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A Work Plan for the Study of non-Smooth Solutions in 3D Fluid Dynamics: The First Regularity Barrier for the Euler Equation

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Abstract

The hope of finding solutions of the fluid dynamic equations developing anomalous behaviors starting from smooth data has stimulated the search for increasingly sophisticated configurations. I too want to add my contribution in these pages, by proposing the construction of possible candidates and experimenting with some numerical schemes. The tests are carried out for the Euler case in a very simple situation, however they may open the way to more complex cases in fluid dynamics, including Navier-Stokes equations. If the intuition is correct, it should not be difficult for those more skilled in theoretical matters to bring home some more concrete mathematical results.

Keywords: fluid dynamics; Euler equations; non regular solutions; blowup.

1 Introduction

Anyone reading these lines knows well that the equations of fluid dynamics in 3D still have aspects that deserve in-depth investigation, especially when it comes to the regularity of the solutions. In the last decades, the analysis has focused more on the “counterexample” version, i.e. the possibility that the solution develops behaviors that lead to some kind of singularities in a finite time, despite starting from smooth data¹. There is no shortage of ideas both with regards to examples taken from natural phenomena and their mathematical modeling [2, 3]. Here too I will follow this path, trying to propose a work program that can provide a

¹I would like to dedicate this article to Richard B. Pelz, who left us prematurely about twenty years ago. He introduced me to the intriguing question of blowup when we were both visiting ICASE at NASA Langley in the 1980s. On that occasion he told me about his desire to study vortex configurations in situations of great stress (see for example [1]).

strategy to address this long-standing issue. I will play with the inviscid Euler equation, but I guess that a similar approach could be used in the case of the Navier-Stokes equations, notably more complicated (I do not deny that I have already made some attempts, with little results at the moment). I will not be able to provide a rigorous mathematical set up, but the reader will realize that the guidelines followed can undoubtedly lead to results that are practicable from a theoretical point of view. I thus leave the completion of the framework proposed here to colleagues who are more versed to pure mathematical aspects.

The battlefield sees the vortex rings as the main reference. I have always loved these stable structures, so much so that I have made extensive use of them in the field of electromagnetism. To gain some publicity I will immediately mention my two books on the subject [4, 5]. To these let me also add a very recent paper of mine [6] on the onset of shocks in Navier-Stokes equations which, although it is aimed to offer interesting new approaches, is surpassed by the contents of this article. Vortexes are also the argument with which the authors in [7, 8] propose a possible mechanism for generating singular solutions of the Euler equations. The dissertations, primarily numerical in nature, provide convincing initial conditions for obtaining blowup in the non viscous case. The discussion has been taken up both from a theoretical and numerical point of view in more recent works [9, 10]. I do not go further in citing publications of a theoretical nature as the list would become infinitely long. Let me just mention [11, 12] and the review papers [13, 14]. Further contributions will be listed later in this article.

The principle on which I will try to work in this paper is based on the following statement: *if we do not know how to reach a singular solution, let the singularity smoothly go back and then return to its initial position.* In practice, I want to start from an appropriate singular solution and solve the equations backward in time in a certain temporal interval. Subsequently, I will use the new displacement thus obtained to return to the singularity. Of course, it is necessary to operate in a purely inviscid context, so I will be able (for the moment) to examine only the case of the Euler equations. However, the idea has chances of being extended to the Navier-Stokes case, although I have only some vague intuition about how to proceed. The concept is outlined in figure 1 and is supported by the observation that singular solutions would be unstable in some sense [15], and therefore difficult to approach from a numerical point of view as it is done classically, that is, starting from an initial guess (with a high success percentage) and proceeding forward with the equation.

The hypotheses in which this “back to the future” procedure has the possibility of working are:

- one must start from a singular guess that is actually “reachable”;
- the equation must be perfectly reversible in time also in presence of singular initial data;
- the backward procedure must lead (instantaneously) to smooth solutions.

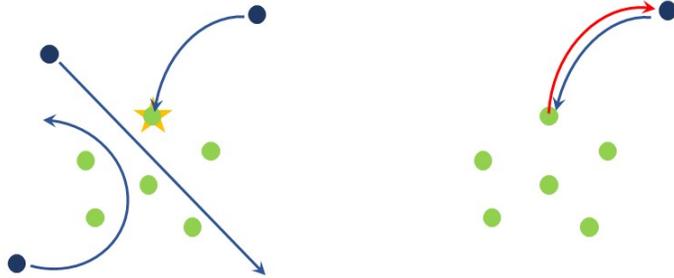


Figure 1: If the green dots denote some possible singular solutions of the Euler equations, it is complicated to choose an initial condition that leads exactly to one of them. The reason for this seems to be attributable to their instability. Instead, you can start from a singular solution, go back in time and use the regularized solution to return (forward) to the singular one.

I will conduct simple numerical experiments just to show empirically that this technique can reasonably work. If the change of the arrow of time is relatively natural in the context of purely hyperbolic equations at special regimes (see for example the impressive educational movie in [16]), the case of Navier-Stokes is certainly not directly approachable in the same way.

2 Preliminary considerations

I will start by writing the set of time-dependent incompressible Navier-Stokes equations. The model expresses the law of momentum conservation through the vector equation:

$$\frac{\partial \mathbf{v}}{\partial t} - \nu \bar{\Delta} \mathbf{v} + (\mathbf{v} \cdot \bar{\nabla}) \mathbf{v} = -\bar{\nabla} p + \mathbf{f} \quad t \in]0, T] \quad (1)$$

where, in order to force mass conservation, the velocity field \mathbf{v} is required to be divergence-free, i.e.: $\bar{\nabla} \cdot \mathbf{v} = 0$. An appropriate initial condition \mathbf{v}_0 is imposed at time $t = 0$. As customary, $\nu \geq 0$ denotes the viscosity parameter. The potential p plays the role of pressure and \mathbf{f} is a given force field, though throughout the paper it will be assumed that $\mathbf{f} = \mathbf{0}$ and (sigh!) $\nu = 0$. The equations are required to be satisfied in a region of the three-dimensional space \mathbf{R}^3 . The symbol $\bar{\Delta}$ denotes the 3D vector Laplacian.

Let me denote by $\boldsymbol{\omega} = \bar{\nabla} \times \mathbf{v}$ the *curl* of the velocity field. Under hypotheses of regularity, we can apply the *curl* to (1), so obtaining an equation for the vorticity:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} - \nu \bar{\Delta} \boldsymbol{\omega} - \bar{\nabla} \times (\mathbf{v} \times \boldsymbol{\omega}) = \mathbf{0} \quad (2)$$

In this way, one gets rid of pressure. It is also possible to get rid of relation $\bar{\nabla} \cdot \mathbf{v} = 0$ by assuming that $\mathbf{v} = \bar{\nabla} \times \mathbf{A}$, for some vector potential \mathbf{A} . At this point, I will work (almost) exclusively with equation (2).

The discussion will be carried out in cylindrical coordinates (r, θ, z) , where no dependence is assumed with respect to the variable θ . For a stream function Ψ , which depends on t, r and z , it is introduced the vector potential $\mathbf{A} = (0, \Psi, 0)$, such that $\bar{\nabla} \cdot \mathbf{A} = 0$ and :

$$\mathbf{v} = (v_1, v_2, v_3) = \bar{\nabla} \times \mathbf{A} = \left(-\frac{\partial \Psi}{\partial z}, 0, \frac{\partial \Psi}{\partial r} + \frac{\Psi}{r} \right) \quad (3)$$

$$\boldsymbol{\omega} = \bar{\nabla} \times \mathbf{v} = (0, -u, 0) \quad (4)$$

where (see, e.g. [17], section 7.5, for the treatment of axisymmetric flows):

$$\Delta \Psi - \frac{\Psi}{r^2} = \frac{\partial^2 \Psi}{\partial r^2} + \frac{1}{r} \frac{\partial \Psi}{\partial r} - \frac{\Psi}{r^2} + \frac{\partial^2 \Psi}{\partial z^2} = u \quad (5)$$

Thus, u is the second component of the vector Laplacian $\bar{\Delta} \mathbf{A}$. Note that, in (5), Δ (without the overbar) indicates the scalar Laplacian. By assuming $\nu = 0$, the equation (2) becomes:

$$\frac{\partial u}{\partial t} - \frac{\partial \Psi}{\partial z} \left(\frac{\partial u}{\partial r} - \frac{u}{r} \right) + \frac{\partial u}{\partial z} \left(\frac{\partial \Psi}{\partial r} + \frac{\Psi}{r} \right) = 0 \quad (6)$$

or, equivalently:

$$\frac{\partial u}{\partial t} + v_1 \left(\frac{\partial u}{\partial r} - \frac{u}{r} \right) + v_3 \frac{\partial u}{\partial z} = 0 \quad (7)$$

Since we are only concerned with what happens near the origin, the domain of interest will be the cylinder C individuated by the inequalities $0 \leq r < 1$, $0 \leq \theta < 2\pi$, $-1 < z < 1$. This is obtained from the rotation about the z -axis of the rectangle $R = \{0 < r < 1, -1 < z < 1\}$ with the addition of the segment $(0, 0, z), -1 < z < 1$. Homogeneous Dirichlet type conditions are assumed for Ψ at the boundary of C . The function is also required to be zero along the z -axis intersected with C . These impositions ensure that the velocity field in (3) will remain tangential to the boundary of R and symmetric with respect to the axis $z = 0$. The fluid will thus remain confined in a torus inside the cylinder.

The implementation starts with an initial datum u_0 . Successively, Ψ is computed by solving the elliptic problem (5). A time discretization scheme allows for an update of u using (6). The specifications will be given in the next section.

In its broad outline, the idea proposed in the introduction develops as follows: For $\epsilon > 0$, find the vorticity $\boldsymbol{\omega}_\epsilon(t)$ solving the backward equation (note the change of sign with respect to (2)):

$$\frac{\partial \boldsymbol{\omega}_\epsilon}{\partial t} + \bar{\nabla} \times (\mathbf{v}_\epsilon \times \boldsymbol{\omega}_\epsilon) = \mathbf{0} \quad t \in]0, T] \quad (8)$$

with $\boldsymbol{\omega}_\epsilon(0) = \boldsymbol{\omega}_{\epsilon,0}$, being $\boldsymbol{\omega}_{\epsilon,0}$ a smooth approximation of a given singular function $\boldsymbol{\omega}_0$ (in order to have $\lim_{\epsilon \rightarrow 0} \boldsymbol{\omega}_{\epsilon,0} = \boldsymbol{\omega}_0$ in some suitable norm). The theoretical analysis of the above problem should present no difficulties, especially

if ϵ takes discrete values that can be associated to the dimension of a finite dimensional space. The second, less trivial, step is to show that the limit exists:

$$\lim_{\epsilon \rightarrow 0} \omega_\epsilon(T) = \omega(T) \quad (9)$$

The final check is to control that the forward equation:

$$\frac{\partial \omega}{\partial t} - \bar{\nabla} \times (\mathbf{v} \times \omega) = \mathbf{0} \quad t \in]T, 2T] \quad (10)$$

with initial datum $\omega(T)$, actually reaches the configuration ω_0 at time $t = 2T$. Clearly, the idea behind this procedure can be made valid in any dimension, as long as the fields are embedded in the correct functional spaces (see, e.g., [18, 19], for the treatment of the 2D case).

3 Discretization of the Euler equations

According to what was said in the previous section, by starting from an initial guess $\omega_0 = (0, -u_0, 0)$ (see (4)), the aim is to solve the two following nonlinear equations one after the other:

$$\frac{\partial \omega}{\partial t} + \bar{\nabla} \times (\mathbf{v} \times \omega) = \mathbf{0} \quad t \in]0, T] \quad (11)$$

$$\frac{\partial \omega}{\partial t} - \bar{\nabla} \times (\mathbf{v} \times \omega) = \mathbf{0} \quad t \in]T, 2T] \quad (12)$$

The goal is reached if the final solution at time $t = 2T$ coincides with the initial one at time $t = 0$. The experiment is crucial if u_0 is somehow singular. Let me try some experiments with the help of the most basic finite-difference method.

Let the even integers N and M be given. Defined $h_r = 1/N$ and $h_z = 2/(M - 1)$, the grid points in the rectangle R are (r_j, z_i) , with $r_j = jh_r$, $j = 0, \dots, N$ and $z_i = -1 + ih_z$, $i = 0, \dots, M - 1$. Note that the point $(0, 0)$ is not included in the grid, although this decision is not mandatory. This is done to avoid having to compute functions at the origin where we expect the singularity to develop. The presence of such a singularity will actually be noticed when the mesh size decreases. A time step Δt is obtained by dividing $2T$ in equal parts by introducing the intermediate points t_k . The approximated solutions at the grid points are now denoted by $u_{j,i}^k \approx u(t_k, r_j, z_i)$ and $\Psi_{j,i}^k \approx \Psi(t_k, r_j, z_i)$. Imposing the initial condition yields: $u(0, r_j, z_i) = u_0(r_j, z_i)$. Each step requires the resolution at the inner nodes of the elliptic problem (compare with (5)):

$$\begin{aligned} & \frac{\Psi_{j+1,i}^k - 2\Psi_{j,i}^k + \Psi_{j-1,i}^k}{h_r^2} + \frac{1}{r_j} \frac{\Psi_{j+1,i}^k - \Psi_{j-1,i}^k}{2h_r} - \frac{\Psi_{j,i}^k}{r_j^2} \\ & + \frac{\Psi_{j,i+1}^k - 2\Psi_{j,i}^k + \Psi_{j,i-1}^k}{h_z^2} = u_{j,i}^k \end{aligned} \quad (13)$$

At the boundary nodes of R the conditions are zero for $\Psi_{j,i}^k$. Following (3), at the inner grid points the discrete velocity field is written as:

$$(v_{1,j,i}^k, v_{2,j,i}^k, v_{3,j,i}^k) = \left(-\frac{\Psi_{j,i+1}^k - \Psi_{j,i-1}^k}{2h_z}, 0, \frac{\Psi_{j+1,i}^k - \Psi_{j-1,i}^k}{2h_r} + \frac{\Psi_{j,i}^k}{r_j} \right) \quad (14)$$

Thus, the advective part (7) is treated with the classical upwind scheme, where for example:

$$\frac{u_{j,i}^{k+1} - u_{j,i}^k}{\Delta t} + v_{1,j,i}^k \left(\frac{u_{j,i}^k - u_{j-1,i}^k}{2h_r} - \frac{u_{j,i}^k}{r_j} \right) + v_{3,j,i}^k \frac{u_{j,i}^k - u_{j,i-1}^k}{2h_z} \quad (15)$$

if $v_{1,j,i}^k > 0$ and $v_{3,j,i}^k > 0$. I won't bother here to enumerate all the other cases which follow from the different combinations of signs that the velocity components assume. For $t > 0$, the values of $u_{j,i}^k$ will remain equal to zero along the z -axis, compatibly with (11)-(12) and the physical requirement that the vorticity must be zero on that axis. No boundary conditions are going to be imposed on the boundary of C , where the upwind scheme is instead implemented. Indeed, considering that the velocity field is tangential to the boundary, it is not difficult to adapt (15) accordingly. The situation would be different in the Navier-Stokes case, where the evaluation of viscosity without imposing boundary conditions requires points that are outside the computational domain.

For a given $\epsilon \geq 0$, an interesting initial guess is the following one:

$$u_{0,\epsilon}(r, z) = \frac{r}{\sqrt{r^2 + z^2 + \epsilon^2}} \quad (16)$$

that reduces to $u_0 = r/\rho$, with $\rho = \sqrt{r^2 + z^2}$, when $\epsilon = 0$. A plot illustrating the behavior of this function is shown in figure 2. A discrete version of the initial guess is obtained by interpolating $u_{0,\epsilon}$ at the grid points.

In a neighbor of the origin, the choice $u_0 = r/\rho$ is in good approximation compatible with the velocity field (see (4)):

$$v_{1,0}(r, z) = -\frac{rz}{4\rho} \quad v_{2,0}(r, z) = 0 \quad v_{3,0}(r, z) = \frac{\rho - 1}{2} + \frac{r^2}{4\rho} \quad (17)$$

The above field is divergence-free and presents an anomalous behavior at the origin. Indeed, u_0 is not continuous, which means that, in three-dimensions, it cannot belong to the functional space $H^s(C)$, with $s > 3/2$. As a matter of fact, the function $v_{3,0}$ does not belong to $H^s(C)$, with $s > 5/2$. Indeed, it does not even belong to $H^{5/2}(C)$ (see, e.g., [20], example 2.28, p.1.54). A rough explanation is obtained by arguing in spherical coordinates, where the $\frac{5}{2}$ -derivative of ρ is $\rho^{-3/2}$. Its $L^2(C)$ norm involves the evaluation of $[\rho^{-3/2}]^2 \rho^2 = 1/\rho$ (observing that the determinant of the Jacobian is proportional to ρ^2), where the last is not a 1D integrable function in the neighborhood of the origin.

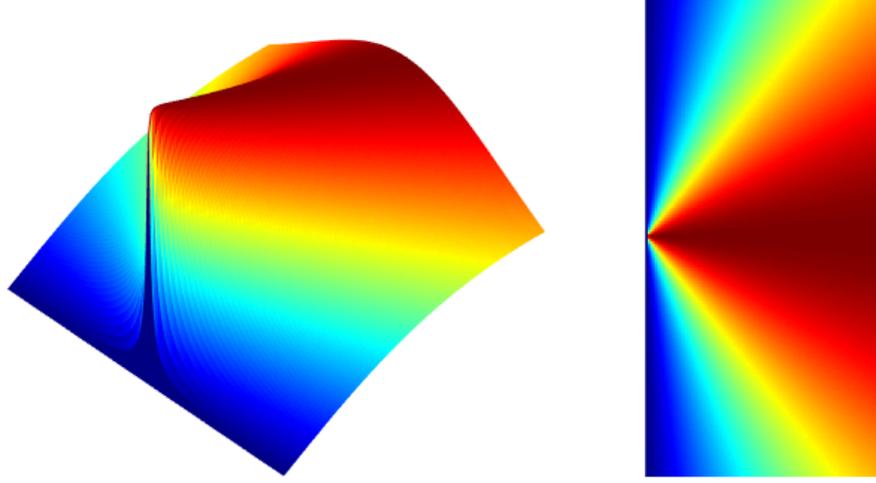


Figure 2: Plot of $u_{0,\epsilon}$ in the domain R , for ϵ suitably small (see (16)). When ϵ tends to zero, the function generates a discontinuity at $(0, 0)$ since $u_0 = 0$ on the z -axis and, at the same time, $u_0 = 1$ on the plane z .

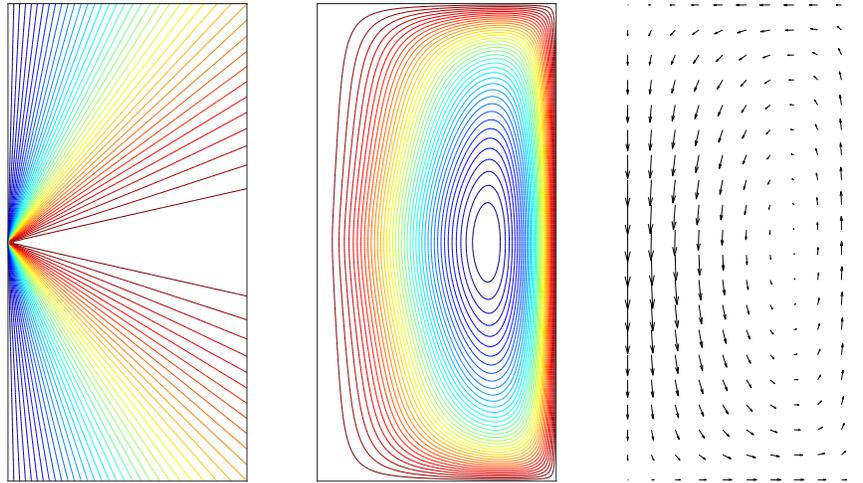


Figure 3: Contour lines of $u_0 = r/\rho$ and the corresponding $r\Psi_0$ in the rectangle $(r, z) \in [0, 1] \times [-1, 1]$. The velocity field (right) is tangential to the contour lines of $r\Psi_0$.

According to the Beale-Kato-Majda criterion [21], we have that the norm of the velocity field having its components in $H^s(C)$, $s > 5/2$, is uniformly

bounded in time if and only if:

$$\int_0^\tau \|\omega(t)\|_{L^\infty(C)} dt < +\infty \quad (18)$$

for $\tau > 0$. In the present situation, the vorticity will be uniformly bounded in the maximum norm (so to satisfy the inequality (18)). At the same time, the velocity field belongs to a Sobolev space with a borderline index. This is referred to as the “first regularity barrier”. Let me mention some theoretical contributions somehow related to the relationship between the boundedness of the vorticity and the regularity of the velocity field: [22, 23, 24, 25].

Near $(0, 0)$ the velocity components in (17) originate from the potential $\Psi_0 = r(\rho-1)/4$, that, for $z = 0$ is compatible with the boundary conditions $\Psi_0(0, z) = \Psi_0(1, z) = 0$. A little more complicated is the expression of Ψ_0 by solving the elliptic problem (5), with the homogeneous boundary constraints imposed on the whole boundary of R . Plots of the initial configurations are provided in figure 3.

4 Numerical tests

We have all the elements to proceed with some numerical experiments. The aim is to show that the process of traveling backward and forward in time can be valid, at least in situations that are very simple for now. Here we limit ourselves to treating the case of the initial guess (16) when ϵ approaches zero. As we will see in the concluding section, the treatment of more sophisticated cases is not so trivial, but requires further efforts of imagination.

Taken $N = 100$ we set $M = 2N = 200$, which amounts to 20200 degrees of freedom including the boundary points. The final time after the backward trip is $T = .5$ and the time-step is $\Delta t = .001$. Various initial data have been taken into account by choosing different values of ϵ in (16), including $\epsilon = 0$. The last choice does not imply that we are in the presence of the initial discontinuous datum u_0 , considering that the interpolation process implies to choosing $\epsilon \approx 1/N$. This means that the value of ϵ is directly associated to the discretization parameter. The graphical results are commented in the captions of figures 4 and 5. As N grows the initial datum approaches u_0 . Convergence is not in the maximum norm, since the limit function is discontinuous. However, it occurs in a fairly acceptable space.

The backward solutions at time T appear quite smooth and tend to a precise limit for $\epsilon \rightarrow 0$. In other words, we are satisfying the relation (9). It is wise, however, to provide some more numerical details, and for this I refer to table 1. In this new context, the solution is computed for different N (with $M = 2N$) and compared to the one obtained for $N = 120$. This is done using the relative errors:

$$\mathcal{E}_N(t) = \frac{\|u_N(t) - u_{120}(t)\|}{\|u_{120}(t)\|} \quad (19)$$

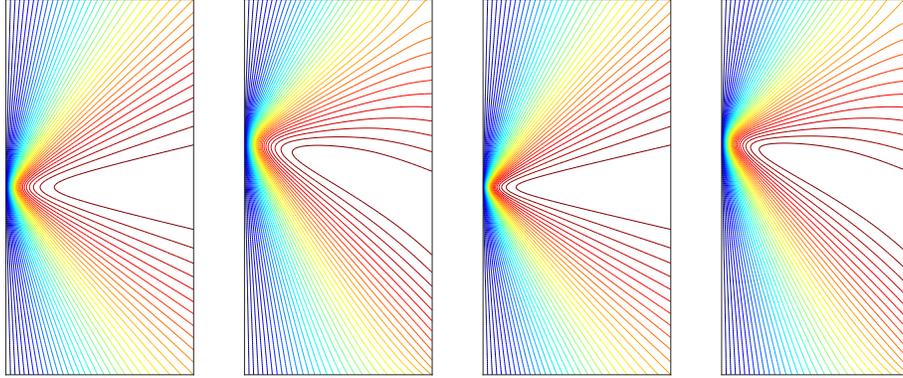


Figure 4: Contour lines of the approximated u for $N = 100$ and $M = 2N$. The subplots 1 and 3 show the initial datum $u_{0,\epsilon}$ for $\epsilon = .6$ and $\epsilon = .4$. The subplots 2 and 4 show the configurations at time $T = .5$.

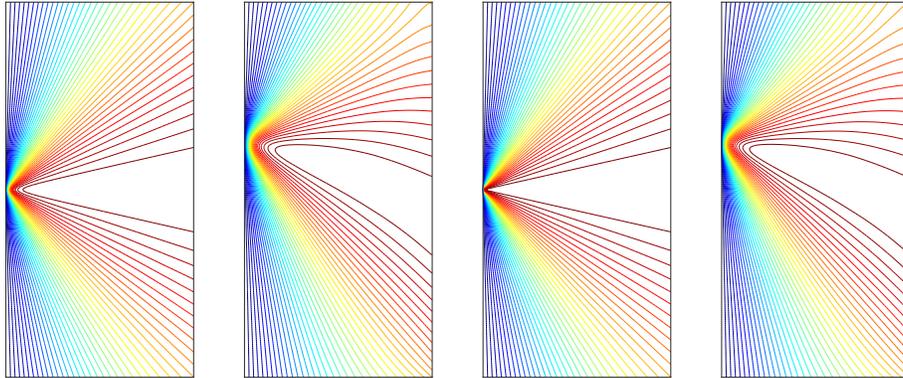


Figure 5: Contour lines of the approximated u for $N = 100$ and $M = 2N$. The subplots 1 and 3 show the initial datum $u_{0,\epsilon}$ for $\epsilon = .2$ and $\epsilon \approx .01$. The subplots 2 and 4 show the configurations at time $T = .5$.

To simplify the notations, in (19) $u_N(t)$ denotes the discrete solution at time t obtained by approximating (11). The norm is either the Euclidean one or the maximum norm. The final time is $T = .5$, the time-step is again $\Delta t = .001$. Instead of fixing N and letting ϵ tend to zero, here it is directly taken $\epsilon = 0$ in (16). Anyway, by increasing N , this is equivalent to automatically reduce ϵ as a consequence of the interpolation procedure of the initial datum. In order to homogenize the quantities involved for $N \neq 120$, the discrete solutions are evaluated in the finer grid corresponding to $N = 120$ through cubic interpolation (Matlab routine `interp2`).

N	$\mathcal{E}_N(0)$ Euc	$u_N(0)$ Max	$\mathcal{E}_N(T)$ Euc	$\mathcal{E}_N(T)$ Max
40	0.00698	1.01619	0.01951	0.18948
60	0.00393	1.01267	0.01696	0.12581
80	0.00226	1.00374	0.01372	0.09155
100	0.00097	0.99999	0.00808	0.05567

Table 1: Relative errors $\mathcal{E}_N(0)$ and $\mathcal{E}_N(T)$ for various N . The vector norms are either of Euclidean type or the maximum ones.

We recognize an error decay for $\mathcal{E}_N(T)$. This is a good symptom that tells us that $u_N(T)$ is stabilizing for $N \rightarrow +\infty$ (i.e.: $\epsilon \rightarrow 0$). It is certainly not a proof of convergence to a limit function, but there is good hope that this will actually happen. As for the maximum norm of the initial data, I preferred to report the values of $u_N(0)$, to emphasize the fact that we are approximating a function discontinuous at the origin (recall that u_N is uniformly zero along the z -axis).

The last thing to check is that the equations are reversible. Since, for a fixed N , we are in finite dimension, we need to verify that $u_N(2T)$ returns to $u_N(0)$, up to an error that depends on the various parameters. To this end let me introduce the relative error:

$$\mathcal{D}_N = \frac{\|u_N(2T) - u_N(0)\|}{\|u_N(0)\|} \quad (20)$$

where the norm is the Euclidean one. In order to save computational time, for the analysis I took a smaller final time, i.e.: $T = .05$, so that the cycle concludes at $2T = .1$. The time-step $\Delta t = .0001$ is small enough not to interfere with the evolution process. Of course, we can get a stable scheme only if Δt is sufficiently small (CFL condition), but here we are well below that threshold. Some results are reported in table 2.

N	$\epsilon = .6$	$\epsilon = .4$	$\epsilon = .2$	$\epsilon = 0$
20	0.00624	0.00804	0.01004	0.01103
40	0.00357	0.00498	0.00738	0.00952
60	0.00249	0.00358	0.00566	0.00837
80	0.00192	0.00279	0.00458	0.00751

Table 2: Relative errors \mathcal{D}_N for some values of ϵ in (16), as well as various N .

If one wants to approach a hypothetical limit situation, ϵ must be progressively reduced. As a consequence, N must increase and Δt must decrease in agreement. If the theoretical analysis is done through a finite-dimensional discretization (this it is not a requirement anyway), the three parameters ϵ , N and Δt need to be linked by appropriate relations.

As for the scheme, I used the upwind method both on the way in and on the way back. The results in the table seem to confirm that the cycle completes with a reasonable accuracy. As ϵ is reduced, it becomes increasingly difficult to return to the original displacement. In this case, it is necessary to considerably increase N and reduce the time-step. However, from a quantitative analysis of the plots in the vicinity of the origin (which I will not show here), the situation appears to be rather poor. Markedly poor! The discontinuity of u_0 is not even sketched and the discrete solution $u_N(2T)$ still appears to be very affected by viscosity. On the other hand, I remember that the numerical scheme used is really basic. A table similar to 2, where the norm used in (20) is the maximum norm would show a much less vigorous trend of convergence. My personal interpretation is that the singularity on the derivatives is highly repulsive and therefore it is easy to escape, but reaching it from the outside is rather difficult.

Impressive results are obtained if the *downwind* method is used on the way back. This means considering for example the formula (15) when $v_{1,j,i}^k < 0$ and $v_{3,j,i}^k < 0$. The error table in this case shows figures ranging between 10^{-5} and 10^{-4} . This implies that we are actually going back correctly. The reader will immediately notice that using the downwind method is something that should not even be thought of. Yet the outcomes are fantastic. I was also very skeptical, but some numerical tests on a simplified 1D model confirmed the trend. It is also important to consider that the fluid is confined within a cavity and that there are no inflow or outflow boundaries. This way the scheme can be implemented without the help of additional data. I will add further considerations on the problem of the correct “back the future” procedure in the next section.

If we proceed with the evolution after time $2T$, the solution develops symmetrically with respect to the plane $z = 0$. The behavior is equivalent to that of changing the sign of the initial datum Ψ_0 , so as to switch the direction of the flow lines. In practice, starting from the instant T , the internal part of the ring, initially smooth, moves so as to form a touch-point of non-regularity at the origin when $t = 2T$. It then proceeds as if there were a rebound, reaching a symmetric configuration at time $3T$. The rapid suction at time $t = 2T$ can be justified by a sort of Venturi effect. The video [26], available in my webpages, offers a good idea of the whole evolution. This was obtained by playing back and forth with the two last plots of figure 5 and their intermediate configurations.

5 Remarks

The main argument for this paper is that it is very difficult to generate an initial datum with the hope that it will produce exactly a singularity. Instead, it is more reasonable to start with an irregular datum and move backwards with

respect to the transport. We will see later on that this second approach should not always be taken for granted.

After these partial results, let me see if it is possible to add some comments to the points raised in the introduction:

- Is u_0 (i.e.: $u_{0,\epsilon}$ in (16) with $\epsilon = 0$) a reasonable initial condition? In other words, is this one of the green dots in figure 1? I really think so. There are so many theoretical articles published that surely someone will be able to confirm the choice or propose more appropriate ones.
- Are the equations reversible? It would seem so. Some more convincing numerical experiments and a good dose of theoretical arguments could confirm the suspicion. I'm pretty confident about this. On the other hand, it will be sufficient to have the reversibility property only in the discrete case, regardless of the number of degrees of freedom taken. The numerical results related to the last part of the previous section are quite crucial, since they show the scarce ability of the approximation methods (at least the less reliable ones) to correctly follow the dynamics along the return path. This confirms how difficult it is to try to arrive by direct paths to solutions that present irregularities, even if the one considered in this paper has a rather low degree of singularity. The build up of artificial viscosity must be kept under strict control.
- Is the solution at time $t \in]0, 2T[$ regular? In my opinion u instantly becomes $C^\infty(R)$ for $t > 0$ by imposing $u = 0$ at the origin $(0, 0)$. Recall that R is the open rectangle $]0, 1[\times]-1, 1[$. Along the z -axis u could be a bit less regular, but a better adjustment of u_0 may improve the situation. When working in finite dimension, all functions can be considered smooth, since they can be represented as a finite expansion of regular basis elements. This aspect also leads to a more likely ability to adapt to time reversal. The singularity point does not exist in this scenario, but manifests itself more and more with the increase of the discretization parameters. However, it is not correct to try to imagine the singular solution that regularizes instantly by going back in time. Instead, it is better to think in the following terms: we go back of an amount of time T to find a candidate which, in its forward evolution, will form a boundary layer and finally degenerates into a singularity. To do this we pass through the discretization of the problem. In reality, in the finite dimension the initial guess will always present a layer at the origin and never a true discontinuity.

The proposed experiments are very elementary from the numerical point of view. Of course one can do better. Fancier discretization schemes are welcome. However, I believe that these partial results are more than enough to point the way. In [7, 8], a component of the velocity field along the variable θ is also considered. I actually think that there is no need for it in the case that has been examined here. However, it is not excluded that this is an essential ingredient in the discussion for the more challenging case of the Navier-Stokes equations.

Finding initial conditions that lead to an effective blowup of the solution might be like trying to throw a ball hoping that it lands exactly on a pin. The singularity basin of attraction may be so small that it would be too difficult to hit the target exactly. By exploiting a sort of time reversal for the Euler equation I was able to provide a workable recipe that, starting from the singularity, produces, going backward, reliable candidates. The procedure has been tested using elementary computational schemes, but it is evident that the world of numerical analysis can suggest more refined and accurate techniques. Theoretical developments could also benefit from this new point of view, although not much space has been devoted to these issues in the present paper.

Inspired by this success we might think of aiming higher and therefore propose a situation in which the velocity field presents a real blowup. A direct extension of what I have done so far has led me to consider the following choice:

$$\begin{aligned} \Psi_0 = r \log \rho \quad \mathbf{v}_0 = \vec{\nabla} \times (0, \Psi_0, 0) &= \left(-\frac{rz}{\rho^2}, 0, \frac{r^2}{\rho^2} + 2 \log \rho \right) \\ u_0 = \Delta \Psi_0 - \frac{\Psi_0}{r^2} &= \frac{3r}{\rho^2} \end{aligned} \quad (21)$$

The situation refers to what happens in a neighborhood of the origin. With the imposition of boundary conditions, the expression of Ψ_0 turns out to be different, while $u_0 = 3r/\rho^2$ will be taken as the initial datum. Now the first component of the velocity field is actually discontinuous and the last one tends logarithmically to minus infinity for ρ tending to zero. The last plot of figure 6 shows the qualitative behavior of \mathbf{v}_0 . The vectors converge to the origin carrying different information depending on the streamline considered, thus producing a discontinuity. A real shock with a blowup flavor!

However, there is no reason to be too optimistic. Things are not going well this time. Let me again refer to the qualitative plots of figure 6 that illustrate the behavior of u_0 (in truth, its projection into a finite dimensional space) and that of u at a certain time T , after going back along the stream lines. The results would seem reasonable. However, they are not quantitatively acceptable. There is too much impetus in singularity. You can play well back and forth with the solutions, but the displacement at time T does not stabilize as the degrees of freedom increase. In fact, as N grows, the intensity of the peak of u_0 at the origin also grows proportionally. This is correct, since u_0 behaves as $1/\rho$. However, it is not correct that the maximum of u at time T also grows in proportion to N , although the intensity of the initial peak undergoes a strong damping, spreading the energy over a wider region. This means that the limit (9) does not exist and that therefore the solution presented in (21) is not “reachable”, according to the sketch of figure 1. Needless to say, I have experimented with other numerical approaches to try to better handle the behavior of this solution near the origin (including some changes of variable). Nevertheless, the problem persisted.

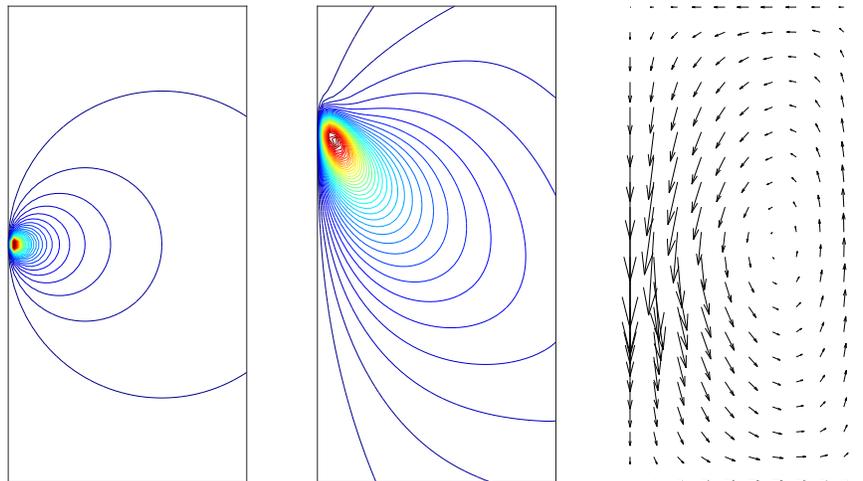


Figure 6: Contour lines of $u_0 = 3r/\rho^2$ and those of u at time $T = .1$ (going backward). The discretization parameters are $N = 40$ and $M = 2N$. Right: qualitative displacement of the initial vector field in (21).

I have other ideas in the works, but I do not intend to discuss them in this article. Among them, also the possible treatment of the Navier-Stokes equations, where, if the singularity is concentrated at the origin, the order of infinity is expected to be much more pronounced than that proposed in (21). Indeed u_0 should behave as $1/\rho^2$, at least (as also confirmed by theoretical considerations). Going back in time in the presence of viscosity is considered an ill-posed problem. However, we have to think about the fact that this has to be done in a finite-dimensional space, where the arrow of time can be reversed instead for short times (it is enough to express the solution in a finite number of eigenfunctions of the Laplacian). At this point, the presence of the nonlinear term is crucial to counterbalance the expansive effects of the heat equation solved in reverse. I have a positive attitude towards the possibility of getting some encouraging outcome, but I am not yet able to provide convincing results (it would be too good to say!).

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