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A weakly nonlinear Love hypothesis for longitudinal waves in elastic rods

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Abstract:	When investigating nonlinear wave propagation in slender hyperelastic rods, the usual stance is to construct a reduced kinematics and then derive a system of coupled nonlinear PDEs for the unknown functions. To make further analytical progress, the linear Love hypothesis, that connects longitudinal and transversal strain, is often reverted to. The viability of this assumption, that was originally proposed within the framework of linear elasticity, remains uncertain. In this paper, \red{a refined Love hypothesis is derived in the weakly nonlinear regime by slow-time perturbation of the motion equations. For the sake of illustration, the simplest two-modal setting is adopted. This refined Love assumption is not equivalent, not even in principle, to that derived by Porubov (1993) by accommodating for the free boundary conditions at the rod mantle. Besides, the perturbation process lends a uni-dimensional model equation which parallels that obtained by Ostrovskii (1977) with the help of the linear Love hypothesis, with yet different coefficients in the dispersive term. The corresponding longitudinal motion is compared numerically against the solution of the bimodal nonlinear system and the transversal motion is contrasted with the linear Love hypothesis. For both motions, excellent agreement is found and the quality of the approximation extends to a wide range of values for the small parameter. Finally, within this setting, the corresponding unimodal Lagrangian is also derived, and it remains accurate regardless of the first correction terms to the linear Love hypothesis.
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MODENA E REGGIO EMILIA

April 9th, 2024

Dear Editor,

Please find enclosed, the revised version of the research paper NLM-D-24-00069

"A weakly nonlinear Love hypothesis for longitudinal waves in elastic rods"
by
A. Nobili

submitted for consideration in the International Journal of Non-Linear Mechanics, for the special issue "Multiscale and microstructure-inspired constitutive models for soft materials".

The paper has been revised according to the Reviewers' suggestions. Changes are highlighted in red color.
The material is original and it has not been submitted for publications in other journals.
The author is aware of the ethical guidelines put forward by the journal and fully adhere to it.
The author is in no conflict of interest regarding the material presented in this paper.

Best regards,
Andrea Nobili
(Corresponding author)

Research paper NLM-D-24-00069

A weakly nonlinear Love hypothesis for longitudinal waves in elastic rods

Reply to Reviewer #1

The author is grateful to the reviewer for his/her scrutiny of the manuscript and for his/her support.

Following upon the Reviewer's suggestion, in the Conclusion section the opportunity to extend the present analysis to rods that are reinforced with fibres, possibly made of soft material. For the Reviewer's convenience, corrections in the manuscript are marked in red.

Reply to Reviewer #2

The author is much appreciative for the honest assessment of the paper and for the many suggestions that are proposed by the Reviewer, especially in terms of literature review. In fact, it clearly appears that the Reviewer competence adds significant value to this work and I tried to make the most out of it. The author has put his best effort in addressing the Reviewer's objections with a fair and honest mindset, while collecting as much literature background as he could manage. The manuscript has been modified according to the Reviewer's suggestion as hereinafter detailed:

1. Overall, the approach is similar to that in

   L. Ostrovsky, A. Sutin, 1977. Nonlinear elastic waves in rods. J. Appl. Math. Mech. (PMM) 41, 543-549.

   In the 1977 paper the full Lagrangian of the problem was simplified using the Love hypothesis, and the reduced Lagrangian was used to derive a two-directional Boussinesq equation, and then the KdV approximation for uni-directional waves..

   Answer.

   Indeed, in this paper the Lagrangian approach is undertaken, much like in Ostrovsky, A. Sutin (1977) or Samsonov (2001) or Porubov (2003) or Dai and Fan (2004) or many more, but other than this the similarity is not great. In fact, it was precisely the point of this paper NOT to adopt the Love hypothesis firsthand but, rather, to (possibly) motivate it, specifically by a slow time perturbation. Indeed, the Love hypothesis is derived from the two-modal kinematics and in this sense it is no longer a hypothesis. Of course, in a weakly nonlinear setting the leading order solution is the linear Love hypothesis and therefore a two-directional Boussinesq equation is arrived at similarly to Ostrovsky, A. Sutin's. Yet, coefficients in the dispersive term (and also in the nonlinear term but this may be a typo in Ostrovsky, A. Sutin's paper) are different.

2. This was then followed by research in A. Samsonov's group, most notably Porubov and Samsonov, see

   A. M. Samsonov, Strain Solitons in Solids and How to Construct Them (Monogr. Surv. Pure Appl. Math., Vol. 117), Chapman and Hall CRC, Boca Raton, Fla. (2001).

   A. V. Porubov, Amplification of Nonlinear Strain Waves in Solids (Ser. Stab. Vibr. Control of Systems Ser. A, Vol. 9), World Scientific, Singapore (2003).

   Porubov and Samsonov have refined the Love hypothesis, using the weakly-nonlinear boundary conditions of the problem formulation, and used this weakly-nonlinear Love hypothesis in order to derive a refined model.

Answer

As pointed out by the Reviewer, the motivation behind this paper parallels that put forward in Samsonov (2001), Sec.2.3.2, where the need for a refined Love assumption is expressed. However, as confirmed by the Reviewer, in Samsonov's work this need is satisfied by matching the free boundary conditions, that is an altogether different approach than it is considered here. Indeed, this paper makes use of slow time perturbation of the motion equations and utterly disregars the free boundary conditions. In fact, it provides an alternative and different expression for the refinement of the Love hypothesis. This is now stressed in the manuscript to avoid confusion and a comparison is drawn. It should be also stressed that the fact that the free boundary conditions are disregarded is not a problem by itself, because, following Samsonov (2001), "generally speaking the identity is not required because an asymptotic solution is to be found". It is therefore important to emphasize that indeed a refined Love assumption is derived in Samsonov (2001) and yet this refinement is very different from the one that is proposed here, which purposefully makes no use of the boundary conditions for it aims to remain within a fully asymptotic solution.

3. Moreover, Garbuzov et al. (2019), mentioned in the paper, have derived Boussinesq-type equations using the multiple-scale analysis of the full problem formulation for a Murnaghan material, having again derived a weakly-nonlinear generalisation of the Love hypothesis.

   Answer

   Garbuzov et al. (2019) proceed in a similar way as Dai and Fan (2004) (and to an extent as Samsonov (2001) and Prubov (2003)) by introducing a multi-modal representation in the radius, then enforcing the motion equation in an asymptotic sense (i.e. term-wise) and finally accounting for force-type boundary conditions, either asymptotically or to linear terms. A second derivation is also presented which solves a set of nonlinear ODEs obtained by asymptotic solution of the boundary conditions.

   Either way, this approach differs under many respects from that considered in this paper (for example and above all the fact that no mention is here made of the boundary conditions) and, indeed, it provides a model equation (in fact two of them) different than Ostrovsky, A. Sutin's in the dispersive term. Still, two interesting observations emerge from comparing Garbuzov et al. (2019)'s work with this paper. First, as observed by the Reviewer, despite its simplicity, this paper presents results that are qualitatively similar to those of other, much more involved works. Second, it appears that, to within this level of approximation, no need is felt to introduce the fulfillment of the boundary conditions.

4. The difference with the aforementioned studies is that in the present manuscript approximations have been built as a two-step process. First, a two-modal approximation (1) was assumed, hugely simplifying the problem at hand, and then the subsequent analysis has followed that in Dai and Fan (2004) and Garbuzov et al. (2019). It will be fair to comment that the derivation of the KdV equation in this hugely simplified setting could be performed by a student.

   Answer

   As stressed in the manuscript, "Spotlight is set on elucidating the procedure as well as on assessing the quality of the approximation in the simplest setting, leaving more sophisticated mechanical models for future developments." Indeed, the simplest possible setting is adopted, with the aim of communicating the method deprived of unnecessary details. For the same reason, the introduction of the boundary conditions is excluded, because, as well-known and also pointed out by Samsonov (2001), they are not required in an asymptotic solution. Clearly, this by no means precludes that more involved situations may be addressed by this approach, also accounting for the boundary conditions. It is nonetheless emphasized that none of the literature contributions, to the best of my knowledge, rely on slow-time perturbation to reduce one of the motion equations to a Love-like assumption. Instead, the general approach is to either introduce some form of Love hypothesis, somehow guessed, or to adopt the reductive perturbation method to the full system, like, for example, in Dai and Fan (2004).

5. Finally, numerical comparisons were made between solutions of the two-modal approximation and the reduced model based on the Love hypothesis, showing good agreement in the range of validity of the model.

   This last part should be put in the context of the study in

   F.E. Garbuzov, Y.M. Beltukov, K.R. Khusnutdinova, Longitudinal bulk strain solitons in a hyperelastic rod with quadratic and cubic nonlinearities, Theor. Math. Phys. 202 (2020) 319-333,

   where numerics was performed within the scope of the full nonlinear problem formulation for a Murnaghan material, and results of the weakly-nonlinear modelling with the KdV and extended KdV (i.e. higher-order model) were compared to that solution, showing excellent agreement for weakly-nonlinear waves, with the extended KdV equation extending the range of validity to the waves of moderate amplitude.

   Answer

   I thank the Reviewer for this reference. The procedure followed in Garbuzov et al (2020) exactly parallels that in Garbuzov et al. (2019) and therefore, on the basis of the previous comments, it is significantly different than that used in this paper. In terms of numerics, the soliton solution is investigated that is not the focus of interest of this paper, which instead compares the longitudinal motion against the wave equation and the nonlinear Mindlin-Herrmann system and the transversal motion against the linear Love hypothesis (which is the focus of this paper). None of the above is related in Garbuzov et al (2020). Still, this paper is now referred to in the manuscript.

6. Minor points: Boussinesq equation can not be integrated exactly, this wording is incorrect. We can construct certain classes of exact solutions, e.g. travelling waves.

   Answer

   This is indeed correct. The incorrect wording is taken verbatim from Soerensen (1984) “The BE contains Uxxxx and U^2xx and is exactly integrable”. This is now better specified in the paper.

7. The other minor points raised by the Reviewer have been dealt with.

Finally, some rearrangements, rewriting and error corrections have been introduced in the manuscript.

For the Reviewer’s convenience, corrections in the manuscript are marked in red color.

Declaration of interests

☒ The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

☐ The author is an Editorial Board Member/Editor-in-Chief/Associate Editor/Guest Editor for *[Journal name]* and was not involved in the editorial review or the decision to publish this article.

☐ The authors declare the following financial interests/personal relationships which may be considered as potential competing interests:

Highlights of the paper
A weakly nonlinear Love hypothesis for longitudinal waves in elastic rods

- A weakly nonlinear Love hypothesis is derived from a two-modal kinematics by the method of multiple scales;
- The corresponding model equation is the Boussinesq equation;
- Comparison with numerical integration reveals that the quality of the approximation is excellent;
- The same unimodal Lagrangian is derived irrespective of the correction to the linear Love hypothesis.

A weakly nonlinear Love hypothesis for longitudinal waves in elastic rods

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Abstract

When investigating nonlinear wave propagation in slender hyperelastic rods, the usual stance is to construct a reduced kinematics and then derive a system of coupled nonlinear PDEs for the unknown functions. To make further analytical progress, the linear Love hypothesis, that connects longitudinal and transversal strain, is often reverted to. The viability of this assumption, that was originally proposed within the framework of linear elasticity, remains uncertain. In this paper, a refined Love hypothesis is derived in the weakly nonlinear regime by slow-time perturbation of the motion equations. For the sake of illustration, the simplest two-modal setting is adopted. This refined Love assumption is not equivalent, not even in principle, to that derived by Porubov and Samsonov (1993) by accommodating for the free boundary conditions at the rod mantle. Besides, the perturbation process lends a uni-dimensional model equation which parallels that obtained by Ostrovskii and Sutin (1977) with the help of the linear Love hypothesis, with yet different coefficients in the dispersive term. The corresponding longitudinal motion is compared numerically against the solution of the bimodal nonlinear system and the transversal motion is contrasted with the linear Love hypothesis. For both motions, excellent agreement is found and the quality of the approximation extends to a wide range of values for the small parameter. Finally, within this setting, the corresponding unimodal Lagrangian is also derived, and it remains accurate regardless of the first correction terms to the linear Love hypothesis.

Keywords: Love hypothesis, Murnaghan materials, Multiple scales, Nonlinear

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longitudinal waves in rods

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1. Introduction

When developing models to describe wave propagation in elastic rods, the starting point is usually represented by a restricted kinematics in which some unknown functions are introduced expressing the longitudinal and the radial strain in the body, see Shatalov et al. (2011) and references therein. Besides the classical wave equation, the simplest case in point is the so-called bimodal representation, where this restricted kinematics takes the form of the Navier–Bernoulli (NB) assumption of plane cross-sections, in which only two functions are used, respectively W for longitudinal and RU for transversal displacement, R being the rod radial position. In the special case $U = 0$, this procedure produces the classical non-dispersive wave equation. Moving from some original intuitions of Rayleigh (1894), this approach was later refined by Love (1927) to encompass for transversal inertia through the well-know *Love hypothesis*, $U = -\nu_0 W_{,Z}$, that relates longitudinal and transversal strain via the Possion's ratio ν_0. In Love's original formulation (Love, 1927, Sec.278), this connection is purposefully assumed *for inertia terms only*, in the so-called Rayleigh-Love theory (on this point see also Hutchinson and Percival (1968)). This ad-hoc procedure, which proves very effective, was later relaxed by Bishop (1952) to include shear deformations, thus leading to the Bishop-Love theory. It may be worth noting that the opposite pathway was taken by Sørensen et al. (1984) to derive the improved Boussinesq equation in a simplified weakly nonlinear setting where transversal inertia is neglected in favor of shear deformation. Mindlin (1951) presents, for the first time, a full bimodal plane section approach and the resulting pair of coupled PDEs is in fact named the Mindlin-Herrmann system (Graff, 2012, Sec.8.3.3). In general, the bimodal approach is especially attractive for its simplicity, although it remains limited in that it cannot accommodate for all three boundary conditions on the mantle. Later, McNiven and Perry (1962) remediated this shortcoming at the expense of introducing extra degrees of freedom, by extending the transversal kinematics through successive odd powers of the radius R multiplied by extra unknown functions.

In the linear framework, the quality of these approximations is usually assessed by comparison with the well-known Pochhammer (1876) solution of the 3D elasticity problem for a rod with circular cross-section. Indeed, as already observed by Graff (2012), the Love hypothesis is the leading order approximation of the Pochhammer-Chree solution in the long-wave low-frequency (LWLF) approximation. This specific feature, that is further detailed in Nobili and Saccomandi (2024), cannot be pursued in the nonlinear framework because of the insurmountable difficulties attached to developing any analytical solution within the full 3D theory. In fact, avoiding such difficulties is precisely the main reason why reduced-dimensional models are introduced in the first place. Still, while facing the formidable task of solving complicated systems of coupled nonlinear PDEs, many contributions appear in the literature that appeal to the original linear Love hypothesis outside the linear framework where it properly belongs.

Sørensen et al. (1984) numerically study soliton interaction in nonlinear elastic rods under many approximations that include the original Love hypothesis. His interest lies in developing soliton solutions of nearly integrable systems. Wright (1985) develops a purely axiomatic (in his words "intrinsic") 1D theory of straight elastic incompressible rods and is able to connect longitudinal and transversal deformation through the incompressibility constraint. As pointed out in Amendola and Saccomandi (2021), the incompressibility constraint is compatible with the Love's hypothesis only to leading order, while the analysis in Wright (1985) is extended to the first correction term in the small deformation, i.e. it is weakly nonlinear. It is noted that a similar "intrinsic" 1D theory was used very recently by Li et al. (2023) to derive nonlinear dispersion curves.

Much research on the topic of nonlinear waves in rods is contributed by Samsonov and his collaborators. Samsonov (1994) assumes the linear Love hypothesis to be valid in a fully nonlinear framework and derives the improved Boussinesq equation. Porubov and Samsonov (1993) propose a multi-modal solution that satisfies the motion equation in the asymptotic sense, as well as the free boundary conditions in the Piola stress. In this process, a refined Love assumption is introduced. According to Samsonov (2001), this analysis is motivated by the desire to “confirm the Love hypothesis formally”, whose limit “is that the boundary conditions on a free lateral surface were not properly taken into account”, although “generally speaking, the identity is not required because an asymptotic solution is to be found”. Precisely in this sense, and as an alternative to Porubov and Samsonov’s approach, this paper adopts a slow time perturbation of the motion equations to derive an asymptotically refined Love hypothesis that is valid regardless of the boundary conditions.

Also Dai and collaborators have much contributed to this topic. Dai and Huo (2000) study propagation of small-but-finite-amplitude (i.e. weakly nonlinear) longitudinal waves in compressible rods by employing an asymptotic form of the Love hypothesis, that is suggested by comparison with the incompressible case. The reductive perturbation method of Jeffrey and Kawahara (1982) is then adopted to derive the model equation valid in the far-field. This procedure is in essence a multiscale analysis similar to that used in this paper. In Dai and Huo (2002), the reductive perturbation method is used in the incompressible context to support the validity of the Navier-Bernoulli approximation through comparing the resulting model equations. Dai and Fan (2004) introduce four different 1D models to study longitudinal waves in a weakly nonlinear Murnaghan material with an enriched kinematics that, departing from the NB assumption, accommodates all free boundary condition on the mantle. However, when developing the far-field model, the linear Love’s assumption is again reverted to. Dai and Fan (2004) are especially critical of Porubov and Samsonov (1993)’s refined model because “still boundary conditions cannot be completely satisfied even for linear terms”. Besides, while discussing Samsonov et al. (1998) and Porubov et al. (1998), they observe “some serious algebraic errors in their derivations [that] led the model equations to be unacceptable”.

More recently, Garbuzov et al. (2019) extend Samsonov (2001)’s approach to the case of general force conditions at the rod mantle and longitudinal prestretch. A family of Boussinesq-type model equations is obtained. The same approach is adopted in Garbuzov et al. (2020) where a stable propagating table-top soliton is observed numerically.

From the above literature review, it appears that several approaches to the problem are possible which lead to widely different model equations. As pointed out by Amendola and Saccomandi (2021), the adoption of the Love hypothesis outside the realm of linear elasticity is questionable and calls for further investigation. In particular, Nobili and Saccomandi (2024) make some progress in this direction by showing that the original Love hypothesis may be equally obtained from slow-time perturbation of the Mindlin-Herrmann model. The advantage of this observation lies in that it provides a pathway to developing the equivalent of the Love hypothesis outside the linear regime and within the asymptotic approach, i.e. without recourse to the boundary conditions. This is precisely the aim of this paper, which sets the spotlight on elucidating the procedure as well as on assessing the quality of the approximation in the simplest possible setting, namely the NB assumption and a bimodal representation (Sec.2). The multiscale analysis is carried out in Sec.3 and the quality of the approximation is numerically investigated in Sec.4. A unimodal asymptotic Lagrangian is illustrated in 5 and conclusions are finally drawn in Sec.6.

2. Mathematical background

Let us consider a rod that, in a reference configuration, is a circular cylinder of radius A and let us introduce cylindrical coordinates in the current configuration $\mathbf{x} = r\mathbf{e}_r + \theta\mathbf{e}_\theta + z\mathbf{e}_z$ and, equally, cylindrical coordinates in the reference configuration $\mathbf{X} = R\mathbf{E}_R + \Theta\mathbf{E}_\Theta + Z\mathbf{E}_Z$, with $0 \leq R \leq A$. Within this framework and in the absence of torsion, the NB hypothesis consists of assuming the following axisymmetric time dependent two-modal deformation (Wright, 1981, Eq.(12))

$$r = R + RU(Z,T), \quad \theta = \Theta, \quad z = Z + W(Z,T). \tag{1}$$

Letting the displacement vector $\mathbf{u} = \mathbf{x} - \mathbf{X}$ with components in our cylindrical reference system

$$u_r(R, Z, t) = RU(Z, T), \quad u_\theta = 0, \quad u_z(Z, T) = W(Z, T), \tag{2}$$

the Lagrangian strain tensor immediately follows $\boldsymbol{E} = \frac{1}{2}\left(\boldsymbol{C} - \boldsymbol{I}\right)$, namely

$$\boldsymbol{E} = \begin{pmatrix} \frac{1}{2}U(U+2) & 0 & \frac{1}{2}R(U+1)U_{,Z} \\ 0 & \frac{1}{2}U(U+2) & 0 \\ \frac{1}{2}R(U+1)U_{,Z} & 0 & \frac{1}{2}\left(R^2U_{,Z}^2 + (W_{,Z}+1)^2 - 1\right) \end{pmatrix}, \tag{3}$$

the linearized version of which is the infinitesimal strain tensor given, among many, in Nobili and Saccomandi (2024)

$$\boldsymbol{\epsilon} = \begin{bmatrix} U & 0 & \frac{1}{2}RU_Z \\ 0 & U & 0 \\ \frac{1}{2}RU_Z & 0 & W_Z \end{bmatrix}. \tag{4}$$

Here, the identity (rank 2) tensor $\boldsymbol{I}$ has been introduced as the invariant element in tensor composition, and it is understood that a comma subscript implies differentiation with respect to the following coordinate, i.e. $U_{,Z} = \partial U / \partial Z$. We let the quadratic (in the deformation components) invariants

$$I_{11} = \mathrm{tr}(\boldsymbol{E}), \quad I_{21} = \mathrm{tr}(\boldsymbol{E}^2), \tag{5}$$

alongside the cubic deformation invariants

$$I_{31} = I_{21}I_{11}, \quad I_{32} = I_{11}^3, \quad I_{34} = \mathrm{tr}(\boldsymbol{E}^3). \tag{6}$$

Here, $\mathrm{tr}\,\boldsymbol{E} = \frac{1}{2}\left(R^2U_{,Z}^2 + (W_{,Z}+1)^2 - 1\right) + U(U+2)$ denotes the usual trace operator. In the nonlinear framework, we take the isotropic Murnaghan strain-energy density per unit volume, see (Dai and Fan, 2004, Eq.(21))

$$\mathcal{W} = \mu I_{21} + \frac{1}{2}\lambda I_{11}^2 + \mu\left(\kappa_1 I_{31} + \frac{1}{3}\kappa_2 I_{32} + \frac{1}{3}\kappa_4 I_{34}\right), \tag{7}$$

where μ and λ are the usual Lamé parameters and $\mu\kappa_i$, $i \in \{1, 2, 4\}$ are Murnaghan material constants (the reason why it is three of them, instead of four, is discussed in Ciarlet (2021)). The form (7) corresponds to the isotropic third-order Landau-Lifshitz constitutive relation.

Hamilton's principle is employed to determine the functions $U(Z,T)$ and $W(Z,T)$ through the Euler-Lagrange equations associated with the Lagrangian density $\mathcal{L} = \mathcal{T} - \mathcal{V}$. The kinetic energy density per unit length remain the same as that given by (Graff, 2012, Eq.(2.5.49)) or (Dai and Fan, 2004, Eq.(24))

$$\mathcal{T} = \int_0^A \int_0^{2\pi} \frac{\rho}{2} \left(W_T^2 + R^2 U_T^2\right) R \,\mathrm{d}\Theta \,\mathrm{d}R = \frac{\pi A^2 \rho}{4} \left(2W_T^2 + A^2 U_T^2\right), \tag{8}$$

where ρ is the mass density in the reference configuration. In light of the assumed deformation (4), the strain-energy density (7) is given by Eq.(A.1) in the Appendix. The weakly nonlinear assumption affords great simplification of this otherwise cumbersome energy density. Within such framework, the displacement is small, i.e. $U \sim W_{,Z} \sim O(\varepsilon)$, with $\varepsilon \ll 1$, and only quadratic and cubic terms in ε are retained in the Lagrangian. We introduce the shorthands

$$k_1 = 2(2\kappa_1 + \frac{4}{3}\kappa_2 + \frac{1}{3}\kappa_4 + \kappa^2 - 1), \qquad k_2 = \kappa^2 - 1 + \kappa_1 + \frac{\kappa_4}{4}, \tag{9}$$

$$k_3 = \kappa^2 - 2 + 2(\kappa_1 + 2\kappa_2), \qquad k_4 = \tfrac{1}{2}\kappa^2 + \tfrac{1}{2}\kappa_1 + \frac{\kappa_4}{4}, \tag{10}$$

$$k_5 = \kappa^2 - 2 + 2(\kappa_1 + \kappa_2), \qquad k_6 = \tfrac{1}{2}\kappa^2 + \kappa_1 + \frac{\kappa_2 + \kappa_4}{3}, \tag{11}$$

where $\kappa = c_L/c_S$ is the ratio of $c_L = \sqrt{(\lambda + 2\mu)/\rho}$ over $c_S = \sqrt{\mu/\rho}$, that are, respectively, the longitudinal and the transversal (shear) wave speed of linear elasticity. It is emphasized that the following constraints hold

$$k_1 - k_5 - 2k_6 = 0, \quad -3k_1 + 8k_2 + 2k_3 = 2(1 + 2\lambda/\mu), \quad -3k_1 + 8k_4 + 4k_5 = 2(1 + \lambda/\mu). \tag{12}$$

Thus, the elastic energy density reads (Dai and Fan, 2004, Eq.(22))

$$\begin{aligned} \mathcal{W} = {} & \mu \left(\frac{1}{2} R^2 U_{,Z}^2 + 2U^2 + W_{,Z}^2\right) + \frac{1}{2}\lambda \left(2U + W_{,Z}\right)^2 \\ & + \mu \left[k_1 U^3 + k_2 R^2 U U_{,Z}^2 + k_3 W_{,Z} U^2 + k_4 R^2 W_{,Z} U_{,Z}^2 + k_5 U W_{,Z}^2 + k_6 W_{,Z}^3\right], \end{aligned} \tag{13}$$

which differs from the form adopted by Dai and Huo (2000) in Eq.(2.9). Integrating over the cross section

$$\mathcal{V} = \int_0^A \int_0^{2\pi} \mathcal{W} R d\Theta dR, \tag{14}$$

we obtain the potential energy per unit length (Dai and Fan, 2004, Eq.(23))

$$\mathcal{V} = \pi A^2 \Big\{ 2(\mu+\lambda)U^2 + 2\lambda U W_{,Z} + \left(\mu + \tfrac{1}{2}\lambda\right) W_{,Z}^2 + \frac{\mu A^2}{4} U_{,Z}^2$$
$$+\mu \left[\tfrac{1}{2} A^2 U_{,Z}^2 \left(k_2 U + k_4 W_{,Z}\right) + k_3 U^2 W_{,Z} + k_5 U W_{,Z}^2 + k_1 U^3 + k_6 W_{,Z}^3 \right] \Big\}. \tag{15}$$

We then form the Lagrangian

$$\mathcal{L} = \mathcal{T} - \mathcal{V}, \tag{16}$$

and the corresponding Euler-Lagrange equations form a pair of nonlinear PDEs

$$A^2 k_4 U_{,Z} U_{,ZZ} + 2U_{,Z}\left(k_3 U + k_5 W_{,Z} + \kappa^2 - 2\right) + W_{,ZZ}\left(2k_5 U + 6k_6 W_{,Z} + \kappa^2\right)$$
$$= c_S^{-2} W_{,TT} \tag{17a}$$
$$-3k_1 U^2 - 2U\left(k_3 W_{,Z} + 2\kappa^2 - 2\right) - W_{,Z}\left(k_5 W_{,Z} + 2\kappa^2 - 4\right)$$
$$+\tfrac{1}{2}A^2\left[k_2 U_{,Z}^2 + 2k_4 U_{,Z} W_{,ZZ} + U_{,ZZ}\left(2k_2 U + 2k_4 W_{,Z} + 1\right)\right] = \tfrac{1}{2} c_S^{-2} A^2 U_{,TT}, \tag{17b}$$

which provide a nonlinear generalization of the well-known Mindlin-Herrmann system, where the Murnaghan strain energy density is used and small terms in the deformation are retained up to $O(\varepsilon^2)$. These equations seem to differ somehow from the corresponding equations (27) given by Dai and Fan (2004) and specifically Eq.(27a) misses the term multiplying k_4 in (17a), while (27b) misses several terms, such as those multiplying k_2 and k_4. Clearly, the original Mindlin-Herrmann system (Mindlin, 1951)

$$\begin{aligned} (\lambda + 2\mu) W_{,ZZ} + 2\lambda U_Z &= \rho W_{,TT}, \\ \mu A^2 U_{,ZZ} - 8(\lambda+\mu)U - 4\lambda W_{,Z} &= \rho A^2 U_{,TT}, \end{aligned} \tag{18}$$

is immediately retrieved when retaining only linear terms in (17). This reveals that the system is a perturbation of a pair of coupled wave equations. This coupled system provides the time evolution of U and W once suitable initial and boundary conditions are given. Since only second order time derivatives appear (linearly), initial conditions take the form

$$U(Z,0) = U_0(Z), W(Z,0) = W_0(Z) \tag{19a}$$

$$U_{,T}(Z,0) = \dot{U}_0(Z), W_{,T}(Z,0) = \dot{W}_0(Z). \tag{19b}$$

Also, in light of the fact that the highest space derivative is two, a pair of boundary conditions is equally required. In general, solving Eqs.(17) calls for numerical methods, as it occurs in Sec.4. A way around the solution of the full system is to impose the Love (L-) *hypothesis*, that assumes a linear relationship between the radial displacement and the longitudinal strain i.e

$$U = -\nu_0 W_{,Z}, \tag{20}$$

where

$$\nu_0 = \frac{\lambda}{2(\lambda+\mu)} = \tfrac{1}{2}\frac{\kappa^2-2}{\kappa^2-1},$$

is the Poisson's ratio. A nice derivation of the Love model in the context of invariant manifolds from nonlinear dynamical systems theory is given by Roberts (1993). As it was originally proposed by Porubov and Samsonov (1993), Samsonov (2001) and later argued by Amendola and Saccomandi (2021), Love's hypothesis belongs to the linear framework and should be suitably generalized to the nonlinear setting. This is precisely the aim of the next Section.

3. Love hypothesis for weakly nonlinear elasticity

Let's begin by assuming that the deformation is small and yet not so small that third order terms in the Lagrangian may be neglected. Next, let's introduce the dimensionless coordinates

$$\zeta = Z/l, \quad t = T/\mathbb{T},$$

where l is a typical wavelength and $\mathbb{T} = l/c_S$ is a reference time. Hence, the dimensionless small parameter $\delta = A^2/l^2$ naturally emerges, cf. Garbuzov et al.

(2019). Next, we need to introduce the weakly nonlinear hypothesis, and specifically its connection to the idea that the solid is slender, namely that $\delta \ll 1$. For this we introduce a second small parameter, ε, which is a measure of the magnitude of the deformation, i.e. $U \sim W_{,Z} = O(\varepsilon)$. As customary, we also assume that differentiation does not affect the asymptotic order of the unknowns. In this form, the problem is multi-parametric and to make further progress we need to assume the reciprocal relation between the small parameters δ and ε. In this paper, we assume $\varepsilon = \delta$, meaning that the deformation gets smaller as the rod becomes slender in linear fashion. This distinct limit amounts to assuming that nonlinearity and dispersion are in balance and small enough (Samsonov, 2001). Indeed, according to Ablowitz (2011), this is the "maximum balance model" wherein nonlinear and dispersive effects are equally important. Other choices are of course possible but we don't pursue them in here. Thus, we let

$$U = \delta u(\zeta, \tau) \quad \text{and} \quad W = \delta l w(\zeta, \tau). \tag{21}$$

The E-L equations may be given to first order in δ

$$-4\left(\kappa^2 - 1\right) u - 2\left(\kappa^2 - 2\right) w_{,\zeta} + \tfrac{1}{2}\delta\left[u_{,\zeta\zeta} - 6k_1 u^2 - 4k_3 u w_{,\zeta} - 2k_5 w_{,\zeta}^2\right] + O\left(\delta^2\right) = \tfrac{1}{2}\delta u_{,tt}, \tag{22a}$$

$$2\left(\kappa^2 - 2\right) u_{,\zeta} + \kappa^2 w_{,\zeta\zeta} + \delta\left[2u_{,\zeta}\left(k_3 u + k_5 w_{,\zeta}\right) + \left(2k_5 u + 6k_6 w_{,\zeta}\right) w_{,\zeta\zeta}\right] + O\left(\delta^2\right) = w_{,tt}. \tag{22b}$$

Following Nobili (2021), the motion equations are perturbed in the slow time $\tau = t\delta$, having let the moving axial coordinate $\xi = \zeta - ct$. Consequently, a straightforward expansion in δ is introduced

$$u = \phi_0(\xi, \tau) + \delta\phi_1(\xi, \tau) + \ldots, \quad w = \psi_0(\xi, \tau) + \delta\psi_1(\xi, \tau) + \ldots.$$

To leading order, the system already obtained in Nobili and Saccomandi (2024) is obtained,

$$\left(4 - 2\kappa^2\right)\psi_{0,\xi} - 4\left(\kappa^2 - 1\right)\phi_0 = 0, \tag{23}$$

$$\left(\kappa^2 - c^2\right)\psi_{0,\xi\xi} + 2\left(\kappa^2 - 2\right)\phi_{0,\xi} = 0. \tag{24}$$

This system lends the trivial solution unless

$$c = \pm \hat{c}_B, \quad \text{with} \quad \hat{c}_B = \sqrt{\frac{3\kappa^2 - 4}{\kappa^2 - 1}},$$

that, multiplied by c_S, gives the dimensional longitudinal wave speed in rods $c_B = \sqrt{E/\rho}$, where $E = \mu \frac{3\lambda + 2\mu}{\lambda + \mu} > 0$ is Young's modulus. Then, the leading order eigenform is obtained

$$\phi_0 = -\nu_0 \psi_{0,\xi}, \tag{25}$$

that is precisely the Love assumption, which remains valid at leading order, with the understanding that a moving coordinate frame is considered.

Moving to the next order, we find

$$\phi_1 + \frac{k_5 - 2\nu_0 k_3 + 3\nu_0^2 k_1}{4(\kappa^2 - 1)} \psi_{0,\xi}^{\,2} - \nu_0 \frac{\hat{c}_B^2 - 1}{8(\kappa^2 - 1)} \psi_{0,\xi\xi\xi} + \nu_0 \psi_{1,\xi} = 0, \tag{26a}$$

$$\left(\kappa^2 - 2\right) \phi_{1,\xi} + \hat{c}_B \psi_{0,\xi\tau} + \left(3k_6 - 2\nu_0 k_5 + \nu_0^2 k_3\right) \psi_{0,\xi} \psi_{0,\xi\xi} + \nu_0 \left(\kappa^2 - 2\right) \psi_{1,\xi\xi} = 0. \tag{26b}$$

Remarkably, this system shows no dependence on k_2 and k_4 which only come at higher order and are therefore disregarded. Clearly, Eq.(26a) immediately lends the first correction to Love hypothesis

$$\phi_1 = -\nu_0 \psi_{1,\xi} + \nu_0 \frac{\hat{c}_B^2 - 1}{8\left(\kappa^2 - 1\right)} \psi_{0,\xi\xi\xi} - \frac{k_5 - 2\nu_0 k_3 + 3\nu_0^2 k_1}{4\left(\kappa^2 - 1\right)} \psi_{0,\xi}^{\,2}, \tag{27}$$

which consists of the linear terms, already appreciated in Nobili and Saccomandi (2024), together with a nonlinear contribution which depends on Murnaghan's moduli. Together, Eq.(25) and (27) give the refined Love hypothesis in the weakly nonlinear setting

$$\phi = -\nu_0 \psi_{,\xi} + \delta \nu_0 \frac{\hat{c}_B^2 - 1}{8\left(\kappa^2 - 1\right)} \psi_{,\xi\xi\xi} - \delta \frac{k_5 - 2\nu_0 k_3 + 3\nu_0^2 k_1}{4\left(\kappa^2 - 1\right)} \psi_{,\xi}^{\,2} + O(\delta^2), \tag{28}$$

and, in particular, we see that the linear Love hypothesis, in its simplicity, only conveys the first of such terms, and certainly misses the nonlinear correction. Plugging Eq.(27) into Eq.(26b), yields the governing equation for the perturbation to leading order,

$$\tfrac{1}{4}\nu_0^2 \left(\hat{c}_B^2 - 1\right) \Psi_{,\xi\xi\xi} + \hat{c}_B \Psi_{,\tau} + 3\beta_1 \Psi \Psi_{,\xi} = 0, \quad \beta_1 = -\nu_0^3 k_1 + \nu_0^2 k_3 - \nu_0 k_5 + k_6, \tag{29}$$

that is the well-known Korteweg-de Vries (KdV) equation for the longitudinal strain $\Psi = \psi_{0,\xi}$. This equation corresponds to the far-field model (4.18) of Dai and Huo (2002) that is valid for an incompressible elastic rod, whence coefficients are different. Space differentiation of the KdV and exploiting the connection

$$\frac{\partial^2}{\partial\xi\partial\tau} = \tfrac{1}{2}\delta^{-1}\hat{c}_B\left(\frac{\partial^2}{\partial\zeta\partial\zeta} - \hat{c}_B^{-2}\frac{\partial^2}{\partial t\partial t}\right) + O(\delta),$$

lends the *Boussinesq equation* (BE)

$$\Psi_{,\zeta\zeta} - \hat{c}_B^{-2}\Psi_{,tt} + \frac{\delta}{\hat{c}_B^2}\left[\tfrac{1}{2}\nu_0^2\left(\hat{c}_B^2 - 1\right)\Psi_{,\zeta\zeta} + 3\beta_1\Psi^2\right]_{,\zeta\zeta} = 0, \tag{30}$$

that admits soliton-like travelling solutions, see Bullough and Caudrey (1980). The BE features poor existence and uniqueness properties and, in particular, no local well-posedness result is available. To the same order of approximation, the asymptotically equivalent form may be obtained (this process, sometimes, is called "regularization")

$$\Psi_{,\zeta\zeta} - \hat{c}_B^{-2}\Psi_{,tt} + \frac{\delta}{\hat{c}_B^2}\left[\tfrac{1}{2}\nu_0^2\left(1 - \hat{c}_B^{-2}\right)\Psi_{,tt} + 3\beta_1\Psi^2\right]_{,\zeta\zeta} = 0, \tag{31}$$

that goes under the name of *improved Boussinesq equation* (IBE). The IBE is far superior to the BE in that it is well-posed.

Eq.(29) may be put to advantage in order to eliminate the third derivative in (28). Indeed, differentiating (28) with respect to ξ and using (29), one finds an asymptotically equivalent form of the refined weakly-nonlinear Love hypothesis

$$\phi_{,\xi} = -\nu_0\psi_{,\xi\xi} - \delta\frac{\hat{c}_B}{\kappa^2 - 2}\psi_{,\xi\tau} - \delta\frac{\nu_0\left(k_3\nu_0 - 2k_5\right) + 3k_6}{\kappa^2 - 2}\psi_{,\xi}\psi_{,\xi\xi} + O(\delta^2). \tag{32}$$

Going back to the original variables, the KdV equation (29) becomes

$$w_{,\zeta\zeta} - \hat{c}_B^{-2}w_{,tt} + \delta\hat{c}_B^{-2}\left[\tfrac{1}{2}\nu_0^2\left(\hat{c}_B^2 - 1\right)w_{,\zeta\zeta\zeta\zeta} + 6\beta_1 w_{,\zeta}w_{,\zeta\zeta}\right] = O(\delta^2), \tag{33}$$

that, to first correction terms, gives

$$W_{,ZZ} - c_B^{-2}W_{,TT} + \nu_0^2K^2\left(1 - \hat{c}_B^{-2}\right)W_{,ZZZZ} + 3\hat{c}_B^{-2}\beta_1(W_{,Z})^2_{,Z} = 0, \tag{34}$$

where $K = A/\sqrt{2}$ is the polar radius of gyration of the cross-section. Furthermore, Eq.(33) may be rewritten, within the same order of accuracy, as

$$w_{,\zeta\zeta} - \hat{c}_B^{-2}w_{,tt} + \delta\hat{c}_B^{-2}\left[\tfrac{1}{2}\nu_0^2\hat{c}_B^{-2}\left(\hat{c}_B^2 - 1\right)w_{,\zeta\zeta tt} + 6\beta_1 w_{,\zeta}w_{,\zeta\zeta}\right] = O(\delta^2), \tag{35}$$

which has the undoubted advantage of being of order 2 in space as well as in time and therefore it requires only one pair of spatial boundary condition and one pair of initial conditions. In the original variables and to first correction, it becomes

$$W_{,ZZ} - c_B^{-2} W_{,TT} + \nu_0^2 K^2 c_B^{-2} \left(1 - \hat{c}_B^{-2}\right) W_{,ZZTT} + 3\hat{c}_B^{-2} \beta_1 (W_{,Z})^2_{,Z} = 0. \quad (36)$$

Recalling that

$$\beta_1 = E/\rho + \kappa_1(1-2\nu_0)(1+2\nu_0^2) + \tfrac{1}{3}\kappa_2(1-2\nu_0)^3 + \tfrac{1}{3}\kappa_4(1-2\nu_0^3), \quad (37)$$

Eq.(36) corresponds to Eq.(1.4) of Ostrovskii and Sutin (1977), provided the coefficient $1 - \hat{c}_B^{-2}$ in the dispersion term is suppressed and the factor $1/2$ added in the nonlinear term (the latter appears in Garbuzov et al. (2019)'s writing of Ostrovskii and Sutin's result). This partial correspondence may not be too surprising, given that Ostrovskii and Sutin obtain their equation via the linear Love hypothesis and therefore miss the correction terms. Conversely, the model equation (34) in Dai and Fan (2004), that is obtained using the linear Love hypothesis, also appears in Porubov and Samsonov (1993) and it is a Boussinesq-type combination of the BE and of the IBE, that is named by Garbuzov et al. (2019) the "doubly dispersive equation" (DDE). In fact, Eq.(45) in Garbuzov et al. (2019) provides yet another model equation. As pointed out by Garbuzov et al. (2019), "the models [by Porubov and Samsonov (1993) and by Ostrovskii and Sutin (1977) (adding the missing 1/2 coefficient)] have different dispersive properties", although, after "regularization", they are all asymptotically equivalent. Furthermore, "it would be interesting to compare the performance of these four nonlinear models with the exact (numerical) solution of the nonlinear problem", that is indeed what is carried out in Sec.4 for the present model. Besides being equivalent, it is also clear that these models are also asymptotically consistent up to $O(\delta)$ terms. In contrast, however, they are not the same to those in Dai and Huo (2000) and Dai and Fan (2004) (where, presumably in Eq.(33) and (34) W should be written in the place of U).

Similarly, the refined Love assumption (32) becomes

$$u_{,\zeta} = -\nu_0 w_{,\zeta\zeta} - \frac{1}{2(\kappa^2-2)}\left(\hat{c}_B^2 w_{,\zeta\zeta} - w_{,tt}\right) - \delta\frac{\nu_0\left(k_3\nu_0 - 2k_5\right) + 3k_6}{\kappa^2 - 2} w_{,\zeta} w_{,\zeta\zeta} + O(\delta^2), \tag{38}$$

that, to first correction terms, gives

$$U_{,Z} = -\nu_0 W_{,ZZ} - \frac{\hat{c}_B^2}{2(\kappa^2-2)}\left(W_{,ZZ} - c_B^{-2} W_{,TT}\right) - \frac{\nu_0\left(k_3\nu_0 - 2k_5\right) + 3k_6}{\kappa^2 - 2} W_{,Z} W_{,ZZ}. \tag{39}$$

The refined Love hypothesis (28), in the original variables, reads

$$U = -\nu_0 W_{,Z} + K^2\nu_0\frac{\hat{c}_B^2 - 1}{4\left(\kappa^2-1\right)} W_{,ZZZ} - \frac{k_5 - 2\nu_0 k_3 + 3\nu_0^2 k_1}{4\left(\kappa^2-1\right)} W_{,Z}^2, \tag{40}$$

and it may be compared with Eq.(2.52) in Samsonov (2001), which, however, emerges from a completely different setting, by enforcing the free boundary conditions. The two equations reveal a similar structure, although coefficients are very different for each correction term (dispersive and nonlinear). In fact, according to Samsonov, the linear correction term sign is opposite and exhibits a quadratic dependence on the radius R (whence it disappears on the rod axis). Nonetheless, this structural correspondence is somewhat remarkable, considering that equations emerge from very different assumptions.

4. Numerical results

We now show how the model equation (33) compares to the nonlinear coupled system (17) and to the solution of the wave equation. Specifically, we are interested in the accuracy of the transversal motion, with special regard to the nonlinear contribution. It is pointed out that this is not the same comparison that is presented in Garbuzov et al. (2020), which instead focuses on propagating solitons. We consider the dimensionless variables $\delta u(\zeta, t)$ and $\delta w(\zeta, t)$, where δ is introduced to properly scale quantities as in (21). We choose the parameter set

$$\kappa = 3, k_1 = 1.1, k_2 = 1, k_3 = 5, k_4 = 2, k_5 = 0.5, k_6 = 0.3,$$

with the aim to produce a significant nonlinear contribution. A few numerical experiments show that this particular choice of parameters is of little importance for the overall picture. In contrast, it is crucial to take for δ a small value: hereinafter we begin with $\delta = 0.1$, which warrants equally small initial conditions to remain within the weakly nonlinear hypothesis. In the following, we assume a periodic system, because our approximation relies on the far-field concept which is at odd with a finite domain. Therefore, we chose a periodic boundary conditions, namely

$$w(-\pi, t) = w(\pi, t), \quad u(-\pi, t) = u(\pi, t). \tag{41}$$

in the period range $\zeta \in [-\pi, \pi]$. For the initial condition on w, we assume

$$w(\zeta, 0) = \sin(\zeta), \tag{42}$$

and, for the sake of definiteness, we assume zero initial velocity, i.e.

$$w_{,t}(\zeta, 0) = u_{,t}(\zeta, 0) \equiv 0. \tag{43}$$

For $u(\zeta, 0)$, we adopt the linear Love assumption, whereby

$$u(\zeta, 0) = -\nu_0 w_{,\zeta}(\zeta, 0) = -\nu_0 \cos \zeta, \tag{44}$$

with the understanding that this choice works in favor of the accuracy of Love's linear hypothesis (25).

To determine the approximating solution, we integrate the Boussinesq-type Eq.(33) by standard methods. Here, it is important to point out that a fourth space derivative appears, which calls for two sets of boundary conditions, precisely as in (41). To bring about periodicity on u, we call upon (28) and thereby impose

$$\begin{aligned} &-\nu_0 u_{,\zeta}(-\pi, t) + \delta \nu_0 \frac{\hat{c}_B^2 - 1}{8\,(\kappa^2 - 1)} u_{,\zeta\zeta\zeta}(-\pi, t) - \frac{k_5 - 2\nu_0 k_3 + 3\nu_0^2 k_1}{4\,(\kappa^2 - 1)} u_{,\zeta}^2(-\pi, t) \\ &= -\nu_0 u_{,\zeta}(\pi, t) + \delta \nu_0 \frac{\hat{c}_B^2 - 1}{8\,(\kappa^2 - 1)} u_{,\zeta\zeta\zeta}(\pi, t) - \frac{k_5 - 2\nu_0 k_3 + 3\nu_0^2 k_1}{4\,(\kappa^2 - 1)} u_{,\zeta}^2(\pi, t). \end{aligned}$$

Finally, for the wave equation, we have the analytic solution

$$w(\zeta, t) = \tfrac{1}{2} \left(\sin(\zeta + c_B t) - \sin(\zeta - c_B t) \right), \tag{45}$$

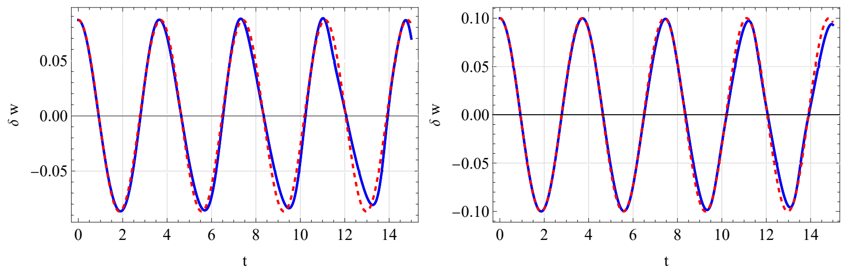

Figure 1: $\delta w(\zeta, t)$ for $\delta = 0.1$ as a function of t for $\zeta = \pi/3$ (left) and $\zeta = \pi/2$ (right) as obtained from the nonlinear Mindlin-Herrmann system (black, solid), the KdV equation (blue, solid) and the solution of the wave equation (red, dashed). The solid curves are indistinguishable.

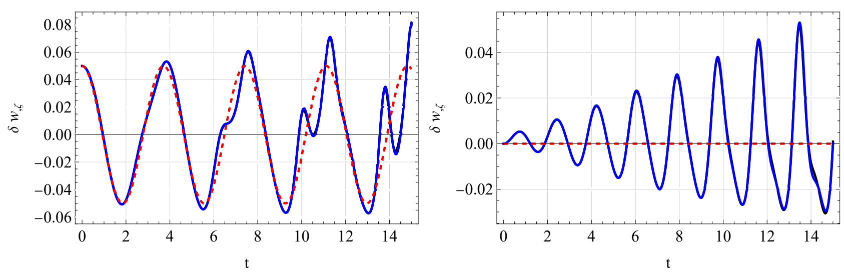

Figure 2: $\delta w_{,\zeta}(\zeta, t)$ for $\delta = 0.1$ as a function of t for $\zeta = \pi/3$ (left) and $\zeta = \pi/2$ (right) as obtained from the nonlinear Mindlin-Herrmann system (black, solid), the KdV equation (blue, solid) and the solution of the wave equation (red, dashed). The solid curves are indistinguishable.

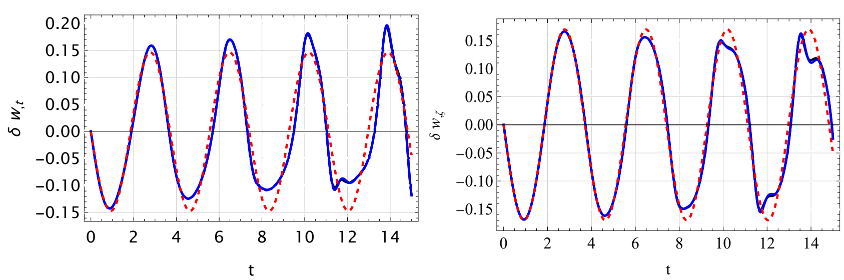

Figure 3: $\delta w_{,t}(\zeta, t)$ for $\delta = 0.1$ as a function of t for $\zeta = \pi/3$ (left) and $\zeta = \pi/2$ (right) as obtained from the nonlinear Mindlin-Herrmann system (black, solid), the KdV equation (blue, solid) and the solution of the wave equation (red, dashed). The solid curves are indistinguishable.

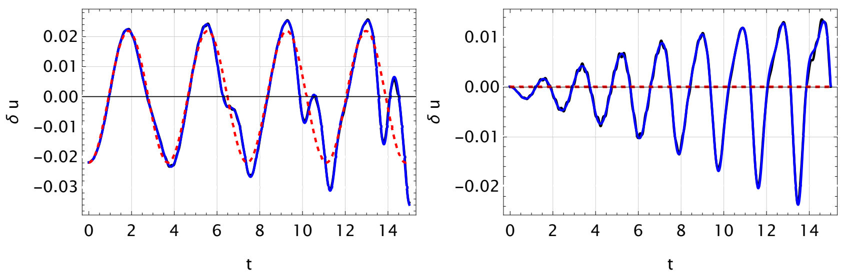

Figure 4: $\delta u(\zeta, t)$ for $\delta = 0.1$ as a function of t for $\zeta = \pi/3$ (left) and $\zeta = \pi/2$ (right) as obtained from the nonlinear Mindlin-Herrmann system (black, solid), the refined nonlinear Love hypothesis (blue, solid) and the linear Love hypothesis applied to the solution of the wave equation (red, dashed). The solid curves are indistinguishable.

which disposes of the initial velocity, according to the first of Eqs.(43).

Figure 1 plots $\delta w(\zeta, t)$ as it emerges from the numerical solution of the nonlinear Mindlin-Herrmann system (17), from the numerical solution of the KdV (33) and finally from solving the wave equation. Clearly, all three solutions appear very close, at least initially, the deviation from the wave equations building up slowly in time. Yet, it is interesting to look at corresponding plots for the space derivative of w, that are shown in Fig.2 where it clearly appears that the wave equation is unable to reproduce the features of the solution. This is especially true for the position $\zeta = \pi/2$ because there the wave equation solution vanishes, thus leaving only the nonlinear term as the leading source for the solution. In contrast, the numerical solution of the KdV offers an outstanding approximation, which can be hardly resolved from the Mindlin-Herrmann system. The same comparison is given in Fig.3 with respect to the time derivative of w and similar conclusions may be drawn.

We now compare the numerical solution for δu obtained from solving the nonlinear Mindlin-Herrmann system (17) with the refined Love hypothesis (28) and with the linear Love hypothesis (25) applied to the solution of the wave equation (namely a $O(1)$ solution). Fig.4 presents this comparison for $\zeta = \pi/3$ and for $\zeta = \pi/2$ and confirms the excellent approximation offered by the refined Love hypothesis, that is indeed capable of reproducing the nonlinear contributions over time. Once again, the location $\zeta = \pi/2$ is expedient to rule

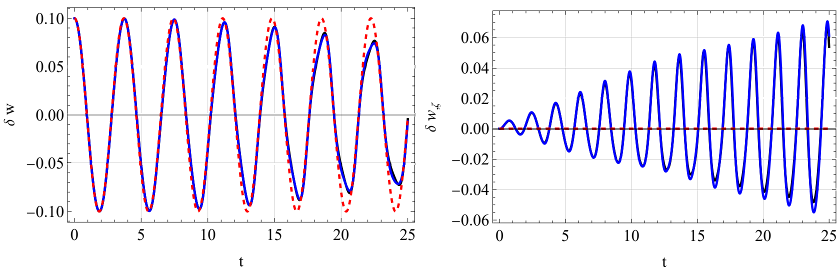

Figure 5: $\delta w(\pi/2, t)$ (left) and $\delta w_\zeta(\pi/2, t)$ (right) as a function of t for $\delta = 0.2$ as obtained from the nonlinear Mindlin-Herrmann system (black, solid), the KdV (blue, solid) and the solution of the wave equation (red, dashed). The solid curves begin to resolve for $t > 10$ in the derivative only.

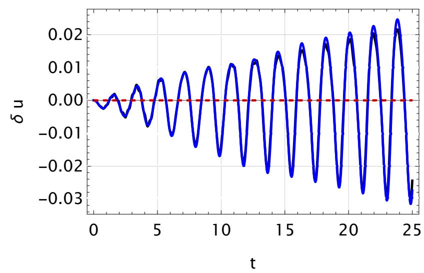

Figure 6: $\delta u(\pi/2, t)$ as a function of t for $\delta = 0.2$ as obtained from the nonlinear Mindlin-Herrmann system (black, solid), the refined nonlinear Love hypothesis (blue, solid) and the linear Love hypothesis applied to the solution of the wave equation (red, dashed). The latter is zero throughout. The solid curves exhibit some small differences at large times.

out any contribution from the linearized solution and single out the outcomes of the nonlinear terms. It should be emphasized that obtaining the numerical solution of the Mindlin-Herrmann system is rather delicate and time consuming, especially for long time frames.

We now investigate the robustness of the approximation in dependence of the parameter δ. Fig.5 shows the displacement δw and its space derivative $\delta w_{,\zeta}$ at $\zeta = \pi/2$, having let $\delta = 0.2$. Some little deviations of the KdV approximation may be appreciated for long times in the space derivative only. A similar comparison, this time for u and the refined Love hypothesis, is plotted in Fig.6 where again little deviations begin to appear at large times. This trend becomes more evident for $\delta = 0.25$ and Fig.7 reveals that the nonlinear approximation deteriorates substantially over time, although it still fares a lot better than the

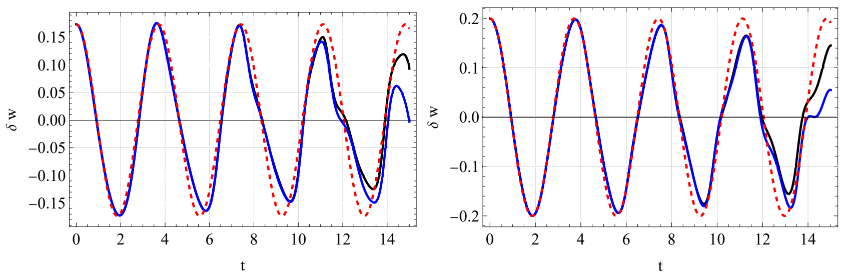

Figure 7: $\delta w(\pi/3, t)$ (left) and $\delta w(\pi/2, t)$ (right) as a function of t for $\delta = 0.25$ as obtained from the nonlinear Mindlin-Herrmann system (black, solid), the KdV (blue, solid) and the solution of the wave equation (red, dashed). The solid curves begin to resolve for $t > 10$ in both locations.

wave equation.

5. Unimodal Lagrangian

Plugging the refined Love hypothesis (40) into the Lagrangian (16) lends a unimodal system whose only variable is W. As already observed in Nobili and Saccomandi (2024), the corresponding Euler-Lagrange equation corresponds to the Love equation (Love, 1927, §278)

$$W_{,ZZ} + \frac{\nu_0^2 K^2}{c_B^2} W_{,ZZTT} = \frac{W_{,TT}}{c_B^2},$$

only to leading order, while different coefficients are obtained in the first correction. This outcome, that is a result of the fact that the Love Lagrangian accommodates for transversal motion only in the inertia term and not in the elastic potential, occurs regardless of the correction terms to the linear Love hypothesis (20). Accordingly, the first correction plays no role in affecting the E-L equation of the unimodal Lagrangian up to first order. Besides, comparing the model equation against the Pochhammer-Chree solution of linear elasticity as in Dai and Fan (2004) to assess its validity brings little value, because we already know that the bimodal kinematics is doomed to fail (and equally so in the case of richer models, provided that they are power series corrections of the bimodal system). Yet, it is even more remarkable that the E-L equation of the unimodal cubic Lagrangian corresponds to the BE (34) *regardless* of the correction terms. In other words, plugging the linear Love hypothesis (20) into the Murnaghan Lagrangian (16) yields the correct model equation, be it (34) or (36). This time, this outcome is a consequence of the first constraint in the Murnaghan material parameters (12). It is concluded that, as it was the case in linear elasticity, plugging a family of Love assumptions, which all differ from the linear Love hypothesis (20) by first correction terms into the nonlinear Lagrangian (16) (and regularizing) always lends the model equations (34) or (36). Indeed, the unimodal Lagrangian corresponding to (34) is given by

$$\mathcal{L}_1 = -\tfrac{1}{2}W_{,Z}^2 + \tfrac{1}{2}c_B^{-2}W_{,T}^2 + \tfrac{1}{2}\nu_0^2K^2\left(1-\hat{c}_B^{-2}\right)W_{,ZZ}^2 - \hat{c}_B^{-2}\beta_1 W_{,Z}^3. \tag{46}$$

Similarly, letting the Love Lagrangian Graff (2012)

$$\mathcal{L}^* = -\tfrac{1}{2}W_{,Z}^2 + \tfrac{1}{2}c_B^{-2}W_{,T}^2 + \tfrac{1}{2}\nu_0^2K^2c_B^{-2}W_{,ZT}^2, \tag{47}$$

we see that the corrected Lagrangian

$$\mathcal{L}_2 = \mathcal{L}^* - \tfrac{1}{2}\nu_0^2K^2c_B^{-2}\hat{c}_B^{-2}W_{,ZT}^2 - \hat{c}_B^{-2}\beta_1 W_{,Z}^3, \tag{48}$$

yields the E-L equation (36).

6. Conclusions

In this paper, slow time perturbation of the motion equations is proposed to systematically derive the (weakly) nonlinear counterpart of the Love hypothesis of linear elasticity. For the sake of illustration, the method is shown in the simplest possible setting. Indeed, a bimodal kinematics is assumed to study longitudinal waves propagating in Murnaghan hyperelastic straight rods and, as a result, a complicated pair of coupled nonlinear PDEs arises, which is the nonlinear generalization of the Mindlin-Herrmann system of linear elasticity. This nonlinear system is difficult to analyze and calls for numerical investigation. Alternatively, a refined weakly nonlinear Love hypothesis is derived by perturbation of the motion equation in slow time. This relation may be seen as a generalization of the linear hypothesis originally proposed by Love (1927) in the context of linear elasticity. In the process, a unidimensional model equation is obtained for longitudinal strain, that, in the moving frame, is the celebrated KdV. In the stationary frame, the KdV turns into the Boussinesq and into the improved Boussinesq equations, which are asymptotically equivalent. In fact, as noted by Garbuzov et al. (2019), the asymptotic approach justifies the presence of many different model equations in the literature. In terms of longitudinal displacement, the Boussinesq-type model of Ostrovskii and Sutin (1977) is retrieved, which was obtained by the linear Love hypothesis, although coefficients in the dispersive term are different (the nonlinear term is also different by a 1/2 factor which, however, may be a typo).

The solution of either Boussinesq equation is compared against the numerical solution of the nonlinear Mindlin-Herrmann system and remarkable accuracy is found, both in terms of longitudinal as well as transversal displacement (and their derivatives alike). In fact, assessment of the accuracy of the transversal displacement, as provided by the refined nonlinear Love equation, is especially interesting and seems not yet explored in the literature. Besides, the accuracy of the approximation seems unexpectedly robust for not-so-small values of the small parameter δ. This procedure may be easily generalized to more involved scenarios, such as the presence of reinforcing fibres or exotic constitutive models, see Amendola et al. (2024). Finally, a unimodal Lagrangian is derived which proves capable of producing (either of) the Boussinesq equations *regardless* of the first correction terms. This surprising outcome emerges in light of the restriction that exists on the Murnaghan material parameters and, possibly, it may be a by-product of the bimodal kinematics.

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Conflict of Interest statement

The author has no conflict of interest to declare.

Data Availability

This paper makes use of no data.

Appendix A. Murnaghan strain energy density

The Murnaghan strain energy density, within the restricted kinematics (1), reads

$$
\begin{aligned}
\mathcal{W} = \mu\left(2U^2 + W_{,Z}^2 + \tfrac{1}{2}R^2U_{,Z}^2\right) + \tfrac{1}{2}\lambda\left[\frac{1}{2}\left(R^2U_{,Z}^2 + (W_{,Z}+1)^2 - 1\right) + U(U+2)\right]^2 \\
+ \frac{1}{24}\mu\Big\{\kappa_2\left(R^2U_{,Z}^2 + 2U^2 + 4U + W_{,Z}(W_{,Z}+2)\right)^3 \\
+3\kappa_1\left(2U^2\left(R^2U_{,Z}^2+4\right) + 2R^2U_{,Z}^2(W_{,Z}+1)^2 + 4R^2UU_{,Z}^2 + R^4U_{,Z}^4 + 2U^4 + 8U^3 + W_{,Z}^2(W_{,Z}+2)^2\right) \\
\times\left(R^2U_{,Z}^2 + 2U^2 + 4U + W_{,Z}(W_{,Z}+2)\right) \\
+\kappa_4\Big[3R^2U^2U_{,Z}^2\left(R^2U_{,Z}^2 + W_{,Z}^2 + 2W_{,Z} + 5\right) + 3U^4\left(R^2U_{,Z}^2 + 8\right) + 4U^3\left(3R^2U_{,Z}^2 + 4\right) \\
+6R^2UU_{,Z}^2\left(R^2U_{,Z}^2 + (W_{,Z}+1)^2\right) + 3R^4U_{,Z}^4(W_{,Z}+1)^2 + 3R^2U_{,Z}^2W_{,Z}(W_{,Z}+1)^2(W_{,Z}+2) \\
+ R^6U_{,Z}^6 + 2U^6 + 12U^5 + W_{,Z}^3(W_{,Z}+2)^3\Big] \\
+6\left(2R^2U_{,Z}^2(W_{,Z}+1)^2 + 4R^2UU_{,Z}^2 + R^4U_{,Z}^4 + 2U^4 + 8U^3 + W_{,Z}^4 + 4W_{,Z}^3\right)\Big\}.
\end{aligned}
\tag{A.1}
$$

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Declaration of interests

☒The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

☐The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: