathbf{a}_{1i} - \bar{\mathbf{a}}_{i'1}\mathbf{a}_{1i'}} . \tag{9.16}$$

We note that the twist D_1 can be further expressed via $\{\mathcal{F}_{ii}\}_{i=1}^{r}$ of (7.12, 7.13) as:

$$D_1 = \tau_1^{-t} \prod_{i=1}^{r} \tau_i^{\mathcal{F}_{ii}} . \tag{9.17}$$

Similarly to (8.27) and (8.68), we should stress right away that the actions of the twists D_1 and D_{K} on the corresponding Fock modules are uniquely determined (up to scalars) by the conditions:

$$DL_1(x)D^{-1} = D_1^{-1}L_1(x)D_1 , \qquad DL_{\mathsf{K}}(x)D^{-1} = D_{\mathsf{K}}^{-1}L_{\mathsf{K}}(x)D_{\mathsf{K}} , \tag{9.18}$$

with D given by:

$$\begin{aligned} &D_r\text{−type:} \quad D = \operatorname{diag}\left(\tau_1, \ldots, \tau_r, \tau_r^{-1}, \ldots, \tau_1^{-1}\right) , \\ &B_r\text{−type:} \quad D = \operatorname{diag}\left(\tau_1, \ldots, \tau_r, 1, \tau_r^{-1}, \ldots, \tau_1^{-1}\right) . \end{aligned} \tag{9.19}$$

Building the monodromy matrices (by considering the N-fold tensor product on the matrix space) from (9.11), taking the normalized trace, and evoking the relation (9.17), the factorisation formula (9.11) implies the following factorisation formula for the transfer matrices $T_{1,t}^{+}(x)$ of (7.40):

$$T_{1,t}^{+}(x) = \mathrm{ch}_{1,t}^{+} \cdot Q_1\left(x - 1 + \tfrac{t}{2} + \tfrac{\mathsf{K}}{4}\right) Q_{\mathsf{K}}\left(x - \tfrac{t}{2} - \tfrac{\mathsf{K}}{4}\right) \tag{9.20}$$

with the character $\mathrm{ch}_{1,t}^{+} = \operatorname{tr} \prod_{i=1}^{r} \tau_i^{\mathcal{F}_{ii}}$ given explicitly by:

$$\begin{aligned} &D_r\text{−type:} \quad \mathrm{ch}_{1,t}^{+} = \tau_1^{t} \prod_{1<\ell\leq r} \frac{1}{\left(1 - \frac{1}{\tau_1\tau_\ell}\right)\left(1 - \frac{\tau_\ell}{\tau_1}\right)} , \\ &B_r\text{−type:} \quad \mathrm{ch}_{1,t}^{+} = \frac{\tau_1^{t}}{\left(1 - \frac{1}{\tau_1}\right)} \prod_{1<\ell\leq r} \frac{1}{\left(1 - \frac{1}{\tau_1\tau_\ell}\right)\left(1 - \frac{\tau_\ell}{\tau_1}\right)} . \end{aligned} \tag{9.21}$$

Following Section 8, we define $2r$ Q-operators using the action of the operators $\hat{B}_k$ in (7.20) as:

$$Q_k(x) = \begin{cases} \left(\hat{B}_k \otimes \cdots \otimes \hat{B}_k\right) Q_1(x) \left(\hat{B}_k^{-1} \otimes \cdots \otimes \hat{B}_k^{-1}\right)\Big|_{\tau_1 \leftrightarrow \tau_k} & \text{for } 1 \leq k \leq r \\ \left(\hat{B}_k \otimes \cdots \otimes \hat{B}_k\right) Q_1(x)\big|_{\tau_i \mapsto \tau_i^{-1}} \left(\hat{B}_k^{-1} \otimes \cdots \otimes \hat{B}_k^{-1}\right)\Big|_{\tau_1 \leftrightarrow \tau_k} & \text{for } r' \leq k \leq 1' \end{cases} . \tag{9.22}$$

The compatibility condition for the $Q_{\mathsf{K}}(x)$ defined as in (9.15) and the above Q-operators follows by employing the natural relation satisfied by the endomorphisms of (7.20) (cf. (8.97)):

$$\hat{B}_k = \hat{B}_{k'}\hat{B}_{1'} , \qquad k \in \{1, \ldots, r\} \cup \{r', \ldots, 1'\} . \tag{9.23}$$

Acting with the endomorphisms $\hat{B}_k$'s on (9.11) and subsequently interchanging the twists, we get:

$$\begin{aligned} T_{k,t}^{+}(x) &= \mathrm{ch}_{k,t}^{+} \cdot Q_k\left(x - 1 + \tfrac{t}{2} + \tfrac{\mathsf{K}}{4}\right) Q_{k'}\left(x - \tfrac{t}{2} - \tfrac{\mathsf{K}}{4}\right) , \\ T_{k',t}^{+}(x) &= \mathrm{ch}_{k',t}^{+} \cdot Q_{k'}\left(x - 1 + \tfrac{t}{2} + \tfrac{\mathsf{K}}{4}\right) Q_k\left(x - \tfrac{t}{2} - \tfrac{\mathsf{K}}{4}\right) , \end{aligned} \tag{9.24}$$

for any $1 \leq k \leq r$. Here, the characters read

$$\mathrm{ch}_{k,t}^{+} = \operatorname{tr} \prod_{i=1}^{r} \tau_i^{\mathcal{F}_{ii}^{k}} , \qquad \mathrm{ch}_{k',t}^{+} = \operatorname{tr} \prod_{i=1}^{r} \tau_i^{\mathcal{F}_{ii}^{k'}} ,$$

see (7.24, 7.25), and are explicitly given by:

$$
\begin{aligned}
D_r\text{−type:}\quad & \mathrm{ch}^+_{k,t} = \frac{\tau_1^{-1}\cdots\tau_{k-1}^{-1}\tau_k^{t+k-1}}{\prod_{1\le\ell<k}\left(1-\frac{\tau_k}{\tau_\ell}\right)\prod_{k<\ell\le r}\left(1-\frac{\tau_\ell}{\tau_k}\right)\prod_{\ell\neq k}\left(1-\frac{1}{\tau_k\tau_\ell}\right)}\,,\\
& \mathrm{ch}^+_{k',t} = \frac{\tau_1^{-1}\cdots\tau_{k-1}^{-1}\tau_k^{k+1-2r-t}}{\prod_{1\le\ell<k}\left(1-\frac{\tau_k}{\tau_\ell}\right)\prod_{k<\ell\le r}\left(1-\frac{\tau_\ell}{\tau_k}\right)\prod_{\ell\neq k}\left(1-\frac{1}{\tau_k\tau_\ell}\right)}\,,\\
B_r\text{−type:}\quad & \mathrm{ch}^+_{k,t} = \frac{\tau_1^{-1}\cdots\tau_{k-1}^{-1}\tau_k^{t+k-1}}{\left(1-\frac{1}{\tau_k}\right)\prod_{1\le\ell<k}\left(1-\frac{\tau_k}{\tau_\ell}\right)\prod_{k<\ell\le r}\left(1-\frac{\tau_\ell}{\tau_k}\right)\prod_{\ell\neq k}\left(1-\frac{1}{\tau_k\tau_\ell}\right)}\,,\\
& \mathrm{ch}^+_{k',t} = \frac{\tau_1^{-1}\cdots\tau_{k-1}^{-1}\tau_k^{k-2r-t}}{\left(1-\frac{1}{\tau_k}\right)\prod_{1\le\ell<k}\left(1-\frac{\tau_k}{\tau_\ell}\right)\prod_{k<\ell\le r}\left(1-\frac{\tau_\ell}{\tau_k}\right)\prod_{\ell\neq k}\left(1-\frac{1}{\tau_k\tau_\ell}\right)}\,,
\end{aligned}
\tag{9.25}
$$

cf. (7.25, 7.26) and (7.47, 7.49).

Combining the factorisation formula (9.24) with Theorem 7.43, we arrive at the following result:

Proposition 9.26. *(a) In type D_r, for $t\in\mathbb{N}$ we have:*

$$
\begin{aligned}
T_{1,t}(x) = & \sum_{k=1}^{r}(-1)^{k-1}\,\mathrm{ch}^+_{k,t}\cdot Q_k\left(x-1+\tfrac{t}{2}+\tfrac{\mathsf{K}}{4}\right)Q_{k'}\left(x-\tfrac{t}{2}-\tfrac{\mathsf{K}}{4}\right)+\\
& \sum_{k=1}^{r}(-1)^{k-1}\,\mathrm{ch}^+_{k',t}\cdot Q_{k'}\left(x-1+\tfrac{t}{2}+\tfrac{\mathsf{K}}{4}\right)Q_{k}\left(x-\tfrac{t}{2}-\tfrac{\mathsf{K}}{4}\right)\,.
\end{aligned}
\tag{9.27}
$$

(b) In type B_r, for $t\in\mathbb{N}$ we have:

$$
\begin{aligned}
T_{1,t}(x) = & \sum_{k=1}^{r}(-1)^{k-1}\,\mathrm{ch}^+_{k,t}\cdot Q_k\left(x-1+\tfrac{t}{2}+\tfrac{\mathsf{K}}{4}\right)Q_{k'}\left(x-\tfrac{t}{2}-\tfrac{\mathsf{K}}{4}\right)+\\
& \sum_{k=1}^{r}(-1)^{k}\,\mathrm{ch}^+_{k',t}\cdot Q_{k'}\left(x-1+\tfrac{t}{2}+\tfrac{\mathsf{K}}{4}\right)Q_{k}\left(x-\tfrac{t}{2}-\tfrac{\mathsf{K}}{4}\right)\,.
\end{aligned}
\tag{9.28}
$$

In the case of B-type, these results are in agreement with the functional relations found by Tsuboi in [Ts2, Ts4]. In the case of D-type, they appeared in [FFK]; see also [ESV] where similar relations were derived from the ODE/IM correspondence [DDT].

10. Further generalizations

One may wonder to which extent our key resolution (1.30) may be further utilized to study spin chains.

Exceptional types

While the present paper covers all examples of $\mathfrak{g}$-resolutions (1.30) with $\lambda = t\omega_i$ which can be further viewed as resolutions of $Y(\mathfrak{g})$-modules for $\mathfrak{g}$ of classical types, there are also three more examples for the case of exceptional $\mathfrak{g}$ as follows from [CGY] (corresponding to the vertices i of the Dynkin diagram of $\mathfrak{g}$ with the label 1, see our Subsection 1.4):

- E_6-type: vertices $i = 1, 5$
- E_7-type: vertices $i = 1$

Type E_6: According to (1.44), the transfer matrices $T_{1,t}(x)$ and $T_{5,t}(x)$ of the corresponding finite-dimensional representations of the highest weights $t\omega_1$ and $t\omega_5$, respectively, may be written as alternating sums of 27 transfer matrices associated to infinite-dimensional representations $M'_\bullet$. Their highest weights are of the form $t\omega + w_\omega(\rho) - \rho$, where ω runs through the set of weights of L_{ω_1} or L_{ω_5}, respectively, and $w_\omega \in W_{E_6}$ are the shortest elements in the Weyl group of E_6 such that $w_\omega(\omega_1) = \omega$ or $w_\omega(\omega_5) = \omega$, respectively (note that the actions of W_{E_6} on the sets of

weights of L_{ω_1} and L_{ω_5} are transitive, as both representations are minuscule, and furthermore the stabilizers of each weight are isomorphic to $W_{D_5} = (\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_5$, the Weyl group of D_5).

Type E_7: According to (1.44), the transfer matrices $T_{1,t}(x)$ of the corresponding finite-dimensional representations of the highest weight $t\omega_1$ may be written as alternating sums of 56 transfer matrices associated to infinite-dimensional representations $M'_\bullet$. The highest weights of these modules are of the form $t\omega + w_\omega(\rho) - \rho$, where ω runs through the set of weights of L_{ω_1} and $w_\omega \in W_{E_7}$ are the shortest elements in the Weyl group of E_7 such that $w_\omega(\omega_1) = \omega$ (note that W_{E_7} acts transitively on the set of the weights of L_{ω_1}, with a stabilizer of each weight isomorphic to W_{E_6}).

It is an interesting problem to realize the aforementioned $Y(\mathfrak{g})$-representations $M'_\bullet$ through the corresponding explicit Lax matrices at generic values of $t \in \mathbb{C}$. To this end, one should construct:

- two polynomial oscillator-type Lax matrices $\mathcal{L}_1(x), \mathcal{L}_5(x)$ of size 27×27 in type E_6
- one polynomial oscillator-type Lax matrix of size 56×56 in type E_7

Not only those Lax matrices are presently unknown (see [CGY, §7.7] for their semiclassical limit), but also explicit formulas for the corresponding R-matrices seem to be missing in the literature.

More general weights

While in the present paper we only treated the examples with λ being a multiple of a fundamental weight from (1.22), thus being exactly in the framework of the weights considered in [CGY], we should stress that there do exist other cases when the $\mathfrak{g}$-module resolution (1.30) does become a resolution of $Y(\mathfrak{g})$-modules. In particular, the oscillator-type Lax matrices of [Fr, (5.24)] give rise to the explicit action of $Y(\mathfrak{so}_{2r})$ on the $\mathfrak{so}_{2r}$-modules $L_{t_1\omega_1+t_2\omega_r}$ and $L_{t_1\omega_1+t_2\omega_{r-1}}$ for any $t_1, t_2 \in \mathbb{N}$.

Trigonometric version

The most interesting continuation of the current work, which shall be discussed elsewhere, is the generalization of the present results to the trigonometric spin chains. To this end, let us recall that the representation theory of the finite quantum group $U_q(\mathfrak{g})$, over the field $\mathbb{C}(q)$, is equivalent to that of the underlying Lie algebra $\mathfrak{g}$. In particular, the description of the weights of singular vectors in the Verma modules over $U_q(\mathfrak{g})$ is precisely the same as for the Verma modules over $\mathfrak{g}$. Thus, the modules $M'_{w\cdot\lambda}$ of (1.36) and our key resolutions (1.30) admit q-analogues. It would be interesting to construct those in a self-contained way. For $(\mathfrak{g}, i)$ as in (1.22), we therefore get resolutions over the quantum loop algebra $U_q(L\mathfrak{g})$, giving rise to the trigonometric version of (1.44) that provides a BGG-type formula for the transfer matrices of finite-dimensional $U_q(L\mathfrak{g})$-representations $L_{t\omega_i}$ (the length $N = 0$ case of which recovers an interesting identity on the corresponding q-characters).

Supersymmetric version

In view of the increasing interest in the integrable structures of supersymmetric gauge theories as well as recent studies of quantum affine superalgebras and Yangians associated to Lie superalgebras, it is desirable to generalize the results of the present paper to the rational (as well as trigonometric) spin chains of super-type. We plan to return to this question in the follow-up work.

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