Measures of edge-uncolorability of cubic graphs

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Abstract

There are many hard conjectures in graph theory, like Tutte's 5-flow conjecture, and the 5-cycle double cover conjecture, which would be true in general if they would be true for cubic graphs. Since most of them are trivially true for 3-edge-colorable cubic graphs, cubic graphs which are not 3-edge-colorable, often called *snarks*, play a key role in this context. Here, we survey parameters measuring how far apart a non 3-edge-colorable graph is from being 3-edge-colorable. We study their interrelation and prove some new results. Besides getting new insight into the structure of snarks, we show that such measures give partial results with respect to these important conjectures. The paper closes with a list of open problems and conjectures.

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1 Introduction

We begin by commenting upon the main motivation of this paper.

1.1 Motivation

There are many hard problems in graph theory which can be solved in the general case if they can be solved for cubic graphs. Examples of such problems are the 4-color-problem (now a theorem), problems concerning cycle- and matching-covers, surface embeddings or flow-problems on graphs. Most of these problems are easy to solve for 3-edge-colorable cubic graphs. By Vizing's theorems [135, 136], or Johnson's [76] for the case $\Delta = 3$, a cubic graph has either chromatic index 3 or 4. Graphs with chromatic index greater than their maximum degree Δ are often called *class 2* graphs, and *class 1* if their chromatic index equals Δ.

Bridgeless cubic class 2 graphs with cycle separating 2- or 3-cuts, or 4-circuits, can be constructed from smaller cubic class 2 graphs by some easy operations. Such substructures are excluded in possible minimal counterexamples for the most of the problems. Thus, possible minimal counterexamples are required to be cyclically 4-edge-connected class 2 cubic graphs with girth at least 5 (see, for instance, Chetwynd and Wilson[23], Isaacs [68], and Watkins [138]). Such graphs were called 'snarks' by Gardner [49], who borrowed the name from a nonsense poem by the famous English author Lewis Carroll [16]. However, the decomposition results given by Cameron, Chetwynd, and Watkins [14] and Hajós [63] showed that this notion of non-triviality may not be appropriate. Thus, some authors adopt the most simple definition stating that a snark is a bridgeless cubic class 2 graph. Moreover, we remark that, in some cases (for instance, in Berge's and Fan-Raspaud's conjectures), it is not known if a minimal counterexample should satisfy the above strong definition of snark. For the sake of clarity, throughout the paper we will call *snark* a class 2 cubic graph satisfying the strong definition, and otherwise we will speak of a *bridgeless cubic class* 2 *graph*.

1.2 Historical remarks

To the authors' knowledge, the history of the hunting of (non-trivial) snarks may be summarized as follows. In 1973 only four snarks were known, the earliest one being the ubiquitous Petersen graph P [106]. The other three, on 18, 210, and 50 vertices, were found by Blanuša [8], Descartes (a pseudonymous of Tutte) [27], and Szekeres [127] respectively. Then, quoting Chetwynd and Wilson [14], "In 1975 the art of snark hunting underwent a dramatic change when Isaacs [68] described two infinite families of snarks." One of these families, called the *BDS class*, included all (three) snarks previously known. In fact, this family is based on a construction also discovered independently by Adelson-Velski and Titov in [3]. The members of the other family are the so-called *flower snarks*. They were also found independently by Grinberg in 1972 [58], although he never published his work. In [71], Jakobsen proposed a method, based on the well-known Hajós-union [63], to construct class 2 graphs. As it was pointed out by Goldberg in [57], some snarks of the BDS class can also be obtained by using this approach. Later, Isaacs [69] described two new infinite sets of snarks found by Loupekhine.

In 1979, Fiol [36] proposed a new method of generating snarks, based on Boolean algebra. This method led to a new characterization of the BDS class, and also to a significant enlargement of it. For instance, Loupekhine's snarks [69] and most of the Goldberg's snarks [56, 57] can be viewed as members of this class. This approach is based on the interpretation of certain cubic graphs as logic circuits and, hence, relates the Tait coloring of such graphs with the SAT problem. Hence, such a method implicitly contains the result of Holyer [64] proving that the problem of determining the chromatic index of a cubic graph is NP-complete. In the same work [36], infinitely many snarks of another family, called by Isaacs the *Q class*, were also given. Apart from the Petersen graph P and the flower snark J_5, in [68] Isaacs had given only one further snark of this class: *the double star graph*. The graphs of this class are all cyclically 5-edge-connected. Subsequently, Cameron, Chetwynd, and Watkins [14] gave a method to construct new snarks belonging to such a family. Other constructions of snarks, most of them belonging to the BDS class, have been proposed by several authors. See, for instance, the papers of Celmins and Swart [18], Fouquet, Jolivet, and Rivière [45], and Watkins [138].

In [80] Kochol used a method that he called *superposition*, to construct snarks with some specific structure (see an overview in [84]). Kochol used the concept of flows (see Section 4) to define superposition. However, the approaches via flows and logical circuits are equivalent since both can be described as 3-edge-colorings with colors from the Klein four group. Thus, Kochol's *supervertices* and *superedges* are special instances of the *multisets* (or, more properly, *multibusses*) used by Fiol in [36, 38]. More generally, in these papers there is a method to construct multisets representing all logic gates of a circuit, which allows us to use 'superposition' in a broader context (see Subsection 2.3). By using this method, Kochol disproved two old conjectures on snarks. First, he constructed snarks of girth greater than 6, disproving a conjecture of Jaeger and Swart [74]. Second, in [85] Kochol disproved a conjecture of Grünbaum [59], that snarks do not have polyhedral embeddings into orientable surfaces.

Later on, stronger criteria of non-triviality and reductions/constructions of (bridgeless) cubic class 2 graphs or snarks are considered in many papers: Chetwynd and Hilton [22], Nedela and Škoviera [105], Brinkmann and Steffen [11], Steffen [118, 120], Grünewald and Steffen [60], Máčajová and Škoviera [93], Chladný and Škoviera [24], Karabáš, Máčajová, and Nedela [78], and Sinclair [117], among others. However, none of them leads to recursive construction of all (bridgeless) cubic class 2 graphs or snarks, as it is known for 3-connected graphs [7, 132]. Intuitively, a bridgeless cubic class 2 graph which is not reducible to a 3-edge-colorable graph seems to be more complicated or of higher complexity than a bridgeless cubic class 2 graph which is reducible to a 3-edge-colorable cubic graph.

One major difficulty in proving theorems for bridgeless cubic class 2 graphs is to find/define appropriate structural parameters for a proof. This leads to the study of invariants that measure 'how far apart' the graph is from being 3-edge-colorable. Isaacs [68] called cubic class 2 graphs *uncolorable*. Hence, these invariants are sometimes called measures of edge-uncolorability in the literature. On one side, these parameters can give new insight into the structure of bridgeless cubic class 2 graphs, on the other side they allow to prove partial results for some hard conjectures.

1.3 Some strong conjectures

The formulation of the 4-Color-Theorem in terms of edge-colorings of bridgeless planar cubic graphs is due to Tait [128] (1880). Tutte generalized the ideas of Tait when he introduced nowhere-zero flows on graphs [130, 131, 133]. He conjectured in 1954 that every bridgeless graph admits a nowhere-zero 5-flow. This conjecture is equivalent to its restriction to cubic graphs.

In a recent paper, Brinkmann, Goedgebeur, Hägglund, and Markström [13] generated a list of all snarks up to 36 vertices and tested whether some conjectures are true for these graphs. They disproved some conjectures. However, the most prominent ones are true for these graphs.

Conjecture 1. [5-Flow Conjecture, Tutte [130]] Every bridgeless graph admits a nowhere-zero 5-flow.

The following conjecture is attributed to Berge (unpublished, see for example [46, 98]).

Conjecture 2. [Berge Conjecture] Every bridgeless cubic graph has five perfect matchings such that every edge is in at least one of them.

Conjecture 2 is true if the following conjecture is true, which is also attributed to Berge in [114]. This conjecture was first published in a paper by Fulkerson [48].

Conjecture 3. [Berge-Fulkerson Conjecture [48]] Every bridgeless cubic graph has six perfect matchings such that every edge is in precisely two of them.

Mazzuoccolo [98] proved that Conjectures 2 and 3 are equivalent. The following conjecture of Fan and Raspaud is true if Conjecture 3 is true.

Conjecture 4. [Fan-Raspaud Conjecture [35]] Every bridgeless cubic graph has three 1-factors such that no edge is in each of them.

A *cycle* is a 2-regular graph, and its components are called *circuits*. A *5-cycle double cover* of a graph G is a set of five cycles, such that every edge is in precisely two of them. The following conjecture was stated by Celmins and Preissmann independently.

Conjecture 5. [5-Cycle-Double-Cover Conjecture, see [140]] Every bridgeless graph has a 5-cycle double cover.

We conclude this section by recalling the outstanding Petersen Coloring Conjecture proposed by Jaeger in [73] which implies both Conjecture 3 and Conjecture 5.

Let G and H be two cubic graphs. If there is a mapping $\phi : E(G) \to E(H)$, such that for each $v \in V(G)$ there is $w \in V(H)$ such that $\phi(E_G(v)) = E_H(w)$ (where $E_G(v)$ denotes the set of edges of G incident the vertex v) then ϕ is called an H-coloring of G.

Conjecture 6. [Petersen Coloring Conjecture [73]] Every bridgeless cubic graph admits a Petersen-coloring.

Again, it is trivial that every 3-edge-colorable cubic graph G admits a Petersen-coloring and the problem is largely open for cubic class 2 graphs. The Petersen conjecture is proved for many classes of snarks in [62]. Furthermore, Mkrtchyan [101] proved that any analogous statement where the Petersen graph is replaced with another bridgeless cubic graph is false.

1.4 Basic notation and first parameters of edge-uncolorability

Let G be a graph with vertex set $V = V(G)$, edge set $E = E(G)$, and no parallel edges. Let $\deg(v)$ be the degree of a vertex $v \in V$, and let $\Delta = \Delta(G)$ denote the maximum degree of G. A mapping $c : E(G) \longrightarrow \mathcal{C} = \{1, 2, \ldots, k\}$ is a *k-edge-coloring* of G. If $c(e) \neq c(e')$ for any two adjacent edges e and e', then c is a *proper k-edge-coloring* of G. The *chromatic index* of G, denoted by $\chi'(G)$, is the minimum integer k such that G has a proper k-edge-coloring. By the well-known theorem of Vizing [135], $\chi'(G)$ must be either $\Delta(G)$ or $\Delta(G) + 1$ (see Johnson [76] for the case $\Delta = 3$). In the former case, G is said to be *class 1*. Otherwise, G is said to be *class 2*. In [41], Fiol and Vilaltella proposed a simple, but empirically efficient, heuristic algorithm for edge-coloring of graphs, which is based on the displacement of conflicting vertices. If c is a proper edge-coloring coloring of G, then we say that $c^{-1}(i)$ is a *color class*. Clearly, $c^{-1}(i)$ is a matching in G. A graph G with $\Delta(G) = 3$ is called *subcubic* and, if G is regular, then G is a *cubic* graph. A proper 3-edge-coloring of a cubic graph is also referred to as a *Tait coloring*.

The following parameters to measure how far apart a graph is from being edge-colorable were defined (they are listed in the historical order they were proposed). As commented above, these are used both to gain information about the structure of class 2 cubic graphs, and to obtain partial results on the aforementioned conjectures.

- (Fiol [36, 38]) $d(G)$: The *edge-coloring degree* $d(G)$ is the minimum number of conflicting vertices (that is, with some incident edges having the same color) in a 3-edge-coloring of G.

- (Huck and Kochol [67]) $\omega'(G)$: The *weak oddness* $\omega'(G)$ is the minimum number of odd components of an even factor F of G. Note, that F may contain vertices of degree 0.

- (Huck and Kochol [67]) $\omega(G)$: The *oddness* $\omega(G)$ is the smallest number of odd components in a 2-factor of G.

- (Steffen [118, 121]) $r(G)$: The *resistance* $r(G)$ is the minimum cardinality of a color class of a proper 4-edge-coloring of G. Clearly, this is precisely the minimum number $\sigma(G)$ of edges whose removal yields a 3-edge-colorable graph. We call a proper 4-edge coloring of G minimal if it has a color class of cardinality $r(G)$. This parameter is called *color number* and denoted by $c(G)$ in [118].

- (Steffen [118], Kochol [87], Mkrtchyan and Steffen [102]) $\rho(G) \equiv r_v(G)$ is the minimum number of vertices to be deleted from G so that the resulting graph has a proper 3-edge-coloring.

Other (more recent) measures or concepts related to edge-uncolorability of cubic graphs, considered in subsequent sections of the paper, are: reduction and decomposition of cubic class 2 graphs (Nedela and Škoviera [105]); maximum 2-edge- and 3-edge-colorable subgraphs of bridgeless cubic class 2 graphs (Steffen [121]); excessive index (Bonisoli and Cariolaro [10]); μ_3 (Steffen [125]); nowhere-zero flows (Tutte [130]); flow resistance (introduced by the authors in [40]); oddness and resistance ratios; etc. Moreover, some measures of edge-uncolorability are closely related to some types of reductions and vice versa, see for example Nedela and Škoviera [105] and Steffen [118].

In this work we give a survey on results on the different measures of edge-uncolorability in cubic graphs, and give some new results on their relation to each other. We also discuss their similarities and differences, and related results in the attempt of a classification of non-edge-colorable graphs, mainly snarks (the case of cubic graphs). The paper is organized as follows. We distinguish coloring, flow and structural parameters in the next sections and relate them to each other. For general terminology and notations on graphs, see for instance, Bondy and Murty [9], Diestel [30], or Chartrand, Lesniak and Zhang [20]. For results on edge-coloring, see for instance Fiorini and Wilson [43] or Stiebitz, Scheide, Toft, and Favrholdt [126].

2 Basic results and multipoles

We begin by considering the first measures of edge-uncolorability that were defined in the Introduction. Basically, they are concerned with the concepts of conflicting vertex, conflicting zone, and oddness. Since their definition does not depend on the regularity of the graph, now we will focus on subcubic graphs. Let c be a 3-edge-coloring of a graph G with $\Delta(G) = 3$. For $v \in V(G)$, let $c(v)$ be the set of colors that appear at v. Vertex v is a *conflicting vertex* if $|c(v)| < \deg(v)$, and it is a *normalized* conflicting vertex if $|c(v)| = \deg(v) - 1$, that is, there is precisely one color that appears twice at v. A *conflicting zone* is a subgraph containing some conflicting vertex for any 3-edge-coloring of G. A *conflicting edge-cut* is a set of edges of G that separates two conflicting zones.

As shown in the following proposition, all the first defined parameters, except the oddness and weak oddness, coincide in the case of graphs with maximum degree 3.

Theorem 7. *The following hold:*

(*i*) *A subcubic graph G has a proper 3-edge-coloring if and only if* $d(G) = r(G) = \sigma(G) = \rho(G) = 0$.

(*ii*) *A cubic graph G with a 2-factor is 3-edge-colorable if and only if* $\omega'(G) = \omega(G) = d(G) = r(G) = \sigma(G) = \rho(G) = 0$.

(*iii*) *For any subcubic graph G we have* $d(G) = r(G) = \rho(G)$.

(*iv*) *A cubic graph G is not edge-colorable (class 2) if and only if $d(G) \geqslant 2$.*

(*v*) *If G is a loopless cubic graph with $d(G) = 2$, then $\omega'(G) = \omega(G) = d(G)$.*

(*vi*) *In general, for any loopless cubic graph G, we have $\omega(G) \geqslant d(G)$, and both parameters can be arbitrarily far apart.*

Proof. The equivalences (*i*) and (*ii*) are simple consequence of the definitions. The result in (*iii*) is a direct consequence of a theorem by Fiol, see [38, Theorem 2.1], stating that, if $d(G) = d$, then there is a 3-edge coloring with exactly d normalized conflicting vertices. This result was rediscovered by Kochol [87] by proving that $\rho(G) = \sigma(G)$. Again, (*iv*) is a consequence of Corollary 1.2 in [38]. The result in (*v*) was proved proved in [38, Theorem 2.2] by using Kempe chains. Concerning (*vi*) notice that, if G has a 2-factor with ω odd components, then there is an obvious 3-edge-coloring with ω conflicting vertices. Just assign alternatively colors 1,2 to the edges of the cycles—Kempe chains—and color 3 to the remaining edges. This proves the inequality. Finally, the fact that $\omega(G)$ and $d(G)$ can be arbitrarily far apart was proved by Steffen in [121, Theorem 2.3]. □

The following results are drawn from [36, 38, 39].

Theorem 8. *The following holds.*

(*i*) *Let G be a class 2 graph. If $H \subset G$ is a conflicting zone with $d(H) = d \geqslant 2$, then there exists a conflicting zone $H' \subset H$ with $d(H') = d - 1$.*

(*ii*) *Let S be a snark with $d(S) \geqslant 2$. Then, for every $d' \in \{1, \dots, d(S)\}$, there exists a conflicting zone $H \subset G$ with $d(H) = d'$.*

(*iii*) *Let S be a snark with $d(S) > 2$. Then S contains a subdivision of a snark S' with $d(S') = 2$ (that is, S' is a minor of S).*

(*iv*) *Let S be a snark with $d(S) \geqslant 2$. Then, its number of vertices satisfy $|V(S)| \geqslant 10 \lfloor (d+1)/2 \rfloor$.*

From Theorem 8(*iv*), the first author [36] also managed to prove the following theorem (the most difficult case was $N = 16$ and it was rediscovered later by Fouquet [44]):

Theorem 9. *There is no snark of order n for each $n \in \{12, 14, 16\}$.*

Some of the aforementioned results were also considered for subcubic graphs by Rizzi [109] and Fouquet and Vanherpe [47].

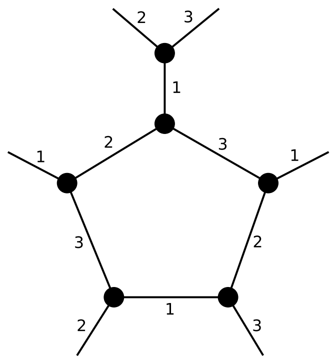

Figure 1: A Tait coloring of a 6-pole.

2.1 Multipoles

In the study of snarks it is useful to think of them as made up by joining two or more graphs with ‘dangling edges’. Following [36, 37], we call these graphs multipoles. More precisely, a *multipole* or *m-pole* $M = (V, E, F)$ consists of a (finite) set of vertices $V = V(M)$, a set of edges $E = E(M)$ or unordered pair of vertices, and a set $F = F(M)$, $|F| = m$, whose elements ϵ are called *semiedges*. Each semiedge is associated either with one vertex or with another semiedge making up what we call an isolated edge. Notice that a multipole can be disconnected or even be ‘empty’, in the sense that it can have no vertices.

The behavior of the semiedges is as expected. For instance, if the semiedge ϵ is associated with vertex u, we say that ϵ is *incident* to u. Then we write $\epsilon = (u)$ following Goldberg’s notation [57]. By joining the semiedges (u) and (v) we obtain the edge (u, v). As for graphs, we define the *degree* of u, denoted by $\deg(u)$, as the number of edges plus the number of semiedges incident to it. Throughout this paper, a multipole will be supposed to be cubic, that is, $\deg(u) = 3$ for all $u \in V$.

As expected, a *Tait coloring* of an m-pole (V, E, F) is an assignment of three colors to its edges and semiedges, that is, a mapping $\phi : E \cup F \to \mathcal{C} = \{1, 2, 3\}$, such that the edges and/or semiedges incident to each vertex have different colors, and each isolated edge has both semiedges with the same color. For example, Figure 1 shows a Tait coloring of a 6-pole. Note that the numbers of semiedges with the same color have the same parity. The following basic lemma states that this is always the case (see Izbicki [70] and also Isaacs [68]):

Lemma 10 (The Parity Lemma). *Let M be a Tait colored m-pole with m_i semiedges having color i for $i \in \{1, 2, 3\}$. Then,*

$$m_1 \equiv m_2 \equiv m_3 \equiv m \pmod{2}. \tag{1}$$

This result has been used extensively in the literature on the subject. See, for instance, Blanuša [8], Descartes [27], Isaacs [68], and Goldberg [57]. Although in these references isolated edges are not allowed, the proof is basically the same. A slightly more general version concerning Boole colorings will be proved in the next subsection.

Given an m-pole M with semiedges $\epsilon_1, \ldots, \epsilon_m$, we define its *set* $\mathrm{Col}(M)$ *of semiedge colorings* as

$$\mathrm{Col}(M) = \{(\phi(\epsilon_1), \phi(\epsilon_2), \ldots, \phi(\epsilon_m)) : \phi \text{ is a Tait coloring of } M\}.$$

Note that $\mathrm{Col}(M)$ depends on the order in which the semiedges are considered. Thus, when referring to such a set we will implicitly assume that this ordering is given.

Of course, $\mathrm{Col}(M) = \emptyset$ if and only if M is not Tait colorable. In this case it is trivial to obtain a class 2 graph from M. Indeed, we can either remove all its semiedges or join them properly in order to achieve regularity (using additional vertices if necessary). By the parity lemma, the simplest example of a non-Tait-colorable m-pole is when $m = 1$, so that any cubic graph with a bridge is trivially class 2.

In the other extreme, we will say that M is *color-complete* if $\mathrm{Col}(M)$ has maximum cardinality. In other words, M is color-complete if it can be Tait colored so that its semiedges have any combination of colors satisfying the parity lemma. For instance, all Tait colorable 2-poles and 3-poles are color-complete because, according to (13), the only possibilities, up to permutation of the colors, are $(\phi(\epsilon_1), \phi(\epsilon_2)) = (a, a)$ and $(\phi(\epsilon_1), \phi(\epsilon_2), \phi(\epsilon_3)) = (a, b, c)$ respectively—here, and henceforth, the letters a, b, c stand for the colors 1, 2, 3 in any order. Clearly, the simplest color-complete 2-pole and 3-pole are respectively an isolated edge and a single vertex with 3 semiedges incident to it. They will be denoted by $\boldsymbol{e}$ and $\boldsymbol{v}$. Besides, a color-complete 4-pole has four different values of $(\phi(\epsilon_1), \phi(\epsilon_2), \phi(\epsilon_3), \phi(\epsilon_4))$. Namely, (a, a, a, a), (a, a, b, b), (a, b, a, b) and (a, b, b, a).

An m-pole M is said to be *reducible* when there exists an m-pole N such that $|V(N)| < |V(M)|$ and $\mathrm{Col}(N) \subseteq \mathrm{Col}(M)$. Otherwise, we say that M is *irreducible*. This concept was first introduced by Fiol in [37], where the following result was proved.

Proposition 11. *For any integer $m \geqslant 1$, there exists a positive integer-valued function $v(m)$ such that any m-pole M contained in a snark, with $|M| > v(m)$, is either not Tait colorable or reducible.*

The known values of $v(m)$ are $v(2) = 0$, $v(3) = 1$ (both are trivial results), $v(4) = 2$ (Goldberg [57]), and $v(5) = 5$ (Cameron, Chetwynd, and Watkins [14]), whereas its exact value is unknown for $m \geqslant 6$. However, in this case, Karabáš, Máčajová, and Nedela [78] proved that $v(6) \geqslant 12$. These are much relevant questions in the decompositions of snarks. According to the Jaeger-Swart's conjecture [74], every snark contains a cycle-separating edge-cut of size at most six. In that case, $v(6)$ would be the most interesting unknown value of $v(m)$. Moreover, the above definition implies that any snark U with a cutset of m edges and $|V(U)| > 2v(m)$ can be 'reduced' to another snark with fewer vertices. See [14] for the cases $m = 4, 5$. More recently, Fiol and Vilaltella [42] proved that the tree and cycle multipoles are irreducible and, as a byproduct, that $v(m)$ has a linear lower bound.

Let M_1 and M_2 be two m-poles with semiedges ϵ_i and ζ_i, $i = 1, 2, \ldots, m$, respectively, and assume that by joining ϵ_i with ζ_i for all $i = 1, 2, \ldots, m$ we obtain a cubic graph G. Then we will say that M_1 and M_2 are *complementary* (*with respect to G*), or that M_2 is the *complement* of M_1, written $M_2 = M_1'$. Moreover, the m-poles M_1 and M_2 are said to be *color-disjoint* if $\mathrm{Col}(M_1) \cap \mathrm{Col}(M_2) = \emptyset$. In particular this is the case when one of the m-poles is not Tait colorable. The analysis and synthesis of snarks is based on the following straightforward result.

Proposition 12. *Let M and M' be two complementary multipoles of a graph G. Then G is a snark if and only if M and M' are color-disjoint.*

Thus, the problem of constructing snarks can be reduced to the problem of finding pairs of color-disjoint multipoles. The main problem to proceed in this way is that, when the number of semiedges increases, the characterization of the set $\mathrm{Col}(M)$ becomes more and more difficult. To overcome this drawback the idea is to group the semiedges in different sets, making the so-called *multibuses* or *connectors*, and give a proper characterization of the 'global' coloring of their elements, as we do in the next section.

2.2 Boole colorings

The construction of cubic graphs which cannot be Tait-colored leads to Boolean algebra, which is commonly used in the study of logic circuits. To this end, the first author [36, 37] introduced a generalization of the concept of 'color', which describes in a simple way the coloring ('$\mathbf{0}$' or '$\mathbf{1}$') of any set of edges or, more abstractly, of any family $\mathcal{F}$ of m colors chosen between three different colors of $\mathcal{C} = \{1, 2, 3\}$, such that color $i \in \mathcal{C}$ appears m_i times. This situation can be represented by the coloring-vector $\boldsymbol{m} = (m_1, m_2, m_3)$, where $m = m_1 + m_2 + m_3$. Then, we say that $\mathcal{F}$ has *Boole-coloring* $\mathbf{0}$, denoted by $Bc(\mathcal{F}) = \mathbf{0}$, if

$$m_1 \equiv m_2 \equiv m_3 \equiv m \pmod 2,$$

whereas $\mathcal{F}$ has *Boole-coloring* $\mathbf{1}$ (more specifically $\mathbf{1}_a$), denoted by $Bc(\mathcal{F}) = \mathbf{1}$ (or $Bc(\mathcal{F}) = \mathbf{1}_a$), if

$$m_a + 1 \equiv m_b \equiv m_c \equiv m + 1 \pmod 2,$$

where, as before, a, b, c represent the colors $1, 2, 3$ in any order. See [39, 37] for more information.

From these definitions, the Boole-coloring of an edge $e \in E$ with color $c(e) = a \in \mathcal{C}$ is $Bc(e) = Bc(\{a\}) = \mathbf{1}_a$, and the Boole-coloring of a vertex $v \in V$, denoted by $Bc(v)$, is defined as the Boole-coloring of its incident edges. In particular, notice that in a Tait coloring of a cubic graph, all its vertices have Boole-coloring $\mathbf{0}$.

Moreover, a natural sum operation can be defined in the set $\mathcal{B} = \{\mathbf{0}, \mathbf{1}_1, \mathbf{1}_2, \mathbf{1}_3\}$ of Boole-colorings in the following way: Given the colorings X_1 and X_2 represented, respectively, by the coloring-vectors $\boldsymbol{m}_1 = (m_{11}, m_{12}, m_{13})$ and $\boldsymbol{m}_2 = (m_{21}, m_{22}, m_{23})$, we define the sum $X = X_1 + X_2$ as the coloring represented by the coloring vector $\boldsymbol{m} = \boldsymbol{m}_1 + \boldsymbol{m}_2$. Then, $(\mathcal{B}, +)$ is isomorphic to the Klein group $\mathbb{K}$, with $\mathbf{0}$ as identity, $\mathbf{1}_a + \mathbf{1}_a = \mathbf{0}$, and $\mathbf{1}_a + \mathbf{1}_b = \mathbf{1}_c$. In fact, note the equivalence between the Boole colorings and the elements of the group $\mathbb{Z}_2 \times \mathbb{Z}_2$, used as values in the theory of flows (see Section 4).

Notice that, since every element coincides with its inverse, $m\mathbf{1}_a = \mathbf{1}_a + \mathbf{1}_a + \overset{(m)}{\cdots} + \mathbf{1}_a$ is $\mathbf{0}$ if m is even and $\mathbf{1}_a$ if m is odd. From this simple fact, we can imply the following result (see Fiol [38]), which is very useful in the further development of the theory, and it can be regarded as another version of the Parity Lemma.

Lemma 13. *Let G be a subcubic graph with n vertices having a 3-edge-coloring, such that n_i vertices have Boole-coloring $\mathbf{1}_i$, for $i \in \mathcal{C}$, with $n' = n_1 + n_2 + n_3 \leqslant n$. Then,*

$$n_1 \equiv n_2 \equiv n_3 \equiv n' \pmod 2. \tag{2}$$

Proof. Indeed, since the Boole-coloring of each vertex is the sum of the Boole-colorings of its incident edges, and recalling that $\sum_{v \in V} \deg(v) = 2|E|$, we can write

$$\sum_{v \in V} Bc(v) = \sum_{i=1}^{3} n_i \mathbf{1}_i + (n - n')\mathbf{0} = \sum_{i=1}^{3} n_i \mathbf{1}_i = \sum_{e \in E} 2Bc(e) = \mathbf{0},$$

but this equality is satisfied if and only if, for every $i \in \mathcal{C}$, $n_i \mathbf{1}_i = \mathbf{0}$ or $n_i \mathbf{1}_i = \mathbf{1}_i$. Then, from $n_1 + n_2 + n_3 = n'$, we get the result. □

Note that, if G is a cubic graph with a given 3-edge-coloring, then the result applies with n_i being the number of conflicting vertices of type $\mathbf{1}_i$, $i = 1, 2, 3$.

As a direct consequence of the lemma, we also get the following:

Corollary 14. *There is no edge-coloring of a graph G having only one vertex with Boole-coloring $\mathbf{1}$ (and the other vertices with Boole-coloring $\mathbf{0}$).*

Similar results are obtained in the context of the resistance in [118]. There it is shown that the uncolored edges can be classified as in Lemma 13. Analogously, there is no proper 4-edge-coloring of a cubic graph with a color class of cardinality 1.

2.3 Max 2- and 3-colorable subgraphs

Maximum 2- and maximum 3-edge-colorable subgraphs of cubic graphs were first studied by Albertson and Haas [4].

The resistance measures the minimum number of uncolored edges in a 3-edge-coloring of a cubic graph G. We can ask a similar question with respect to 2-edge-colorable subgraphs of G, since if there is a 2-edge-colorable subgraph H with $|E(H)| = \frac{2}{3}|E(G)|$, then G is a class 1 graph. Let $c_2(G) = \max\{|E(H)| : H$ is a subgraph of G and $\chi'(H) = 2\}$, and $r_2(G) = \frac{2}{3}|E(G)| - c_2(G)$. The following theorem shows that $r_2(G)$ is also a measure of edge-uncolorability.

Theorem 15 ([121])**.** *If G is a bridgeless cubic graph, then*

(i) $r_2(G) = 1$ *if and only if* $r(G) = 2$.

(ii) $\frac{1}{2}r(G) \leqslant r_2(G) \leqslant \min\{\frac{2}{3}r(G), \frac{1}{2}\omega(G)\}$*, and the bounds are attained.*

3 Factors

In this section, we review some measures of edge-uncolorability of a cubic graph G which depend on the properties of the sets of its 1-factors and 2-factors.

3.1 1-factors

We start from an unusual statement of Vizing's Theorem for cubic graphs:

The edge-set of every cubic graph can be written as a union of at most four of its matchings.

What happens if we would like to replace matchings with perfect matchings (that is, 1-factors)? Can we prove an analogous theorem?

First of all, we must remark that this question is only relevant in the class of bridgeless cubic graphs. Indeed, if a cubic graph G has a bridge then it has to be contained in every 1-factor. So, the edges adjacent to a bridge cannot be covered, and hence we cannot obtain the edge-set of G by union of 1-factors.

Let G be a bridgeless cubic graph, by Petersen's Theorem from 1891 [106], G has a 1-factor. Schönberger [113] refined this result by proving that for every edge e, there is a 1-factor of G which contains e. Schönberger's result implies that the edge-set of every bridgeless cubic graph G can be obtained as a union of a finite set of 1-factors. We denote by $\chi'_e(G)$ the minimum cardinality among all such sets of 1-factors. The parameter $\chi'_e(G)$ is called *excessive index* in [10] and *perfect matching index* in [46]. Note that if a cubic graph has two disjoint 1-factors, then $\chi'(G) = \chi'_e(G) = 3$. Hence, $\chi'(G) = 4$ if and only if any two 1-factors of G have a non-empty intersection.

In an attempt to mimic Vizing's result, one could try to prove that $\chi'_e(G) \leqslant 4$ for every bridgeless cubic graph G, but such an attempt is guaranteed to fail. Indeed, the union of five distinct 1-factors of the Petersen graph is necessary to obtain its edge-set.

If Berge's conjecture (Conjecture 2) holds, then bridgeless cubic graphs which are not 3-edge-colorable are divided into two classes depending on whether they have excessive index 4 or 5. Hence, we can consider the excessive index as a measure of 3-edge-uncolorability finer than chromatic index. It is interesting that cubic graphs G with $\chi'_e(G) = 4$ share some properties with 3-edge-colorable cubic graphs. For instance a shortest cycle cover (that is, a set of cycles covering all the edges) of length $\frac{4}{3}|E(G)|$ [125]. Moreover, the 5-cycle double cover conjecture is true for such graphs (see Hou, Lai, and Zhang [65], and Steffen [125]). In other words, a possible counterexample for the 5-cycle double cover conjecture would be in the class of cubic graphs with excessive index at least 5. Abreu, Kaiser, Labbate, and Mazzuoccolo [1] suggest a relation between excessive index and circular flow number of a cubic graph (see Section 4 for a definition). In particular, it is remarked that all known snarks with excessive index 5 have circular flow number at least 5. In other words, if a snark is critical with respect to Conjecture 2, then it seems to be critical also for the 5-flow conjecture. We believe that proving a relation between these two famous conjectures could be a very interesting result.

All previous arguments stress the fact that the class of snarks having excessive index 5 has a particular relevance and computational evidence shows that these snarks are quite rare. The smallest example apart from the Petersen graph, has order 34 and it was found by Hägglund [61]. Starting from Hägglund's example, some infinite families of snarks with excessive index 5 have been recently constructed by Esperet and Mazzuoccolo [32], Abreu, Kaiser, Labbate, and Mazzuoccolo [1], and Chen [21]. Note that all such snarks are not cyclically 5-edge-connected, it remains open the question about the existence of a cyclically 5-edge-connected snark with excessive index 5 different from the Petersen graph (see Problem 41). Such graphs had already been studied by Sinclair in [117]. He proves some decomposition results for graphs which cannot be covered by four perfect matchings. That paper is written in the language of weights and cycle covers. The equivalence to the cover with perfect matchings follows from the fact that a snark G can be covered by four perfect matchings if and only if it has a cycle cover such that each edge is in one or in two circuits and the edges which are in two circuits form a 1-factor of G, see Theorem 3.5 of Hou, Lai, and Zhang [65].

Conjecture 2 is largely open and it remains open even if we replace 5 with an arbitrary larger constant k. It is not hard to show, by using Edmonds' matching polytope theorem [31], that the edges of any bridgeless cubic graph of order n can be obtained as a union of $\log(n)$ perfect matchings. As far as we know, the best bound known (still logarithmic in the order of G) is the one given by Mazzuoccolo in [99] as a corollary of the technique introduced by Kaiser, Král and Norine in [77].

In the remaining part of this section, we review known results about the cardinality and the structure of the union of a prescribed number of 1-factors in a bridgeless cubic graph. More precisely, let G be a bridgeless cubic graph. Consider a list of k 1-factors of G. Let E_i be the set of edges contained in precisely i members of the k 1-factors. Let $\mu_k(G)$ be the smallest $|E_0|$ over all lists of k 1-factors of G and set $m_k(G) = 1 - \frac{\mu_k(G)}{|E(G)|}$, that is, the maximum possible fraction of edges in a union of k 1-factors of G.

We can restate some of the previous results about the excessive index in terms of these parameters. For instance, a bridgeless cubic graph G is 3-edge-colorable if and only if $m_3(G) = 1$ (or equivalently $\mu_3(G) = 0$); and $m_5(G) = 1$ for all G is exactly Conjecture 2. Furthermore, if G is a bridgeless cubic class 2 graph, then $\mu_3(G) \geqslant 3$, see [125].

Kaiser, Král and Norine [77] proved that $m_2(G) \geqslant \frac{3}{5}$ and this result is the best possible, since the union of any two 1-factors of the Petersen graph P contains 9 of the 15 edges of the graph. It is also proved that $m_3(G) \geqslant \frac{27}{35}$, but it is conjectured that $m_3(G) \geqslant \frac{4}{5} = m_3(P)$ for every bridgeless cubic graph G.

Let G be a cubic graph and S_3 be a list of three 1-factors M_1, M_2, M_3 of G. Let $\mathcal{M} = E_2 \cup E_3$, $\mathcal{U} = E_0$ and $|\mathcal{U}| = k$. The edges of E_0 are also called the uncovered edges. The k-core of G with respect to S_3 (or to M_1, M_2, M_3) is the subgraph G_c of G which is induced by $\mathcal{M} \cup \mathcal{U}$; that is, $G_c = G[\mathcal{M} \cup \mathcal{U}]$. If the value of k is irrelevant, then we say that G_c is a core of G. If $M_1 = M_2 = M_3$, then $G_c = G$. A core G_c is proper if $G_c \neq G$. If G_c is a cycle, then we say that G_c is a cyclic core. In [125] it is shown that every bridgeless cubic graph has a proper core and therefore, every $\mu_3(G)$-core is proper. Cores of cubic graphs have bee studied by Jin and Steffen [75, 125].

Let $\gamma_2(G) = \min\{|M_1 \cap M_2| : M_1 \text{ and } M_2 \text{ are 1-factors of } G\}$. Then, $\mu_2(G) = \gamma_2(G) + \frac{1}{3}|E(G)|$.

Theorem 16 ([75]). *Let G be a bridgeless cubic graph. If G is not 3-edge-colorable, then $\omega(G) \leqslant 2\gamma_2(G) \leqslant \mu_3(G) - 1$. Furthermore, if G has cyclic $\mu_3(G)$-core, then $\gamma_2(G) \leqslant \frac{1}{3}\mu_3(G)$.*

Theorem 17 ([75]). *If G is a bridgeless cubic graph, then $\omega(G) \leqslant \frac{2}{3}\mu_3(G)$.*

In [75] it is also proved that there are graphs G with $\omega(G) = \frac{2}{3}\mu_3(G)$, and that those graphs have very specific structural properties. Indeed, if G is a bridgeless cubic graph with $\omega(G) = \frac{2}{3}\mu_3(G)$, then it satisfies Conjecture 4.

3.2 2-factors

Let G be a bridgeless cubic graph. Obviously, $\omega'(G) \leqslant \omega(G)$. Note that an even factor of a bridgeless cubic graph is a spanning subgraph of G having all vertices of degree either 2 or 0, that is, a union of circuits and isolated vertices.

In several papers, over the last few decades, weak oddness and oddness of a cubic graph appear as interchangeable definitions, implicitly assuming that they should be equal for every bridgeless cubic graph. But, the long standing discussion whether $\omega(G) = \omega'(G)$ for all bridgeless cubic graphs G was recently finished by the following negative result of Lukot'ka and Mazák.

Theorem 18 ([91]). *There exists a graph G with $r(G) = 12$, $\omega'(G) = 14$, and $\omega(G) = 16$.*

Theorem 18 gives rise to infinite families of cubic graphs where oddness and weak oddness differ. Moreover, it is observed in [91] that there exist cubic graphs having arbitrarily large difference between oddness and weak oddness.

Here, we improve Theorem 18, showing an example of a cubic graph having $\omega'(G) = 6$ and $\omega(G) = 8$.

In order to give a general approach to the problem we introduce the following definitions:

A *minimal* 2-factor (even factor) of a bridgeless cubic graph G is a 2-factor (even factor) of G with the minimum number of odd circuits (odd components). In other words, a minimal 2-factor has $\omega(G)$ odd circuits and a minimal even factor has $\omega'(G)$ odd components.

We follow the terminology introduced by Esperet and Mazzuoccolo in [33] for a standard operation on cubic graphs. Namely, given two cubic graphs G and H and two edges xy in G and uv in H, the *gluing*, or 2-cut-connection, of (G, xy) and (H, uv) is the graph obtained from G and H by removing the edges xy and uv, and adding the new edges xu and yv. In the resulting graph, we call these two new edges the *clone edges* of xy (or uv). Note that if G and H are cubic and bridgeless, then the resulting graph is also cubic and bridgeless.

In what follows H will always be the Petersen graph P, which is arc-transitive. Hence, the choice of uv and the order of each pair (x, y) and (u, v) are not relevant, so that we will simply say that we glue P on the edge xy of G.

Lemma 19. *Let (G, xy) be a pair where G is a bridgeless cubic graph and xy is an edge of G. If xy belongs to an odd circuit of a minimal even factor of G,* ***and*** *xy does not belong to a minimal 2-factor of G, then the graph G^* obtained by gluing a copy of the Petersen graph on the edge xy of G has the following properties:*

(*i*) $\omega(G^*) \geqslant \omega(G) + 2$.

(*ii*) $\omega'(G^*) = \omega'(G)$.

Proof. Denote by uv the edge of P used to perform the gluing (as already observed, the choice of uv is not relevant since P is arc-transitive). Denote by F' a minimal even factor of G having an odd circuit C' which contains the edge xy. Select one of the minimal even factors of P which consists of a circuit of length 9 passing through uv and an isolated vertex. Consider the even factor of G^* obtained by gluing together F' and the selected minimal even factor of P. The gluing of C' and the 9-circuit of P (see Figure 2) produces an even circuit of G^*, then this even factor of G^* has the same number of odd components of F', that is, $\omega'(G^*) = \omega'(G)$.
Moreover, every 2-factor of P consists of two 5-circuits. Hence, if no minimal 2-factor of G contains the edge xy, then the graph G^* has at least two odd components more than G in every of its 2-factors, that is, $\omega(G^*) \geqslant \omega(G) + 2$ as claimed. □

Now, our aim is to construct a pair (G, xy) which satisfies the assumptions of previous lemma. Moreover, we would like to find G such that their oddness and weak oddness be as small as possible. In particular, we will produce an example with oddness 6 and we leave as an open problem the existence of an example with oddness 4.

In order to construct such an example, we will implicitly use several times the two following properties of the graph G^* described in Lemma 19:

P1. If a circuit C of a 2-factor (even factor) F of G^* passes through the two clone edges of xy, then F has exactly (at least) one odd component distinct from C in P.

P2. If a 2-factor (even factor) F does not contain a circuit C passing through the clone edges of xy, then F has exactly (at least) two odd components in P.

Finally, we recall in the next lemma some well-known properties of 2-factors and circuits of the Petersen graph which are useful in the proof of our main result (see also Figure 2).

Lemma 20. *Let $M = \{e_1, e_2, e_3\}$ be a set of three edges of the Petersen graph P, which induce a maximal matching of P. Then,*

(*i*) *Every 2-factor F of P consists of two 5-circuits C_1, C_2. Moreover, $|F \cap M| = 2$, $|C_1 \cap M| = 1$ and $|C_2 \cap M| = 1$.*

(*ii*) *There exists a minimal even factor F' of P such that $M \subset F'$.*

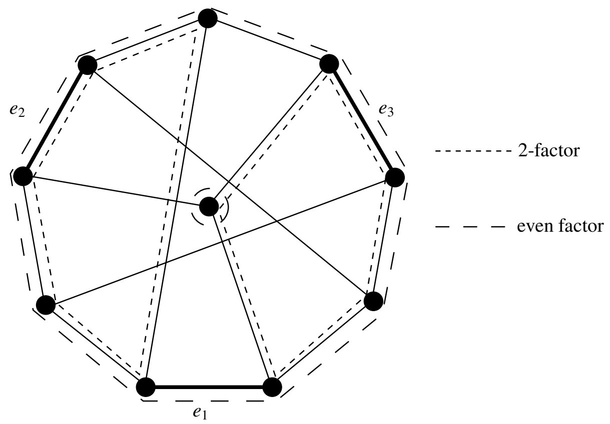

Figure 2: A 2-factor F and an even factor F' of the Petersen graph which satisfy the conditions in Lemma 20.

We denote by K the graph obtained starting from the Petersen graph P and glueing two further copies of the Petersen graph on each of two, say e_2, e_3, of the three edges of a maximal matching $M = \{e_1, e_2, e_3\}$ (see Figure3).

Remark 21. Every even factor and every 2-factor either contains both edges or no edge of a given 2-edge-cut. Since clone edges form a 2-edge-cut, we can naturally reconstruct from each factor F of K the *underlying factor of* F in P.

Now, we are in position to prove our main result.

Theorem 22. *Let K^* be the graph obtained by gluing a copy of the Petersen graph on the edge e_1 of K (see Figure 3). Then, $\omega(K^*) = 8$ and $\omega'(K^*) = 6$.*

Proof. We only need to prove that $\omega(K) = \omega'(K) = 6$ and (K, e_1) satisfies the assumptions of Lemma 19. Consider a 2-factor F of K. The underlying factor of F in P has two 5-circuits. If only one between e_2 and e_3, say e_2, lies in the underlying factor of F, then F consists of eight odd circuits: two in each copy of the Petersen graph on e_3, one in each copy of the Petersen graph on e_2 and the two original odd circuits of P (which still correspond to odd circuits in K). On the other hand, if both e_2 and e_3 are edges of the two 5-circuits, then F has only six odd circuits: one in each of the four copies of the Petersen graph and the two original circuits of P (which again correspond to odd circuits in K). Then, $\omega(K) = 6$. Moreover, by Lemma 20(i), every minimal 2-factor of G does not contain the edge e_1. Since every even factor has at least one odd component in each of the four copies of the Petersen graph, and at least two odd components arising from the odd components in P, then also $\omega'(K) = 6$ holds. Furthermore, it follows from property **P2** that there exists an even factor of K having an isolated vertex, four 5-circuits (one in each copy of the Petersen) and a circuit of length 29 passing through e_1. Hence, by Lemma 19, the graph K^* has $\omega' = 6$ and $\omega \geqslant 8$. Finally, it is routine to check that, in fact, $\omega = 8$ (see Figure 3). □

As already observed, $\omega'(G) = 2$ implies $\omega(G) = 2$, then the remaining open question

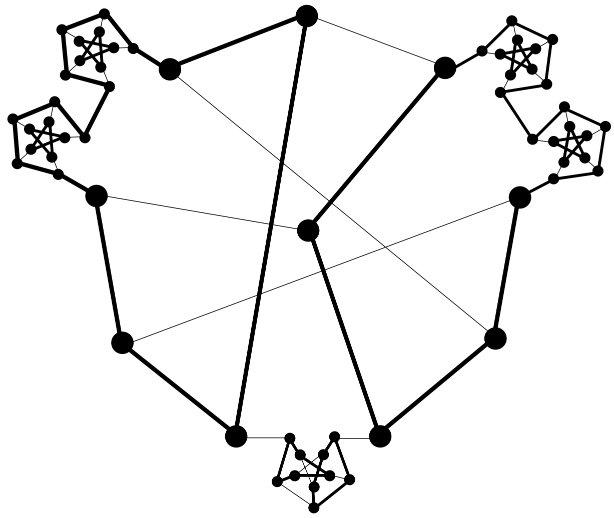

Figure 3: The graph K^* such that $\omega(K^*) = 8$ and $\omega'(K^*) = 6$.

is whether $\omega'(G) = 4$ if and only if $\omega(G) = 4$ (see Problem 47). The next two theorems study the relation of the $\omega'(G)$ and $r(G)$, when $r(G) \in \{3, 4\}$.

Theorem 23. *Let G be a bridgeless cubic graph. If $r(G) = 3$, then $\omega'(G) = 4$.*

Proof. Let $c : E(G) \to \{0, 1, 2, 3\}$ be a minimal proper 4-edge-coloring of G with $|c^{-1}(0)| = r(G)$. For $i \in \{1, 2, 3\}$, denote by A_i the set of edges e in $c^{-1}(0)$ such that exactly two of the edges adjacent to e are colored i. By Lemma 1.1 in [121] and $r(G) = 3$, it follows that $c^{-1}(0) = \{e_1, e_2, e_3\}$ and $e_i \in A_i$. Let $e_i = x_i y_i$, where we denote by x_i (y_i) the vertex of e_i incident to an edge of color the smallest (largest) positive value different from i. There is a $(2, 3)$-chain $P_{(2,3)}(x_1, y_1)$ from x_1 to y_1 which starts with an edge of color 2 and ends with an edge of color 3, otherwise, as standard, if such edges are in two distinct $(2, 3)$-chains we can switch the colors in one of these two chains and color e_1 with the available color. Hence $r(G) < 3$, a contradiction. The chains $P_{(1,3)}(x_2, y_2)$ and $P_{(1,2)}(x_3, y_3)$ are defined analogously. By Lemma 2.4 of [118] the three paths $P_{(2,3)}(x_1, y_1)$, $P_{(1,3)}(x_2, y_2)$ and $P_{(1,2)}(x_3, y_3)$ are pairwise disjoint.

Furthermore, there are chains $P_{(1,3)}(x_1, y_3)$ and $P_{(2,3)}(x_2, x_3)$. We have to consider two cases.

Case (*a*) $P_{(1,3)}(x_1, y_3)$ and $P_{(2,3)}(x_2, x_3)$ are disjoint.

Interchange the colors on the chains, that is, consider $P_{(3,1)}(x_1, y_3)$ and $P_{(3,2)}(x_2, x_3)$, to obtain a proper 4-edge-coloring c' of G. Then e_3 can be colored with color 3, and we still have a proper coloring. Hence $r(G) < 3$, a contradiction.

Case (*b*) $P_{(1,3)}(x_1, y_3)$ and $P_{(2,3)}(x_2, x_3)$ are not disjoint.

Let z be the first vertex of $P_{(1,3)}(x_1, y_3)$ which is incident to an edge of $P_{(2,3)}(x_2, x_3)$. Then $P_{(1,3)}(x_1, z)$ is of odd length, that is, its last edge is colored with color 1.

Let $x \in \{x_2, x_3\}$ and let $P_{(2,3)}(z, x)$ be the subpath of $P_{(2,3)}(x_2, x_3)$ which starts at z with an edge of color 2.

$(b1)$ $x = x_2$. Replace $P_{(2,3)}(x_2, z)$ by $P_{(3,2)}(x_2, z)$, and note that the coloring of the edges of $P_{(1,3)}(x_1, z)$ is unchanged. Replace $P_{(1,3)}(x_1, z)$ by $P_{(3,1)}(x_1, z)$, to obtain a 4-edge-coloring ϕ where all edges incident to z are colored with color 3, for all $v \in V(G) - \{z\}$, the three edges incident to v receive pairwise different colors. Furthermore, $\phi^{-1}(0) = \{e_1, e_2, e_3\}$ and each e_i is incident to two edges of color 3 in ϕ. Now, $G - \phi^{-1}(3)$ has (at most) 4 odd components. Hence $\omega'(G) = 4$.

$(b2)$ $x = x_3$. Let z' be the neighbor of z in $P_{(1,3)}(x_1, y_3)$ such that $c(zz') = 3$, that is, $zz' \in E(P_{(1,3)}(x_1, y_3)) \cap E(P_{(2,3)}(x_2, x_3))$. Let $P_{(2,3)}(x_2, z')$ be the subpath of $P_{(2,3)}(x_2, x_3)$. Obtain a proper 4-edge-coloring ϕ of G by replacing $P_{(2,3)}(x_2, z')$ by $P_{(3,2)}(x_2, z')$, and $P_{(1,3)}(x_1, z)$ by $P_{(3,1)}(x_1, z)$, and color edge zz' with color 0. Then

$$\phi^{-1}(0) = \{e_1, e_2, e_3, zz'\} = A_3.$$

Now, $\phi^{-1}(3)$ is a 1-factor of G and $G - \phi^{-1}(3)$ has (at most) 4 odd components. Hence $\omega'(G) = 4$. $\square$

Theorem 24. *There is a bridgeless cubic graph G with $r(G) = 4$ and $\omega'(G) = \omega(G) = 6$.*

Proof. Let P^- be the Petersen graph minus a vertex and x, y, z be the three divalent vertices. For $i \in \{1, 2, 3\}$ there is minimal proper 4-edge-coloring c_i of P^- such that color 1 is missing at x, z and color i is missing at y. Let 0 be the fourth color and the edges of color 0 be the minimal color class. For $i \in \{1, 2, 3\}$ let P_i be a copy of P^- with divalent vertices x_i, y_i, z_i and a minimal proper 4-edge-coloring c_i. Let v be a vertex and add edges $y_i v$ (for each $i \in \{1, 2, 3\}$), $x_1 z_2$, $x_2 z_3$, $x_3 z_1$ to obtain a 3-edge-connected cubic graph H with 4-edge-coloring with three edges colored with color 0. Since H contains three pairwise disjoint 3-critical graphs it follows that $r(H) = 3$. (It is easy to see that $\omega'(H) = \omega(H) = 4$.) Let H_1 and H_2 be two copies of H. Remove an edge e_i of color 0 from $H_i[V(P_3)]$. Let $e_i = v_i w_i$ and add edges $v_1 v_2$ and $w_1 w_2$ to $H_1[V(P_3)] - e_1$ and $H_2[V(P_3)] - e_2$, to obtain a bridgeless cubic graph G and a 4-edge-coloring of G with minimal color class of cardinality 4. Since G contains 4 pairwise disjoint 3-critical graphs it follows that $r(G) = 4$. It is easy to see that $\omega(G), \omega'(G) \leqslant 6$. Suppose to the contrary that $\omega'(G) = 4$. Then $G[V(H_1)]$ or $G[V(H_2)]$ (say $G[V(H_1)]$) contains at most two odd components of an even factor of G. But this implies that H_1 has an even factor with at most three odd components, a contradiction. We similarly deduce that $\omega(G) = 6$. $\square$

4 Nowhere-zero flows

If v is a vertex of a graph G, then $E_G(v)$ denotes the set of edges which are incident to v. If there is no harm of confusion, we sometimes use $E(v)$ instead of $E_G(v)$. An orientation D of a graph G is an assignment of a direction to each edge, and for $v \in V(G)$, $E^-(v)$ is the set of edges of $E(v)$ with head v and $E^+(v)$ is the set of edges with tail v. The oriented graph is denoted by $D(G)$. If there is no harm of confusion we write D instead of $D(G)$.

Let A be an Abelian group. An *A-flow* (D, ϕ) on G is an orientation D of G together with a function $\phi : E(G) \to A$ such that

$$\sum_{e \in E^+(v)} \phi(e) = \sum_{e \in E^-(v)} \phi(e), \text{ for all } v \in V(G).$$

The set $\{e : \phi(e) \neq 0\}$ is the *support* of (D, ϕ) and it is denoted by $\operatorname{supp}(D, \phi)$. Furthermore, (D, ϕ) is a *nowhere-zero* A-flow, if $\operatorname{supp}(D, \phi) = E(G)$. If $A = \mathbb{R}$, the real numbers, then we say that (D, ϕ) is an *r-flow* if $1 \leqslant |\phi(e)| \leqslant r - 1$ or $\phi(e) = 0$ for each $e \in E(G)$, and $\sum_{e \in E^+(v)} \phi(e) = \sum_{e \in E^-(v)} \phi(e)$, for all $v \in V(G)$. Furthermore, if (D, ϕ) only uses elements of $\mathbb{Z}$, the integers, and $1 \leqslant |\phi(e)| \leqslant k - 1$, then (D, ϕ) is an integer flow and it is called a k-flow. The following theorem of Tutte relates integer flows and group flows to each other.

Theorem 25 ([130])**.** *Let A be an Abelian group. A graph admits a nowhere-zero A-flow if and only if it admits a nowhere-zero $|A|$-flow.*

If we reverse the orientation of an edge e and replace the flow value by $-\phi(e)$, then we obtain another nowhere-zero A-flow on G. Hence, if there exist an orientation of the edges of G such that G has a nowhere-zero A-flow, then G has a nowhere-zero A-flow for any orientation. Thus, the question for which values r (k) a graph admits a nowhere-zero r-flow (nowhere-zero k-flow) is a question about graphs, not directed graphs. The *circular flow number* of G is $\inf\{r | G$ admits a nowhere-zero r-flow$\}$, and it is denoted by $F_c(G)$. It is known that $F_c(G)$ is always a minimum and that it is a rational number (see Goddyn, Tarsi, and Zhang [50]). Furthermore, if $F_c(G)$ is an integer, say k, then G admits an integer nowhere-zero k-flow, see [120, Thm. 1.1]. If G does not admit any nowhere-zero flow, then we define $F(G) = \infty$. Let $F(G)$ be the smallest number k such that G admits a nowhere-zero k-flow. Clearly, $F_c(G) \leqslant F(G)$. In this context, Tutte conjectured that every bridgeless graph admits a nowhere-zero 5-flow (Conjecture 1).

Seymour [115] proved that every bridgeless graph admits a nowhere-zero 6-flow, see [29] for alternative proofs. Conjecture 1 is equivalent to its restriction on cubic graphs. The following result shows that it suffices to prove it for bridgeless cubic class 2 graphs (and with further easy arguments for snarks).

Theorem 26 (Tutte [129, 130])**.** *The following holds.*

(*i*) *A cubic graph G is bipartite if and only if $F_c(G) = 3$.*

(*ii*) *A cubic graph G is class 1 if and only if $F_c(G) \leqslant 4$.*

The following statement is a combination of results of the third author and Lukot'ka and Škoviera.

Theorem 27 ([89, 119])**.** *For every s of the interval $(3, 4)$, there is no cubic graph G with $F_c(G) = s$, and for every $r \in \{3\} \cup [4, 5]$, there is a cubic graph H with $F_c(H) = r$.*

Clearly, if $F_c(G) > 4$, then G is a bridgeless cubic class 2 graph. However, it is not clear whether such graphs with circular flow number close to 4 are somehow less complex than those with circular flow number close to or equal to 5. For instance, the Petersen graph has circular flow number 5, c.f. [119]. There are infinitely many snarks with circular flow number 5 (see Máčajova and Raspaud [92], Esperet, Mazzuoccolo and Tarsi [34] and Goedgebeur, Mattiolo and Mazzuoccolo [53]) and also with further properties (see Abreu, Kaiser, Labbate, and Mazzuoccolo [1]). From the results of Kochol [82, 86] and Mazzuoccolo and Steffen [100] it follows that a minimal counterexample to Conjecture 1 is a cyclically 6-edge connected snark with girth at least 11 and with oddness at least 6. So far no such snark is known.

Tutte [134] conjectured that every graph G with $F_c(G) > 4$ has a Petersen minor. This conjecture is still open, see Robertson, Seymour, and Thomas [111] for the current status of the work on that conjecture. However, in [110] (by the same authors) it is shown that every cubic graph with girth at least 6 has a Petersen minor. Hence, the existence of a Peterson minor is not a complexity measure for snarks with girth at least 6.

Jaeger and Swart [74] conjectured that if G is a cyclically k-edge connected cubic graph and $k > 6$, then $F(G) \leqslant 4$. Having in mind that the best known upper bound for the flow number is 6, the following result of Steffen [122] can also be seen as a first approximation to this conjecture.

Theorem 28 ([122])**.** *Let G be a cyclically k-edge connected cubic graph. If $k \geqslant \frac{5}{2}\omega(G)-3$, then G admits a nowhere-zero 5-flow.*

As commented in the Introduction, Kochol [80] used the method of superposition, which glues multibuses (that is, multipoles with different sets of terminals) together to construct snarks. As described in Section 2.2, such constructions strongly rely on the Klein four group $\mathbb{K}$ of Boole colorings. Then, Kochol noted that it is useful to consider nowhere-zero flows in $\mathbb{K}$. Indeed, since every element of the Klein group is self inverse, we do not have to take care of the orientation of the edges. Moreover, a nowhere-zero $\mathbb{K}$-flow on a cubic graph directly gives a 3-edge-coloring because of the isomorphism ϕ, between the sets of colors and Boole-colorings, such that $\phi(i) = \mathbf{1}_i$ for $i \in \{1, 2, 3\}$.

4.1 Flow resistance

By Theorem 26 we have that a bridgeless cubic graph G is not 3-edge-colorable if and only if $F(G) > 4$. However, every cubic graph has a subgraph H such that $E(H)$ is the support of a 4-flow. The following parameter measures how far apart a cubic graph is from admitting a nowhere-zero 4-flow. Let G be a cubic graph and $r_f(G) = \min\{|E(G) - \mathrm{supp}(D, \phi)| : (D, \phi) \text{ is a 4-flow on } G\}$. We call $r_f(G)$ the flow-resistance of G. This parameter was introduced by the authors during the preparation of the present paper (see also [40]) and a bit later also studied by Máčajová and Škoviera in [96].

For the Petersen graph P we have $r_f(P) = \gamma_2(P) = 1$, $r(P) = \omega'(P) = \omega(P) = 2$, and $\mu_3(P) = 3$. Recall that $\gamma_2(G) = \min\{|M_1 \cap M_2| : M_1 \text{ and } M_2 \text{ are 1-factors of } G\}$.

Proposition 29. *If G is a bridgeless cubic graph, then $r_f(G) \leqslant \gamma_2(G)$.*

Proof. Let M_1 and M_2 be two 1-factors such that $|M_1 \cap M_2| = \gamma_2(G)$, and F_1 and F_2 be the complementary 2-factors, respectively. For $i \in \{1,2\}$ let (D_i, ϕ_i) be a nowhere-zero i-flow on F_i. The sum of (D_1, ϕ_1) and (D_2, ϕ_2) is a 4-flow (D, ϕ) on G with $|E(G) - \mathrm{supp}(D, \phi)| = \gamma_2(G)$. □

In [124] Steffen showed that if G is a cyclically 6-edge-connected cubic graph with $\gamma_2(G) \leqslant 2$, then G admits a nowhere-zero 5-flow. This result was generalized by Li, Hou, Hong, and Ding [88] to cyclically 6-edge-connected cubic graphs having two spanning parity subgraphs that intersect in at most two edges. In view of Theorem 17, the bound of Proposition 29 might not be the best upper bound, see Conjecture 51 in Section 5.

Jaeger [73] defined a graph G to be a *deletion nowhere-zero* 4*-flow graph* if it does not admit a nowhere-zero 4-flow but it has an edge e, such that $F(G - e) \leqslant 4$. He remarked that every deletion nowhere-zero 4-flow graph admits a nowhere-zero 5-flow. Note, that $F(G - e)$ is not always smaller than $F(G)$. For instance $F(K_{3,3}) = 3$, but $K_{3,3} - e$ is isomorphic to the complete graph on four vertices, K_4, with two subdivided non-adjacent edges, and $F(K_4) = 4$; that is, $F(K_{3,3} - e) > F(K_{3,3})$, for all $e \in E(K_{3,3})$. A cubic graph G is 4*-flow-critical* if it does not admit a nowhere-zero 4-flow but $G - e$ has a nowhere-zero 4-flow for every $e \in E(G)$. It is easy to see that the flower snarks are 4-flow-critical. According to [118] we say that a bridgeless cubic graph G is *edge-irreducible* if for any two adjacent vertices $x, y \in V(G)$, the graph $G - \{x, y\}$ cannot be extended to a bridgeless cubic class 2 graph by adding edges.

Theorem 30. *Let G be a bridgeless cubic graph. The following three statements are equivalent.*

- *(i) G is* 4*-flow critical.*
- *(ii) G is cyclically 4-edge-connected and for every $e \in E(G)$ there is a 2-factor F_e of G with precisely two odd circuits which are connected by e.*
- *(iii) G is edge-irreducible.*

Proof. The equivalence between items *(ii)* and *(iii)* was proved by Steffen in [118]. It remains to prove the equivalence of the first two statements. Let G be 4-flow-critical and $e = xy$. Let G^* be obtained from $G - e$ by suppressing the two divalent vertices, and let e_x and e_y be the two edges where the divalent edges x and y are suppressed. The graph $G - e$ admits a nowhere-zero 4-flow. With the Klein four group as flow values it follows that G^* is 3-edge-colorable. Hence, there is a 2-factor F, where all cycles are even since G is 3-edge-colorable, which contains e_x and e_y. Subdividing e_x and e_y by x and y to adding e to reconstruct G gives a 2-factor F_e of G with two odd circuits which are connected by e. It is easy to see that G is cyclically 4-edge-connected.

If condition *(ii)* is satisfied, the G^* has an even 2-factor. Therefore G^* and $G - e$ have a nowhere-zero 4-flow. □

Let G be a cubic graph and u, v be two vertices of G. The graph $G/\{u, v\}$ is the graph which is obtained from G by identifying the vertices u and v. If u and v are adjacent and $e = uv$, then G/e arises from $G/\{u, v\}$ by removing the loop resulting from e. If $E = \{e_1, \ldots, e_k\}$ is a subset of $E(G)$, then $G/E = (\cdots((G/e_1)/e_2)\cdots/e_k)$. We also say that G/e and G/E are obtained from G by contracting e and E, respectively.

Máčajová and Škoviera [96] remarked that da Silva and Lucchesi [25] and da Silva, Pesci, and Lucchesi [26] introduced the following definition. A cubic class 2 graph G is 4-edge-critical if G/e admits a nowhere-zero 4-flow for every $e \in E(G)$. Then, da Silva and Lucchesi [25] proved that a cubic graph G is 4-edge-critical if and only if $G - e$ admits a nowhere-zero 4-flow for every $e \in E(G)$. Máčajová and Škoviera [96] proved the following far-reaching generalization of this result.

Theorem 31 ([96])**.** *Let G be a cubic class 2 graph and u and v be two distinct vertices of G. The following statements (i)–(iii) are equivalent. If, in addition, u and v are adjacent and joined by an edge e, then all statements (i)–(vi) are equivalent.*

(*i*) $\chi'(G - \{u, v\}) = 3$.

(*ii*) $G - \{u, v\}$ *admits a nowhere-zero* 4*-flow.*

(*iii*) $G/\{u, v\}$ *admits a nowhere-zero* 4*-flow.*

(*iv*) $G - e$ *admits a nowhere-zero* 4*-flow.*

(*v*) G/e *admits a nowhere-zero* 4*-flow.*

(*vi*) $\chi'(G^*) = 3$, *where G^* is obtained from $G - e$ by suppressing the two divalent vertices of $G - e$.*

From Theorem 31 a further characterization of 4-flow-critical graphs can be deduced.

Theorem 32 ([96])**.** *A cubic class* 2 *graph G is* 4*-edge-critical if and only if $\chi'(G - \{u, v\}) = 3$ for any two adjacent vertices u, v of G.*

Carneiro, da Silva, and McKay [19] generated all 4-edge-critical snarks with at most 36 vertices. The operation of contracting edges allows to define a further measure of edge-uncolorability of cubic graphs in terms of flows. For a cubic graph G let $r'_f(G)$ be the minimum number of edges that have to be contracted in G to obtain a graph that admits a nowhere-zero 4-flow.

Theorem 33. *Let G be a graph with $F(G) > 4$. A set E is a minimal set of edges such that G/E admits a nowhere-zero* 4*-flow if and only if E is a minimal set of edges such that $G - E$ admits a nowhere-zero* 4*-flow.*

Proof. For trivial reasons we may assume that G contains a circuit. By Theorem 25 it suffices to prove the statement for $\mathbb{Z}_4$-flows. The theorem is true if the following two statements hold.

(*i*) If $E \subseteq E(G)$ is a minimal set such that $G - E$ admits a nowhere-zero 4-flow, then G/E admits a nowhere-zero 4-flow.

(ii) If $E \subseteq E(G)$ is a minimal set such that G/E admits a nowhere-zero 4-flow, then $G - E$ admits a nowhere-zero 4-flow.

Proof of (i): Let E be a minimal set of edges such that $G-E$ admits a nowhere-zero 4-flow (D, ϕ). Clearly, (D, ϕ) can be considered as 4-flow on G with $\text{supp}(D, \phi) = E(G)-E$. Hence, (D, ϕ) induces a nowhere-zero 4-flow on G/E.

Proof of (ii): We proceed by induction on $|E|$.

$|E| = 1$: Let $E = \{e\}$, $e = xy$ and z be the vertex of G/e which is obtained by contracting e. For $w \in \{x, y\}$ let $E^w_{G/e}(z) = \{zv : wv \in E(G) \text{ and } v \neq w\}$, and (D, ϕ) be a nowhere-zero 4-flow on G/e. Clearly, $\sum_{e \in E^x_{G/e}(z)} \phi(e) = \sum_{e \in E^y_{G/e}(z)} \phi(e)$, and if $\sum_{e \in E^x_{G/e}(z)} \phi(e) \neq 0$, then it is easy to see that (D, ϕ) can be extended to a nowhere-zero 4-flow on G, a contradiction. Hence, $\sum_{e \in E^x_{G/e}(z)} \phi(e) = 0$, and (D, ϕ) induces a nowhere-zero 4-flow on $G - e$.

$|E| = k > 1$: Let $e \in E$, $E' = E - \{e\}$, and $G' = G/E'$. Since $G'/e = G/E$, it follows that $F(G') > 4$ and that G'/e has a nowhere-zero 4-flow. As the statement is true for $k = 1$ (by induction hypothesis) it follows that $G' - e$ has a nowhere-zero 4-flow.

Note that $G' - e = (G - e)/E'$. We claim that E' is a minimal set of edges of $G - e$ such that $(G - e)/E'$ has a nowhere-zero 4-flow. Since the graph property of admitting a nowhere-zero 4-flow is invariant under contraction, it follows that if there is an edge set E'' of $G - e$ with $|E''| < |E'|$ such that $(G - e)/E''$ admits a nowhere-zero 4-flow, then there is a set $E^* = E'' \cup \{e\}$ with $|E^*| < |E|$ such that G/E^* admits a nowhere-zero 4-flow, which is a contradiction to the minimality of $|E|$.

Since $|E'| < k$, it follows by induction hypothesis, that $(G-e)-E'$ has a nowhere-zero 4-flow, and therefore, $G - E$ has a nowhere-zero 4-flow. □

Corollary 34. *If G is a cubic graph, then $r'_f(G) = r_f(G)$.*

As remarked in [96], further notions of criticality of cubic class 2 graphs are considered in [28, 112].

4.2 Extensions

Another parameter which measures the complexity of a cubic graph is due to Jaeger [73]. A graph G is a *nearly nowhere-zero 4-flow graph* if it is possible to add an edge in order to obtain a graph admitting a nowhere zero 4-flow. Note that a deletion nowhere-zero 4-flow graph is also a nearly nowhere-zero 4-flow graph. In [123] Steffen extended this approach to nowhere-zero r-flows ($r \in \mathbb{Q}$). Jaeger's approach can be generalized as follows. Let $\Phi^+_k(G)$ be the minimum number of edges that have to be added to a cubic graph G in order to obtain a graph with nowhere-zero k-flow ($k \in \{3, 4, 5\}$). This parameter is studied by Mohar and Škrekovski in [103].

Theorem 35 ([103])**.** *Let G be a loopless cubic graph. If $|V(G)| = n$, then $\Phi^+_3(G) \leqslant \lfloor \frac{n}{4} \rfloor$ and $\Phi^+_4(G) \leqslant \lceil \frac{1}{2} \lfloor \frac{n}{5} \rfloor \rceil$.*

We will give some upper bounds for Φ^+_4 in terms of oddness and flow resistance.

Theorem 36. *If G be a bridgeless cubic graph, then $\Phi_4^+(G) \leqslant \min\{\frac{1}{2}\omega(G), r_f(G)\}$.*

Proof. Let G be a cubic graph with $\omega(G) = 2t$. Let F be a 2-factor of G with $\omega(G)$ odd circuits. Let G/F be the multigraph which is obtained from G by contracting the elements of F to vertices. Note that chords in circuits of F will become loops in G/F. Clearly, every odd circuit of F corresponds to a vertex of odd degree in G/F. Let $C_1, \dots, C_{2t}$ be the odd circuits of F and $c_i \in V(C_i)$. Let $G^* = (G + \{c_{2i-1}c_{2i} : i \in \{1, \dots, t\}\})/F$. The multigraph G^* is Eulerian and therefore, it has a nowhere-zero $\mathbb{Z}_2$-flow. Hence, G has a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow, and therefore a nowhere-zero 4-flow. Hence $\Phi_4^+(G) \leqslant t = \frac{1}{2}\omega(G)$.

Let (D, ϕ) be a nowhere-zero 4-flow with $|\operatorname{supp}(D, \phi)| = |E(G)| - r_f$. Replace every edge $e \notin \operatorname{supp}(D, \phi)$ by a double-edge to obtain a graph G'' which admits a nowhere-zero 4-flow. □

Mohar and Škrekovski also studied the parameter Φ_5^+. Since every bridgeless cubic graph with oddness 2 admit a nowhere-zero 5-flow it follows as above that $\Phi_5^+(G) \leqslant \min\{\frac{1}{2}\omega(G) - 1, r_f(G) - 1\}$.

5 Final remarks and conjectures

5.1 Partial results on the hard conjectures

Besides the objective to gain new insight into the structure of snarks, complexity measures of bridgeless cubic class 2 graphs also allow us to deduce partial results with respect to the aforementioned conjectures. In the following we list the current status of the results with respect to the conjectures formulated in the Introduction.

Theorem 37. *If G is a possible minimum counterexample to Conjecture 1* (*5-flow conjecture*)*, then*

(*i*) *G is a cubic graph [115].*

(*ii*) *G is cyclically 6-edge connected [82].*

(*iii*) *the cyclic connectivity of G is at most $\frac{5}{2}\omega(G) - 4$ [122].*

(*iv*) *G has girth at least 11 [86].*

(*v*) *G has oddness $\omega(G) \geqslant 6$. [100]*

Theorem 38. *If G is a possible minimum counterexample to Conjecture 2* (*Berge conjecture*)*, then*

(*i*) *$\omega(G) \geqslant 2$.*

(*ii*) *if G does not have a non-trivial 3-edge-cut, then $\mu_3(G) \geqslant 5$ [125].*

As far as we know, there are no partial results for Conjecture 3 (Berge-Fulkerson conjecture), besides the trivial ones that a possible minimum counterexample is cyclically 4-edge connected and it has girth at least 5.

Theorem 39. *If G is a possible minimum counterexample to Conjecture 4 (Fan-Raspaud conjecture), then*

(i) $\omega(G) \geqslant 4$ *[95].*

(ii) $\mu_3(G) \geqslant 7$ *[125].*

(iii) $\chi_e(G) \geqslant 5$.

Theorem 40. *If G is a possible minimum counterexample to Conjecture 5 (5-cycle-double-cover conjecture), then*

(i) $\omega'(G) \geqslant 6$ *[66].*

(ii) $\chi_e(G) \geqslant 5$ *[125].*

5.2 Conjectures and problems for bridgeless cubic uncolorable graphs

Petersen graph

We start with some conjectures and problems which are related to the Petersen graph.

Problem 41. ([61]) Is the Petersen graph the only cyclically 5-edge-connected snark with excessive index 5?

The Petersen graph has circular flow number 5, see for example [119]. All other known snarks with circular flow number 5 have cyclic connectivity 4 (see Esperet, Mazzuoccolo, and Tarsi [34], Máčajová and Raspaud [92]), and Goedgebeur, Mattiolo, and Mazzuoccolo [53]).

Problem 42. Is the Petersen graph the only cyclically 5-edge-connected snark with circular flow number 5?

A bridgeless cubic class 2 graph G is *vertex-irreducible*, if for any two vertices $x, y \in V(G)$, the graph $G - \{x, y\}$ cannot be extended to a bridgeless cubic class 2 graph by adding edges. Notice that, according to the definitions in Section 2, if G has two disjoint conflicting zones, then it is neither edge- nor vertex-irreducible.

Conjecture 43. ([118]) The Petersen graph is the only vertex-irreducible bridgeless cubic class 2 graph.

In [118] it is proved that a vertex-irreducible cubic graph is cyclically 5-edge connected. Conjecture 43 is true for all bridgeless cubic graphs with at most 36 vertices [51].

Bridgeless cubic uncolorable graphs

Conjectures on general properties of bridgeless cubic class 2 graphs are the following.

Conjecture 44 ([74])**.** If G is a bridgeless cubic class 2 graph, then its cyclic connectivity is at most 6.

The following problem relates problems 41 and 42 to each other.

Problem 45 ([1])**.** Let G be a bridgeless cubic graph. Is it true that if $\chi_e(G) = 5$, then G has circular flow number 5?

Goedgebeur [55] computationally showed that the above statement holds for bridgeless cubic class 2 graphs with girth at least 4 up to at least 32 vertices.

We propose the following conjecture:

Conjecture 46. There is $\epsilon > 0$, such that $\chi_e(G) \leqslant 4$ for every cubic graph G with circular flow number smaller than $4 + \epsilon$.

Measures of edge-uncolorability

We start with three specific problems of Lukot'ka and Mazák [91].

Problem 47 ([91])**.** Does there exist a cubic graph with weak oddness 4 and oddness at least 6?

Problem 48 ([91])**.** For which integers $k \geqslant 3$ does there exist a cyclically k-edge-connected cubic class 2 graph G with $\omega(G) \neq \omega'(G)$?

Problem 49 ([91])**.** In a 3-edge-connected cubic class 2 graph, can the expansion of a vertex into a triangle decrease the oddness?

A graph G is *hypohamiltonian* if it is not hamiltonian but $G - v$ is hamiltonian for every vertex $v \in V(G)$. For instance, the Petersen graph and the flower snarks are hypohamiltonian. Máčajová and Škoviera [94] showed that cubic hypohamiltonian class 2 graphs are cyclically 4-edge-connected and have girth at least 5 (that is, they are snarks), and that there are cyclically 6-edge-connected hypohamiltonian snarks with girth 6. If G is a hypohamiltonian snark, then $r(G) = \omega(G) = 2$, and G satisfies Conjecture 4. If the following conjecture is true, then hypohamiltonian snarks have excessive index at most 5, that is, they satisfy Conjecture 2.

Conjecture 50 ([125])**.** Let G be a cubic class 2 graph. If G is hypohamiltonian, then $\mu_3(G) = 3$.

Conjecture 50 is verified for all hypohamiltonian class 2 graphs with at most 36 vertices by Goedgebeur and Zamfirescu [54].

Conjecture 51. If G is a bridgeless cubic graph, then $r_f(G) \leqslant r(G)$.

connectivity	lower bound	current upper bound
2	5.41	7.5
3	5.52	9
4	5.52	13
5	5.83	25
6	7	99

Table 1: Lower and upper bounds for the oddness ratio [90]

Let $\mathcal{S}$ be the set of bridgeless cubic class 2 graphs, and τ be one of the complexity measures which are discussed in this paper. Let

$$q(\tau, k) = \max\left\{\frac{k}{|V(G)|} : G \in \mathcal{S} \text{ and } \tau(G) \geqslant k\right\}.$$

Problem 52. Let $k \geqslant 1$ be an integer. Determine $q(\tau, k)$.

Problem 53. What is the largest value c such that $q(\tau, k) \geqslant c$ for all $k > 0$?

These two questions were asked by Hägglund [61] for the oddness. Clearly, for the oddness it suffices to consider even numbers. The Petersen graph is the smallest snark. Hence, $q(r_f, 1) = q(\gamma_2, 1) = \frac{1}{10}$ and $q(r, 2) = q(\omega, 2) = q(\omega', 2) = \frac{1}{5}$, and $q(\mu_3, 3) = \frac{3}{10}$.

The smallest snark with oddness 4 has at least 38 vertices (see Brinkmann, Goedgebeur, Hägglund, and Markström [13]) and at most 44 vertices (see Lukot'ka, Máčajová, Mazák, and Škoviera [90]). These results are further refined by Goedgebeur, Máčajová, and Škoviera [52], who showed that smallest snarks with oddness 4 and cyclic connectivity 4 have order 44, but note that they cannot exclude the existence of a snark of oddness 4, cyclic connectivity 5 and order less than 44. Hence, $\frac{2}{19} \leqslant q(\omega, 4) \leqslant \frac{1}{11}$. Hägglund [61] proved that $q(\omega, k) \geqslant \frac{1}{18}$ and for multiples of 6 he improved this result to $q(\omega, 6k) \geqslant \frac{1}{15}$.

In [90] the reciprocal parameter *oddness ratio* $|V(G)|/\omega(G)$ and the *resistance ratio* $|V(G)|/r(G)$ are also studied. They adopt the asymptotical approach of Steffen [121] for a sophisticated analysis of these parameters in the case of cyclically k-edge-connected snarks, $k \in \{2, \ldots, 6\}$. More precisely, they study the parameters

$$A_\omega = \lim_{|V(G)| \to \infty} \inf \frac{|V(G)|}{\omega(G)} \qquad \text{and} \qquad A_r = \lim_{|V(G)| \to \infty} \inf \frac{|V(G)|}{r(G)}.$$

In [90], Lukot'ka, Máčajová, Mazák, and Škoviera gave lower and upper bounds for the oddness ratio, see Table 1. Moreover, the lower bounds were improved by Candráková and Lukot'ka in [15].

Conjecture 54 ([90])**.** Let G be a snark. If $\omega(G) = 4$, then $|V(G)| \geqslant 44$.

If Conjecture 54 is true, then it is best possible as remarked above.

Conjecture 55 ([90])**.** If G is a bridgeless cubic class 2 graph, then

$$|V(G)| \geqslant \left(7 + \frac{1}{2}\right)\omega(G) - 5.$$

Problem 56. Let τ_1 and τ_2 be two complexity measures such that $\tau_1(G) \leqslant \tau_2(G)$ for all cubic bridgeles graphs G. Does there exist a function f such that $\tau_2(G) \leqslant f(\tau_1(G))$?

That question was studied in [121], where it is asked whether there is a function f such that $\omega(G) \leqslant f(r(G))$ for each bridgeless cubic graph. Furthermore it is proved that there is no constant c such that $1 \leqslant c < 2$ and $\omega(G) \leqslant cr(G)$, for every bridgeless cubic class 2 graph G. We conjecture the following two statements to be true.

Conjecture 57. If G is a bridgeless cubic graph, then $\omega'(G) \leqslant 2r(G)$.

Conjecture 58. If G is a bridgeless cubic graph, then $\omega(G) \leqslant 2r(G)$.

It might be interesting to extend the definition of Mohar and Škrekovski in [103]. Let $\Phi_r^+(G)$ be the minimum number of edges that have to be added to a cubic graph G in order to obtain a graph with nowhere-zero r-flow ($r < 6$).

Problem 59. Does there exist $s \in (4, 5)$ such that $\Phi_r^+(G) = \Phi_4^+(G)$ for all $r \in (4, s)$?

Note added in proof: Conjecture 58 is disproved by I. Allie in [5]. That paper also gives affirmative answers to Problem 47 and to Problem 48 for the cases $k \in \{3, 4\}$.

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