Potential Analysis
https://doi.org/10.1007/s11118-022-09987-8

A BMO-Type Characterization of Higher Order Sobolev Spaces

Serena Guarino Lo Bianco[1] **· Roberta Schiattarella**[2]

Received: 28 October 2021 / Accepted: 19 January 2022

Abstract
We obtain a new characterization of the higher Sobolev space $W^{m,p}(\mathbb{R}^n)$, $m \in \mathbb{N}$ and $p \in (1, +\infty)$ and of the space BV^m, the space of functions of higher order bounded variation. The characterizations are in term of BMO-type seminorms. The results unify and substantially extend previous results in Fusco et al. (ESAIM Control Optim. Calc Var., **24**(2), 835–847 2018) and Farroni et al. (J. Funct. Anal., **278**(9), 108451 2020).

Keywords Higher order Sobolev spaces · Higher order bounded variation · BMO-type seminorms

Mathematics Subject Classification (2010) 46E35 · 26B30

1 Introduction

Let $W^{m,p}_{\text{loc}}(\mathbb{R}^n)$ ($m \in \mathbb{N}$, $1 \leq p < \infty$), denote the Sobolev space of functions belonging to $L^p_{\text{loc}}(\mathbb{R}^n)$ whose distribution derivatives up to order m belong to $L^p_{\text{loc}}(\mathbb{R}^n)$.

In [3], the Authors studied a characterization of $W^{m,p}$ based on J. Bourgain, H. Brezis and P. Mironescu's approach introduced in [5] (see also [7]). In particular they prove that if $f \in W^{m-1,p}(\Omega)$, $1 < p < \infty$ and Ω is a smooth bounded domain in $\mathbb{R}^n$ then f belongs to $W^{m,p}(\Omega)$ if and only if,

$$\liminf_{\varepsilon \to 0} \int_\Omega \int_\Omega \frac{|R^{m-1} f(x, y)|^p}{|x - y|^{mp}} \rho_\varepsilon(|x - y|)\, dx\, dy < \infty \tag{1}$$

✉ Roberta Schiattarella
roberta.schiattarella@unina.it

Serena Guarino Lo Bianco
serena.guarinolobianco@unina.it

[1] Dipartimento di Agraria, Università di Napoli “Federico II”, via Università 100, Portici (NA), 80055, Italy

[2] Dipartimento di Matematica e Applicazioni “R. Caccioppoli”, Università di Napoli “Federico II”, Via Cintia, Napoli, 80126, Italy

where ρ_ε, with $\varepsilon > 0$, are radial mollifiers and $R^{m-1} f$ is the Taylor $(m-1)$ remainder of f. For $p = 1$, the condition (1) describes BV^m.

Here we say that a $W^{m-1,1}(\Omega)$ is of m-th order bounded variation BV^m if its m-th order partial derivatives in the sense of distributions are finite Radon measures. Spaces of this kind have been studied in [10] as applications in mathematical imaging in the setting of isotropic and anisotropic variants of the TV-model (see also [13]).

Another characterization of $W^{m,p}$, $1 < p < \infty$, (BV^m for $p = 1$) formulated in terms of the m-th differences has been presented in [4].

In this article we are concerned with a characterization of $W^{m,p}$ $1 < p < \infty$, (BV^m for $p = 1$) as the limit of certain BMO–type seminorms similar to the one introduced by J. Bourgain, H. Brezis, P. Mironescu in [6].

In [15] the Authors showed that a function $f \in L^p_{\rm loc}(\mathbb{R}^n)$ belongs to the Sobolev space $W^{1,p}_{\rm loc}(\mathbb{R}^n)$, $1 < p < +\infty$, if and only if

$$\lim_{\varepsilon \to 0^+} K(\varepsilon, 1, p) < +\infty$$

where

$$K_\varepsilon(f, 1, p) := \varepsilon^{n-p} \sup_{\mathcal{G}_\varepsilon} \sum_{Q' \in \mathcal{G}_\varepsilon} \fint_{Q'} \left| f(x) - \fint_{Q'} f \right|^p dx\,, \tag{2}$$

and the supremum on the right hand side is taken over all families $\mathcal{G}_\varepsilon$ of disjoint ε-cubes $Q' = Q'(x_0, \varepsilon)$ of side length ε, centered in x_0, with arbitrary orientation. Moreover, if $f \in W^{1,p}_{\rm loc}(\mathbb{R}^n)$ and $p \geq 1$ then

$$\lim_{\varepsilon \to 0^+} K_\varepsilon(f, 1, p) = \gamma(n, p) \int_{\mathbb{R}^n} |\nabla f|^p\, dx \tag{3}$$

where

$$\gamma(n, p) := \max_{\nu \in \mathbb{S}^{n-1}} \int_Q |x \cdot \nu|^p\, dx \tag{4}$$

where $Q = \left(-\frac{1}{2}, \frac{1}{2}\right)^n$.

Following some ideas in [1], an analogous representation formula is obtained for the total variation of SBV functions in [14] (see also [11]). For related results see also [9, 12].

Here, given a function $f \in W^{m-1,p}_{\rm loc}(\mathbb{R}^n)$, $p \geq 1$, for any $\varepsilon > 0$, we consider

$$K_\varepsilon(f, m, p) := \varepsilon^{n-mp} \sup_{\mathcal{G}_\varepsilon} \sum_{Q' \in \mathcal{G}_\varepsilon} \fint_{Q'} \left| f(x) - P^{m-1}_{Q'}[f](x) \right|^p dx\,,$$

where the families $\mathcal{G}_\varepsilon$ are as above and $P^{m-1}_{Q'}[f]$ is the polynomial of degree $m-1$ centered at x_0, given by

$$P^{m-1}_{Q'}[f](x) = \sum_{|\alpha| \leq m-1} (x - x_0)^\alpha \fint_{Q'} \left(D^\alpha f\right)(s)\, ds. \tag{5}$$

In particular, for $m = 1$ and $m = 2$ we have:

$$P^0_{Q'}[f](x) = \fint_{Q'} f; \qquad P^1_{Q'}[f](x) = \fint_{Q'} f + \sum_{i=1}^n (x_i - x_{0_i}) \left(\fint_{Q'} \frac{\partial f}{\partial y_i}(y)\, dy \right).$$

Our main Theorem reads as follows:

Theorem 1 *Let $p > 1$ and $f \in W^{m-1,p}_{loc}(\mathbb{R}^n)$, then*

$$|\nabla^m f| \in L^p_{loc}(\mathbb{R}^n) \iff \liminf_{\varepsilon \to 0} K_\varepsilon(f, m, p) < \infty. \tag{6}$$

Moreover, if $f \in W^{m,p}_{loc}(\mathbb{R}^n)$ *and* $p \geq 1$ *we have also*

$$\lim_{\varepsilon \to 0} K_\varepsilon(f, m, p) = \beta(n, m, p) \int_{\mathbb{R}^n} |\nabla^m f|^p \, dx. \tag{7}$$

The constant in Eq. 7 is given by

$$\beta(n, m, p) := \max_{\nu \in \mathbb{S}^{N-1}} \left(\frac{1}{m!}\right)^p \int_Q \left| \nu \cdot x^m - \int_Q \nu \cdot y^m \, dy \right|^p dx. \tag{8}$$

where $N = n^m$ and we refer to Section 2 for the notation.

Note that this Theorem is exactly an extension of Theorem 2.2 in [15] to the higher order case; indeed, in the case $m = 1$, since $\int_Q x \cdot \nu \, dx = 0$, the constant $\beta(n, 1, p)$ coincides with the one defined in Eq. 4.

A drawback of the formula Eq. 7 is that one does not recover the function in BV^m. However, we are able to show that it is possible to characterize the functions in $BV^m(\mathbb{R}^n)$ as the functions $f \in W^{m-1,1}_{\text{loc}}(\mathbb{R}^n)$ such that $\limsup_{\varepsilon \to 0} K_\varepsilon(f, m, 1) < +\infty$.

2 Notation and Preliminaries

We denote by $Q = \left(-\frac{1}{2}, \frac{1}{2}\right)^n \subset \mathbb{R}^n$ the unit cube with faces parallel to coordinate axes in $\mathbb{R}^n$. For any $z \in \mathbb{R}^n$ and $\varepsilon > 0$ we denote by $Q_\varepsilon(z) = z + \varepsilon Q$ the cube of sidelenght ε centered in z.

For $m, n \geq 1$, we denote by $N_j = n^{m-j}$ for $j = 0, \ldots, m$. Given $\nu \in \mathbb{R}^{N_0}$ we denotes its components by $\nu_{i_1,\ldots,i_k,\ldots,i_m}$ with $i_k = 1 \ldots n$. Taking $x \in \mathbb{R}^n$, $x = (x_{i_k})_{i_k \in \{1,\ldots,n\}}$ we define the product $\nu \cdot x$ as the element of $\mathbb{R}^{N_1}$ given by

$$(\nu \cdot x)_{i_1,\ldots,i_{k-1},i_{k+1},\ldots,i_m} = \sum_{i_k=1}^{n} \nu_{i_1,i_2,\ldots,i_m} x_{i_k}.$$

The product of $\nu \in R^{N_0}$ and m times the vector $x \in \mathbb{R}^n$, $\nu \cdot x \cdot x \cdot \cdots \cdot x$ is an element of $\mathbb{R}^{N_m} = \mathbb{R}$ and it is denoted for brevity by $\nu \cdot x^m$.

For a multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$, $\alpha_i \geq 0$ and a point $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, we denote by

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$$

the monomial of degree $|\alpha| = \sum_{i=1}^n \alpha_i$.

In the same way,

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$$

is a weak partial derivative of order $|\alpha|$.

Sometimes, we use the convention that $D^0 u = u$. Moreover, let $\nabla^m u$ be a vector with the components $D^\alpha u$, $|\alpha| = m$.

2.1 The Sobolev Space $W^{m,p}$

Definition 1 Let $\Omega \subset \mathbb{R}^n$ be an open set, let $m \in \mathbb{N}$, and let $1 \leq p < \infty$. The Sobolev space $W^{m,p}(\Omega)$ is the space of all functions $u \in L^p(\Omega)$ which admit α−th weak derivative $D^\alpha u$ in $L^p(\Omega)$ for every $\alpha \in \mathbb{N}^n$ with $1 \leq |\alpha| \leq m$.

The space $W^{m,p}(\Omega)$ is endowed with the norm

$$\|u\|_{W^{m,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \sum_{1\le|\alpha|\le m} \|D^\alpha u\|_{L^p(\Omega)}$$

Definition 2 Let $\Omega \subset \mathbb{R}^n$ be an open set, let $m \in \mathbb{N}$, and let $1 \le p < \infty$. The homogeneous Sobolev space $\dot{W}^{m,p}(\Omega)$ is the space of all functions $u \in L^1_{loc}(\Omega)$ whose $\alpha-$th weak derivative $D^\alpha u$ belongs to $L^p(\Omega)$ for every $\alpha \in \mathbb{N}^n$ with $|\alpha| = m$.

Note that the inclusion

$$W^{m,p}(\Omega) \subseteq \dot{W}^{m,p}(\Omega)$$

holds. Moreover, as a consequence of Poincarè's inequality for sufficiently regular domains of finite measure the spaces $\dot{W}^{m,p}(\Omega)$ and $W^{m,p}(\Omega)$ actually coincide.

The space $\dot{W}^{m,p}(\Omega)$ is equipped with the seminorm

$$|u|_{\dot{W}^{m,p}(\Omega)} = \|\nabla^m u\|_{L^p(\Omega)}.$$

Sometimes we will also use the equivalent seminorm $u \mapsto \sum_{|\alpha|=m} \|D^\alpha u\|_{L^p(\Omega)}$.

The equivalence of the norm permit to have a useful density result as in [17, Remark 11.28]. Indeed, if $u \in \dot{W}^{m,p}(\Omega)$ then for every $\sigma > 0$ there exists $v \in C^\infty(\Omega) \cap \dot{W}^{m,p}(\Omega)$ such that $\|u - v\|_{W^{m,p}(\Omega)} \le \sigma$.

Let $\Omega \subset \mathbb{R}^n$ be an open bounded set and let $E \subset \Omega$ be a Lebesgue measurable set with finite positive measure. Let $1 \le p \le +\infty$ and let $m \in \mathbb{N}$. Then, for every $u \in W^{m,p}(\Omega)$, there exists a polynomial $P_E^{m-1}[u]$ of degree $m-1$ such that for every multi-index $\alpha \in \mathbb{R}^n$ with $0 \le |\alpha| \le m-1$ (see [17, Exercise 13.26]),

$$\int_E \left(D^\alpha u(x) - D^\alpha P_E^{m-1}[u](x) \right) dx = 0. \tag{9}$$

Theorem 2 (Poincarè inequality in $W^{m,p}$ [17, Theorem 13.27]) *Let $m \in \mathbb{N}$, let $1 \le p < +\infty$ and let $\Omega \subset \mathbb{R}^n$ be an open, bounded and convex set. Then there exists a positive constant $C = C(m, n, p, \Omega) > 0$ such that,*

$$\sum_{k=0}^{m-1} \|\nabla^k (u - P_\Omega^{m-1}[u])\|_{L^p(\Omega)} \le C \|\nabla^m u\|_{L^p(\Omega)},$$

for every $u \in W^{m,p}$ and for every $k = 0, \dots, m-1$.

Notice that for $m = 1$ the previous Theorem is the classical Poincarè inequality and the polynomial $P_\Omega[u]$ is the mean of u over Ω. In particular, if $u \in W^{m,p}(Q')$ with $Q' = Q'(x_0, \varepsilon)$, then there exists a unique polynomial $P_{Q'}^{m-1}[u]$ of degree $m-1$ such that Eq. 9 holds and there exists a constant $C = C(n, m, p)$ such that

$$\int_{Q'} |u - P_{Q'}^{m-1}[u]|^p \le C \varepsilon^{mp} \int_{Q'} |\nabla^m u|^p. \tag{10}$$

Next, we consider the Sobolev–Gagliardo–Nirenberg's embedding in $W^{m,p}$(see Lemma 2.1 in [18]).

Let $n > mp$, $1 \le p < \frac{n}{m}$. Let $u \in W^{m,p}(Q')$ with $Q' = Q'(x_0, \varepsilon)$. Then there exists a unique polynomial $P_{Q'}^{m-1}[u]$ of degree $m-1$ such that Eq. 9 holds and there exists a constant $C = C(n, m, p)$ such that

$$\left(\fint_{Q'} |u - P^{m-1}_{Q'}[u]|^{p^\star}\right)^{\frac{1}{p^\star}} \le C\left(\frac{1}{\varepsilon^{n-mp}}\int_{Q'} |\nabla^m u|^p\right)^{\frac{1}{p}} \tag{11}$$

where $p^\star = \frac{np}{n-mp}$.

Moreover, the following easy properties of $P_\Omega[u]$ holds:

- Linearity:
$$P_\Omega[u](x) + P_\Omega[v](x) = P_\Omega[u+v](x).$$
- Scaling:
$$P_{\varepsilon\Omega}[u](\varepsilon x) = P_\Omega[u_\varepsilon](x),$$
where $u_\varepsilon(x) := u(\varepsilon x)$.

We write

$$T^m_y u(x) = \sum_{|\alpha|\le m} D^\alpha u(y)\frac{(x-y)^\alpha}{\alpha!}$$

for the Taylor polynomial of order m and

$$R^m u(x, y) = u(x) - T^m_y u(x)$$

for the Taylor remainder of order m.

2.2 Functions of Higher-Order Bounded Variation

Let $\Omega \subset \mathbb{R}^n$ be an open set. A function $u \in L^1(\Omega)$ is of bounded variation (for short $u \in BV(\Omega)$) if u has a distributional gradient in form of a Radon measure of finite total mass and write

$$|\nabla u|(\Omega) = \sup\left\{u \ \operatorname{div}\varphi : \varphi \in C^1_0(\Omega), \|\varphi\|_{L^\infty} \le 1\right\}.$$

We define

$$BV^m(\Omega) = \{u \in W^{m-1,1}(\Omega),\ \ \nabla^{m-1}u \in BV(\Omega, S^{m-1}(\mathbb{R}))\}$$

the space of (real valued) functions of m-th order bounded variation, i.e. the set of all functions, whose distributional gradients up to order $m-1$ are represented through 1-integrable tensor-valued functions and whose m-th distributional gradient is a tensor-valued Radon measure of finite total variation. Here $S^k(\mathbb{R}^n)$ denotes the set of all symmetric tensors of order k with real components, which is naturally isomorphic to the set of all k-linear symmetric maps $(\mathbb{R}^n)^k \to \mathbb{R}$ (see [10]).

It becomes a Banach space with the norm

$$\|u\|_{BV^m(\Omega)} = \|u\|_{W^{m-1,1}(\Omega)} + |\nabla^m u|(\Omega).$$

Here the total variation of $\nabla^{m-1}u$ is denoted by $|\nabla^m u|(\Omega)$ and defined by

$$|\nabla^m u|(\Omega) = \sup\left(\sum_{\alpha_1,\dots,\alpha_m=1}^{n}\int_\Omega D_{\alpha_1,\dots,\alpha_{m-1}}u \cdot \partial_{\alpha_m}\varphi_{\alpha_1,\dots,\alpha_m}\,dx\right),$$

where the supremum is taken over all $\varphi \in C^1_0(\Omega, \mathbb{R}^n)$ with $\|\varphi\|_\infty = 1$.

Obviously, $W^{m,1}(\Omega)$ is a subspace of $BV^m(\Omega)$.

The definition of BV^m generalizes that of the classical space of functions of bounded variation and many results about BV can be obtained in BV^m similarly (see [16]). We recall a higher-order variant of the famous Poincaré inequality, which will be useful throughout the sequel:

Theorem 3 (Poincarè inequality in BV^m [13, Lemma 2.2]) *Let $\Omega \subset \mathbb{R}^n$ be an open and bounded subset with Lipschitz boundary, $m \in \mathbb{N}$, $1 \le p < \infty$. Then there exist a constant $C > 0$, depending only on Ω, m and n such that for all $u \in BV^m(\Omega)$*

$$\|u\|_{BV^m(\Omega)} \le C|\nabla^m u|(\Omega).$$

In particular, the following version of Poincare's inequality holds.

Let $u \in BV^m(Q')$ with $Q' = Q'(x_0, \varepsilon)$, then there exists a unique polynomial $P_{Q'}^{m-1}[u]$ of degree $m-1$ such that Eq. 9 holds and there exists a constant $C = C(n, m)$ such that

$$\int_{Q'} |u - P_{Q'}^{m-1}[u]| \le C\varepsilon^m |\nabla^m u|(Q') \tag{12}$$

By the nature of its definition, the space BV^m inherits the Poincare-Wirtinger inequality which can be proved exactly as the corresponding first order result.

Let $n > m$, $u \in BV^m(Q')$ with $Q' = Q'(x_0, \varepsilon)$. Then there exists a unique polynomial $P_{Q'}^{m-1}[u]$ of degree $m-1$ such that Eq. 9 holds and there exists a constant $C = C(n, m)$ such that

$$\left(\fint_{Q'} |u - P_{Q'}^{m-1}[u]|^{\frac{n}{n-m}}\right)^{\frac{n-m}{n}} \le C\frac{1}{\varepsilon^{n-m}}|\nabla^m u|(Q'). \tag{13}$$

We end this subsection with a higher- order variant of the compactness result in BV (Theorem 3.23 in [2]).

Proposition 1 (Compactness result in BV^m [16, Lemma 2.1]) *Let $\Omega \subset \mathbb{R}^n$ be open and bounded with Lipschitz boundary, and let $(u_k)_{k=1}^{\infty}$ be a sequence of BV^m functions such that*

$$\|u_k\|_{BV^m(\Omega)} \le M$$

for some constant $M > 0$. Then there is a subsequence $(u_{k_l})_{l=1}^{\infty}$ and a function $u \in BV^m(\Omega)$ such that

$$\|u - u_{k_l}\|_{W^{m-1,1}(\Omega)} \to 0 \quad \textit{for } l \to \infty \quad \textit{and} \quad \|u\|_{BV^m(\Omega)} \le M.$$

2.3 Other Useful Inequalities

The following tools will be useful in the sequel.

Given $\delta \in (0, 1)$, from the convexity of the function $t \to |t|^p$ we get for every $a, b \in \mathbb{R}$

$$|a+b|^p = \left|\frac{1}{(1+\delta)}(1+\delta)a + \frac{\delta}{1+\delta}\frac{1+\delta}{\delta}b\right|^p \le (1+\delta)^p|a|^p + \frac{(1+\delta)^p}{\delta^p}|b|^p \tag{14}$$

Taking into account Eq. 14, we also obtain the following pointwise inequality

$$|a-b|^p \ge \frac{1}{(1+\delta)^p}|a|^p - \frac{1}{\delta^p}|b|^p \tag{15}$$

for every $a, b \in \mathbb{R}$. Given $\xi, \eta \in \mathbb{R}^n$ it holds

$$\left||\xi|^p - |\eta|^p\right| \le p\,(|\xi| + |\eta|)^{p-1}\,|\xi - \eta| \tag{16}$$

and, given $\xi, \eta \in \mathbb{R}^n \setminus \{0\}$ it holds

$$\left|\frac{\xi}{|\xi|} - \frac{\eta}{|\eta|}\right| \le 2\frac{|\xi - \eta|}{|\xi|}. \tag{17}$$

2.4 The Local Version of the Functional $K_\varepsilon(f, m, p)$

We define the following local counterpart of Eq. 2 which will be use in Step 3 of proof of Theorem 1

$$K_\varepsilon(f, m, p, \Omega) = \varepsilon^{n-mp} \sup_{\mathcal{G}_\varepsilon} \sum_{Q' \in \mathcal{G}_\varepsilon} \fint_{Q'} \left| f(x) - P_{Q'}^{m-1}[f](x) \right|^p dx \,, \tag{18}$$

where the supremum on the right hand side is taken over all families $\mathcal{G}_\varepsilon$ of disjoint open cubes of sidelenght ε and arbitrary orientation contained in Ω.

This quantity is strictly related to the L^p norm of $\nabla^m f$. Indeed, for $p < \frac{n}{m}$ with $p^\star = \frac{np}{n-mp}$, by using Hölder inequality, we have

$$\|f\|_{L^p(Q)} \leq \|f\|_{L^{p^\star}(Q)} \tag{19}$$

Thus, there exists a constant C depending only on Q, m, p such that for $Q' = \varepsilon Q + x_0$, by Eqs. 19 and 11, we get

$$\varepsilon^{n-mp} \fint_{Q'} |f(x) - P_{Q'}^{m-1}[f]|^p \, dx \leq C \int_{Q'} |\nabla^m f|^p. \tag{20}$$

Summing over all sets Q' in $\mathcal{G}_\varepsilon$, we obtain

$$\varepsilon^{n-mp} \sum_{Q' \in \mathcal{G}_\varepsilon} \fint_{Q'} |f(x) - P_{Q'}^{m-1}[f]|^p \, dx \leq C \|\nabla^m f\|^p_{L^p(\Omega)}$$

and therefore

$$K_\varepsilon(f, m, p, \Omega) \leq C \|\nabla^m f\|^p_{L^p(\Omega)}.$$

We conclude this subsection, by observing that if $\bar{\nu} \in \mathbb{S}^{N-1}$ is a vector maximizing the integral in Eq. 8, $x_0 \in \mathbb{R}^n$ and $Q_\eta(x_0)$ is a cube of side length η with center in x_0 then

$$\frac{1}{(m!)^p} \int_{Q_\eta(x_0)} \left| (x - x_0)^m \cdot \bar{\nu} - \int_{Q_\eta(x_0)} (y - x_0)^m \cdot \bar{\nu} \, dy \right|^p dx = \beta(n, m, p) \cdot \eta^{n+mp}. \tag{21}$$

3 The Case $m = 2$

In this section we deal with the case $m = 2$. In this case it is easier to make some explicit computations. Moreover we give an estimates on the constant $\beta(n, 2, p)$ in terms of the Laplacian of the function $f \in W^{2,p}$.

We prove the following

Proposition 2 *Let $f \in W^{2,p}$ and $\beta(n, 2, p)$ as in Eq. 8. Then the following estimate from below holds true*

$$\beta(n, 2, p) \geq C_{n,p} |\Delta f(0)|^p. \tag{22}$$

First, by virtue of Eq. 9, it is possible to characterize $P_\Omega[u]$ for $m = 2$. Fixed $x_0 \in \Omega$, a generic polynomial of degree 1 centered in x_0 is given by

$$P^1_\Omega[u](x) = \langle a, x - x_0 \rangle + b, \qquad a \in \mathbb{R}^n, b \in \mathbb{R}.$$

By Eq. 9 with $|\alpha| = 0$, we have

$$b|\Omega| = \int_\Omega (u(x) - \langle a, x - x_0 \rangle) \, dx$$

which implies

$$b = \fint_{\Omega} (u(x) - \langle a, x - x_0 \rangle)\, dx.$$

Moreover, for every $i = 1, \ldots, n$, again Eq. 9 for $|\alpha| = 1$ gives

$$a_i = \fint_{\Omega} \frac{\partial u}{\partial x_i}(x)\, dx$$

and we write

$$a = \fint_{\Omega} \nabla u(x)\, dx.$$

Then the polynomial $P^1_\Omega(u)$ is

$$P^1_\Omega[u](x) = \fint_{\Omega} \left(u(y) - \langle \fint_{\Omega} \nabla u, y \rangle \right) dy + \langle \fint_{\Omega} \nabla u(y)\, dy, x - x_0 \rangle \tag{23}$$

where, with a slight abuse of notation, we mean

$$\langle \fint_{\Omega} \nabla u(y)\, dy, x - x_0 \rangle = \sum_{j=1}^{n} (x_i - x_{0_i}) \fint_{\Omega} \frac{\partial u}{\partial y_i}(y)\, dy.$$

Remark 1 We observe that if Ω is symmetric with respect to x_0, the polynomial $P^1_\Omega[u]$ has a simpler form, indeed

$$\fint_{\Omega} \langle \fint_{\Omega} \nabla u, y \rangle\, dy = 0,$$

and then

$$P^1_\Omega[u](x) = \fint_{\Omega} u(y)\, dy + \langle \fint_{\Omega} \nabla u(y)\, dy, x - x_0 \rangle \tag{24}$$

Proof of Proposition 5 We observe that when $m = 2$, $p \geq 1$, Eq. 8 reads as

$$\beta(n, 2, p) := \max_{\nu \in \mathbb{S}^{n^2-1}} \frac{1}{4} \int_Q \left| \nu \cdot x^2 - \int_Q \nu \cdot y^2\, dy \right|^p dx. \tag{25}$$

In this case $\nu \cdot x^2$ can equivalently be write as

$$\langle Ax, x \rangle$$

where $A \in \mathcal{M}(n)$ is a matrix $n \times n$ and $\langle \cdot, \cdot \rangle$ denote the usual scalar product in $\mathbb{R}^n$.

It is worth to remark that

$$\beta(n, 2, p) \geq \frac{1}{2^p} \fint_Q \left| \langle \nabla^2 f(0)x, x \rangle - \fint_Q \langle \nabla^2 f(0)y, y \rangle \right|^p dx\,. \tag{26}$$

Firstly we observe that denoting by e_i the canonical basis of $\mathbb{R}^n$, by $O \in \mathcal{O}(n)$ an orthogonal matrix and by $\mathcal{R} \in SO(n)$ a rotation around the origin taking $O^{-1}(Q)$ into Q we have

$$\int_{O^{-1}(Q)} y_i^2\, dy = \int_{O^{-1}(Q)} (y \cdot e_i)^2\, dy = \int_{R \circ O^{-1}(Q)} (Rw \cdot e_i)^2\, dw = \int_Q (w \cdot R^{-1} e_i)^2\, dy = \frac{1}{12}$$

Moreover, given $A \in \mathcal{S}(n)$ a symmetric matrix there exist $O \in \mathcal{O}(n)$ and $D \in \mathcal{D}(n)$ such that $A = ODO^{-1}$. Thus we have

$$\begin{aligned} \textstyle\int_Q \langle Az, z \rangle\, dz &= \textstyle\int_Q \langle (ODO^{-1})z, z \rangle\, dz = \int_Q \langle (DO^{-1})z, O^{-1}z \rangle\, dz = \int_{O^{-1}(Q)} \langle Dy, y \rangle\, dy \\ &= \textstyle\int_{O^{-1}(Q)} \sum_{i=1}^n \lambda_i y_i^2\, dy = \sum_{i=1}^n \lambda_i \int_{O^{-1}(Q)} y_i^2\, dy = \frac{1}{12} \sum_{i=1}^n \lambda_i \end{aligned} \tag{27}$$

Then we can estimate from below $\beta(n,2,p)$ using Eqs. 26 and 27, proving Eq. 22. Indeed, setting $\nabla^2 f(0)=A$ we have

$$\fint_Q \langle Ax, x\rangle = \frac{\Delta f(0)}{12}.$$

Moreover setting $\overline{y}=\min y_i$, we have

$$\begin{aligned}\frac{1}{2^p}\fint_Q\left|\langle Ax,x\rangle-\fint_Q\langle Ay,y\rangle\right|^p dx &= \frac{1}{2^p}\fint_Q\left|\langle (DO^{-1})x, O^{-1}x\rangle-\frac{\Delta f(0)}{12}\right|^p dx\\ =\frac{1}{2^p}\fint_Q\left|\sum\lambda_i y_i^2-\frac{\Delta f(0)}{12}\right|^p dx &\geq \frac{1}{2^p}\fint_{O^{-1}(Q)}\left|\sum\lambda_i\overline{y}^2-\frac{\Delta f(0)}{12}\right|^p dx\\ &=\frac{1}{2^p}|\Delta f(0)|^p\fint_{O^{-1}(Q)}\left|\overline{y}-\frac{1}{12}\right|^p dx = C_{n,p}|\Delta f(0)|^p. \quad (28)\end{aligned}$$

□

4 A Characterization of $W^{m,p}$

Proof of Theorem 1 We divide the proof in three steps, proving first the limsup and liminf inequalities in Eq. 7 and then the validity of Eq. 6.

As a starting point we fix a bounded open set $\Omega\subset\mathbb{R}^n$ and $f\in W^{m,p}(\Omega)$. Given $\sigma>0$, there exists a function $g\in C_c^\infty(\Omega)$ such that $\|f-g\|_{W^{m,p}(\Omega)}<\sigma$ and we choose $\varepsilon>0$ such that

$$|\nabla^m g(x)-\nabla^m g(y)|\leq\sigma,\qquad \forall x,y,\ \ |x-y|\leq\frac{\sqrt{n}\varepsilon}{2} \quad (29)$$

Let us take now a family $\mathcal{G}_\varepsilon$ of disjoint open cubes Q' of side length ε and arbitrary orientation and let us denote by $\mathcal{G}'_\varepsilon$ the subfamily of $\mathcal{G}_\varepsilon$ made by all cubes $Q'\in\mathcal{G}_\varepsilon$ such that $Q'\subset\Omega$.

Step1 (limsup inequality) We are going to show that

$$\limsup_{\varepsilon\to0^+}K_\varepsilon(f,m,p)\leq\beta(n,m,p)\int_{\mathbb{R}^n}|\nabla^m f|^p\,dx.$$

We may assume, without loss of generality, that $|\nabla^m f|\in L^p(\Omega)$. Using Eq. 14 and the linearity of $P^{m-1}_{Q'}[f]$, for any $Q'\in\mathcal{G}'_\varepsilon$ we have:

$$\fint_{Q'}\left|f-P^{m-1}_{Q'}[f]\right|^p dx \leq(1+\delta)^p\fint_{Q'}\left|g-P^{m-1}_{Q'}[g]\right|^p dx+M_\delta\fint_{Q'}\left|(f-g)-P^{m-1}_{Q'}[f-g]\right|^p dx \quad (30)$$

where $M_\delta=(1+\delta)^p/\delta^p$.

We recall the notation in Section 2, so denoting by x_0 the center of the cube Q' and for all $x\in Q'$ we write

$$g(x)=T^m_{x_0}g(x)+R^m g(x,x_0),$$

where $|R^m g(x,y)|<(n^{\frac{m}{2}}\sigma\varepsilon^m)/2^m=C_1\sigma\varepsilon^m$.

We now estimate the two terms in Eq. 30. Let us focus on the first addendum: using again Eq. 14 we have

$$
\begin{aligned}
&\fint_{Q'} \left|g - P_{Q'}^{m-1}[g]\right|^p \, dx \\
&= \fint_{Q'} \left|\frac{1}{m!}\nabla^m g(x_0)\cdot(x-x_0)^m + R^m g(x,x_0) - \left[\fint_{Q'} \frac{1}{m!}\nabla^m g(x_0)\cdot(y-x_0)^m \, dy + \fint_{Q'} R^m g(y,x_0)\, dy\right]\right|^p dx \\
&\le (1+\delta)^p \tfrac{1}{(m!)^p} \fint_{Q'} \left|\nabla^m g(x_0)\cdot(x-x_0)^m - \fint_{Q'} \nabla^m g(x_0)\cdot(y-x_0)^m \, dy\right|^p dx + 2^p M_\delta \fint_{Q'} \left|R^m g(x,x_0)\right|^p dx \\
&\le (1+\delta)^p \beta(n,m,p)\varepsilon^{mp}|\nabla^m g(x_0)|^p + C_2 M_\delta \sigma^p \varepsilon^{mp}.
\end{aligned} \tag{31}
$$

Moreover, applying again Eqs. 14 and 29 we have

$$
|\nabla^m g(x_0)|^p \le (1+\delta)^p \fint_{Q'} \left|\nabla^m g(x)\right|^p dx + C_3 M_\delta \sigma^p.
$$

Hence

$$
\fint_{Q'} \left|g - P_{Q'}^{m-1}[g]\right|^p dx \le \beta(n,m,p)(1+\delta)^{2p}\varepsilon^{mp} \fint_{Q'} |\nabla^m g(x)|^p \, dx + C_4 M_\delta \varepsilon^{mp}\sigma^p. \tag{32}
$$

Let us focus now on the second addendum in Eq. 30. By Poincaré inequality in $W^{m,p}$ (see Theorem 2), we have

$$
\fint_{Q'} \left|(f-g) - P_{Q'}^{m-1}[f-g]\right|^p dx \le C_p \varepsilon^{mp-n} \int_{Q'} |\nabla^m (f-g)|^p \, dx \tag{33}
$$

where C_p is the Poincaré constant for cubes.

Observe now that $\sharp(\mathcal{G}'_\varepsilon) \le \varepsilon^{-n}|\Omega|$ and set $\Omega_\varepsilon = \{x \in \Omega \,:\, \mathrm{dist}(x, \partial\Omega) > \varepsilon\sqrt{n}\}$. Using Eqs. 30, 32 and 33 we have

$$
\begin{aligned}
\varepsilon^{n-mp} \; & \textstyle\sum_{Q'\in\mathcal{G}_\varepsilon} \fint_{Q'} \left|f - P_{Q'}^{m-1}[f]\right|^p dx \\
&\le \varepsilon^{n-mp} \textstyle\sum_{Q'\in\mathcal{G}'_\varepsilon} \fint_{Q'} \left|f - P_{Q'}^{m-1}[f]\right|^p dx + C_6 \displaystyle\sum_{Q'\in\mathcal{G}_\varepsilon\setminus\mathcal{G}'_\varepsilon} \int_{Q'} |\nabla^m f|^p \\
&\le (1+\delta)^p \varepsilon^{n-mp} \textstyle\sum_{Q'\in\mathcal{G}'_\varepsilon} \fint_{Q'} \left|g - P_{Q'}^{m-1}[g]\right|^p dx + C_p M_\delta \displaystyle\int_\Omega |\nabla^m (f-g)|^p + C_6 \int_{\mathbb{R}^n\setminus\Omega_\varepsilon} |\nabla^m f|^p dx \\
&\le (1+\delta)^{3p} \beta(n,m,p) \textstyle\sum_{Q'\in\mathcal{G}'_\varepsilon} \int_{Q'} |\nabla^m g(x)|^p \, dx + C_4 M_\delta \varepsilon^n \sigma^p + C_p M_\delta \sigma^p + C_6 \int_{\mathbb{R}^n\setminus\Omega_\varepsilon} |\nabla^m f|^p dx \\
&\le (1+\delta)^{3p} \beta(n,m,p) \textstyle\int_\Omega |\nabla^m f(x)|^p \, dx + C_4 M_\delta \varepsilon^n \sigma^p + C_p M_\delta \sigma^p + C_6 \int_{\mathbb{R}^n\setminus\Omega_\varepsilon} |\nabla^m f|^p dx
\end{aligned} \tag{34}
$$

where the constants depend only on n, p and $|\Omega|$. Then, taking the supremum over all the families of cubes $\mathcal{G}_\varepsilon$, and then letting first $\varepsilon \to 0^+$, $\sigma \to 0$, $\delta \to 0$ and $\Omega \uparrow \mathbb{R}^n$ we conclude.

Step2 (liminf inequality) We fix $\Omega \subset \mathbb{R}^n$, we assume again that $f \in W^{m,p}_{loc}(\Omega)$ and we fix $\sigma > 0$ and $g \in C_c^\infty(\Omega)$ as in the previous Step. We prove that

$$
\liminf_{\varepsilon\to 0^+} K_\varepsilon(f,m,p) \ge \beta(n,m,p) \int_{\mathbb{R}^n} |\nabla^m f|^p \, dx. \tag{35}
$$

So, for $\eta \in (0,1)$ we consider the set

$$
U_\eta = \{x \in \Omega \,:\, |\nabla^m g(x)| > \eta\}
$$

With a clever use of Lemma 2.95 of [2] (as in Proposition 3.6 of [14]) it is possible to find k sufficiently small pairwise disjoint open sets $S_j \subset \mathbb{S}^{N-1}$ covering $\mathbb{S}^{N-1}$. Precisely,

$$\bigcup_{j=1}^{k} \bar{S}_j = \mathbb{S}^{N-1}$$

$$\text{diam } S_j < \eta \quad \text{for all } j = 1...k$$

$$\left| \bigcup_{j=1}^{k} \left\{ x \in U_\eta : \frac{\nabla^m g(x)}{|\nabla^m g(x)|} \in \partial S_j \right\} \right| = 0.$$

For all $j = 1, \dots, k$ we denote

$$A_j = \left\{ x \in U_\eta \ : \ \frac{\nabla^m g(x)}{|\nabla^m g(x)|} \in S_j \right\},$$

which are open sets with the property

$$\left| U_\eta \setminus \bigcup_{j=1}^{k} A_j \right| = 0. \tag{36}$$

For $\varepsilon > 0$ we consider the family $\mathcal{F}_\varepsilon$ of all open cubes with faces parallel to the coordinate planes, side length ε, centered at all points of the form εv, with $v \in \mathbb{Z}^n$. Then for all $j = 1, \dots, k$ we choose $M_j \in S_j$ and we denote by $R_j \in SO(n)$ a rotation that takes e_1 into M_j.

Note that in this way, denoting by x' the center of the cube $Q' \in \mathcal{F}_\varepsilon$, we have (see Eq. 21),

$$\frac{1}{(m!)^p} \int_{R_j(Q')} \left| (x - x')^m \cdot \bar{v} - \int_{R_j(Q')} (y - x')^m \cdot \bar{v} \, dy \right|^p dx = \beta(n, m, p) \cdot \varepsilon^{n+mp}.$$

For all $j = 1, \dots, k$ we denote by $R_j(Q_{h,j})$, $Q_{h,j} \in \mathcal{F}_\varepsilon$, $h = 1, \dots, m_j$, the elements of $\mathcal{G}_\varepsilon$ contained in A_j. By Eq. 36 there exists $\varepsilon(\sigma, \eta)$ such that if $\varepsilon < \varepsilon(\sigma, \eta)$ then

$$\left| U_\eta \setminus \bigcup_{j=1}^{k} \bigcup_{h=1}^{m_j} \mathcal{R}_j(Q_{h,j}) \right| \le \eta^p.$$

We denote by $x_{h,j}$ the center of the cube $\mathcal{R}_j(Q_{h,j})$ and we argue as in Step 1. Indeed we have

$$\begin{aligned}
&\fint_{R_j(Q_{h,j})} \left| g - P^{m-1}_{R_j(Q_{h,j})}[g] \right|^p dx \\
&\ge \tfrac{1}{(1+\delta)^p} \tfrac{1}{(m!)^p} \fint_{R_j(Q_{h,j})} \left| \nabla^m g(x_{h,j}) \cdot (x - x_{h,j})^m - \fint_{R_j(Q_{h,j})} \nabla^m g(x_{h,j}) \cdot (x - x_{h,j})^m \right|^p dx \\
&\quad - \tfrac{2^p}{\delta^p} \fint_{R_j(Q_{h,j})} |R^m g(x, x_{h,j})|^p \, dx \\
&\ge \tfrac{1}{(1+\delta)^{2p}} \tfrac{|\nabla^m g(x_{h,j})|^p}{(m!)^p} \fint_{R_j(Q_{h,j})} \left| M_j \cdot (x - x_{h,j})^m - \fint_{R_j(Q_{h,j})} M_j \cdot (x - x_{h,j})^m \right|^p dx \\
&\quad - \tfrac{2^p}{\delta^p} \tfrac{|\nabla^m g(x_{h,j})|^p}{(m!)^p} \fint_{R_j(Q_{h,j})} \left| (\nabla^m g(x_{h,j}) - M_j) \cdot (x - x_{h,j})^m - \fint_{R_j(Q_{h,j})} (\nabla^m g(x_{h,j}) - M_j) \cdot (x - x_{h,j})^m \right|^p dx \\
&\quad - C_7 \tfrac{\sigma^p \varepsilon^{mp}}{\delta^p} \\
&\ge \tfrac{\varepsilon^{mp} \beta(n, m, p) |\nabla^m g(x_{h,j})|^p}{(1+\delta)^{2p}} - \tfrac{C_8 \eta^p \varepsilon^{mp}}{\delta^p} \|\nabla^m g\|^p_{L^\infty} - C_7 \tfrac{\sigma^p \varepsilon^{mp}}{\delta^p}.
\end{aligned} \tag{37}$$

Now, adding on j and h the previous inequality, recalling Eq. 36, we have

$$\begin{aligned}
&\varepsilon^{n-mp} \sum_{R_j(Q_{h,j})\in\mathcal{G}'_\varepsilon} \fint_{R_j(Q_{h,j})} \left|g - P^{m-1}_{Q'}[g]\right|^p \, dx \\
&\geq \varepsilon^{n-mp} \sum_{j=1}^{k} \sum_{h=1}^{m_j} \tfrac{\varepsilon^{mp}\beta(n,m,p)|\nabla^m g(x_{h,j})|^p}{(1+\delta)^{2p}} - \tfrac{C_8\eta^p\varepsilon^{mp}}{\delta^p}\|\nabla^m g\|^p_{L^\infty} - C_7\tfrac{\sigma^p\varepsilon^{mp}}{\delta^p} \\
&\geq \tfrac{\beta(n,m,p)}{(1+\delta)^{3p}} \sum_{j=1}^{k} \sum_{h=1}^{m_j} \int_{R_j(Q_{h,j})} |\nabla^m g|^p - \tfrac{C_8\eta^p\varepsilon^{n}}{\delta^p}\|\nabla^m g\|^p_{L^\infty} - C_7\tfrac{\sigma^p\varepsilon^{n}}{\delta^p} \\
&\geq \tfrac{\beta(n,m,p)}{(1+\delta)^{3p}} \int_\Omega |\nabla^m g|^p - \tfrac{C\eta^p}{\delta^p}(1+\|\nabla^m g\|^p_{L^\infty}) - C\tfrac{\sigma^p}{\delta^p} \\
&\geq \tfrac{\beta(n,m,p)}{(1+\delta)^{4p}} \int_\Omega |\nabla^m f|^p - \tfrac{C\eta^p}{\delta^p}(1+\|\nabla^m g\|^p_{L^\infty}) - C\tfrac{\sigma^p}{\delta^p},
\end{aligned}$$

where the constants may change from line to line and depend only on p, n and $|\Omega|$. We conclude choosing η small enough and consequently ε small,

$$\begin{aligned}
&\varepsilon^{n-mp} \sum_{Q'\in\mathcal{G}_\varepsilon} \fint_{Q'} \left|f - P^{m-1}_{Q'}[f]\right|^p \, dx \\
&\qquad\geq \tfrac{1}{(1+\delta)^p}\varepsilon^{n-mp} \sum_{Q'\in\mathcal{G}_\varepsilon} \fint_{Q'} \left|g - P^{m-1}_{Q'}[g]\right|^p \, dx - \frac{1}{\delta^p}\int_\Omega |\nabla^m(f-g)|^p \\
&\qquad\geq \tfrac{\beta(n,m,p)}{(1+\delta)^{5p}} \int_\Omega |\nabla^m f|^p - \tfrac{C\sigma^p}{\delta^p},
\end{aligned}$$

where again C may change from line to line and depend on p, n and $|\Omega|$. To conclude we take the supremum over all the families $\mathcal{G}_\varepsilon$ and let first $\varepsilon \to 0$, $\sigma \to 0$, $\delta \to 0$ and $\Omega \uparrow \mathbb{R}^n$, proving Eq. 35.

Step3 (proof of Eq. 6) Now let $p > 1$, $f \in W^{m-1,p}_{\text{loc}}(\mathbb{R}^n)$ and $\liminf_{\varepsilon\to 0} K_\varepsilon(f, m, p) < \infty$. We fix $\sigma > 0$, $\Omega \subset \mathbb{R}^n$ and observe that there exist $r > 0$ and a finite family of pairwise disjoint open cubes $Q(x_i, r)$ such that

$$\left|\Omega \setminus \bigcup_{i=1}^{m} Q(x_i, r)\right| < \sigma. \tag{38}$$

$$|\nabla^m f(x) - \nabla^m f(y)| < \sigma \tag{39}$$

Moreover we fix $0 < \varepsilon < r$ and we set $f_\varepsilon(x) = (\varrho_\varepsilon * f)(x)$, where ϱ is a standard mollifier with compact support in the unit cube Q and $\varrho_\varepsilon(x) = \varepsilon^{-n}\varrho(x/\varepsilon)$.

For every $Q(x_i, r)$ we consider a family $\mathcal{H}_\varepsilon$ of pairwise disjoint cubes $Q_j = z_j + \varepsilon Q \subset Q(x_i, r)$, for $j = 1, \ldots, k$.

We compute now

$$\begin{aligned}
|\nabla^m f_\varepsilon(z_j)|^p &= \left|\int_{\mathbb{R}^n} f(y)\nabla^m\rho_\varepsilon(z_j - y)\, dy\right|^p = \left|\int_{\mathbb{R}^n} \left(f(y) - P^{m-1}_{Q_j}[f](y)\right)\nabla^m\rho_\varepsilon(z_j - y)\, dy\right|^p \\
&\leq \varepsilon^{(-m-n)p+np-n} \int_{Q_j} \left|f(y) - P^{m-1}_{Q_j}[f](y)\right|^p \, dy = \varepsilon^{-mp} \fint_{Q_j} \left|f(y) - P^{m-1}_{Q_j}[f](y)\right|^p \, dy.
\end{aligned}$$

noindent Moreover, by Eqs. 29 and 14, we have

$$|\nabla^m f_\varepsilon(z_j)|^p \geq \frac{1}{1+\delta}\varepsilon^{-n}\int_{Q_j} |\nabla^m f_\varepsilon(x)|^p \, dx - \frac{C}{\delta^p}\sigma^p$$

Then

$$\frac{1}{1+\delta}\int_{Q_j} |\nabla^m f_\varepsilon(x)|^p \, dx \leq \varepsilon^{n-mp} \fint_{Q_j} \left|f(y) - P^{m-1}_{Q_j}[f](y)\right|^p \, dy + \frac{C}{\delta^p}\sigma^p\varepsilon^n.$$

Summing up all the cubes in $\mathcal{H}_\varepsilon$, we obtain

$$\begin{aligned}&\frac{1}{1+\delta}\int_{Q(x_i,r)}|\nabla^m f_\varepsilon(x)|^p\,dx\\ &\le \frac{1}{1+\delta}\sum_{j=1}^k\int_{Q_j}|\nabla^m f_\varepsilon(x)|^p\,dx \le \varepsilon^{n-mp}\sum_{j=1}^k \fint_{Q_j}\left|f(y)-P^{m-1}_{Q_j}[f](y)\right|^p dy+\frac{C}{\delta^p}\sigma^p\varepsilon^n\\ &\le \varepsilon^{n-mp}\sum_{j=1}^k \fint_{Q_j}\left|f(y)-P^{m-1}_{Q_j}[f](y)\right|^p dy+\frac{C}{\delta^p}\sigma^p r^n, \quad (40)\end{aligned}$$

where the last inequality follows since $k\varepsilon^n \le r^n$. Taking the supremum with respect to all families $\mathcal{H}_\varepsilon$ and the liminf with respect to ε, we have

$$\frac{1}{1+\delta}\int_{Q(x_i,r)}|\nabla^m f(x)|^p\,dx \le \liminf_{\varepsilon\to 0} K_\varepsilon(f,m,p,Q(x_i,r))+\frac{C}{\delta^p}\sigma^p r^n.$$

Summing up with respect to i and using Eq. 38 we have

$$\frac{1}{1+\delta}\int_{\Omega}|\nabla^m f(x)|^p\,dx \le \liminf_{\varepsilon\to 0} K_\varepsilon(f,m,p,\Omega)+\frac{C}{\delta^p}\sigma^p |\Omega|.$$

Letting $\sigma\to 0$, $\delta\to 0$ and $\Omega\uparrow\mathbb{R}^n$, we conclude. □

Remark 2 We observe that Theorem 7 hold also in an open set Ω with the same proof replacing $K_\varepsilon(f,m,p)$ by the quantity $K_\varepsilon(f,m,p,\Omega)$ defined in Eq. 18.

Corollary 1 *Let $p>1$, $n>mp$, $p^\star=\frac{np}{n-mp}$, $\Omega\subset\mathbb{R}^n$ and $\mathcal{G}_\varepsilon$ a pairwise disjoint family of translations Q' of εQ contained in Ω. Then, the following three statements are equivalent:*

i) $f\in W^{m,p}(\Omega)$;

ii)

$$\sup_{\mathcal{G}_\varepsilon}\sum_{Q'\in\mathcal{G}_\varepsilon}\varepsilon^{n-mp}\fint_{Q'}\left|f-P^{m-1}_{Q'}[f]\right|^p<+\infty;$$

iii)

$$\sup_{\mathcal{G}_\varepsilon}\sum_{Q'\in\mathcal{G}_\varepsilon}\|f-P^{m-1}_{Q'}[f]\|^p_{L^{p^\star}(Q')}<+\infty$$

.

Proof In this proof the constant C may change from line to line.

We prove that $iii)\Rightarrow ii)$. By Hölder's inequality it holds

$$\varepsilon^{n-mp}\fint_{Q'}\left|f-P^{m-1}_{Q'}[f]\right|^p dx\le\frac{\varepsilon^{n-mp}}{\varepsilon^n}\left(\int_{Q'}\left|f-P^{m-1}_{Q'}[f]\right|^{\frac{np}{n-mp}}\right)^{\frac{n-mp}{n}}|Q'|^{\frac{mp}{n}}=\left\|f-P^{m-1}_{Q'}[f]\right\|^p_{L^{p^\star}(Q')}. \quad (41)$$

Summing over all sets Q' in $\mathcal{G}_\varepsilon$ and passing to the supremum, we conclude.

We prove that $i)\Rightarrow iii)$. Using the Sobolev-Gagliardo-Nirenberg inequality Eq. 11, we obtain that there exists a constant $C=C(n,m,p)$ such that

$$\left\|f-P^{m-1}_{Q'}[f]\right\|_{L^{p^\star}(Q')}\le C\left\|\nabla^m f\right\|_{L^p}. \quad (42)$$

Summing over Q' in $\mathcal{G}_\varepsilon$ and passing to the supremum over all families G_ε the proof is completed.

The equivalence $i) \Leftrightarrow ii)$ is proved in [8]. □

5 A Characterization of Higher Order Bounded Variation

In this section we deal with the case $p = 1$. This case is not included in Theorem 1 since Eq. 6 hold only for $p > 1$.

The case $m = 1$ was treated in [15]. They proved that (see Proposition 2.4 of [15]) if $f \in L^1_{loc}(\mathbb{R}^n)$ then

$$f \in BV(\mathbb{R}^n) \quad \Longleftrightarrow \quad \liminf_{\varepsilon \to 0} K_\varepsilon(f, 1, 1) < +\infty \tag{43}$$

Precisely, they prove that for $f \in L^1_{loc}(\mathbb{R}^n)$ it holds

$$\frac{1}{4}|\nabla f|(\mathbb{R}^n) \le \liminf_{\varepsilon \to 0^+} K_\varepsilon(f, 1, 1) \le \limsup_{\varepsilon \to 0^+} K_\varepsilon(f, 1, 1) \le \frac{1}{2}|\nabla f|(\mathbb{R}^n),$$

where the total variation of f in $\Omega \subset \mathbb{R}^n$, possibly equal to $+\infty$, is defined by setting

$$|\nabla f|(\Omega) := \sup\left\{\int_\Omega f(x)\,\mathrm{div}\,\varphi(x)\,dx \quad : \varphi \in C^1_c(\Omega),\ \|\varphi\|_\infty \le 1\right\}$$

We prove a similar characterization for the case $m > 1$. Now an equivalence similar to Eq. 43 involve the space $BV^m(\mathbb{R}^n)$ of functions of m-th order bounded variation (see Section 2).

Precisely, we prove the following

Proposition 3 *Let $f \in W^{m-1,1}_{loc}(\mathbb{R}^n)$. Then*

$$f \in BV^m(\mathbb{R}^n) \quad \Longleftrightarrow \quad \liminf_{\varepsilon \to 0} K_\varepsilon(f, m, 1) < +\infty$$

Moreover, there is a positive constants C, independent of f, such that

$$|\nabla^m f|(\mathbb{R}^n) \le \liminf_{\varepsilon \to 0^+} K_\varepsilon(f, m, 1) \le \limsup_{\varepsilon \to 0^+} K_\varepsilon(f, m, 1) \le C|\nabla^m f|(\mathbb{R}^n). \tag{44}$$

Proof To prove the first inequality in Eq. 44 we argue as in Step 3 of Theorem 1. In particular, we have

$$\frac{1}{1+\delta}\int_{Q(x_i,r)} |\nabla^m f_\varepsilon(x)|\,dx \le \varepsilon^{n-m}\sum_{j=1}^{k} \fint_{Q_j} \left|f(y) - P^{m-1}_{Q_j}[f](y)\right| dy + \frac{C}{\delta}\sigma r^n, \tag{45}$$

Taking the supremum with respect to all families $\mathcal{H}_\varepsilon$ and the liminf with respect to ε, we have

$$\frac{1}{1+\delta}\liminf_{\varepsilon \to 0}\int_{Q(x_i,r)} |\nabla^m f_\varepsilon(x)|\,dx \le \liminf_{\varepsilon \to 0} K_\varepsilon(f, m, Q(x_i, r)) + \frac{C}{\delta}\sigma r^n.$$

By the compactness in BV^m (Proposition 4), we get

$$\frac{1}{1+\delta}|\nabla^m f|(Q(x_i, r))\,dx \le \liminf_{\varepsilon \to 0} K_\varepsilon(f, m, Q(x_i, r)) + \frac{C}{\delta}\sigma r^n$$

Summing up with respect to i and using Eq. 38 we obtain

$$\frac{1}{1+\delta}\,|\nabla^m f|(\Omega)\,dx \le \liminf_{\varepsilon\to 0} K_\varepsilon(f,m,\Omega) + \frac{C}{\delta}\sigma r^n|\Omega|$$

We conclude letting $\sigma \to 0$, $\delta \to 0$, $\Omega \uparrow \mathbb{R}^n$.

In order to prove the estimate from above in Eq. 44, it is is sufficient to apply the Poincare' inequality in BV^m (see Section 2). □

Corollary 2 *Let $n > m$, $1^\star = \frac{n}{n-m}$, $\Omega \subset \mathbb{R}^n$ and $\mathcal{G}_\varepsilon$ is any pairwise disjoint family of translations Q' of εQ contained in Ω. Then, the following three statements are equivalent:*

i) $f \in BV^m(\Omega)$;

ii)

$$\sup_{\mathcal{G}_\varepsilon} \sum_{Q'\in\mathcal{G}_\varepsilon} \varepsilon^{n-m} \fint_{Q'} \left|f - P_{Q'}^{m-1}[f]\right| < +\infty$$

iii)

$$\sup_{\mathcal{G}_\varepsilon} \sum_{Q'\in\mathcal{G}_\varepsilon} \|f - P_{Q'}^{m-1}[f]\|_{L^{1^\star}(Q')} < +\infty$$

Proof We prove that $iii) \Rightarrow ii)$. By Hölder's inequality it holds

$$\varepsilon^{n-m}\fint_{Q'} \left|f - P_{Q'}^{m-1}[f]\right|\,dx \le \left|f - P_{Q'}^{m-1}[f]\right|_{L^{1^\star}(Q')}. \tag{46}$$

The conclusion follows by summing over all sets Q' in $\mathcal{G}_\varepsilon$.

We prove that $i) \Rightarrow iii)$. By using Eq. 12 there exists a constant $C = C(n.m)$ such that

$$\left\|f - P_{Q'}^{m-1}[f]\right\|_{L^{1^\star}(Q')} \le C\left\|\nabla^m f\right\|_{L^p}(Q') \tag{47}$$

The conclusion follows again by summing over all sets Q' in $\mathcal{G}_\varepsilon$.

The equivalence $i) \Leftrightarrow ii)$ is proved in [8]. □

Author Contributions The Authors contributed equally to this work.

Funding The authors are members of Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of INdAM. The research of S.G.L.B. has been funded by PRIN Project 2017AYM8XW and the research of R.S. has been funded by PRIN Project 2017JFFHSH.

Declarations

Conflict of Interests (There is no conflict of interest)

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