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Intransitive collineation groups of ovals fixing a triangle☆

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Abstract

We investigate collineation groups of a finite projective plane of odd order fixing an oval and having two orbits on it, one of which is assumed to be primitive. The situation in which there exists a fixed triangle off the oval is considered in detail. Our main result is the following. **Theorem**. *Let* π *be a finite projective plane of odd order n containing an oval* Ω. *If a collineation group G of* π *satisfies the properties*:

(a) *G fixes* Ω *and the action of G on* Ω *yields precisely two orbits* Ω_1 *and* Ω_2,
(b) *G has even order and a faithful primitive action on* Ω_2,
(c) *G fixes neither points nor lines but fixes a triangle ABC in which the points A*, *B*, *C are not on the oval* Ω,

then $n \in \{7, 9, 27\}$, *the orbit* Ω_2 *has length* 4 *and G acts naturally on* Ω_2 *as* A_4 *or* S_4.

Each order $n \in \{7, 9, 27\}$ does furnish at least one example for the above situation; the determination of the planes and the groups which do occur is complete for $n = 7$, 9; the determination of the planes is still incomplete for $n = 27$.

Keywords: Finite projective plane; Oval; Collineation group; Homology; Baer involution

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1. Introduction

A collineation group of a projective plane is said to be *irreducible* if it fixes no point, line or triangle of the plane. The possible structures and actions of irreducible collineation groups of finite projective planes were investigated by Hering who was able to prove a classification theorem under the hypothesis that the group contains non-trivial perspectivities, see [12–14].

Hering's results have often played a relevant role in the study of collineation groups of finite projective planes of odd order fixing an oval. For example, Hering's classification theorem was the main tool in the study of transitive ovals: as a matter of fact, the sole assumption of transitivity on the points of the oval does imply both irreducibility and the existence of non-trivial perspectivities in several meaningful instances, see [3,4].

Some recent papers have singled out interesting situations in which the given collineation group is neither transitive on the oval nor irreducible on the plane, see [1,9–11], with examples both in desarguesian and non-desarguesian planes.

It is the purpose of the present paper to investigate the case of a reducible collineation group fixing an oval and a triangle. More specifically, let G be a collineation group of a finite projective plane π of odd order n and let Ω be an oval of π. The group G is assumed to satisfy the following properties:

(a) G fixes Ω and the action of G on Ω yields precisely two orbits Ω_1 and Ω_2;
(b) G has even order and a faithful primitive action on Ω_2;
(c) G fixes neither points nor lines but fixes a triangle ABC in which the points A, B, C are not on the oval Ω.

Our main result is the following.

Theorem 1. *Let π be a finite projective plane of odd order n containing an oval Ω. If a collineation group G of π satisfies properties* (a)–(c), *then $n \in \{7, 9, 27\}$, the orbit Ω_2 has length* 4 *and G acts naturally on Ω_2 as A_4 or S_4.*

The only examples of the above situation in which the group G is isomorphic to A_4 occur in the desarguesian plane of order 7, in the desarguesian plane of order 9 and in the Hughes plane of order 9. Examples in which the group G is isomorphic to S_4 occur in the desarguesian plane of order 9, in the Hughes plane of order 9, in the desarguesian plane of order 27 and in the Figueroa plane of order 27. The only order for further possible examples is $n = 27$: we have not attempted a complete search yet.

2. Preliminaries

We begin by recalling that a group action on a given set is said to be *faithful* if the kernel of the action is trivial.

Lemma 1 (Wielandt [19]). *Let H be a collineation group of a finite projective plane π and let Δ be one of its point-orbits. Let N be a normal subgroup of H. If N fixes a point of Δ then Δ is pointwise fixed by N. If H acts primitively on Δ then either N fixes Δ pointwise or N is transitive on Δ. In the latter case if N is an abelian minimal normal subgroup of H then it is regular on Δ.*

If H is a collineation group of a finite projective plane and X is a subset of H, we denote by $\mathrm{Fix}(X)$ the substructure consisting of the points and lines which are fixed by every collineation in X.

Lemma 2. *Let H be a collineation group of a finite projective plane π fixing an oval Ω. Assume H fixes at least three points on Ω. Then $\mathrm{Fix}(H)$ is a subplane π_0 of π and the fixed points of H on Ω form an oval Ω_0 in π_0. If the order of π is odd then so is the order of π_0.*

Proof. The tangents through two points of Ω which are fixed by H meet at a point which is also fixed by H and so there exists a quadrangle which is pointwise fixed by H. Then we know that $\mathrm{Fix}(H)$ is a subplane π_0. Let Ω_0 be the subset of Ω consisting of the points which are fixed by H. Clearly Ω_0 is an arc in π_0. If P lies on Ω_0 and ℓ_P denotes the unique tangent to Ω in π, then H fixes ℓ_P and so ℓ_P is a line of π_0. If ℓ is another line in π_0 through P, then ℓ must be a secant to Ω through P; since ℓ and P are fixed by H, we see that H must also fix the further point Q at which ℓ meets Ω; consequently Q lies in π_0 and ℓ is a secant to Ω_0 in π_0. We have shown that Ω_0 is an arc in π_0 with a unique tangent at each one of its points, which means Ω_0 is an oval in π_0.

Assume π has odd order and let P, Q, R be three distinct points on Ω_0. If the order of π_0 is even, then the tangents to Ω_0 at these points pass through the nucleus of Ω_0 in π_0. At least one of these three lines, say ℓ, must therefore be a secant to Ω in π. Let X denote the further point of intersection of ℓ with Ω; since ℓ is fixed by H, so is X, which means X is in π_0 and consequently ℓ is a secant to Ω_0 in π_0, a contradiction. □

Useful information can usually be obtained on a collineation group of a finite projective plane which is known to contain central collineations. A non-trivial perspectivity fixing an oval in a projective plane of odd order is an involutory homology, [3, Proposition 2.1].

Lemma 3. *An involutory homology of a finite projective plane of odd order fixing a subplane must induce an involutory homology on the subplane.*

For the rest of this section π denotes a finite projective plane of odd order n with an oval Ω and G denotes a collineation group of π satisfying the following properties:

(a) G fixes Ω and the action of G on Ω yields precisely two orbits Ω_1 and Ω_2;
(b) G has even order and a faithful primitive action on Ω_2.

The following lemma is an adjustment of [4, Lemma 4.1] to our situation.

Lemma 4. *Assume* $|\Omega_1| \geqslant 2$. *If* G *possesses an elementary abelian normal subgroup* M *of odd order*, *then* G *contains some involutory homology.*

Proof. Write $|M| = p^e$ with p an odd prime and e a positive integer. By assumption (b) the group M acts non-trivially on Ω_2 and consequently M is regular on Ω_2 by Lemma 1, yielding in particular $|\Omega_2| = p^e$.

Assume M has no fixed point in Ω_1. Then M is fixed-point-free on Ω and M can therefore fix only interior points and exterior lines. In addition, p divides the length of each M-orbit on Ω_1, hence p divides $|\Omega_1|$ and consequently p divides $|\Omega| = n+1$. Since n is odd we obtain the relations $\gcd(p,n) = \gcd(p,n-1) = 1$ which in turn imply $\gcd(p, n(n-1)/2) = 1$. We conclude that M fixes at least one interior point L and at least one exterior line ℓ.

Suppose that M fixes an exterior line ℓ' different from ℓ; then M fixes $\ell \cap \ell'$; furthermore, M fixes at least two more points on each line ℓ and ℓ' because $\gcd(p,n) = \gcd(p,n-1) = 1$; hence $\mathrm{Fix}(M)$ is a subplane in this case and the same conclusion holds if M fixes an interior point other than L. We have shown that $\mathrm{Fix}(M)$ either consists of a single interior point and a single exterior line or is a subplane consisting entirely of interior points and exterior lines. In either case an involutory collineation h in G must fix some line ℓ in $\mathrm{Fix}(M)$ because h normalizes M and therefore fixes $\mathrm{Fix}(M)$ setwise. If h is a Baer involution, then since $|\Omega_2|$ is odd, h must fix some point in Ω_2 and so Case II of [3, Proposition 2.2] occurs, in particular no point of $\mathrm{Fix}(h)$ is interior and no line is exterior, a contradiction.

Assume that M fixes some point of Ω_1. Lemma 1 shows that M fixes Ω_1 pointwise. Since $|\Omega_2| = p^e$ is odd and $|\Omega| = |\Omega_1| + |\Omega_2|$ is even, we have that $|\Omega_1|$ is odd, in particular $|\Omega_1| \geqslant 3$. Lemma 2 yields now that $\mathrm{Fix}(M)$ is a subplane of odd order and Ω_1 is an oval in $\mathrm{Fix}(M)$, whence $|\Omega_1|$ is even, a contradiction. □

Proposition 1. *If* $|\Omega_1| \geqslant 2$ *and* $|\Omega_2| \geqslant 3$ *then* G *contains some involutory homology. If* $|\Omega_1| = 3$ *and* $|\Omega_2| \geqslant 3$ *then* $n = 5$ *and the group* G *is isomorphic to* S_3 *fixing an interior point and acting sharply* 2*-transitively on each orbit* Ω_1 *and* Ω_2.

Proof. Assume all involutions in G are Baer involutions. By [3, Proposition 2.5] the Sylow 2-subgroups of G are cyclic. If M is a minimal normal subgroup of G, then by [15, IV Satz 2.8] it is elementary abelian and its order is either odd or equal to 2. By assumption (b) the group M acts non-trivially on Ω_2 and consequently M is regular on Ω_2 by Lemma 1.

The case $|M| = 2$ yields $|\Omega_2| = 2$, which is excluded. If $|M|$ is odd then Lemma 4 yields the assertion.

Assume now $|\Omega_1| = 3$ and put $\Omega_1 = \{A_1, A_2, A_3\}$. Since $|\Omega| = |\Omega_1| + |\Omega_2|$ is even we see that $|\Omega_2|$ is odd. Since an involutory homology fixes 0 or 2 points on Ω, each involutory homology in G fixes exactly one point in Ω_1: since this point cannot be the center of the involutory homology, it must lie on its axis. Let h and h' be two distinct involutory homologies of G fixing the same point in Ω_1, say A_1. Since both h and h'

interchange the tangents to Ω through A_2 and A_3, we see that the common point P of these tangents is fixed by h and h'. Both involutory homologies h and h' have thus the line PA_1 as axis, and that contradicts the fact that distinct involutory homologies fixing Ω have distinct axes, see [3, Proposition 2.1 (2)].

Since G acts transitively on Ω_1, we conclude that there exist in G exactly three involutory homologies h_1, h_2, h_3 such that each one of them fixes precisely one point in Ω_1 and distinct involutory homologies fix distinct points. No two such involutory homologies commute, as the center of one of them does not lie on the axis of the other and we may assume the labelling to be such that A_i is the fixed point of h_i on Ω_1 for $i = 1, 2, 3$.

Consider the normal subgroup $H = \langle h_1, h_2, h_3 \rangle$ of G. The involutory homologies h_1, h_2, h_3 induce the 2-cycles (A_2, A_3), (A_1, A_3), (A_1, A_2) on Ω_1 respectively and consequently H induces the symmetric group S_3 on Ω_1.

The next step consists in showing that H acts faithfully on Ω_1. The relation $h_1h_2h_1 = h_3$ implies that $\langle h_1, h_2 \rangle$ contains h_3 and therefore $H = \langle h_1, h_2 \rangle$ is a dihedral group as it is generated by two involutions. The relations $h_1h_2h_1 = h_3$, $h_2h_1h_2 = h_3$ yield now that h_1h_2 has order 3 and so H has order 6, consequently H is isomorphic to S_3 and the action of H on Ω_1 is faithful.

Each involutory homology has precisely one fixed point in Ω_2, hence H cannot fix Ω_2 pointwise and so H must be transitive on Ω_2 by Lemma 1. If $X \in \Omega_2$ is fixed by an involutory homology in H, we have that the stabilizer H_X has order at least 2, consequently $|\Omega_2| = |H : H_X| \leqslant 3$. We have thus $|\Omega_2| = 3$, which together with $|\Omega_1| = 3$ yields $n = 5$.

We have $S_3 \cong H \leqslant G$ and G is faithful on Ω_2, hence $H = G \cong S_3$ and since each collineation of order 3 of $PG(2, 5)$ fixes precisely one point, we see that G has one fixed point which is an interior point. □

3. Main result

Let π be a finite projective plane of odd order n with an oval Ω and let G be a collineation group of π satisfying properties (a)–(c) stated in the Introduction. The pointwise stabilizer of $\{A, B, C\}$ in G will be denoted by G_0. We note that since G fixes no point, it must be transitive on $\{A, B, C\}$ and so G_0 is a proper normal subgroup of G.

Assumption (c) excludes the possibilities $|\Omega_1| = 1$, 2, $|\Omega_2| = 1$, 2 and so we may assume that both relations $|\Omega_1| \geqslant 3$, $|\Omega_2| \geqslant 3$ hold. Proposition 1 shows that if $|\Omega_1| = 3$ then G has a fixed point and so this case is excluded as well.

From now on we shall thus assume $|\Omega_1| > 3$, $|\Omega_2| \geqslant 3$. It follows then from Proposition 1 that G contains involutory homologies. We recall that an involutory homology fixing an oval in a finite projective plane of odd order is uniquely determined by its center or by its axis respectively, see [3, Proposition 2.1]. In our situation, it is immediately seen that if the center is one of the vertices of the triangle then the axis is the opposite side and, similarly, if the axis is one of the sides of the triangle then the center is the opposite vertex. The next basic step is to show the existence of involutory homologies in G_0.

Proposition 2. *The group G contains involutory homologies with center and axis on the fixed triangle ABC.*

Proof. Assume G contains no involutory homology with center or axis on ABC; each involutory homology in G then fixes ABC setwise but not pointwise and therefore each such involutory homology will have its axis through one of the vertices and its center on the opposite side. Since the vertex is not fixed by G, we have that G induces at least two distinct involutions on $\{A, B, C\}$. We have shown $G/G_0 \cong S_3$.

Assume $G_0 = \{\mathrm{id}\}$. We have then $G \cong S_3$, whence $|\Omega_1| \leqslant 6$ and since the action of G on Ω_2 is primitive we get $|\Omega_2| = 3$. Altogether we have $n + 1 = |\Omega| \leqslant 9$ and so $n \in \{3, 5, 7\}$. If $n = 3$, then $|\Omega_2| = 3$ yields $|\Omega_1| = 1$ which is excluded. If $n = 5$, then $|\Omega_2| = 3$ yields $|\Omega_1| = 3$ and Proposition 1 shows that G fixes a point, which is excluded by assumption (c). Assume $n = 7$; from $|\Omega_2| = 3$ and $|\Omega| = n + 1 = 8$ we obtain $|\Omega_1| = 5$ and so Ω_1 cannot be an orbit under S_3, a contradiction. We have shown the property $G_0 \neq \{\mathrm{id}\}$.

Let M be a minimal normal subgroup of G. We want to prove that we may assume $M \leqslant G_0$. If not we have $|M| = 3$; then since M is regular on Ω_2 we have $|\Omega_2| = 3$. If M fixes some point in Ω_1, then M fixes Ω_1 pointwise by Lemma 1; the relation $|\Omega_1| > 3$ and Lemma 2 yield then that Ω_1 is an oval in the proper subplane $\mathrm{Fix}(M)$ of odd order. It follows from $|\Omega_1| = n - 2$ that the order of this subplane is $n - 3$, which is even, a contradiction. We have shown that M is fixed-point-free on Ω_1. In particular each M-orbit on Ω_1 has length 3 and the relation $n \equiv 2 \bmod 3$ holds. It follows from $n^2 + n + 1 \equiv 1 \bmod 3$ that M fixes some point of π. The group M cannot have a unique fixed point in π, otherwise, since M is normal in G, such a point would also be fixed by G, which is excluded by assumption (c). Let P, Q be distinct fixed points of M on π. The relation $n \equiv 2 \bmod 3$ shows that M fixes at least two lines through P other than PQ; similarly, M fixes at least two lines through Q other than PQ; consequently M fixes a quadrangle pointwise, that is $\mathrm{Fix}(M)$ is a proper subplane of π. Since M has no fixed points on Ω, each point of $\mathrm{Fix}(M)$ is interior and each line of $\mathrm{Fix}(M)$ is exterior. Since M is normal in G, we see that $\mathrm{Fix}(M)$ is left invariant by G and so, in particular, $\mathrm{Fix}(M)$ is left invariant by any involutory homology h in G, which therefore induces an involutory homology on $\mathrm{Fix}(M)$ by Lemma 3, in particular the center and the axis of h lie in $\mathrm{Fix}(M)$. Furthermore, setting $\Omega_2 = \{X, Y, Z\}$, we have that h must fix at least one of the points of Ω_2, say Z. As the center of h cannot lie on the oval Ω, we see that Z must be on the axis of h, which is thus a secant line, contradicting the fact that each line of $\mathrm{Fix}(M)$ is exterior. We conclude that the possibility $|M| = 3$ is also ruled out.

Since G acts faithfully on Ω_2, the group M does not fix Ω_2 pointwise and so Lemma 1 shows that M is transitive on Ω_2. Furthermore, since G_0 contains no involutory homologies, [3, Proposition 2.5] applies and so the Sylow 2-subgroups of

G_0 are cyclic. It follows from [15, IV Satz 2.8] that M is elementary abelian and $|M|$ is either 2 or p^e for some odd prime p and positive integer e. In addition, M is regular on Ω_2.

If $|M| = 2$ then we have $|\Omega_2| = 2$ which is not the case. Hence $|\Omega_2| = p^e$ and $|\Omega_1| = n + 1 - p^e$ is also odd. If M fixes a point of Ω_1, then Lemma 1 shows that Ω_1 is pointwise fixed by M. The relation $|\Omega_1| > 3$ and Lemma 2 yield then that Ω_1 is an oval in the proper subplane $\mathrm{Fix}(M)$ of odd order, contradicting the fact that $|\Omega_1|$ is odd. We conclude that M is fixed-point-free on Ω_1, whence p divides $|\Omega_1|$ and consequently p also divides $|\Omega| = n + 1$. Since n is odd, that yields $\gcd(p, n) = \gcd(p, n - 1) = 1$. Furthermore, M is fixed-point-free on Ω and so each fixed point of M is interior and each fixed line of M is exterior.

We can now state that $\mathrm{Fix}(M)$ is a subplane. As a matter of fact M fixes $\{A, B, C\}$ pointwise; since $\gcd(p, n - 1) = 1$ we see that M must fix at least one line ℓ through A other than AB, AC; similarly if P is the point of intersection of ℓ with BC, we see that M must fix at least one point D on ℓ other than A and P. The points A, B, C, D form a quadrangle and are in $\mathrm{Fix}(M)$, showing that $\mathrm{Fix}(M)$ is a subplane.

Since M is normal in G, each involutory homology h in G fixes $\mathrm{Fix}(M)$ setwise and hence induces an involutory homology on $\mathrm{Fix}(M)$ by Lemma 3, in particular the center and the axis of h lie in $\mathrm{Fix}(M)$ and so the center is an interior point and the axis is an exterior line, respectively. On the other hand, since both $|\Omega_1|$ and $|\Omega_2|$ are odd, we have that h fixes at least one point in Ω_1 and at least one point in Ω_2, whence precisely one point in Ω_1 and precisely one point in Ω_2. The axis of h is thus the secant line joining these two points, a contradiction. □

Proposition 2 shows that we may assume the existence of an involutory homology in G with center and axis on ABC, say h_A with center$A$ and axis BC. Since G acts transitively on $\{A, B, C\}$, we see that G also contains the involutory homology h_B with center B and axis AC and the involutory homology h_C with center C and axis AB. Furthermore, the involutions h_A, h_B, h_C are conjugate in G. Clearly the subgroup $K = \langle h_A, h_B, h_C \rangle$ is normal in G and is elementary abelian of order 4 by the quoted property [3, Proposition 2.1] that an involutory homology in our context is uniquely determined by its center or by its axis respectively and by [7, 3.1.8(a) p. 120]. The unique fixed points of K in the plane π are A, B and C, hence K has no fixed points on Ω_2 and so K is regular on Ω_2, in particular $|\Omega_2| = 4$. By assumption (b) the group G acts faithfully and primitively on Ω_2. The primitive permutation groups on four objects are A_4 and S_4 and so these are the unique possibilities for G.

Proposition 3. *The group G is either isomorphic to A_4 or to S_4.*

Proposition 4. *If the sides of the triangle ABC (that is the axes of the involutory homologies h_A, h_B h_C) are secant lines, then π is either the desarguesian plane of order* 9 *or the Hughes plane of order* 9. *In either case Ω_2 is an oval in a Baer subplane and both possibilities $G \cong A_4$ or $G \cong S_4$ do occur.*

Proof. The points of intersection of each axis with Ω lie in the same K-orbit, for instance, h_A exchanges the points of intersection of AB with Ω. These six points are pairwise distinct and they all lie in the same G-orbit as G is transitive on $\{A, B, C\}$. Since $|\Omega_2| = 4$ we conclude that the G-orbit containing these six points is Ω_1.

On the other hand K is normal in G and the K-orbits on Ω_1 form blocks of imprimitivity for G on Ω_1. In particular these orbits must have the same length. The six points mentioned above form three K-orbits on Ω_1 of length two each: if there were a further point in Ω_1, its K-orbit should have length four, as such a point is not fixed by any one of the involutory homologies in K. We conclude that Ω_1 consists precisely of these six points and so the plane π has order $n = 9$.

By [16] there are only four isomorphism classes of planes of order 9: the desarguesian plane, the Hall plane, its dual plane and the Hughes plane. The Hall plane has a unique isomorphism class of ovals and so does its dual plane, see [6, Sections 3 and 4]; in either case the automorphism group leaving an oval invariant has order 32 and thus contains neither A_4 nor S_4.

Assume π is desarguesian. Then Ω is a conic. The stabilizer S of Ω in the full collineation group of $PG(2, 9)$ is isomorphic to $P\Gamma L(2, 9)$. A subgroup of $P\Gamma L(2, 9)$ isomorphic to A_4 or S_4 must normalize its unique Klein subgroup K. As we have seen, this subgroup contains three involutory homologies with secant axes. That means K is contained in the unique subgroup U of S which is isomorphic to $PSL(2, 9)$ and K is uniquely determined up to conjugation in S. The normalizer in S of K has order 48 and contains a unique subgroup M isomorphic to S_4, which already lies in U. This subgroup M and its unique subgroup N isomorphic to A_4 both fix a triangle off Ω setwise and yield two orbits on Ω, one of length six and one of length four with 2-transitive action on the latter one. The fact that the orbit of length four is a conic in a Baer subplane can be best seen by choosing the equation for Ω to have coefficients in $GF(3)$.

The Hughes plane of order 9 has two isomorphism classes of ovals as described in [8,17]. The ovals in one class are called *central extension ovals* in [6] because they can be obtained by extending a conic of the Baer subplane which is fixed by the automorphism group of the plane. They are also known as Room ovals [18]. If Ω is one such oval, then its stabilizer S in the full collineation group of the plane has order 48 and contains a subgroup M which is isomorphic to S_4. Both M and its subgroup N isomorphic to A_4 yield two orbits on Ω of length six and four, respectively; the latter one is obviously the oval Ω_2 in the fixed Baer subplane, from which the extension started: both M and N fix setwise the triangle formed by the three points of the fixed Baer subplane which are interior with respect to Ω_2. The ovals of the other class are called *new ovals* in [6]; the stabilizer of each such oval in the full collineation group of the plane has order 16 and thus contains neither A_4 nor S_4. □

Proposition 5. *If the sides of the triangle ABC (that is the axes of the involutory homologies h_A, h_B h_C) are exterior lines, then π is either the plane of order 7, or one of the planes of order 27.*

Proof. Since any one of the involutory homologies h_A, h_B, h_C is the product of the other two, we have that at least one such involutory homology induces a fixed-point-free even permutation on Ω, whence $(n+1)/2$ is even, which means $n \equiv 3 \bmod 4$.

The case $|\Omega_1| = 4$ yields $n = 7$, hence π is desarguesian and Ω is a conic. The stabilizer S of Ω in the full collineation group of $PG(2,7)$ is isomorphic to $PGL(2,7)$. A subgroup of $PGL(2,7)$ isomorphic to S_4 or A_4 must normalize its unique Klein subgroup K, which already lies in the unique subgroup U of S which is isomorphic to $PSL(2,7)$. The normalizer N in U of the subgroup K has order 24 and it is isomorphic to S_4. The group N is transitive on Ω, but its unique subgroup M isomorphic to A_4 has two orbits on Ω of length four each and fixes a triangle off Ω setwise.

Assume $G \cong S_4$. Then since we are assuming $|\Omega_1| > 3$, the remaining possibilities for Ω_1 are 6, 8, 12 or 24.

The case $|\Omega_1| = 6$ yields $n = 9 \equiv 1 \bmod 4$ which does not occur.

The case $|\Omega_1| = 8$ yields $n = 11$. Consider a collineation g of order 3 in G; we see that g has at least two fixed points on Ω_1 and at least one fixed point in Ω_2. It follows from Lemma 2 that $\pi_0 = \mathrm{Fix}(g)$ is a proper subplane of odd order m in π; since π_0 cannot be a Baer subplane we necessarily have $11 \geqslant m^2 + m \geqslant 3^2 + 3 = 12$, a contradiction.

The case $|\Omega_1| = 12$ yields $n = 15$. The stabilizer in G of each point in Ω_1 has order 2; the involutions in K are fixed-point-free on Ω and so the unique involution j in such a stabilizer lies in $G \backslash K$; furthermore, $|\Omega_1| = 12$ shows that j fixes at least one more point in Ω_1. Since G acts on Ω_2 as S_4 in its natural action, we see that j fixes two points in Ω_2 and thus, altogether, at least four points in Ω. As no three of these points are collinear, we conclude that j is a planar collineation, namely a Baer involution, which is impossible as 15 is not a square.

The case $|\Omega_1| = 24$ yields $n = 27$. There are examples for the given situation both in the desarguesian plane and in the Figueroa plane. In either case, consider the irreducible conic Ω_2 with equation $X_0^2 = X_1 X_2$ in the canonical subplane of order 3; clearly Ω_2 is a subset of the irreducible conic of $PG(2,27)$ with the same equation; it has been shown in [5] that Ω_2 can also be extended to an oval in the Figueroa plane of order 27. In either case a copy of $PGL(2,3) \cong S_4$ fixing Ω_2 in the subplane of order 3 can be extended to a collineation group G fixing the oval in the plane of order 27. It is not hard to see that G satisfies the required properties; in particular the three pairwise commuting involutions in the normal Klein subgroup K of G are involutory homologies, hence K fixes the triangle formed by their centers pointwise, and this triangle will be setwise fixed by G. Other planes of order 27 are known, but a full classification is not yet available: we have not tried to see if the constraints imposed by our situation possibly exclude further examples.

Assume $G \cong A_4$. Then since we are assuming $|\Omega_1| > 3$, the remaining possibilities for $|\Omega_1|$ are 6 and 12.

Again, $|\Omega_1| = 6$ yields $n = 9 \equiv 1 \bmod 4$ which does not occur.

The case $|\Omega_1| = 12$ yields $n = 15$. Differently from the situation described above in which $G \cong S_4$, we were not able to discard this possibility on the basis of purely theoretical considerations. A deeper combinatorial analysis was necessary along with some computer calculations: the full treatment is given in [2]. □

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