

Article

# The Space-Time Metric Outside a Pulsating Charged Sphere

Daniele Funaro 

Dipartimento di Scienze Chimiche e Geologiche, Università di Modena e Reggio Emilia, Via Campi 103, 41125 Modena, Italy; daniele.funaro@unimore.it

**Abstract:** We consider the problem of determining the dynamics of the electromagnetic field generated outside a ball whose charge changes depending on time. We are in conditions of perfect symmetry and the electric field is considered to be radial. This is not a simplification since, under such a hypothesis, the magnetic field does not develop. Thus, it is first necessary to find out the appropriate modeling equations. These are obtained by writing a suitable energy tensor that combines the classical electromagnetic stress-energy tensor with a special kind of mass tensor. The next step is to show that it is possible to solve Einstein's equations by plugging the new tensor on the right-hand side. Interesting connections with some classical solutions related to black holes are finally established.

**Keywords:** electrostatics; stress-energy tensor; Einstein's equations; black holes

## 1. Preliminary Considerations

We discuss an elementary electrostatics problem: the surface of a given ball is subject to variation of charge and we would like to study how the signal develops at its exterior. We will work in simplified conditions of perfect radial symmetry. For this reason, the approach may sound mainly academic, though there are examples of applications in several areas. From the theoretical viewpoint, the analysis of black holes is a natural referring topic. From the practical viewpoint, we just mention the case of our Sun. This has an effective small charge (less than 100 Coulombs [1]) and presents periodic movements on its surface (see, e.g., [2]). Real-life problems are indeed very challenging; however, we believe that it is important to understand the difficulties hidden in the study of the most basic formulation. As a matter of fact, finding the modeling equations ruling the behavior of the fields outside the ball is not a trivial question, as one would expect, especially if the surrounding space is void (i.e.: the ball is immersed in a vacuum). It is implicitly assumed that we look for solutions that develop at a finite speed (commonly that of light). Expedients suggested by Gauss's law are not taken into account, since they are based on an infinite velocity of propagation.

We briefly formalize here the problem as follows. We work in the spherical coordinate system  $(r, \theta, \phi)$ , and we assume that the fields do not depend on the variables  $\theta$  and  $\phi$ . This will ensure perfect central symmetry. As usual, we denote by  $c$  the speed of light. Time-dependent boundary conditions are imposed on a ball of fixed radius. These are homogeneously distributed on the surface in such a way that the electric field  $\vec{E} = (E, 0, 0)$  is of radial type. As we specified, outside the ball, the function  $E$  will continue to depend only on  $t$  and  $r$ . Clearly, no magnetic field can be generated in this circumstance. This means that, if the ball is surrounded by a pure vacuum, we cannot invoke the help of Maxwell's equations. Indeed, the Ampère law:  $\partial \vec{E} / \partial t = c^2 \text{curl} \vec{B} = 0$  implies that only a stationary electric field can be taken into account. In this trivial circumstance, we get the expression  $Q/r^2$ , where  $Q$  is the fixed charge. A solution of the type  $Q(t)/r^2$  is not acceptable because the whole three-dimensional space would be simultaneously affected by what happens at the surface of the ball. We claimed instead that the information has to develop with finite velocity. Thus, it is necessary to build up the proper modeling equations in order to study the phenomenon.



**Citation:** Funaro, D. The Space-Time Metric Outside a Pulsating Charged Sphere. *Appl. Sci.* **2022**, *12*, 7290. <https://doi.org/10.3390/app12147290>

Academic Editor: Cristian De Santis

Received: 26 June 2022

Accepted: 19 July 2022

Published: 20 July 2022

**Publisher's Note:** MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.



**Copyright:** © 2022 by the author. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

We start by introducing the divergence  $\rho$  in spherical coordinates:

$$\rho = \frac{\partial E}{\partial r} + \frac{2E}{r} = \frac{1}{r^2} \frac{\partial(Er^2)}{\partial r} = \text{div} \vec{E} \tag{1}$$

We then assume that the field  $\vec{V} = (v, 0, 0)$  represents the radial velocity of propagation of the signal. The vectors  $\vec{E}$  and  $\vec{V}$  are parallel, therefore the information develops longitudinally. This sounds unconventional, but represents the reason why the investigation is not so easy. On the other hand, there are no other ways to proceed.

For simplicity, we take  $v$  constant with  $v \leq c$ . If  $\rho$  is allowed to be different from zero, the Ampère law (with  $\vec{B} = 0$ ) now becomes:

$$\frac{\partial \vec{E}}{\partial t} = -\rho \vec{V} \quad \Rightarrow \quad \frac{\partial E}{\partial t} = -\rho v \tag{2}$$

This admits solutions of the type  $r^{-2}g(vt - r)$ , where  $g$  is totally arbitrary. When  $g = Q$  is a constant we return to the classical stationary case. These positions are not extraordinary if the transmission of the signal happens inside a certain medium allowing for  $\rho$  to be different from zero. They may be considered unorthodox in empty space. We show in Section 2 that (2) descends naturally from the construction of a suitable stress-energy tensor, and needs to be coupled with another equation (see (15)).

What we are showing here is another piece of evidence that the standard set of Maxwell’s equation cannot fully describe the development of electromagnetic fields in several situations [3–5]. This is particularly true in the inter-space between bodies (not infinitesimal), dynamically modifying their charge, or in movement (or both the situations) [6]. Dating back to the pioneering paper in [7], plenty of variants have been proposed in the search for valuable alternatives. The existing literature is rather rich, so we limit ourselves to the citation of a few papers: [8–14]. A very recent publication [15] further remarks on these discrepancies and offers new solutions.

The main message we would like to divulge in the present paper is the following. If we want the information to travel at finite speeds comparable to that of light, it is necessary to assume regions of space (also in a perfect vacuum) where  $\text{div} \vec{E} \neq 0$ , even if there are no physical charges. The elementary case of a capacitor consisting of two infinite parallel plates represents a viable exercise to verify this statement [16,17].

## 2. Devising the Modeling Equations

We begin with observing that the following continuity equation holds:

$$\frac{\partial \rho}{\partial t} = -\text{div}(\rho \vec{V}) = -\frac{1}{r^2} \frac{\partial(\rho v r^2)}{\partial r} \tag{3}$$

which is obtained by applying the divergence operator to both terms of Equation (2). The next step is to express the model equations in tensor form, by the use of four vectors. Among many others, we suggest [18–20], as possible referring books in general relativity. We proceed by introducing the system of coordinates in space-time:  $x_\alpha = (ct, r, \theta, \phi)$ . We then need to consider the electromagnetic tensor  $F_{\alpha\beta}$ . In the special case we are examining, its non-vanishing entries are  $F_{10} = E, F_{01} = -E, F^{10} = -E, F^{01} = E$ . We also recall that spherical coordinates are characterized by the metric tensor  $g_{\alpha\beta} = \text{diag}(1, -1, -r^2, -r^2 \sin^2 \theta)$ . In other terms, we have:

$$(ds)^2 = c^2(dt)^2 - (dr)^2 - r^2(d\theta)^2 - r^2 \sin^2 \theta (d\phi)^2 \tag{4}$$

Moreover, we get:  $\sqrt{-g} = r^2 \sin \theta$ , where  $g$  denotes the determinant of the metric tensor. In this space-time, it is straightforward to compute Christoffel’s symbols

$$\Gamma_{22}^1 = -r \quad \Gamma_{33}^1 = -r \sin^2 \theta \quad \Gamma_{31}^3 = \Gamma_{13}^3 = \Gamma_{12}^2 = \Gamma_{21}^2 = 1/r$$

$$\Gamma_{33}^2 = -\sin\theta \cos\theta \qquad \Gamma_{23}^3 = \Gamma_{32}^3 = \frac{\cos\theta}{\sin\theta} \tag{5}$$

In addition, the Ricci’s tensor  $R_{\alpha\beta}$  is identically zero, as well as the scalar curvature  $R$ . These calculations and the following ones have been also checked with software for symbolic manipulation.

It is also worthwhile to define the electromagnetic stress-energy tensor  $U_{\alpha\beta}$ , that, in the simplified case we are examining, is diagonal and can be written as a function of  $E$  as follows:

$$U_{00} = \frac{1}{2}E^2, \quad U_{11} = -\frac{1}{2}E^2, \quad U_{22} = \frac{1}{2}E^2r^2, \quad U_{33} = \frac{1}{2}E^2r^2 \sin^2\theta \tag{6}$$

$$U^{00} = \frac{1}{2}E^2, \quad U^{11} = -\frac{1}{2}E^2, \quad U^{22} = \frac{E^2}{2r^2}, \quad U^{33} = \frac{E^2}{2r^2 \sin^2\theta} \tag{7}$$

It will be useful to introduce the velocity 4-vector  $V_\alpha = (c, -v, 0, 0)$ , with its contravariant version  $V^\alpha = (c, v, 0, 0)$ . We also need a kind of mass tensor  $M_{\alpha\beta}$ , akin to that of a perfect fluid:

$$M_{\alpha\beta} = \mu^{-1}\rho V_\alpha V_\beta + \epsilon_0^{-1} g_{\alpha\beta} \Pi_{\alpha\beta} \tag{8}$$

with  $\Pi_{\alpha\beta} = \text{diag}(\mathcal{E}, -p, -p, -p)$ . Here,  $\mathcal{E}$  denotes an energy density per unit of volume and  $p$  a pressure density per unit of surface. We are assuming to be in a vacuum so that  $\epsilon_0$  is the dielectric constant. If we want to study the case of a different isotropic medium, it will be enough to change the dielectric constant accordingly. Finally, there is a constant  $\mu$  which is dimensionally equivalent to Coulomb/Kg. The magnitude of this constant depends on the type of application one has in mind. An estimate of  $\mu$  was provided for example in [5], Appendix H, and, in that circumstance, it turned out to be approximately of the order of the ratio between the elementary charge and the electron mass. The multiplicative term  $\mu$  is necessary because  $\rho$  is not a density of mass, but a density of charge per unit of volume. This setting is justified by the fact that we want to remain within a pure electromagnetic context. The information is not transported through the dust of massive particles as it may happen in some plasma but travels by means of compression and rarefaction waves altering the divergence of the electric field. By the way, if we also have matter at the exterior of our ball, appropriate corrections can be easily taken into account.

A global stress-energy tensor, including all the quantities so far examined is arranged in the following fashion:

$$T_{\alpha\beta} = \epsilon_0 (U_{\alpha\beta} - M_{\alpha\beta}) \tag{9}$$

We can write it in contravariant version, by taking into account that both  $\vec{E}$  and  $\vec{V}$  are radial, so obtaining:

$$\begin{aligned} T^{00} &= \frac{\epsilon_0}{2}E^2 - \frac{\epsilon_0 c^2}{\mu}\rho - \mathcal{E} & T^{11} &= -\frac{\epsilon_0}{2}E^2 - \frac{\epsilon_0 v^2}{\mu}\rho - p \\ T^{01} = T^{10} &= -\frac{\epsilon_0 cv}{\mu}\rho & T^{22} &= \frac{1}{r^2} \left( \frac{\epsilon_0}{2}E^2 - p \right) & T^{33} &= \frac{T^{22}}{\sin^2\theta} \end{aligned} \tag{10}$$

We are now ready to recover the modeling equation. These are obtained by requiring (see [18], Formula (41.25)):

$$\nabla_\beta T^{\alpha\beta} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x_\beta} \left( \sqrt{-g} T^{\alpha\beta} \right) + \Gamma_{\gamma\delta}^\alpha T^{\gamma\delta} = 0 \tag{11}$$

where  $\partial/\partial x_\beta = \left( \frac{1}{c} \frac{\partial}{\partial t}, \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi} \right)$ . An explicit computation brings to the four equations:

$$\nabla_\beta T^{0\beta} = \frac{\epsilon_0}{2c} \frac{\partial E^2}{\partial t} - \frac{c\epsilon_0}{\mu} \left( \frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial(\rho vr^2)}{\partial r} \right) - \frac{1}{c} \frac{\partial \mathcal{E}}{\partial t} \tag{12}$$

$$\begin{aligned} \nabla_\beta T^{1\beta} &= -\frac{\epsilon_0}{2r^2} \frac{\partial(E^2 r^2)}{\partial r} - \frac{\epsilon_0}{\mu} \left( \frac{\partial(\rho v)}{\partial t} + \frac{1}{r^2} \frac{\partial(\rho v^2 r^2)}{\partial r} \right) - \frac{1}{r^2} \frac{\partial(pr^2)}{\partial r} - \frac{2}{r} \left( \frac{\epsilon_0}{2} E^2 - p \right) \\ &= -\epsilon_0 \rho E - \frac{\epsilon_0}{\mu} v \left( \frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial(\rho v r^2)}{\partial r} \right) - \frac{\epsilon_0}{\mu} \rho \left( \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial r} \right) - \frac{\partial p}{\partial r} \end{aligned} \tag{13}$$

$$\nabla_\beta T^{2\beta} = 0 \qquad \nabla_\beta T^{3\beta} = 0 \tag{14}$$

Therefore, in (14), the requested relations follow automatically. We also want (12) and (13) to be zero. The property is obtained thanks to the following impositions. We start by setting:  $\mathcal{E} = \frac{1}{2}\epsilon_0 E^2 = \frac{1}{2}\epsilon_0 |\vec{E}|^2$  and recalling (3). In this way  $\nabla_\beta T^{0\beta} = 0$ . In the last expression in (13), we can still use (3), so that  $\nabla_\beta T^{1\beta} = 0$  implies:

$$\rho \left( \mu^{-1} \frac{D\vec{V}}{Dt} + \vec{E} \right) = -\epsilon_0^{-1} \vec{\nabla} p \tag{15}$$

This is equivalent to an Euler’s equation for non-viscous fluids, with a forcing term represented by the electric field. An extension of this equation will be examined in Section 4. Here,  $D\vec{V}/Dt = (\partial v/\partial t + v \partial v/\partial r, 0, 0)$  denotes the substantial derivative. Since we supposed that  $v$  is constant, such a derivative is zero. In particular, Equation (15) takes the simplified form  $\partial p/\partial r = -\epsilon_0 \rho E$ . In principle, it is not necessary to enforce that  $v$  must be constant, though in the present paper we will continue to stay under this hypothesis.

Finally, we claim that the two modeling equations are represented by (2) and (15), where by definition  $\rho = \text{div} \vec{E}$ . They actually couple the two unknowns  $\vec{E}$  and  $\vec{V}$ . From direct computation one finds out that  $E(t, r) = r^{-2}g(vt - r)$  is actually a solution whatever is  $g$ , that  $\mathcal{E} = \epsilon_0 g^2/2r^4$ , and that  $p$  is deduced by integrating the expression:  $\partial p/\partial r = -\epsilon_0 \rho E = -\epsilon_0 g g'/r^4$ .

Another useful equation is obtained from (2) after scalar multiplication by  $\vec{E}$ . This is:

$$\frac{\partial \mathcal{E}}{\partial t} = -\epsilon_0 \rho \vec{E} \cdot \vec{V} \tag{16}$$

A further interesting relation is obtained by taking the trace of the tensor in (9). Recalling that the trace of the electromagnetic stress tensor  $U_{\alpha\beta}$  is zero, we get:

$$\text{tr}(T_{\alpha\beta}) = -\frac{\epsilon_0}{\mu} \rho (c^2 - v^2) - \mathcal{E} + 3p \tag{17}$$

### 3. Solving Einstein’s Equations

We can now express the tensor  $T_{\alpha\beta}$  in (9) in a generic metric  $g_{\alpha\beta}$  and plug it on the right-hand side of Einstein’s equation:

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = \chi T_{\alpha\beta} \tag{18}$$

where, as usual,  $R_{\alpha\beta}$  is the Ricci’s tensor and  $R = g^{\alpha\beta}R_{\alpha\beta}$  denotes the scalar curvature. The magnitude of the dimensional constant  $\chi$  (meters/joules) is suggested by the type of application. Note, however, that  $\chi$  is not in relation with the gravitational constant  $G$ , since, as already remarked in the previous section, there are no physical masses involved in our study. The sign of  $\chi$  will be discussed later on. It is known that the trace of  $G_{\alpha\beta}$  is equal to  $-R$ . This suggest interesting relations between the scalar curvature and the trace of  $T_{\alpha\beta}$  (see (17) for the case of the metric (4)).

We look for a metric of the form:

$$(ds)^2 = c^2 \tau^2(t, r)(dt)^2 - \sigma^2(t, r)(dr)^2 - r^2(d\theta)^2 - r^2 \sin^2 \theta (d\phi)^2 \tag{19}$$

where  $\tau$  and  $\sigma$  should be determined as a function of the forcing term  $E$ . In other words, the time-varying electric field emanated by the ball produces alterations of the space-time geometry that are linked to the solution of (18). We may call them gravitational waves, though there is no “gravity” in our exercise. Similar settings were considered in [21–23], for non-static spheres of charged dust. Other related papers are for instance [24,25].

We continue our analysis by observing that  $\sqrt{-g} = r^2 \sin \theta \tau(t,r)\sigma(t,r)$ , and by computing the (non-vanishing) Christoffel’s symbols:

$$\begin{aligned} \Gamma_{00}^0 &= -\frac{1}{\sigma} \frac{\partial \sigma}{\partial t} & \Gamma_{11}^0 &= \sigma^3 \frac{\partial \sigma}{\partial t} & \Gamma_{01}^0 &= \Gamma_{10}^0 = -\frac{1}{\sigma} \frac{\partial \sigma}{\partial r} \\ \Gamma_{00}^1 &= -\frac{1}{\sigma^5} \frac{\partial \sigma}{\partial r} & \Gamma_{11}^1 &= \frac{1}{\sigma} \frac{\partial \sigma}{\partial r} & \Gamma_{01}^1 &= \Gamma_{10}^1 = \frac{1}{\sigma} \frac{\partial \sigma}{\partial t} \\ \Gamma_{22}^1 &= -\frac{r}{\sigma^2} & \Gamma_{33}^1 &= -\frac{r \sin^2 \theta}{\sigma^2} & \Gamma_{31}^3 &= \Gamma_{13}^3 = \Gamma_{12}^2 = \Gamma_{21}^2 = \frac{1}{r} \\ \Gamma_{33}^2 &= -\sin \theta \cos \theta & \Gamma_{23}^3 &= \Gamma_{32}^3 = \frac{\cos \theta}{\sin \theta} \end{aligned} \tag{20}$$

The nonzero entries of Einstein’s tensor  $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R$  take the form:

$$\begin{aligned} G_{00} &= -\frac{\tau^2}{\sigma^2} \left( \frac{1}{r^2} - \frac{\sigma^2}{r^2} - \frac{2}{r\sigma} \frac{\partial \sigma}{\partial r} \right) & G_{11} &= \frac{1}{r^2} - \frac{\sigma^2}{r^2} + \frac{2}{r\tau} \frac{\partial \tau}{\partial r} \\ G_{22} &= -\frac{r}{\sigma^3 \tau^3} \left( \tau^3 \frac{\partial \sigma}{\partial r} - \sigma \tau^2 \frac{\partial \tau}{\partial r} - r \sigma \tau^2 \frac{\partial^2 \tau}{\partial r^2} + r \sigma^2 \tau \frac{\partial^2 \sigma}{\partial t^2} + r \tau^2 \frac{\partial \tau}{\partial r} \frac{\partial \sigma}{\partial r} - r \sigma^2 \frac{\partial \tau}{\partial t} \frac{\partial \sigma}{\partial t} \right) \\ G_{01} &= G_{10} = \frac{2}{r\sigma} \frac{\partial \sigma}{\partial t} & G_{33} &= G_{22} \sin^2 \theta \end{aligned} \tag{21}$$

For a given  $\vec{E} = (E, 0, 0)$ , we build up the electromagnetic stress-energy tensor  $U_{\alpha\beta}$ :

$$U_{00} = \frac{E^2}{2\sigma^2} \quad U_{11} = -\frac{E^2}{2\tau^2}, \quad U_{22} = \frac{r^2 E^2}{2\sigma^2 \tau^2} \quad U_{33} = U_{22} \sin^2 \theta \tag{22}$$

Moreover, we need the velocity 4-vectors:  $V^\alpha = (c, v, 0, 0)$  and  $V_\alpha = (c\tau^2, -v\sigma^2, 0, 0)$ . Thus, we are ready to set up the mass tensor:

$$M_{\alpha\beta} = \frac{\rho}{\mu} \begin{pmatrix} c^2 \tau^4 & -cv\tau^2 \sigma^2 & 0 & 0 \\ -cv\tau^2 \sigma^2 & v^2 \sigma^4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \frac{1}{\epsilon_0} \begin{pmatrix} \mathcal{E} \tau^2 & 0 & 0 & 0 \\ 0 & p\sigma^2 & 0 & 0 \\ 0 & 0 & pr^2 & 0 \\ 0 & 0 & 0 & pr^2 \sin^2 \theta \end{pmatrix} \tag{23}$$

In the end, we arrive at a set of five differential equations to be solved. To go on, it is worthwhile to set  $\tau = 1/\sigma$ . In this way we are left with only one unknown, for which we must have:

$$\begin{aligned} G_{00} &= \frac{1}{r^2 \sigma^5} \left( 2r \frac{\partial \sigma}{\partial r} - \sigma + \sigma^3 \right) = \chi \left( \frac{\epsilon_0 E^2}{2\sigma^2} - \frac{\mathcal{E}}{\sigma^2} - \frac{\epsilon_0 \rho c^2}{\mu \sigma^4} \right) & G_{01} &= G_{10} = \frac{2}{r\sigma} \frac{\partial \sigma}{\partial t} = \chi \frac{\epsilon_0 \rho c v}{\mu} \\ G_{11} &= -\frac{1}{r^2 \sigma} \left( 2r \frac{\partial \sigma}{\partial r} - \sigma + \sigma^3 \right) = \chi \left( -\frac{\epsilon_0 E^2}{2} \sigma^2 - p\sigma^2 - \frac{\epsilon_0 \rho v^2}{\mu} \sigma^4 \right) \\ G_{22} &= -\frac{r}{\sigma^4} \left( 2\sigma \frac{\partial \sigma}{\partial r} - 3r \left( \frac{\partial \sigma}{\partial r} \right)^2 + r\sigma^4 \left( \frac{\partial \sigma}{\partial t} \right)^2 + r\sigma^5 \frac{\partial^2 \sigma}{\partial t^2} + r\sigma \frac{\partial^2 \sigma}{\partial r^2} \right) = \chi \left( \frac{\epsilon_0 E^2}{2} r^2 - pr^2 \right) \end{aligned} \tag{24}$$

The equation relative to  $G_{33}$  is trivially obtained from the one corresponding to  $G_{22}$  after multiplication by  $\sin^2 \theta$ . We successively evaluate  $\mathcal{E}$  from the first equation:

$$\mathcal{E} = \frac{\epsilon_0 E^2}{2} - \frac{1}{\chi} \left[ \frac{c}{v} \frac{2}{r\sigma^3} \frac{\partial \sigma}{\partial t} + \frac{1}{r^2 \sigma^3} \left( 2r \frac{\partial \sigma}{\partial r} - \sigma + \sigma^3 \right) \right] \tag{25}$$

By suitably combining  $G_{00}$  and  $G_{11}$  we may recover  $p$ :

$$p = -\mathcal{E} + \frac{\epsilon_0 \rho}{\mu} \left( \frac{c^2}{\sigma^2} + v^2 \sigma^2 \right) = -\frac{\epsilon_0 E^2}{2} + \frac{1}{\chi} \left[ \frac{1}{r^2 \sigma^3} \left( 2r \frac{\partial \sigma}{\partial r} - \sigma + \sigma^3 \right) - \frac{v}{c} \frac{2\sigma}{r} \frac{\partial \sigma}{\partial t} \right] \tag{26}$$

In such a way,  $\rho$ ,  $\mathcal{E}$ , and  $p$  are evaluated as functions of the given  $E$  and the unknown  $\sigma$ . These will be substituted into the equation relative to  $G_{22}$ . The equation for  $G_{33}$  is equivalent. After implementing the substitution:  $\sigma(t, r) = 1/\sqrt{1 + \omega(t, r)}$ , and using the expression of  $p$  in (26), we get from the last equation in (24):

$$\frac{1}{r} \left( -\frac{\omega}{r} - r\sigma^6 \left( \frac{\partial \omega}{\partial t} \right)^2 + \frac{r\sigma^4}{2} \frac{\partial^2 \omega}{\partial t^2} + \frac{r}{2} \frac{\partial^2 \omega}{\partial r^2} + \frac{v}{c} \sigma^4 \frac{\partial \omega}{\partial t} \right) = \chi \epsilon_0 E^2 \tag{27}$$

where  $\sigma^{2n} = 1/(1 + \omega)^n$ . This is the final equation to be solved. Unfortunately, we do not have a closed expression for  $\omega$  as a function of  $E$ . We can, however, make some considerations. If  $\omega$  does not depend on  $t$  and  $E = Q/r^2$ , equation (27) reduces to:

$$-\frac{\omega}{r^2} + \frac{1}{2} \frac{\partial^2 \omega}{\partial r^2} = \chi \epsilon_0 \frac{Q^2}{r^4} \tag{28}$$

The general solution in this case is  $\omega(r) = -K/r + \chi \epsilon_0 Q^2 / 2r^2$ , where  $K$  is an arbitrary real number. We have  $\rho = 0$ ,  $\mathcal{E} = 0$  and  $p = 0$ . Therefore  $M_{\alpha\beta} = 0$ . The final expression of  $\sigma$  exactly corresponds to that of the Reissner–Nordström metric, where the parameter  $K = 2M > 0$  is usually related to an effective mass  $M$ . This stationary metric simulates a charged black hole. In the phenomenon we are currently studying there are no physical masses, and  $K$  is just a constant to be fixed according to some boundary conditions. Our “mass” tensor  $M_{\alpha\beta} = 0$  comes into place in the case of time-dependent fields and somehow simulates the transfer of information from the surface of the ball towards the exterior.

When  $E$  actually depends on time, Equation (27) can be approached with the help of numerical simulations. By the way, this issue is not within the goals of this paper. Theoretical considerations can be done for small-varying fields in the neighborhood of the stationary solution.

Note that Birkhoff’s theorem states that solutions of the vacuum Einstein’s equations, displaying spherical symmetry, are locally isometric to the Schwarzschild solution, so they will not generate pure gravitational waves. The term “vacuum” in this context means the absence of other massive-like bodies. Here the situation is different. Einstein’s equations are solved with a forcing right-hand side (depending on  $E$ ) that simulates the oscillating presence of the electric field. Indeed, any form of energy should in principle be able to produce a deformation of space-time. Exact gravitational plane waves sharing the same support as an electromagnetic one were computed in ([4], Section 4.3). This approach is different from that followed in the historical paper [26], where the equations were solved in the pure gravitational vacuum. The final solution in that case was rather complicated (and of difficult interpretation). The authors actually end up with gravitational waves that were not of plane type, so, part of the paper was devoted to the discussion of the concept of “planeness”. In the present case, the right-hand side  $T_{\alpha\beta}$  is a dynamical forcing tensor, and, although an explicit solution of (27) is not available, the attached gravitational waves look perfectly spherical.

#### 4. Comments

We can provide further observation concerning the stationary solution presented at the end of the previous section. In this case, Einstein’s equation reduces to:

$$G_{\alpha\beta} = \chi\epsilon_0 U_{\alpha\beta} \tag{29}$$

in which only the electromagnetic stress tensor  $U_{\alpha\beta}$  appears on the right-hand side. As we noted, a general solution has the form:  $\tau(r) = \sqrt{1 - K/r + \chi\epsilon_0 Q^2/2r^2}$ ,  $\sigma(r) = 1/\tau(r)$ . This effectively corresponds to the Reissner–Nordström metric ( $K = 2M$ ) when  $\chi > 0$  (note that the signature of our space is  $(+, -, -, -)$ ). The existence of  $\tau$  and  $\sigma$  is guaranteed if  $r$  is large enough and  $M^2 - \chi\epsilon_0 Q^2 \geq 0$ . This is an effective restriction when  $\chi$  is positive, but can be totally removed by taking  $\chi$  as negative. As a matter of fact, there is no real reason to choose a specific sign accompanying the electromagnetic stress-energy tensor, since unlike astronomical or cosmological applications, there is no sign actually involved in electromagnetic phenomena. This issue was already discussed in ([4], p. 128), and can be further reinforced after examining the Kerr–Newman metric [27], simulating a rotating massive black hole. The metric generalizes the above one and uses a particular coordinates framework, namely the Boyer–Lindquist coordinates. The explicit expression is:

$$c^2(ds)^2 = -\left(\frac{(dr)^2}{\Delta} + (d\theta)^2\right)x^2 + \left(c dt - a \sin^2 \theta d\phi\right)^2 \frac{\Delta}{x^2} - \left((r^2 + a^2)d\phi - ac dt\right)^2 \frac{\sin^2 \theta}{x^2} \tag{30}$$

where  $x^2 = r^2 + a^2 \cos^2 \theta$ ,  $a$  is a parameter and  $\Delta = r^2 - 2Mr + a^2 + Q^2$ . For brevity, we omitted various dimensional constants. It turns out that, if  $\chi$  is positive in (29), we get restrictions on  $r$ , bringing to the definition of an “event horizon”. It is also required that  $M^2 \geq Q^2 + a^2$  (see [19], p. 879), which may enforce the mass  $M$  to be rather large. On the other hand, if we now define  $\Delta = r^2 - 2Mr + a^2 - Q^2$ , we still get solutions to Einstein’s equation. They correspond to negative values of  $\chi$ , and the restriction becomes milder, i.e.:  $M^2 + Q^2 \geq a^2$ . Our suggestion is to adopt the negative sign concerning all tensors that involve the electromagnetic portion of the phenomenon under study.

As a final remark, we propose the full set of modeling equations, including the presence of the magnetic field. They are obtained as in (11) by imposing  $\nabla_\beta T^{\alpha\beta} = 0$ , with  $T^{\alpha\beta}$  built on the most general electromagnetic tensor  $F_{\alpha\beta}$ . We have:

$$\frac{\partial \vec{E}}{\partial t} = c^2 \text{curl} \vec{B} - \rho \vec{V} \qquad \frac{\partial \vec{B}}{\partial t} = -\text{curl} \vec{E} \tag{31}$$

with  $\rho = \text{div} \vec{E}$  and  $\text{div} \vec{B} = 0$ . In addition, we get:

$$\rho \left( \mu^{-1} \frac{D\vec{V}}{Dt} + \vec{E} + \vec{V} \times \vec{B} \right) = -\epsilon_0^{-1} \vec{\nabla} p \tag{32}$$

Note that the term  $\vec{E} + \vec{V} \times \vec{B}$  recalls Lorentz’s force. Thus, we are coupling Maxwell’s equations with those ruling fluid dynamics. This is a very classical approach, especially in the framework of plasma physics (see, e.g., [28]). The crucial difference is that the equations above do not necessarily require the presence of massive particles, and they can survive in a perfect vacuum. One of the advantages is to be able to solve the problem of the radially pulsating charged ball, which cannot otherwise be dealt with with more standard tools.

#### 5. Conclusions

With the help of a suitable electromagnetic stress-energy tensor combining the classical one with a special mass tensor, we were able to get a differential model that extends the usual Maxwell’s one. We applied this to the study of the electromagnetic development of the fields outside a charged ball displaying time-dependent radial boundary conditions. By the same equations, the analysis may be extended to other more complex situations.

We showed that is possible to solve Einstein's equations by having the new tensor on the right-hand side. The so obtained space-time metric, which extends already known solutions, may constitute a further step in the investigation of the properties of black holes.

**Funding:** This research received no external funding.

**Conflicts of Interest:** The author declares no conflict of interest.

## References

- Neslusan, L. On the global electrostatic charge of stars. *Astron. Astrophys.* **2001**, *372*, 913–915. [[CrossRef](#)]
- Gough, D.O.; Kosovichev, A.G.; Toomre, J.; Anderson, E.; Antia, H.M.; Basu, S.; Chaboyer, B.; Chitre, S.M.; Christensen-Dalsgaard, J.; Dziembowski, W.A.; et al. The seismic structure of the Sun. *Science* **1996**, *272*, 1296–1300. [[CrossRef](#)] [[PubMed](#)]
- Engelhardt, W. On the Solvability of Maxwell's Equations. *Ann. Fond. Louis Broglie* **2012**, *37*, 3–14.
- Funaro, D. *Electromagnetism and the Structure of Matter*; World Scientific: Singapore, 2008.
- Funaro, D. *From Photons to Atoms, The Electromagnetic Nature of Matter*; World Scientific: Singapore, 2019.
- Zhakatayev, A. Divergence of electric field of continuous and of a point charge for relativistic and non-relativistic motion. *arXiv* **2016**, arXiv:1608.02898
- Born, M.; Infeld, L. Foundations of the new field theory. *Proc. R. Soc. Lon. A* **1934**, *144*, 425–451. [[CrossRef](#)]
- Arbab, I. Extended electrodynamics and its consequences. *Mod. Phys. Lett. B* **2017**, *31*, 1750099. [[CrossRef](#)]
- Benci, V.; Fortunato, V. Towards a unified field theory for classical electrodynamics. *Arch. Rat. Mech. Anal.* **2004**, *173*, 379–414. [[CrossRef](#)]
- Coclite, G.M.; Georgiev, V. Solitary waves for Maxwell Schrödinger equations. *Electron. J. Differ. Eq.* **2004**, *94*, 1–31.
- Harmuth, H.F.; Barrett, T.W.; Meffert, B. *Modified Maxwell Equations in Quantum Electrodynamics*; SCCP; World Scientific: Singapore, 2001; Volume 19.
- Hively, L.M.; Giakos, G C. Toward a more complete electrodynamic theory. *Int. J. Signal Imaging Syst. Eng.* **2012**, *5*, 3–10. [[CrossRef](#)]
- Lehnert, B.; Roy, S. *Extended Electromagnetic Theory—Space Charge in Vacuo and the Rest Mass of the Photon*; SCCP; World Scientific: Singapore, 1998; Volume 16.
- Munz, C.-D.; Schneider, R.; Sonnendrücker, E.; Voss, U. Maxwell's equations when the charge conservation is not satisfied. *C. R. Acad. Sci. Paris* **1999**, *328*, 431–436. [[CrossRef](#)]
- Wang, Z.L. On the expanded Maxwell's equations for moving charged media system—General theory, mathematical solutions and applications. *TENG Mater. Today* **2022**, *52*, 348–363. [[CrossRef](#)]
- Funaro, D. Charging capacitors according to Maxwell's equations: impossible. *Ann. Fond. Louis Broglie* **2014**, *39*.
- Funaro, D. High frequency electrical oscillations in cavities. *Math. Model. Anal.* **2018**, *23*, 345–358. [[CrossRef](#)]
- Fock, V. *The Theory of Space, Time and Gravitation*; Pergamon Press: London, UK, 1959.
- Misner, C.W.; Thorne, K.S.; Wheeler, J.A. *Gravitation*; W. H. Freeman and Company: San Francisco, CA, USA, 1973.
- Atwater, R.A. *Basic Relativity*; Springer: New York, NY, USA, 1994.
- Hamoui, A. Deux nouvelles classes de solutions non-statiques à symétrie sphérique des équations d'Einstein-Maxwell. *Ann. l'IHP Phys. Théorique* **1969**, *10*, 195–227.
- Kuchowicz, B. A non-static sphere of charged dust. *Phys. Lett. A* **1970**, *31*, 220. [[CrossRef](#)]
- Vickers, P.A. Charged dust spheres in general relativity *Ann. l'IHP Phys. Théorique* **1973**, *18*, 137–145.
- Tikekar, R. A source for the Reissner-Nordström metric *J. Astrophys. Astron.* **1984**, *5*, 273–276. [[CrossRef](#)]
- Beltracchi, P.; Gondolo, P. An exact time-dependent interior Schwarzschild solution. *arXiv* **2019**, arXiv:1902.07827
- Bondi, H.; Pirani, F. A. E.; Robinson, I. Gravitational waves in general relativity III. Exact plane waves. *Proc. R. Soc. Lond. A* **1959**, *251*, 519–533.
- Newman, E.T.; Couch, R.; Chinnapared, K.; Exton, A.; Prakash, A.; Torrence, R. Metric of a Rotating, Charged Mass. *J. Math. Phys.* **1965**, *6*, 918–919. [[CrossRef](#)]
- Jackson, J.D. *Classical Electrodynamics*, 2nd ed.; John Wiley & Sons: New York, NY, USA, 1975.