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Research paper

Evolution equations with nonlocal multivalued Cauchy problems

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ABSTRACT

We consider evolution equations in Banach spaces. Their linear parts generate a strongly continuous C_0-semigroup of contractions. The nonlinear term is a Carathéodory function. When the semigroup is not compact the nonlinearity has an additional restriction, involving the Hausdorff measure of noncompactness. We provide solutions satisfying nonlocal, multivalued Cauchy conditions. Our approach involves a suitable degree argument. The duality mapping is used for guaranteeing the lack of fixed points of the associated homotopic fields along the boundary of their domain. We apply our results for the investigation of transport and diffusion equations for which we provide the existence of nonlocal solutions.

1. Introduction

The discussion in this paper is motivated by the study of nonlocal solutions to partial differential equations. Our techniques apply to transport equations of the type

$$u_t(t,y) + a \cdot \nabla u(t,y) = g\left(t, u(t,y), \int_{\mathbb{R}^n} |u(t,\xi)|^p \, d\xi\right), \; t \in [0,T] \text{ and } y \in \mathbb{R}^n \tag{1.1}$$

with $a \in \mathbb{R}^n$, $1 < p < \infty$ and some appropriate function g (see Section 5). Our methods are also suitable for treating diffusion equations such as

$$u_t(t,y) = \Delta u(t,y) - bu(t,y) + g\left(t, u(t,y), \int_{\Omega} \eta(y,\xi) u(t,\xi) \, d\xi\right), \; t \in [0,T] \tag{1.2}$$

with $y \in \Omega \subset \mathbb{R}^n$ bounded, $b > 0$, $\eta \in L^\infty(\Omega \times \Omega)$ and a sufficiently regular g (see Section 6).

Typical examples of nonlocal Cauchy conditions for Eqs. (1.1) and (1.2) are the multipoint condition

$$u(0,y) = \sum_{i=1}^{m} \beta_i u(t_i, y), \quad y \in \mathbb{R}^n$$

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for some $t_i \in [0,T]$ and $\beta_i \in \mathbb{R}$, $i = 1, \dots, m$ and the mean value condition

$$u(0,y) = \frac{1}{T}\int_0^T u(t,y)\,dt, \quad y \in \mathbb{R}^n.$$

In some models nonlocal solutions can better describe the behavior of the related process. This motivates their interest and the study of a great variety of them. A nonlocal condition can be also nonlinear and even multivalued as in the recent paper [1]. The search for solutions satisfying some nonlocal Cauchy condition started around the nineties of previous century and many contributions are already available. In the following we limit ourselves to mentioning only the most recent and refer to those papers for further references. The results in [2,3] apply to parabolic diffusion equations such as (1.2), the nonlinear part is sublinear and globally continuous in [3] while in [2] it can even be, for instance, a cubic polynomial; an approximation solvability method is exploited in [4–6]; the controllability of nonlocal solutions is then obtained in [7], with a similar technique. The solutions in [6] satisfy a variational inequality. The results in [8,9] concern second order inclusions. A functional term appears in [10], whereas the linear part also depends on t both in [10] and in [11]. The results in [12] applies to systems.

The partial differential equations as (1.1) or (1.2) are usually written in their abstract form

$$x'(t) = Ax(t) + f(t, x(t)) \qquad t \in [0, T] \tag{1.3}$$

with x in some function space and A generator of a C_0−semigroup. The results on the existence of nonlocal Cauchy solutions are consequently stated for the associated Eq. (1.3). Eq. (1.3) becomes an inclusion in abstract setting when the original dynamic is multivalued. Instead of A it appears $A(t)$ when the linear part of the original partial differential equation is non-autonomous. In this case $A(t)$ is the generator of an evolution operator. Here we persue the same strategy and following [1], we assume a multivalued, possibly nonlinear, nonlocal Cauchy condition (see (3.1)). In Section 3 we consider the case when the semigroup generated by A is not compact and the main result is Theorem 3.2 (see also Corollary 3.3). In Section 4 we assume that A generates a compact semigroup and the main result is Theorem 4.2. In both sections we consider a semigroup of contractions. We search for mild solutions to (1.3) (see Definition 3.1).

The investigation of nonlocal conditions is always conducted with topological methods, frequently by some fixed point theorem. The topological technique is then combined with the theory of semigroups and some measure of non-compactness can be involved, when the semigroup is not compact. All these topics can be found in the books [13–17] (see also Section 2).

The main tool of our investigation is a degree argument instead of a fixed point theorem. Starting from (1.3) and the notion of mild solution we introduce a solution operator depending on a real parameter λ which varies on the compact interval $[0,1]$. In such a way we obtain a family of homotopic fields and we check that no fixed point appears on the boundary of their domain. Hence, we derive the solvability of our problem by means of a continuation principle based on the Leray–Schauder degree or on the degree for condensing maps (see Theorems 2.7 and 2.8, respectively). Only very recently this method was introduced in this context. Here we improve and generalize it (see the end of this section). This approach originates from Hartman [18] and was first introduced for studying the two-point boundary value problem for second order ordinary differential equations. It was then developed by Mawhin [19] and Bebernes [20] who introduced, in particular, the notion of *bounding function* (see also [21]), i.e. a Lyapunov-like function. Several problems for ordinary differential equations were then solved and we refer to [22] for a survey on this topic. The same approach was also recently used in [23], for treating second order ordinary differential equations with the ϕ−laplacian operator. The method was then generalized in abstract setting for integro-differential equations [24] and more recently for partial differential equations in [2,3,5]. In this paper we make use of the simplest possible bounding function i.e. $\frac{1}{2}(\|x\|^2 - R^2)$ with x in a suitable function space and so we introduce the duality mapping (see Section 2 and (3.2)) in order to guarantee the lack of fixed points on the boundary of our solution operator. We need to differentiate our technique in the two cases when the associate semigroup is not compact, as for the group generated by Eq. (1.1), and when it is, such as in (1.2).

We improve the results in [2,3,5] since we consider more general nonlocal conditions. In addition both in [2,3] only the case when the semigroup is compact is considered; the function f appearing in (1.3) is globally continuous in [3] and it has some strong restrictions in [2] (see [2, (A2) and (f4)]) for letting some superlinear growths. The uniqueness of solutions is also investigated in [2]. An arbitrary semigroup of contraction is assumed in [5]; but their nonlocal condition is linear and a stronger regularity restriction is assumed (see [5, (A5)]) on f.

2. Notation and preliminary results

We denote by X an infinite dimensional real Banach space with norm $\|\cdot\|$ and by B the open unit ball of X centered at the origin, $B = \{x \in X : \|x\| < 1\}$. Moreover, let $C([a,b], X)$ be the space of continuous functions on the real interval $[a,b]$ with values in X.

If the dual space X^* of X is uniformly convex, then the duality mapping $J : X \to X^*$ defined by

$$J(x) = \left\{x^* \in X^* : \|x^*\| = \|x\| \text{ and } \langle x^*, x\rangle = \|x\|^2\right\} \tag{2.1}$$

is single-valued and continuous. Moreover, the map $\Phi : X \to \mathbb{R}$, $\Phi(x) = \frac{1}{2}\|x\|^2$, is Fréchet differentiable and $\Phi'(x) = J(x)$.

Example 2.1. If $X = L^p(\Omega)$, $\Omega \subset \mathbb{R}^n$, $1 < p < +\infty$, the map $\Phi : L^p(\Omega) \to \mathbb{R}$

$$\Phi(u) = \frac{1}{2}\|u\|_p^2 = \frac{1}{2}\left(\int_\Omega |u(x)|^p\,dx\right)^{\frac{2}{p}}$$

is Fréchet differentiable and $\Phi'(u) = J(u)$, where the duality mapping $J : L^p(\Omega) \to L^{p'}(\Omega)$, $\frac{1}{p} + \frac{1}{p'} = 1$ is defined by

$$\langle J(u), v\rangle = \frac{1}{\|u\|_p^{p-2}} \int_\Omega |u(x)|^{p-2} u(x)v(x)\,dx \tag{2.2}$$

see e.g. [25, Example 1.4.4].

The *normalized upper semi-inner product* on X is the function $[\cdot,\cdot]_+ : X \times X \to \mathbb{R}$ defined by

$$[x, y]_+ = \lim_{h\to 0^+} \frac{\|x + hy\| - \|x\|}{h}, \quad x, y \in X,$$

moreover, if X^* is uniformly convex,

$$[x, y]_+ = \begin{cases} \|y\| & \text{if } x = 0 \\ \frac{1}{\|x\|}\langle J(x), y\rangle & \text{if } x \neq 0 \end{cases} \quad x, y \in X,$$

(see [25, Lemma 1.4.1, Definition 1.4.2, Lemma 1.4.3]). Consider the following linear, nonhomogeneous Cauchy problem

$$\begin{cases} x'(t) = Ax(t) + h(t) \\ x(0) = \xi_0 \end{cases} \tag{2.3}$$

where $A : D(A) \subset X \to X$ generates a strongly continuous semigroup (C_0-semigroup) $\{S(t)\}_{t\geq 0}$ of contractions i.e. such that $\|S(t)\| \leq 1$ for all $t \geq 0$ (see e.g. [16]), $\xi_0 \in X$ and $h \in L^1([0,T],X)$. Following [25, Definition 1.7.4] we give the following definitions of generalized solutions of the problem (2.3).

Definition 2.2. A function $x \in C([0,T],X)$ is said

- a *mild solution* of (2.3) if
$$x(t) = S(t)\xi_0 + \int_0^t S(t-s)h(s)ds,$$
for every $t \in [0,T]$;
- an *integral solution* of (2.3) if $x(0) = \xi_0$ and
$$\|x(t) - \xi\| \leq \|x(s) - \xi\| + \int_s^t [x(\tau) - \xi, h(\tau) + A\xi]_+\, d\tau, \tag{2.4}$$
for every $\xi \in D(A)$ and $s, t \in [0,T]$, $0 \leq s \leq t \leq T$.

Theorem 2.3. *A function $x \in C([0,T],X)$ is a mild solution of (2.3) if and only if it is an integral solution of (2.3) and $x(0) = \xi_0$.*

Proof. It is a consequence of [17, Theorem 3.4.2], [25, Theorem 1.7.3 and Theorem 1.8.2]. □

In our discussion we will use measures of non-compactness (m.n.c. for short) in Banach spaces. We recall that, given a non empty subset C of the Banach space X, the *Hausdorff m.n.c.* of C (see e.g. [14, Chap. 2]) is the function $\chi : \mathcal{P}(X) \to [0,+\infty]$ defined by

$$\chi(C) = \inf\left\{ \epsilon > 0 : \exists x_1, \ldots x_k \in X \text{ such that } C \subset \bigcup_{i=1}^k B_\epsilon(x_i) \right\}.$$

if C is bounded and $\chi(C) = +\infty$ if C is unbounded.

The following properties of χ easily follow from the definition.

Proposition 2.4. *If $\chi : \mathcal{P}(X) \to [0,+\infty]$ is the Hausdorff m.n.c. defined above then*

(i) *$\chi(C) = 0$ if and only if C is relatively compact;*
(ii) *if $C_1 \subset C_2 \subset X$ then $\chi(C_1) \leq \chi(C_2)$;*
(iii) *for every $C_1, C_2 \subset X$, $\chi(C_1 \cup C_2) \leq \max\{\chi(C_1), \chi(C_2)\}$;*
(iv) *for every $C_1, C_2 \subset X$, $\chi(C_1 + C_2) \leq \chi(C_1) + \chi(C_2)$;*
(v) *if Y is a Banach space and $\Phi : X \to Y$ is a Lipschitz function with constant L, then for every $C \subseteq X$, $\chi(\Phi(C)) \leq L\chi(C)$;*
(vi) *for every set $C \subset X$, $\chi(C) = \chi\left(\cup_{\lambda\in[0,1]}\lambda C\right)$.*

The following is a consequence of a more general result [14, Theorem 4.2.2 and Corollary 4.2.4] in the case of a C_0-semigroup of contractions.

Theorem 2.5. *Let $\{S(t)\}_{t\geq 0}$ be a C_0-semigroup of contractions and F be the linear operator from $L^1([0,T],X)$ to $C([0,T],X)$ defined by*

$$F(f)(t) = \int_0^t S(t-s)f(s)\,ds, \quad f \in L^1([0,T],X) \text{ and } t \in [0,T]. \tag{2.5}$$

Let $q \in L^1(0,T)$ and $\{f_n\}_n \subset L^1([0,T],X)$ be such that

$$\chi\left(\{f_n(t)\}_n\right) \le q(t), \qquad \text{for a.e. } t \in [0,T].$$

Then

$$\chi\left(\{F(f_n)(t)\}_n\right) \le 2\int_0^t q(s)\,ds, \qquad \text{for every } t \in [0,T].$$

If the Banach space X is separable, then

$$\chi\left(\{F(f_n)(t)\}_n\right) \le \int_0^t q(s)\,ds, \qquad \text{for every } t \in [0,T].$$

In the sequel we will consider the following m.n.c. on the subsets of continuous function (see [14, Ex. 2.1.4]). For every bounded set $\Omega \subset C([a,b],X)$.

$$\nu(\Omega) = \max_{\{x_n\}_n \subset \Omega}\left(\sup_{t\in[a,b]} \chi\left(\{x_n(t)\}_n\right), \mathrm{mod}_C(\{x_n\}_n)\right) \in \mathbb{R}^2_+ \tag{2.6}$$

where the maximum is taken with respect to the ordering induced by the cone $\mathbb{R}^2_+$ and mod_C is the modulus of equicontinuity defined by

$$\mathrm{mod}_C(\Omega) = \lim_{\delta\to 0}\sup_{x\in\Omega}\max_{|t_1-t_2|<\delta}\|x(t_1)-x(t_2)\|.$$

The m.n.c. ν is regular, that is $\nu(C)=0$ if and only if C is a relatively compact subset of $C([a,b],X)$

Definition 2.6. Given a Banach space E, a set $F \subset E$, a m.n.c. β, the multivalued mapping $T : F\times[0,1] \multimap E$ is called *condensing* with respect to β, *β-condensing* for short, if for every $\Omega \subseteq F$

$$\beta(T(\Omega,[0,1])) \ge \beta(\Omega) \;\Rightarrow\; \Omega \text{ is relatively compact.}$$

Our existence results are based on the following fixed-point theorems.

Theorem 2.7. *Assume that Q is a closed and convex subset of $C([a,b],X)$ with non empty interior and $\mathcal{T} : Q\times[0,1] \multimap C([a,b],X)$ is such that:*

(1) *$\mathcal{T}(q,\lambda)$ is convex, for every $q \in Q$ and $\lambda \in [0,1]$;*
(2) *the graph of $\mathcal{T}$ is closed;*
(3) *$\mathcal{T}$ is compact;*
(4) *$\{x \in Q : x \in \mathcal{T}(x,\lambda)$ for some $\lambda \in [0,1)\} \cap \partial Q = \emptyset$;*
(5) *$\mathcal{T}(\cdot,0) \equiv \{\bar{x}\}$, $\bar{x} \in \mathring{Q}$.*

Then there exists $x \in Q$ such that $x \in \mathcal{T}(x,1)$.

Proof. It is a particular case of Theorem 2.23 in [26]. □

Theorem 2.8. *Assume that Q is a closed and convex subset of $C([a,b],X)$ with non empty interior and $\mathcal{T} : Q\times[0,1] \multimap C([a,b],X)$ is such that:*

(1) *$\mathcal{T}(q,\lambda)$ is compact and convex, for every $q \in Q$ and $\lambda \in [0,1]$;*
(2) *$\mathcal{T}$ is u.s.c.;*
(3) *$\mathcal{T}$ is $\beta-$condensing, where β is a nonsingular, monotone m.n.c. on $C([a,b],X)$;*
(4) *$\{x \in Q : x \in \mathcal{T}(x,\lambda)$ for some $\lambda \in [0,1)\} \cap \partial Q = \emptyset$;*
(5) *$\mathcal{T}(\cdot,0) \equiv \{\bar{x}\}$, $\bar{x} \in \mathring{Q}$.*

Then there exists $x \in Q$ such that $x \in \mathcal{T}(x,1)$.

Proof. It follows from the definition of a topological degree for β-condensing multifields (see [14, Chap. 3] and in particular Theorem 3.3.2). □

3. Existence results for strongly continuous semigroups of contractions

In this section we consider the abstract Cauchy problem given by (1.3) with nonlocal initial condition

$$x(0) \in x_0 + g(x) \tag{3.1}$$

in a Banach space X with X^* uniformly convex.

Throughout this section we assume:

(A) $A : D(A) \subset X \to X$ is a linear, not necessarily bounded, operator that generates a C_0-semigroup of contractions $\{S(t)\}_{t\geq 0}$ on X;

(f) $f : [0,T] \times X \to X$ is a function satisfying the following assumptions:

- (f_1) the function $f(\cdot, x) : [0,T] \to X$ is measurable with respect to the Lebesgue measure on $[0,T]$ for every $x \in X$;
- (f_2) the function $f(t, \cdot) : X \to X$ is continuous, for almost every $t \in [0,T]$;
- (f_3) for every $\rho > 0$ there exists a function $\ell_\rho \in L^1(0,T)$ such that $\|f(t,x)\| \leq \ell_\rho(t)$ for a.e. $t \in [0,T]$ and every $x \in X$ with $\|x\| \leq \rho$;

(g) $g : C([0,T], X) \multimap X$ is a multivalued function such that

- (g_1) $g(q)$ is convex, for every $q \in C([0,T], X)$;
- (g_2) there exists $R > 0$ such that $\|x_0\| < R$ and $x_0 + g\left(C([0,T], R\bar{B})\right) \subseteq R\bar{B}$;
- (g_3) g has a closed graph;
- (g_4) there exists $k_g \geq 0$ such that for every bounded subset $C \subset C([0,T], X)$, $\chi(g(C)) \leq k_g \sup_{t\in[0,T]} \chi\{q(t) : q \in C\}$.

In order to apply topological degree methods, we consider the following transversality condition: there exists $\varepsilon > 0$ such that for a.e. $t \in [0,T]$

$$\langle J(x), f(t,x)\rangle \leq 0, \qquad \text{for every } x,\ R-\varepsilon < \|x\| < R, \tag{3.2}$$

where $J : X \to X^*$ is defined in (2.1) and R is the radius of the ball in (g_2).

Definition 3.1. We say that $x \in C([0,T], X)$ is a *mild solution* of the nonlocal problem (1.3), (3.1) if there exists $\gamma \in g(x)$ such that

$$x(t) = S(t)(x_0 + \gamma) + \int_0^t S(t-s) f(s, x(s)) ds.$$

for every $t \in [0,T]$.

In this general setting, we will prove the following existence result.

Theorem 3.2. *Let conditions (A), (f), (g) and (3.2) be satisfied. Suppose in addition that*

(H1) *there exists $k_f \in L^1(0,T)$ such that $\chi(f(t,E)) \leq k_f(t)\chi(E)$ for almost every $t \in [0,T]$ and every $E \subset R\bar{B}$, where R appears in (g_2);*

(H2) $k_g + 2\|k_f\|_1 < 1$.

Then the problem (1.3)–(3.1) has a mild solution.

Proof. The set $Q = C([0,T], R\bar{B})$ is a closed convex subset of $C([0,T], X)$ with non empty interior. Let us consider the homotopy $\mathcal{T} : Q \times [0,1] \multimap C([0,T], X)$ defined by

$$\mathcal{T}(q,\lambda)(t) = S(t)[x_0 + \lambda g(q)] + \lambda \int_0^t S(t-s) f(s, q(s)) ds, \quad t \in [0,T].$$

A mild solution of (1.3)–(3.1) is a fixed point of $\mathcal{T}(\cdot, 1)$. If $\mathcal{T}$ satisfies assumptions (1)–(5) of Theorem 2.8, then problem (1.3)–(3.1) has a mild solution.

(1) By (g_1), (g_3) and (g_4), for every $q \in C([0,T], X)$, $g(q)$ is compact and convex. Since $S(t)$ is continuous and linear, also $\mathcal{T}(q,\lambda)$ is compact and convex for every $q \in C([0,T], X)$ and $\lambda \in [0,1]$.

(2) If $\mathcal{T}$ is quasicompact (i.e. $\mathcal{T}$ maps compact sets into relatively compact sets) and its graph is closed, then by [14, Theorem 1.1.12], $\mathcal{T}$ is upper semicontinuous.

Consider $y_n \in \mathcal{T}(q_n, \lambda_n)$, $\{q_n\} \subset Q$ and $\lambda_n \in [0,1]$, such that $\lambda_n \to \bar{\lambda}$, $y_n \to \bar{y}$ and $q_n \to \bar{q}$ in $C([0,T], X)$. We have to prove that $\bar{y} \in \mathcal{T}(\bar{q}, \bar{\lambda})$. By the definition of $\mathcal{T}$, for every n there exists $\mu_n \in g(q_n)$ such that

$$y_n(t) = S(t)[x_0 + \lambda_n \mu_n] + \lambda_n \int_0^t S(t-s) f(s, q_n(s)) ds, \quad t \in [0,T]. \tag{3.3}$$

Since $\chi\left(\{q_n(t)\}_n\right) = 0$ for every $t \in [0,T]$, by (g_4) we have

$$\chi(\{\mu_n\}_n) \leq \chi\left(g(\{q_n\}_n)\right) \leq k_g \sup_{t\in[0,T]} \chi\left(\{q_n(t)\}_n\right) = 0 \tag{3.4}$$

hence, up to a subsequence, $\mu_n \to \bar{\mu}$ and $\bar{\mu} \in g(\bar{q})$ by the closure of the graph of g. Now, by (f_3),

$$\|S(t-\cdot) f(\cdot, q_n(\cdot))\| \leq \ell_R(\cdot) \in L^1(0,t) \tag{3.5}$$

for every $t \in [0,T]$; hence, passing to the limit as $n \to \infty$ in (3.3), we have

$$\bar{y}(t) = S(t)[x_0 + \bar{\lambda}\bar{\mu}] + \bar{\lambda} \int_0^t S(t-s) f(s, \bar{q}(s)) ds, \qquad \text{for every } t \in [0,T], \tag{3.6}$$

proving that $\bar{y} \in \mathcal{T}(\bar{q}, \bar{\lambda})$, so the graph of $\mathcal{T}$ is closed.

To prove that $\mathcal{T}$ is quasicompact it is enough to prove that given $\{q_n\}_n \subset Q$ and $\lambda_n \in [0,1]$, such that $\lambda_n \to \bar{\lambda}$ and $q_n \to \bar{q}$ in $C([0,T],X)$, for every $\{y_n\}_n$, $y_n \in \mathcal{T}(q_n, \lambda_n)$, there exists a subsequence $\{y_{n_k}\}_k$ converging in $C([0,T],X)$.

As before, we deduce from (3.3) and (3.4) that there exists $\mu_{n_k} \to \bar{\mu} \in g(\bar{q})$ and y_{n_k} converges pointwise to $\bar{y}$ defined by (3.6). Moreover, the sequence $\{S(\cdot)(x_0 + \lambda_n\mu_n)\}_n$ is relatively compact in $C([0,T],X)$.

As in (3.5) we obtain that $\|f(t, q_n(t))\| \le \ell_R(t)$ for a.e. $t \in [0,T]$ and every $n \in \mathbb{N}$; hence the sequence $\{f(\cdot), q_n(\cdot)\}_n$ is integrably bounded. Moreover, by (H1),

$$\chi(\{f(t, q_n(t))\}_n) \le k_f(t)\chi(\{q_n(t)\}_n) = 0, \quad \text{for a.e. } t \in [0,T].$$

Therefore $\{f(\cdot, q_n(\cdot))\}_n$ is a semicompact sequence. According to [14, Theorem 5.1.1] and the convergence of $\{\lambda_n\}_n$ we obtain that $\left\{\lambda_n \int_0^{\cdot} S(\cdot - s) f(s, q_n(s))ds\right\}_n$ is relatively compact in $C([0,T],X)$. In conclusion $\mathcal{T}$ is quasicompact.

(3) Let Ω be a subset of Q such that $\nu(\mathcal{T}(\Omega, [0,1])) \ge \nu(\Omega)$, where ν is defined in (2.6). We aim to prove (see Definition 2.6) that Ω is relatively compact.

By (2.6) there exists a sequence $\{x_n\}_n \subset \mathcal{T}(\Omega, [0,1])$ such that

$$\nu(\mathcal{T}(\Omega, [0,1])) = \left(\sup_{t\in[0,T]} \chi\left(\{x_n(t)\}_n\right), \mathrm{mod}_C\left(\{x_n\}_n\right) \right).$$

Therefore for every $\{w_n\}_n \subset \Omega$

$$\begin{cases} \sup_{t\in[0,T]} \chi\left(\{x_n(t)\}_n\right) \ge \sup_{t\in[0,T]} \chi\left(\{w_n(t)\}_n\right) \\ \mathrm{mod}_C\left(\{x_n\}_n\right) \ge \mathrm{mod}_C\left(\{w_n\}_n\right). \end{cases} \tag{3.7}$$

By the definition of $\mathcal{T}$, for every $n \in \mathbb{N}$ there exist $q_n \in \Omega$, $\mu_n \in g(q_n)$ and $\lambda_n \in [0,1]$ such that

$$x_n(t) = S(t)[x_0 + \lambda_n\mu_n] + \lambda_n \int_0^t S(t-s)f(s, q_n(s))ds, \quad t \in [0,T].$$

Applying properties of χ stated in Proposition 2.4, Theorem 2.5, (H1), (f_3) and (g_4) we have for every $t \in [0,T]$ and $s \in [0,t]$

$$\chi\left(\left\{S(t-s)f(s, q_n(s))\right\}_n\right) \le k_f(s)\chi\left(\left\{q_n(s)\right\}_n\right) \le Rk_f(s)$$

and

$$\begin{aligned} &\chi\left(\{x_n(t)\}_n\right) \\ &\quad \le \chi\left(\bigcup_{\lambda\in[0,1]} \lambda\{S(t)\mu_n\}_n \right) + \chi\left(\bigcup_{\lambda\in[0,1]} \lambda \left\{ \int_0^t S(t-s)f(s, q_n(s))ds \right\}_n \right) \\ &\quad \le \chi\left(\{S(t)g(q_n)\}_n\right) + \chi\left(\left\{ \int_0^t S(t-s)f(s, q_n(s))ds \right\}_n \right) \\ &\quad \le k_g \sup_{t\in[0,T]} \chi\left(\{q_n(t)\}_n\right) + 2\int_0^t k_f(s)\,ds \sup_{t\in[0,T]} \chi\left(\left\{q_n(t)\right\}_n\right) \\ &\quad = \left(k_g + 2\|k_f\|_1\right) \sup_{t\in[0,T]} \chi\left(\left\{q_n(t)\right\}_n\right). \end{aligned}$$

Finally, by (3.7) and (H2),

$$\begin{aligned} \sup_{t\in[0,T]} \chi\left(\left\{q_n(t)\right\}_n\right) &\le \sup_{t\in[0,T]} \chi\left(\{x_n(t)\}_n\right) \\ &\le \left(k_g + 2\|k_f\|_1\right) \sup_{t\in[0,T]} \chi\left(\left\{q_n(t)\right\}_n\right) < \sup_{t\in[0,T]} \chi\left(\left\{q_n(t)\right\}_n\right) \end{aligned} \tag{3.8}$$

that is $\sup_{t\in[0,T]} \chi\left(\left\{q_n(t)\right\}_n\right) = 0$ and $\sup_{t\in[0,T]} \chi\left(\{x_n(t)\}_n\right) = 0$.

We will show that the functions $\{x_n\}_n$ are equicontinuous. As in (3.4) we have that $\{\mu_n\}_n$ and hence $\{x_0 + \lambda_n\mu_n\}_n$ is relatively compact. Therefore the maps $t \mapsto S(t)[x_0 + \lambda_n\mu_n]$, $t \in [0,T]$, $n \in \mathbb{N}$ form a relatively compact set in $C([0,T],X)$. Setting for $n \in \mathbb{N}$

$$y_n(t) = \int_0^t S(t-s)f(s, q_n(s))ds, \quad t \in [0,T],$$

as in (3.5) we have that $f(\cdot, q_n(\cdot))$ are integrably bounded in $[0,T]$, moreover $\chi\{f(t, q_n(t))\}_n \le L(t)\chi\left\{q_n(t)\right\}_n = 0$ for every $t \in [0,T]$, hence, by [14, Theorem 5.1.1], $\{y_n\}_n$ and $\{\lambda_n y_n\}_n$ are relatively compact in $C([0,T],X)$. Therefore $\{x_n\}_n$ is relatively compact and by Ascoli–Arzelà theorem the functions x_n are equicontinuous, so we have $\mathrm{mod}_C\left(\{x_n\}_n\right) = 0$.

By (3.7) for every $\{w_n\}_n \subset \Omega$

$$\begin{cases} \sup_{t\in[0,T]} \chi\left(\{w_n(t)\}_n\right) = 0 \\ \mathrm{mod}_C\left(\{w_n\}_n\right) = 0 \end{cases}$$

that is $\nu(\Omega) = 0$, proving that Ω is relatively compact in $C([0,T], X)$.
(4) Let $q_0 \in Q$ and $\lambda_0 \in [0,1)$ be such that $q_0 \in \mathcal{T}(q_0, \lambda_0)$, that is there exists $\mu_0 \in g(q_0)$ such that

$$q_0(t) = S(t)[x_0 + \lambda_0\mu_0] + \lambda_0 \int_0^t S(t-s)f(s, q_0(s))ds, \quad t \in [0,T]. \tag{3.9}$$

Notice that, by (g_2) and $\lambda_0 < 1$,

$$\|q_0(0)\| = \|x_0 + \lambda_0\mu_0\| \le \lambda_0\|x_0 + \mu_0\| + (1-\lambda_0)\|x_0\| < \lambda_0 R + (1-\lambda_0)R = R. \tag{3.10}$$

We have to prove that $\|q_0(t)\| < R$ for every $t \in [0,T]$.

If $\lambda_0 = 0$, $\|q_0(t)\| = \|S(t)x_0\| \le \|x_0\| < R$, for every $t \in [0,T]$.

If $0 < \lambda_0 < 1$, suppose by contradiction that $\max_{t\in[0,T]} \|q_0(t)\| = R$. Let define $t_0 = \min\{t \in [0,T] : \|q_0(t)\| = R\}$. By (3.10), $t_0 > 0$, moreover $\|q_0(t_0)\| = R$ and $\|q_0(t)\| < R$ for every $t \in [0,t_0)$. By continuity there exists $\delta > 0$ such that $R - \varepsilon < \|q_0(t)\| < R$ for every $t \in [t_0 - \delta, t_0)$, where ε is the positive constant defined in (3.2).

By (3.9), q_0 is a mild solution of the linear Cauchy problem (2.3), with $\xi_0 = x_0 + \lambda_0\mu_0$ and $h(\cdot) = f(\cdot, q_0(\cdot))$. Therefore, by Theorem 2.3, q_0 satisfies (2.4). In particular for $\xi = 0$, $t = t_0$ and $s = t_0 - \delta$ we obtain

$$\begin{aligned} 0 < R - \|q_0(t_0-\delta)\| = \|q_0(t_0)\| - \|q_0(t_0-\delta)\| &\le \int_{t_0-\delta}^{t_0} \left[q_0(\tau), f(\tau, q_0(\tau))\right]_+ d\tau \\ &\le \int_{t_0-\delta}^{t_0} \frac{1}{\|q_0(\tau)\|} \langle J(q_0(\tau)), f(\tau, q_0(\tau))\rangle\, d\tau \le 0 \end{aligned}$$

and we get a contradiction. Therefore we have proved that $\mathcal{T}$ has no fixed points on ∂Q. Notice that the last inequality is due to condition (3.2).
(5) For every $q \in Q$, $t \in [0,T]$, since $S(t)$ is a contraction and $\|x_0\| < R$ (see (g_2)), $\mathcal{T}(q,0)(t) = \{S(t)x_0\} \subset \mathring{Q}$. □

When X is separable, the previous result hold true with a condition weaker than (H2).

Corollary 3.3. *Let conditions* (A), (f), (g), (3.2) *and (H1) hold. Suppose in addition that* X *is separable and*

$$k_g + \|k_f\|_1 < 1. \tag{3.11}$$

Then the problem (1.3)–(3.1) *has a mild solution.*

Proof. In view of Theorem 2.5, if X is separable inequalities (3.8) become

$$\begin{aligned} \sup_{t\in[0,T]} \chi\left(\{q_n(t)\}_n\right) &\le \sup_{t\in[0,T]} \chi\left(\{x_n(t)\}_n\right) \\ &\le \left(k_g + \|k_f\|_1\right) \sup_{t\in[0,T]} \chi\left(\{q_n(t)\}_n\right) < \sup_{t\in[0,T]} \chi\left(\{q_n(t)\}_n\right). \end{aligned}$$

So the proof is the same as that of the previous theorem. □

Remark 3.4. Notice that, if the multimap g is compact, then $k_g = 0$. In this case, to prove the condensivity of $\mathcal{T}$ (step (3) in the proof of Theorem 3.2), it is possible to use the same techniques as in [14, Theorem 5.1.3] concluding that in Theorem 3.2 and in Corollary 3.3 the result holds true without assuming (H2) and (3.11) respectively.

Remark 3.5. Assumption (H1) is satisfied, for instance, il f is Lipschitz continuous in its second variable on bounded sets:

(f_2') for every $\rho > 0$ there exists $L_\rho \in L^1(0,T)$ such that $\|f(t,x) - f(t,y)\| \le L_\rho(t)\|x-y\|$ for a.e. $t \in [0,T]$ and for every $x, y \in \rho B$.

In this case $k_f = L_R$ with R introduced in (g_2).

Nevertheless, there are functions satisfying (H1), but not (f_2'). For example if $f = f_1 + f_2$, where only f_1 satisfies (f_2'), but f_2 is compact. In this latter case the growth of $\|f(t,x)\|$ need not be sublinear with respect to $\|x\|$. Consider, for example, $X = L^p(a,b)$, $1 \le p \le +\infty$, and $f : X \to X$ defined by

$$f(\eta)(y) = \left(\int_a^y \eta(s)\,ds\right)^2, \quad y \in [a,b].$$

The map f is compact in $L^p(a,b)$. In fact, if $\{\eta_n\}_n$ is bounded in $L^p(a,b)$, the corresponding sequence $\{f(\eta_n)\}_n$ is in $C([a,b])$ and it satisfies Ascoli Arzelà theorem. Therefore $\{f(\eta_n)\}_n$ is relatively compact in $C([a,b])$ and also in $L^p(a,b)$.

Consequently (H1) holds with $k_f = 0$, and along some directions (for instance along constant functions) f has a quadratic growth at infinity. Hence we showed that (H1) does not imply that the growth at infinity of $\|f(t,x)\|$ is dominated by a linear function of $\|x\|$.

A further family of functions satisfying (H1) but not (f_2') can be found in the following example

Example 3.6. Let $h : [0,T]\times X \to X$ be a Carathéodory function satisfying (H1) (with L^1-function k_h), $\Psi : \mathbb{R} \to \mathbb{R}$ be a continuous and bounded function and x^* be a given element in the dual X^* of X. Consider the map $f : [0,T]\times X \to X$ defined by

$$f(t,x) = \Psi(\langle x^*, x\rangle)h(t,x)$$

This is obviously a Carathéodory function but, in general, it is not locally Lipschitz continuous in its second variable. Moreover, for a.e. $t \in [0,T]$ and for every set $E \subseteq R\bar{B}$,

$$f(t,E) \subseteq \cup_{\lambda\in[0,1]}\lambda\bar{\psi}\, h(t,E) \cup \cup_{\lambda\in[0,1]}\lambda(-\bar{\psi})h(t,E),$$

where $\bar{\psi} = \sup_{\mathbb{R}} |\Psi|$. Therefore, by (iii) and (vi) in Proposition 2.4 we have that

$$\chi(f(t,E)) \le \bar{\psi}\, \chi(h(t,E)) \le \bar{\psi}\, k_h(t)\chi(E)$$

and hence f satisfies (H1).

When f is Lipschitz continuous in its second variable, less restrictive conditions are required for studying (1.3)–(3.1) (see also Remark 3.8).

Theorem 3.7. *Let conditions* (A), (f_1), (f_2'), (g_1), (g_2), (g_3) *and* (3.2) *hold. Suppose in addition that*

(H3) $f(\cdot,0) \in L^1(0,T)$;
(H4) *there exists* $k_g \ge 0$ *such that for every bounded subset* $C \subset C([0,T],X)$, $\chi(g(C)) \le k_g\chi(C)$;
(H5) $k_g + \|L_R\|_1 < 1$, *where* R *appears in* (g_2).

Then the problem (1.3)–(3.1) *has a mild solution.*

Remark 3.8. It is easy to prove ([14, Ex. 2.1.3]) that assumption (g_4) is stronger than (H4).

Proof of Theorem 3.7. Assumption (f_2') implies (f_2). Moreover, by (f_2') and (H3) we have

$$\|f(t,x)\| \le \|f(t,x) - f(t,0)\| + \|f(t,0)\| \le L_\rho(t)\|x\| + \|f(t,0)\|$$

for every $x \in \rho\bar{B}$ and almost every $t \in [0,T]$, therefore assumption (f_3) holds for $\ell_\rho(\cdot) = L_\rho(\cdot)\rho + \|f(\cdot,0)\|$.

In Theorem 3.2, assumptions (g_4), (H1) and (H2) are involved in the proof of points (1) and (3).

Notice that (g_3) and (H4) imply that $g(q)$ is compact for every $q \in C([0,T],X)$, so also step (1) in the proof of Theorem 3.2 can be proved in the same way.

Therefore we have only to prove (3), that is the condensivity of the solution operator $\mathcal{T}$ in the present hypotheses.

Let Ω be a subset of Q such that $\chi(\mathcal{T}(\Omega,[0,1])) \ge \chi(\Omega)$. By Proposition 2.4(iv) we deduce

$$\chi(\mathcal{T}(\Omega,[0,1])) \le \chi(S(\cdot)g(\Omega)) + \chi\left(\left\{\int_0^\cdot S(\cdot - s)f(s,y(s))\,ds :\ y \in \Omega\right\}\right).$$

By (H4) and (v) of Proposition 2.4

$$\chi(S(\cdot)g(\Omega)) \le k_g\chi(\Omega). \tag{3.12}$$

As to the second summand we claim that

$$\chi\left(\left\{\int_0^\cdot S(\cdot - s)f(s,y(s))\,ds :\ y \in \Omega\right\}\right) \le \|L_R\|_1\chi(\Omega). \tag{3.13}$$

In fact, let $\{q_1,\dots,q_m\}$ be a ε-net for Ω. Let us consider the functions y_k in $C([0,T],X)$ defined by

$$y_k(t) = \int_0^t S(t-s)f(s,q_k(s))\,ds, \qquad t \in [0,T],\ k = 1,\dots,m.$$

By (f_2') we easily deduce that $\{y_1,\dots,y_m\}$ is a $\left(\|L_R\|_1\varepsilon\right)$-net for the set

$$\left\{\int_0^\cdot S(\cdot - s)f(s,y(s))\,ds :\ y \in \Omega\right\},$$

proving the claim.

Summing (3.12) and (3.13) we obtain

$$\chi(\Omega) \le \chi(\mathcal{T}(\Omega,[0,1])) \le (k_g + \|L_R\|_1)\chi(\Omega).$$

By (H5) this is possible only if $\chi(\Omega) = 0$, therefore we conclude that Ω is relatively compact and $\mathcal{T}$ is χ-condensing. □

4. Existence results for compact semigroups

As in the previous section, let X be a Banach space with X^* uniformly convex. We remark that X^*, and hence X is a reflexive space. In this section we will discuss the nonlocal Cauchy problem (1.3)–(3.1) with the additional assumption that A generates a compact semigroup.

Throughout this section we will consider (A), (f) and the following assumptions on g:

(g') $g : C([0,T],X) \multimap X$ is a multivalued function satisfying the conditions (g_1), (g_2) and

(g_3') for every $\{x_n\}_n \subset C([0,T],R\bar{B})$ such that $x_n \to x \in C([0,T],R\bar{B})$ pointwise in $(0,T]$ and for every $\mu_n \in g(x_n)$, there exists a subsequence $\{\mu_{n_k}\}_k$, $\mu_{n_k} \to \mu \in g(x)$; the value R appears in (g_2).

As in [3, Theorem 3.1] we will consider the family of homotopies $\mathcal{T}_m : Q\times[0,1] \multimap C([0,T],X)$, for every integer m greater than $1/T$, defined by

$$\mathcal{T}_m(q,\lambda)(t) = \begin{cases} S\left(\frac{1}{m}\right)[x_0 + \lambda g(q)] & \text{if } t \in \left[0,\frac{1}{m}\right] \\ S(t)[x_0+\lambda g(q)] + \lambda \int_{\frac{1}{m}}^{t} S(t-s)f(s,q(s))ds & \text{if } t \in \left(\frac{1}{m},T\right] \end{cases}$$

where $Q = C([0,T],R\bar{B})$. Then we will obtain the solution to (1.3)–(3.1) by passing to the limit in a sequence $\{x_m\}_m$ of fixed points of $\mathcal{T}_m(\cdot,1)$.

The following proposition is a fixed point result for $\mathcal{T}_m(\cdot,1)$.

Proposition 4.1. *Let conditions (A), (f), (g') and (3.2) hold. Suppose in addition that A generates a compact semigroup. Then there exists $x_m \in \mathcal{T}_m(x_m,1)$.*

Proof. We aim to prove that all the homotopies $\mathcal{T}_m$ satisfy conditions (1)–(5) of Theorem 2.7.

Fix $m \in \mathbb{N}$. The proof of condition (5) is as in Theorem 3.2.

(1) By (g_1), for every $q \in C([0,T],X)$, $g(q)$ is convex. Since $S(t)$ is linear, $\mathcal{T}_m(q,\lambda)$ is convex for every $q \in C([0,T],X)$ and $\lambda \in [0,1]$.

(2) We have to prove that the graph of $\mathcal{T}_m$ is closed. Consider, for $n\in\mathbb{N}$, $y_n \in \mathcal{T}_m(q_n,\lambda_n)$, $q_n \in Q$ and $\lambda_n \in [0,1]$, such that $y_n \to \bar{y}$ and $q_n \to \bar{q}$ in $C([0,T],X)$ and $\lambda_n \to \bar{\lambda}$. We have to show that $\bar{y} \in \mathcal{T}_m(\bar{q},\bar{\lambda})$.

Since $y_n \in \mathcal{T}_m(q_n,\lambda_n)$, there exists $\mu_n \in g(q_n)$ such that

$$y_n(t) = \begin{cases} S\left(\frac{1}{m}\right)[x_0 + \lambda_n\mu_n] & \text{if } t \in \left[0,\frac{1}{m}\right] \\ S(t)[x_0+\lambda_n\mu_n] + \lambda_n \int_{\frac{1}{m}}^{t} S(t-s)f(s,q_n(s))ds & \text{if } t \in \left(\frac{1}{m},T\right]. \end{cases} \tag{4.1}$$

Since $\mu_n \in g(q_n)$ and $q_n \to \bar{q}$ in $C([0,T],X)$, by (g_3') there exists a subsequence $\mu_{n_k} \to \bar{\mu} \in g(\bar{q})$. Therefore

$$S(t)[x_0 + \lambda_{n_k}\mu_{n_k}] \to S(t)[x_0+\bar{\lambda}\bar{\mu}] \qquad \text{for every } t \in (0,T],$$

in particular

$$S\left(\frac{1}{m}\right)[x_0+\lambda_{n_k}\mu_{n_k}] \to S\left(\frac{1}{m}\right)[x_0+\bar{\lambda}\bar{\mu}].$$

As to the integral term, by (f_2) and the continuity of $S(t-s)$

$$\lim_{n\to\infty} S(t-s)f(s,q_n(s)) = S(t-s)f(s,\bar{q}(s)) \qquad \text{for a.e. } s \in [1/m,t],$$

for every $t \in [1/m,T]$. Moreover, by (f_3),

$$\|S(t-s)f(s,q_n(s))\| \le \ell_R(s) \qquad \text{for a.e. } s \in [1/m,t].$$

Therefore, by the dominated convergence theorem,

$$\lim_{n\to\infty}\int_{\frac{1}{m}}^{t} S(t-s)f(s,q_n(s))ds = \int_{\frac{1}{m}}^{t} S(t-s)f(s,\bar{q}(s))ds$$

for every $t\in[1/m,T]$. Summing up the previous results we have

$$y_{n_k}(t) \to z(t) = \begin{cases} S\left(\frac{1}{m}\right)[x_0+\bar{\lambda}\bar{\mu}] & \text{if } t \in \left[0,\frac{1}{m}\right] \\ S(t)[x_0+\bar{\lambda}\bar{\mu}] + \bar{\lambda}\int_{\frac{1}{m}}^{t} S(t-s)f(s,\bar{q}(s))ds & \text{if } t\in\left(\frac{1}{m},T\right] \end{cases}$$

for every $t\in[0,T]$. Since $y_n \to \bar{y}$ in $C([0,T],X)$, we have $\bar{y}(t) = z(t)$ for every $t \in [0,T]$, that is $\bar{y} \in \mathcal{T}_m(\bar{\lambda},\bar{q})$.

(3) We shall prove that $\mathcal{T}_m(Q\times[0,1])$ is relatively compact in $C([0,T],X)$. Let $\{y_n\}_n$ be a sequence in $\mathcal{T}_m(Q\times[0,1])$, that is there exist $\{q_n\}_n \subset Q$, $\{\lambda_n\}_n \subset [0,1]$ and $\{\mu_n\}_n$, $\mu_n \in g(q_n)$ such that (4.1) holds. $S\left(\frac{1}{m}\right)$ is compact and, by (g_2), $x_0+\lambda_n\mu_n \subset R\bar{B}$. Therefore $\left\{S\left(\frac{1}{m}\right)[x_0+\lambda_n\mu_n]\right\}_n$ is relatively compact in X.

By the same argument, for every $t \in \left(\frac{1}{m}, T\right]$, $\left\{S(t)[x_0 + \lambda_n \mu_n]\right\}_n$ is relatively compact.

For a fixed $t \in \left(\frac{1}{m}, T\right]$, let $w_n \in L^1(\left[\frac{1}{m}, t\right], X)$ be defined by

$$w_n(s) = \lambda_n S(t-s) f(s, q_n(s)) \qquad \text{for a.e. } s \in [1/m, t].$$

By (f_3) the sequence $\{w_n\}_n$ is uniformly integrable. Since X is reflexive, $\{w_n\}_n$ is relatively weakly compact in $L^1(\left[\frac{1}{m}, t\right], X)$ (see [27, p. 101]); therefore there exists a subsequence, still denoted by $\{w_n\}_n$, such that $\left\{\int_{1/m}^t w_n(s)\, ds\right\}_n$ converges in X.

Summing up the previous results, we have that $\{y_n(t)\}_n$ is relatively compact in X.

In order to apply Ascoli–Arzelà theorem, it remain to prove that $\{y_n\}_n$ is equicontinuous.

In $[0, 1/m]$ the maps y_n are constant.

For every $t_1, t_2 \in [1/m, T]$, $t_1 < t_2$, by (g_2) we have

$$\|S(t_2)[x_0 + \lambda_n \mu_n] - S(t_1)[x_0 + \lambda_n \mu_n]\| \le R\|S(t_2) - S(t_1)\|.$$

The semigroup $\{S(t)\}_{t\ge 0}$ is compact, hence $S(\cdot)$ is uniformly continuous in $[1/m, T]$ (see [16, Theorem 3.2]) and then $\{S(\cdot)[x_0 + \lambda_n \mu_n]\}_n$ is equicontinuous in $[1/m, T]$.

Let us define the sequence $\{z_n\}_n$ in $C([1/m, T], X)$,

$$z_n(t) = \int_{\frac{1}{m}}^t w_n(s)ds \qquad \text{for every } t \in [1/m, T].$$

Consider $t_1, t_2 \in [1/m, T]$, $t_1 < t_2$. Since $\lambda_n \in [0, 1]$

$$\begin{aligned}\|z_n(t_1) - z_n(t_2)\| &\le \left\|\int_{\frac{1}{m}}^{t_1} [S(t_2 - s) - S(t_1 - s)] f(s, q_n(s)) ds\right\| \\ &\quad + \left\|\int_{t_1}^{t_2} S(t_2 - s) f(s, q_n(s)) ds\right\| \\ &\le \int_{\frac{1}{m}}^{t_1} \left\|S(t_2 - s) - S(t_1 - s)\right\| \ell_R(s) ds + \int_{t_1}^{t_2} \ell_R(s) ds.\end{aligned}$$

For a fixed $\varepsilon > 0$, let $\sigma > 0$ be such that, for every measurable set $E \subset [0, T]$

$$|E| \le \sigma \qquad \Rightarrow \qquad \int_E \ell_R(s) ds < \frac{\varepsilon}{4}.$$

Moreover, by the uniform continuity of $S(\cdot)$ in $[\sigma, T]$, there exists δ, $0 < \delta < \sigma$, such that for every $\tau_1, \tau_2 \in [\sigma, T]$

$$|\tau_2 - \tau_1| \le \delta \qquad \Rightarrow \qquad \|S(\tau_1) - S(\tau_2)\| < \frac{\varepsilon}{4\|\ell_R\|_1}.$$

Notice that, if $t_1, t_2 \in [1/m + \sigma, T]$, $0 < t_2 - t_1 < \delta$, $s \in [1/m, t_1 - \sigma]$, setting $\tau_1 = t_1 - s$ and $\tau_2 = t_2 - s$ we have $\tau_1, \tau_2 \in [\sigma, T]$ and $0 < \tau_2 - \tau_1 < \delta$.

We will show that for any $t_1, t_2 \in [1/m, T]$

$$0 < t_2 - t_1 < \delta \qquad \Rightarrow \qquad \|z_n(t_1) - z_n(t_2)\| < \varepsilon, \quad \text{for every } n.$$

Indeed, consider $t_1, t_2 \in [1/m, T]$, $0 < t_2 - t_1 < \delta$.

If $t_1 \in [1/m, 1/m + \sigma]$, we have $t_1 - 1/m < \sigma$ and

$$\begin{aligned}\|z_n(t_1) - z_n(t_2)\| &\le \int_{\frac{1}{m}}^{t_1} \left\|S(t_2 - s) - S(t_1 - s)\right\| \ell_R(s) ds + \int_{t_1}^{t_2} \ell_R(s) ds \\ &\le 2\int_{\frac{1}{m}}^{t_1} \ell_R(s) ds + \int_{t_1}^{t_2} \ell_R(s) ds < 2\frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \varepsilon.\end{aligned}$$

On the other hand, if $t_1 \in (1/m + \sigma, T]$ we have

$$\begin{aligned}\|z_n(t_1) - z_n(t_2)\| &\le \int_{\frac{1}{m}}^{t_1 - \sigma} \left\|S(t_2 - s) - S(t_1 - s)\right\| \ell_R(s) ds + 2\int_{t_1 - \sigma}^{t_1} \ell_R(s) ds \\ &\quad + \int_{t_1}^{t_2} \ell_R(s) ds < \frac{\varepsilon}{4\|\ell_R\|_1} \int_{\frac{1}{m}}^{t_1} \ell_R(s) ds + 2\frac{\varepsilon}{4} + \frac{\varepsilon}{4} \le \varepsilon.\end{aligned}$$

Therefore $\{z_n\}_n$ and $\{y_n\}_n$ are equicontinuous.

Finally, by Ascoli–Arzelà theorem $\{y_n\}_n$ is relatively compact in $C([0, T], X)$.

(4) Let $q_0 \in Q$ and $\lambda_0 \in [0, 1)$ be such that $q_0 \in \mathcal{T}_m(q_0, \lambda_0)$. We have to prove that $\|q_0(t)\| < R$ for every $t \in [0, T]$. For $t \in [0, 1/m]$, there exists $\mu_0 \in g(q_0)$ such that $q_0(t) = S\left(\frac{1}{m}\right)[x_0 + \lambda_0 \mu_0]$. Since $S\left(\frac{1}{m}\right)$ is a contraction, by (g_2) we can conclude, as in (3.10), that $\|q_0(t)\| < R$, for every $t \in [0, 1/m]$.

For $t \in (1/m, T]$, the same reasoning as in step (4) of the proof of Theorem 3.2 leads to $\|q_0(t)\| < R$, for every $t \in (1/m, T]$.

We have proved that $\mathcal{T}_m$ satisfies all the assumptions of Theorem 2.7, therefore the proof is complete. □

Now we can prove our existence result for compact semigroups.

Theorem 4.2. *Let conditions* (A), (f), (g') *and* (3.2) *hold. Suppose in addition that* A *generates a compact semigroup. Then the problem* (1.3)–(3.1) *has a mild solution.*

Proof. By the previous proposition, there exists a sequence $\{x_m\}_m$ in $C([0,T], R\bar{B})$ such that for every m

$$x_m(t) = \begin{cases} S\left(\frac{1}{m}\right)[x_0 + \mu_m] & \text{if } t \in \left[0, \frac{1}{m}\right] \\ S(t)[x_0 + \mu_m] + \int_{\frac{1}{m}}^t S(t-s)f(s, x_m(s))ds & \text{if } t \in \left(\frac{1}{m}, T\right], \end{cases} \tag{4.2}$$

with $\mu_m \in g(x_m)$; the value R was introduced in (g_2).

We want to prove that a subsequence of $\{x_m\}_m$ converges pointwise to a mild solution of (1.3)–(3.1). By the same reasonings as in step (3) of the proof of Proposition 4.1 we have that $\{x_m\}_m$ is relatively compact in $C([a,T], X)$, for every $a > 0$. Therefore, for every integer i sufficiently large, we recursively define a subsequence $\{x_n^i\}_n$ of $\{x_m\}_m$ such that $\{x_n^i\}_n$ is a subsequence of $\{x_n^{i-1}\}_n$ converging uniformly in $[1/i, T]$. By a standard digonalization method we obtain a subsequence $\{x_{m_k}\}_k$ of $\{x_m\}_m$, pointwise converging to $\bar{x} \in C((0,T], X)$.

The space X is reflexive and, by (g_2), $\mu_m \in -x_0 + R\bar{B}$, therefore there is a subsequence $\mu_{m_k} \rightharpoonup \bar{\mu}$. Since $S(t)$ is compact, for every $t > 0$, $S(t)\mu_{m_k} \to S(t)\bar{\mu}$.

By (4.2), for every fixed $t \in (0, T]$

$$x_{m_k}(t) = S(t)[x_0 + \mu_{m_k}] + \int_{\frac{1}{m_k}}^t S(t-s)f(s, x_{m_k}(s))ds$$

for sufficiently large k.

Therefore by (f_2), (f_3) and the dominated convergence theorem

$$\bar{x}(t) = S(t)[x_0 + \bar{\mu}] + \int_0^t S(t-s)f(s, \bar{x}(s))ds \tag{4.3}$$

for every $t \in (0, T]$. Since, by (4.3), $\bar{x} \in C([0,T], X)$ and, by (g_3'), $\bar{\mu} \in g(\bar{x})$, we conclude that $\bar{x}$ is a mild solution of (1.3)–(3.1), proving the theorem. □

5. Nonlocal transport equations

In this section we consider a semilinear, nonlocal transport equation of the form

$$\begin{cases} u_t(t, y) + a \cdot \nabla u(t, y) = \Phi\left(\int_{\mathbb{R}^n} |u(t, \xi)|^p d\xi\right) \ell(t, u(t, y)) \\ u(0, y) = u_0(y) + \sum_{i=1}^m \beta_i u(t_i, y) \end{cases} \tag{5.1}$$

$y \in \mathbb{R}^n$, $t \in [0, T]$, where $a \in \mathbb{R}^n$, $t_1, \dots, t_m \in [0, T]$, $\beta_1, \dots, \beta_m \in \mathbb{R}$ and $u_0 \in L^p(\mathbb{R}^n)$, $1 < p < \infty$.

Equation in (5.1) is a nonlinear version of the known transport equation (see e.g. [17,28]) which is still intensely studied because of its several applications such as the transport of particles, the study of traffic flows etc. In some cases, as in the equation in (5.1), the nonlinear term contains a nonlocal part (see e.g. [29,30]).

Consider the following assumptions on maps $\Phi : \mathbb{R} \to \mathbb{R}$ and $\ell : [0, T] \times \mathbb{R} \to \mathbb{R}$:

(Φ) Φ is continuous and $\Phi(\mathbb{R}) \subseteq [0, 1]$;
(ℓ_1) the map $\ell(\cdot, \eta) : [0, T] \to \mathbb{R}$ is Lebesgue measurable for every $\eta \in \mathbb{R}$;
(ℓ_2) there exists $L \in L^1(0, T)$ such that $|\ell(t, \eta_1) - \ell(t, \eta_2)| \le L(t)|\eta_1 - \eta_2|$ for a.e. $t \in [0, T]$ and for every $\eta_1, \eta_2 \in \mathbb{R}$;
(ℓ_3) $\eta\, \ell(t, \eta) \le 0$, for a.e. $t \in [0, T]$ and for every $\eta \in \mathbb{R}$.

Using Corollary 3.3 we will prove the following existence result for (5.1).

Theorem 5.1. *Suppose that* Φ *and* h *satisfy the assumptions* (Φ), (ℓ_1)–(ℓ_3). *Therefore, if*

$$\sum_{i=1}^m |\beta_i| + \|L\|_1 < 1 \tag{5.2}$$

the problem (5.1) *admits a solution* $u \in C([0, T], L^p(\mathbb{R}^n))$, $1 < p < \infty$.

Proof. In order to apply Corollary 3.3 we will define the abstract formulation of (5.1) as a nonlocal Cauchy problem (1.3)–(3.1).

The separable Banach space X is $L^p(\mathbb{R}^n)$, $1 < p < \infty$.

The operator $A : D(A) \to L^p(\mathbb{R}^n)$, with $D(A) = W^{1,p}(\mathbb{R}^n)$, is the linear operator $Az = -a \cdot \nabla z$; A is the generator of the C_0 group of contractions

$$S(t)x(y) = x(y - ta) \tag{5.3}$$

for $x \in L^p(\mathbb{R}^n)$, $y \in \mathbb{R}^n$ and $t \in \mathbb{R}$ ([17, Example 4.4.1 and Theorem 4.4.1]); therefore, (A) is proved, moreover this group is not compact.

The function $f : [0,T] \times L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is

$$f(t,x)(y) = \Phi\left(\int_{\mathbb{R}^n} |x(\xi)|^p d\xi\right) \ell(t, x(y))$$

and f is well defined. In fact, by (ℓ_2), the map $y \longmapsto f(t,x)(y)$ is measurable in Ω for all $t \in [0,T]$ and $x \in L^p(\Omega)$; moreover, by (Φ), (ℓ_2) and (ℓ_3) we have that $\ell(t,0) = 0$ for a.e. $t \in [0,T]$ and

$$\int_{\mathbb{R}^n} |f(t,x)(y)|^p\, dy \le \int_{\mathbb{R}^n} |\ell(t,x(y))|^p\, dy \le L(t)^p \int_{\mathbb{R}^n} |x(y)|^p\, dy$$

for every $t \in [0,T]$ and $x \in L^p(\mathbb{R}^n)$.

Concerning the nonlocal condition, $x_0 = u_0(\cdot)$ and $g : C([0,T], L^p(\mathbb{R}^n)) \to L^p(\mathbb{R}^n)$ is the single-valued function defined by

$$g(u)(y) = \sum_{i=1}^m \beta_i u(t_i)(y), \qquad y \in \mathbb{R}^n.$$

It remains to prove that (f), (g), (3.2), (H1) and (3.11) are satisfied.

Assumption (f_1). Since $L^p(\mathbb{R}^n)$, $1 < p < \infty$ is separable, by Petty's Measurability Theorem it is enough to show that, for every $x \in L^p(\mathbb{R}^n)$ and $z \in L^{p'}(\mathbb{R}^n)$, $\frac{1}{p} + \frac{1}{p'} = 1$, the map $t \mapsto \langle f(t,x), z\rangle$ is measurable in $[0,T]$. We have

$$\langle f(t,x), z\rangle = \Phi\left(\int_{\mathbb{R}^n} |x(\xi)|^p d\xi\right) \int_{\mathbb{R}^n} \ell(t, x(y))\, z(y)\, dy.$$

By (ℓ_1) and (ℓ_2) the map $(t,y) \mapsto \ell(t,x(y))$ is measurable. Hence, also the map $\psi : [0,T]\times\mathbb{R}^n \to \mathbb{R}$ defined by $\psi(t,y) = \ell(t,x(y))z(y)$ is measurable. Moreover, by (ℓ_2) and (ℓ_3)

$$|\psi(t,y)| = |\ell(t,x(y))z(y)| \le L(t)|x(y)||z(y)|, \quad t \in [0,T],\ y \in \mathbb{R}^n;$$

therefore ψ is integrable in $[0,T]\times\mathbb{R}^n$. Hence, by Fubini's Theorem, the map

$$t \mapsto \int_{\mathbb{R}^n} \ell(t, x(\xi))\, z(\xi)\, d\xi$$

is measurable in $[0,T]$, proving the measurability of $t \mapsto \langle f(t,x), z\rangle$.

Assumption (f_2). Let $\{x_n\}_n \subset L^p(\mathbb{R}^n)$ be a sequence converging to $\bar{x}$ in L^p-norm. Therefore, for a.e. $t \in [0,T]$,

$$|f(t,x_n)(y) - f(t,\bar{x})(y)| \le \Phi\left(\|x_n\|_p{}^p\right)\left|\ell\left(t,x_n(y)\right) - \ell(t,\bar{x}(y))\right| + \left|\Phi\left(\|x_n\|_p{}^p\right) - \Phi\left(\|\bar{x}\|_p{}^p\right)\right| |\ell(t,\bar{x}(y))|$$

and, by (Φ) and (ℓ_2), we obtain that

$$\|f(t,x_n) - f(t,\bar{x})\|_p \le L(t)\|x_n - \bar{x}\|_p + \left|\Phi\left(\|x_n\|_p{}^p\right) - \Phi\left(\|\bar{x}\|_p{}^p\right)\right| \|\ell(t,\bar{x}(\cdot))\|_p.$$

Since Φ is continuous, $\|f(t,x_n) - f(t,\bar{x})\|_p \to 0$ as $n \to \infty$.

Assumption (f_3). By (Φ), (ℓ_2) and (ℓ_3) we have that, for a.e. $t \in [0,T]$ and for every $x \in L^p(\mathbb{R}^n)$,

$$\|f(t,x)\|_p \le \Phi\left(\int_{\mathbb{R}^n} |x(\xi)|^p d\xi\right) \|\ell(t,x(\cdot))\|_p \le L(t)\|x\|_p.$$

Therefore (f_3) holds with $\ell_\rho = \rho L$.

Assumption (g_1). It is obviously satisfied, since g is singlevalued.

Assumption (g_2). If $u \in C([0,T], L^p(\mathbb{R}^n))$, $\|u\| \le R$, we have

$$\|u_0 + g(u)\|_p \le \|u_0\|_p + R\sum_{i=1}^m |\beta_i|.$$

By (5.2), $\sum_{i=1}^m |\beta_i| < 1$, therefore, there exists $\bar{R}$ such that, for every $R \ge \bar{R}$, $\|u_0 + g(u)\|_p < R$.

Assumption (g_3). The map g is Lipschitz continuous, in fact,

$$\|g(u_2) - g(u_1)\|_p \le \sum_{i=1}^m |\beta_i| \|u_2(t_i) - u_1(t_i)\|_p \le \sum_{i=1}^m |\beta_i| \|u_2 - u_1\|$$

for every $u_1, u_2 \in C([0,T], L^p(\mathbb{R}^n))$. In particular, the graph of g is closed.

Assumption (g_4). Let C be a bounded subset of $C([0,T], L^p(\mathbb{R}^n))$. For every $t \in [0,T]$, we set $C(t) = \{u(t) : u \in C\}$. By the definition of g, $g(C) \subseteq \sum_{i=1}^m \beta_i C(t_i)$, therefore from (iv) and (v) of Proposition 2.4 it follows that

$$\chi(g(C)) \le \sum_{i=1}^m \chi(\beta_i C(t_i)) \le \sum_{i=1}^m |\beta_i| \chi(C(t_i)) \le \sum_{i=1}^m |\beta_i| \sup_{t\in[0,T]} \chi(C(t)).$$

We conclude that (g_4) holds with $k_g = \sum_{i=1}^{m} |\beta_i|$.

Assumption (3.2). By (2.2) and (ℓ_3), for every $x \in L^p(\mathbb{R}^n)$ we have

$$\langle J(x)\,, f(t,x)\rangle = \frac{1}{\|x\|_p^{p-2}} \Phi\left(\int_{\mathbb{R}^n} |x(\xi)|^p d\xi\right) \int_{\mathbb{R}^n} |x(\xi)|^{p-2} x(\xi)\ell(t\,,x(\xi))\, d\xi \leq 0,$$

so (3.2) holds for every $R > 0$.

Assumption (H1). Consider a bounded subset E of $L^p(\mathbb{R}^n)$. As in Example 3.6 we can prove that $\chi(f(t\,,E)) \leq L(t)\chi(E)$ for a.e. $t \in [0\,,T]$. Therefore assumption (H1) holds with $k_f = L$.

Since (3.11) is equivalent to (5.2), all the assumptions of Corollary 3.3 are satisfied and we conclude that problem (5.1) admits a mild solution. □

6. Nonlocal parabolic equations

We will show how our results apply to the Dirichlet problem

$$\begin{cases} u_t(t\,,y) = \Delta u(t\,,y) - bu(t\,,y) + \int_\Omega \eta(y\,,\xi)u(t\,,\xi)\, d\xi + h(t\,,u(t\,,y)) & \text{in } \Omega \\ u(t\,,y) = 0 & \text{in } \partial\Omega \end{cases} \tag{6.1}$$

where $t \in [0\,,T]$, $\Omega \subset \mathbb{R}^n$ is open, bounded, with C^2 boundary, $b > 0$, $\eta \in L^\infty(\Omega \times \Omega)$ and $h : [0\,,T] \times \Omega \to \mathbb{R}$.

We will consider an initial condition of the form

$$u(0\,,y) \in K(\|u(t^*\,,\cdot)\|_1) \int_0^T u(t\,,y)\, d\nu(t) \qquad \text{for a.e. } y \in \Omega, \tag{6.2}$$

where $t^* \in (0\,,T]$ is fixed, $K : [0\,,+\infty) \multimap \mathbb{R}$ is a u.s.c. multimap and ν is a signed Borel measure in $[0\,,T]$ with total variation $|\nu| \leq 1$.

Remark 6.1. Notice that if $K \equiv \{1\}$ and ν is the normalized Lebesgue measure we obtain the mean value condition

$$u(0\,,y) = \frac{1}{T}\int_0^T u(t\,,y)\, dt \qquad \text{for a.e. } y \in \Omega.$$

If $K \equiv \{1\}$ and ν is a linear combination of Dirac masses, $\nu = \sum_{i=1}^{\bar n} \alpha_i \delta_{t_i}$, $|\nu| = \sum_{i=1}^{\bar n} |\alpha_i| \leq 1$, we obtain the multipoint condition

$$u(0\,,y) = \sum_{i=1}^{\bar n} \alpha_i u(t_i\,,y) \qquad \text{for a.e. } y \in \Omega.$$

In order to rewrite problem (6.1)–(6.2) as an abstract Cauchy problem, we identify u with function $t \mapsto u(t,\cdot)$. In the following we denote by A the Laplace operator with domain $D(A) = W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$ for $1 \leq p < +\infty$. It is known that A is the generator of a compact C_0-semigroup of contractions $\{S(t)\}_{t\geq 0}$ (see e.g. [17, Theorem 4.1.3]) which does not depend on p (see [17, Lemma 7.2.1]).

In order to apply the results in Section 4, we take $X = L^p(\Omega)$, $1 < p < +\infty$. Nevertheless, since Ω is a bounded set, we will be able to find solutions of problem (6.1) in $C([0\,,T]\,,L^1(\Omega))$.

We look for a mild solution $u \in C([0\,,T]\,,L^p(\Omega))$ of the problem (1.3)–(3.1), where

$$f(t\,,x)(y) = -bx(y) + \int_\Omega \eta(y\,,\xi)x(\xi)\, d\xi + h(t\,,x(y)) \qquad \text{for a.e. } y \in \Omega \text{ and } t \in [0\,,T]$$

and

$$g(u) = K(\|u(t^*)\|_1) \int_0^T u(t)\, d\nu(t)$$

for every $u \in C([0\,,T]\,,L^p(\Omega))$ (the last integral is a Bochner integral).

Remark 6.2. We remark that the multifunction $g : C([0\,,T]\,,L^p(\Omega)) \multimap L^p(\Omega)$ is well-defined, since Ω is bounded. Moreover, in general g does not admit a continuous selection, so our initial condition is really multivalued. To show this let the multimap $K : [0\,,+\infty) \multimap \mathbb{R}$ be defined by

$$K(s) = \begin{cases} \{0\} & \text{if } 0 \leq s < \bar r \\ [0\,,1] & \text{if } s = \bar r \\ \{1\} & \text{if } \bar r < s \end{cases}$$

for a fixed $\bar r > 0$ and let the measure ν be the normalized Lebesgue measure. It is easy to see that for every selection σ of g, if $\bar u \in C([0\,,T]\,,L^p(\Omega))$ is such that $\|\bar u(t^*)\|_1 = \bar r$ and $\int_0^T \bar u(t)\, dt \neq 0$, then σ is discontinuous in $\bar u$. In fact

$$\left(1 \pm \frac{1}{n}\right)\bar u \to \bar u \qquad \text{as } n \to \infty,$$

in $C([0,T],L^p(\Omega))$ but, $\sigma\left(\left(1-\frac{1}{n}\right)\bar{u}\right)=0$ for every $n\in\mathbb{N}$ and

$$\sigma\left(\left(1+\frac{1}{n}\right)\bar{u}\right)=\frac{1}{T}\int_0^T\left(1+\frac{1}{n}\right)\bar{u}(t)dt\to\frac{1}{T}\int_0^T\bar{u}(t)\,dt\neq 0\qquad\text{as } n\to\infty.$$

Theorem 6.3. *Consider the problem (6.1)–(6.2). Suppose that the multimap K and the maps h and η satisfy the following assumptions*

(K_1) $K(s)$ is closed and convex, for every $s\in[0,+\infty)$;
(K_2) $K(s)\subseteq[-1,1]$, for every $s\in[0,+\infty)$
(K_3) $K:[0,+\infty)\multimap\mathbb{R}$ is u.s.c.;
(h_1) the function $h(\cdot,\eta):[0,T]\to\mathbb{R}$ is measurable with respect to the Lebesgue measure on $[0,T]$ for every $\eta\in\mathbb{R}$;
(h_2) the function $h(t,\cdot):\mathbb{R}\to\mathbb{R}$ is continuous for almost every $t\in[0,T]$;
(h_3) there exist two constants $L>0$ and $c\geq 0$ such that $|h(t,\zeta)|\leq L|\zeta|+c$ for almost every $t\in[0,T]$ and for every $\zeta\in\mathbb{R}$;
(η_1) $\eta\in L^\infty(\Omega\times\Omega)$, $0\leq\eta(\cdot,\cdot)\leq 1$ a.e. in $\Omega\times\Omega$.

Then, if

$$b>L+|\Omega|\tag{6.3}$$

the problem (6.1)–(6.2) admits a solution $u\in C([0,T],L^1(\Omega))$.

Proof. We will prove that, in this framework, problem (1.3)–(3.1) satisfies all the assumptions of Theorem 4.2 for all $1<p<\infty$.

We have already remarked that A is the generator of a compact C_0 semigroup of contractions on $L^p(\Omega)$.

For every $x\in L^p(\Omega)$ and for a.e. $t\in[0,T]$, $f(t,x)\in L^p(\Omega)$. In fact, by (h_2) and (η_1), the map $y\longmapsto f(t,x)(y)$ is measurable in Ω for all $t\in[0,T]$ and $x\in L^p(\Omega)$. Moreover, by (h_3),

$$|f(t,x)|(y)\leq(b+L)|x(y)|+c+\|x\|_1\in L^p(\Omega).\tag{6.4}$$

Assumption (f_1). Reasoning as in the same step in Section 5, we can prove that the map

$$t\mapsto\langle f(t,x),z\rangle=\int_\Omega\left[-bx(y)+\int_\Omega\eta(y,\xi)x(\xi)\,d\xi+h(t,x(y))\right]z(y)\,dy$$

is measurable in $[0,T]$ for every $x\in L^p(\Omega)$ and $z\in L^{p'}(\Omega)$, $\frac{1}{p}+\frac{1}{p'}=1$; hence ($f_1$) is satisfied.

Assumption (f_2). We have to prove that, given a sequence $\{x_n\}_n\subset L^p(\Omega)$ convergent to $\bar{x}$, $f(t,x_n)\to f(t,\bar{x})$ as $n\to\infty$, for a.e. $t\in[0,T]$. By the definition of f and Hölder inequality

$$\begin{aligned}\|f(t,x_n)-f(t,\bar{x})\|_p&\leq b\|x_n-\bar{x}\|_p+\left\|\int_\Omega\eta(\cdot,\xi)(x_n(\xi)-\bar{x}(\xi))\,d\xi\right\|_p\\&+\|h(t,x_n(\cdot))-h(t,\bar{x}(\cdot))\|_p\leq(b+|\Omega|)\|x_n-\bar{x}\|_p+\|h(t,x_n(\cdot))-h(t,\bar{x}(\cdot))\|_p.\end{aligned}$$

Obviously $\|x_n-\bar{x}\|_p\to 0$. As to the second summand in the r.h.s. every subsequence of $\{x_n\}_n$ admits a subsequence $\{x_{n_k}\}_k$ converging a.e. to $\bar{x}$ in Ω and the convergence is dominated by $\lambda\in L^p(\Omega)$ ([31, Theorem 4.9]). Since, for a.e. $t\in[0,T]$, $h(t,\cdot)$ is continuous, $h(t,x_{n_k})\to h(t,\bar{x})$ a.e. in Ω. Moreover, by (h_3)

$$|h(t,x_{n_k}(y))|\leq 2^{p-1}\left(L^p|x_{n_k}(y)|^p+c^p\right)\leq 2^{p-1}\left(L^p\lambda(y)^p+c^p\right)\qquad\text{for a.e. } y\in\Omega.$$

Therefore $h(t,x_{n_k})\to h(t,\bar{x})$ in $L^p(\Omega)$ By the arbitrariness of the subsequence of $\{x_n\}_n$, $h(t,x_n)\to h(t,\bar{x})$ in $L^p(\Omega)$ for a.e. $t\in[0,T]$.

Assumption (f_3). By (6.4)

$$\|f(t,x)\|_p\leq(b+L)\|x\|_p+(c+\|x\|_1)|\Omega|^{1/p}\leq(b+L+|\Omega|)\|x\|_p+c|\Omega|^{1/p}$$

therefore (f_3) is true for $\ell_\rho(t)=(b+L+|\Omega|)\rho+c|\Omega|^{1/p}$ for every $t\in[0,T]$.

Assumption (g_1). Since K is convex valued, also $g(u)$ is convex, for every $u\in C([0,T],L^p(\Omega))$.

Assumption (g_2). In this case $x_0=0$. Given $R>0$, let us consider $u\in C([0,T],L^p(\Omega))$ with $\|u(t)\|_p\leq R$ for every $t\in[0,T]$. By (K_2), for every $\lambda\in K(\|u(t^*)\|_1)$ we have that

$$\left\|\lambda\int_0^T u(t)\,d\nu(t)\right\|_p\leq\int_0^T\|u(t)\|_p\,d\nu(t)\leq R$$

and (g_2) is proved.

Assumption (g_3'). For every $R>0$, consider a sequence $\{u_n\}_n\subset C([0,T],L^p(\Omega))$ and $\bar{u}\in C([0,T],L^p(\Omega))$ such that $\max_{0\leq t\leq T}\|u_n(t)\|_p\leq R$ for every $n\in\mathbb{N}$ and

$$\lim_{n\to\infty}\|u_n(t)-\bar{u}(t)\|_p=0$$

for every $t \in (0, T]$. We have to prove that, for every sequence $\{x_n\}_n$, $x_n \in g(u_n)$ for every $n \in \mathbb{N}$, there exists a subsequence $\{x_{n_k}\}_k$ such that $x_{n_k} \to \bar{x} \in g(\bar{u})$ as $k \to \infty$. By the definition of g, for every $n \in \mathbb{N}$ there exists $\lambda_n \in K(\|u_n(t^*)\|_1)$ such that

$$x_n = \lambda_n \int_0^T u_n(t)\, d\nu(t).$$

Since $\Omega \subset \mathbb{R}^n$ is bounded, $u_n(t^*) \to \bar{u}(t^*)$ in $L^p(\Omega)$, $1 \le p < \infty$; in particular $\|u_n(t^*)\|_p \to \|\bar{u}(t^*)\|_p$, $1 \le p < \infty$ as $n \to \infty$, so, by (K_3), there exists a convergent subsequence $\{\lambda_{n_k}\}$ such that $\lambda_{n_k} \to \bar{\lambda} \in K(\|\bar{u}(t^*)\|_1)$. Setting

$$\bar{x} = \bar{\lambda} \int_0^T \bar{u}(t)\, d\nu(t) \in g(\bar{u})$$

we have

$$\begin{aligned}\|x_{n_k} - \bar{x}\|_p &= \left\| \int_0^T \lambda_{n_k} u_{n_k}(t) - \bar{\lambda}\bar{u}(t)\, d\nu(t) \right\|_p \le \int_0^T \|\lambda_{n_k} u_{n_k}(t) - \bar{\lambda}\bar{u}(t)\|_p\, d\nu(t) \\ &\le \int_0^T |\lambda_{n_k}| \|u_{n_k}(t) - \bar{u}(t)\|_p + |\lambda_{n_k} - \bar{\lambda}| \|\bar{u}(t)\|_p\, d\nu(t) \\ &\le \int_0^T \|u_{n_k}(t) - \bar{u}(t)\|_p + R|\lambda_{n_k} - \bar{\lambda}|\, d\nu(t)\end{aligned}$$

and $\|u_{n_k}(t) - \bar{u}(t)\|_p + R|\lambda_{n_k} - \bar{\lambda}| \le 4R$ for every $t \in [0, T]$. Therefore (g_3') follows by Dominated Convergence Theorem.

Finally, it remain to prove the transversality condition (3.2). The duality mapping in $L^p(\Omega)$ is defined in (2.2), therefore, by (h_3), for every $x \in L^p(\Omega)$

$$\begin{aligned}\langle J(x), f(t,x)\rangle &= \frac{1}{\|x\|_p^{p-2}} \int_\Omega |x(y)|^{p-2} x(y) f(t,x)(y)\, dy \\ &= -b\|x\|_p^2 + \frac{1}{\|x\|_p^{p-2}} \int_\Omega |x(y)|^{p-2} x(y) \left[\int_\Omega \eta(y,\xi) x(\xi)\, d\xi + h(t, x(y)) \right] dy \\ &\le -b\|x\|_p^2 + \frac{\|x\|_1}{\|x\|_p^{p-2}} \int_\Omega |x(y)|^{p-1}\, dy + \frac{1}{\|x\|_p^{p-2}} \int_\Omega L|x(y)|^p + c|x(y)|^{p-1}\, dy.\end{aligned}$$

Now, by Hölder inequality,

$$\int_\Omega |x(y)|^{p-1}\, dy \le |\Omega|^{1/p} (\|x\|_p)^{p-1} \quad \text{and} \quad \|x\|_1 \le |\Omega|^{1-1/p} \|x\|_p,$$

therefore

$$\langle J(x), f(t,x)\rangle \le (-b + L + |\Omega|)(\|x\|_p)^2 + c|\Omega|^{\frac{1}{p}} \|x\|_p.$$

By (6.3), $(b - L - |\Omega|) > 0$, therefore, choosing $R > \frac{c|\Omega|^{\frac{1}{p}}}{b - L - |\Omega|}$, for every $x \in L^p(\Omega)$, $\|x\|_p = R$,

$$\langle J(x), f(t,x)\rangle \le (-b + L + |\Omega|) R^2 + c|\Omega|^{\frac{1}{p}} R < 0.$$

By continuity, there exists $\varepsilon > 0$ such that, for every x, $R - \varepsilon < \|x\|_p < R$, $\langle J(x), f(t,x)\rangle < 0$. Therefore condition (3.2) is satisfied.

We have proved that for every $p > 1$ all the assumptions of Corollary 3.3 are satisfied, therefore the abstract problem admits a mild solution $u \in C([0, T], L^p(\Omega))$, where $L^p(\Omega) \subset L^1(\Omega)$. Since the semigroup generated by A does not depend on p, we conclude that u is a solution also in $L^1(\Omega)$, proving the theorem. □

Our techniques also applies to more general parabolic equations, where the Laplace operator is replaced by a strongly elliptic differential operator in divergence form (see e.g. [3]).

CRediT authorship contribution statement

Luisa Malaguti: Conceptualization, Investigation, Methodology, Writing – original draft, Writing – review & editing. **Stefania Perrotta:** Conceptualization, Investigation, Methodology, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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References

[1] Pinaud MF, Henríquez HR. Controllability of systems with a general nonlocal condition. J Differential Equations 2020;269:4609–42.
[2] Benedetti I, Ciani S. Evolution equations with nonlocal initial conditions and superlinear growth. J Differential Equations 2022;318:270–97.
[3] Benedetti I, Malaguti L, Taddei V. Nonlocal solutions of parabolic equations with strongly elliptic differential operators. J Math Anal Appl 2019;473:421–43.
[4] Benedetti I, Loi NV, Malaguti L, Obukhovskii V. An approximation solvability method for nonlocal differential problems in Hilbert spaces. Commun Contemp Math 2017;19. 1650002, 34 pp.
[5] Benedetti I, Loi NV, Taddei V. An approximation solvability method for nonlocal semilinear differential problems in Banach spaces. Discrete Contin Dyn Syst 2017;37:2977–98.
[6] Lu L, Liu Z, Guo X. Existence results for a class of semilinear differential variational inequalities with nonlocal boundary conditions. Topol Methods Nonlinear Anal 2020;55:429–49.
[7] Malaguti L, Perrotta S, Taddei V. L^p-Exact controllability of partial differential equations with nonlocal terms. Evol Equ Control Theory 2022;11:1533–64.
[8] Cardinali T, Duricchi G. Further study on second order nonlocal problems monitored by an operator: an approach without compactness. Electron J Q Theory Differ Equ 2023;Paper No. 13:34.
[9] Pavlačková M, Taddei V. Mild solutions of second-order semilinear impulsive differential inclusions in Banach spaces. Mathematics 2022;10, 672:25 pp..
[10] Cardinali T, Precup R, Rubbioni P. A unified existence theory for evolution equations and systems under nonlocal conditions. J Math Anal Appl 2015;432:1039–57.
[11] Colao V, Muglia L. Solutions to nonlocal evolution equations governed by non-autonomous forms and demicontinuous nonlinearities. J Evol Equ 2022;22. Paper No. 77, 27 pp.
[12] Infante G, Maciejewski M. Multiple positive solutions of parabolic systems with nonlinear, nonlocal initial conditions. J Lond Math Soc (2) 2016;94:859–82.
[13] Denkowski Z, Migórski S, Papageorgiou NS. An introduction to nonlinear analysis: theory. Boston, MA: Kluwer Academic Publishers; 2003.
[14] Kamenskii M, Obukhovskii V, Zecca P. Condensing multivalued maps and semilinear differential inclusions in Banach spaces. Grundlehren math. wiss., Berlin: W. de Gruyter; 2001.
[15] Obukhovskii V, Gel'man B. Multivalued maps and differential inclusions—Elements of theory and applications. Hackensack, NJ: World Scientific Publishing Co. Pte. Ltd.; 2020.
[16] Pazy A. Semigroups of linear operators and applications to partial differential equations. Applied mathematical sciences, (no. 44). New York: Springer; 1983.
[17] Vrabie II. C_0 semigroups and applications. North-Holland mathematics studies, (no. 191). Amsterdam: North-Holland Publishing Co.; 2003.
[18] Hartman P. On boundary value problems for systems of ordinary, nonlinear, second order differential equations. Trans Amer Math Soc 1960;96:493–509.
[19] Mawhin J. Boundary value problems for nonlinear second-order vector differential equations. J Differ Equ 1974;16:257–69.
[20] Bebernes JW. A simple alternative problem for finding periodic solutions of second order ordinary differential systems. Proc Amer Math Soc 1974;42:121–7.
[21] Gaines RE, Mawhin J. Coincidence degree, and nonlinear differential equations. Lecture notes in mathematics, no. 568, Berlin-New York: Springer-Verlag; 1977.
[22] Mawhin J, Szymańska-Dębowska K. Convexity, topology and nonlinear differential systems with nonlocal boundary conditions: a survey. Rend Istit Mat Univ Trieste 2019;51:125–66.
[23] Feltrin G, Zanolin F. Bound sets for a class of ϕ-Laplacian operators. J Differential Equations 2021;297:508–35.
[24] Andres J, Malaguti L, Pavlačková M. On second-order boundary value problems in Banach spaces: a bound sets approach. Topol Methods Nonlinear Anal 2011;37:303–41.
[25] Vrabie II. Compactness methods for nonlinear evolutions. Pitman monographs and surveys in pure and applied mathematics, (no. 32). Harlow, Essex, England: Longman Scientific & Technical; 1987.
[26] Andres J, Gabor G, Górniewicz L. Boundary value problems on infinite intervals. Trans Amer Math Soc 1999;351:4861–903.
[27] Diestel J, Uhl Jr JJ. Vector measures. Math. surveys, (no. 15). Providence, RI: AMS; 1977.
[28] DiPerna RJ, Lions PL. Ordinary differential equations, transport theory and Sobolev spaces. Invententiones Mathematicae 1989;98:511–7.
[29] Colombo RM, Corli A, Rosini MD. Non local balance laws in traffic models and crystal growth. ZAMM Z Angew Math Mech 2007;87:449–61.
[30] De Lellis C, Gwiazda P, Świerczewska-Gwiazda A. Transport equations with integral terms: existence, uniqueness and stability. Calc Var Partial Differential Equations 2016;55:128:1–17.
[31] Brezis H. Functional analysis, Sobolev spaces and partial differential equations. Universitext, New York: Springer; 2011.