Communications on Pure and Applied Analysis

doi:10.3934/cpaa.2026043

AN ALLEN-CAHN TUMOR GROWTH MODEL WITH TEMPERATURE

Stefania Gatti[1], Erica Ipocoana[2] and Alain Miranville[*3,4]

[1]Universitá degli Studi di Modena e Reggio Emilia,
Dipartimento di Scienze Fisiche, Informatiche e Matematiche, via Campi 213/b, I-41125 Modena, Italy

[2]Freie Universität Berlin, Department of Mathematics and Computer Science,
Arnimallee 9, 14195 Berlin, Germany

[3]Henan Normal University, School of Mathematics and Statistics, Xinxiang, Henan, China

[4]Université Le Havre Normandie, Laboratoire de Mathématiques Appliquées du Havre (LMAH)
25, rue Philippe Lebon BP 1123, 76063 Le Havre cedex, France

(Communicated by Wei Feng)

Abstract. In this paper, we propose a new non-isothermal Allen-Cahn (Ginzburg-Landau) model for tumor growth. After deriving it using a microforces approach, we study its well-posedness. In particular, we are able to prove the existence and uniqueness of a local and global-in-time solution to our PDE system.

1. **Introduction.** An increasing number of mathematical models have been proposed to support the medical community in better understanding and treating cancers (see [3] and the review papers [9, 28, 35]). Indeed, mathematical models might provide further insights in tumor growth, focusing on the reality-matching mathematical assumptions on parameters and analysis of the long-time behavior. Among these, phase-field models have been investigated intensively. In this framework, the evolution of the tumor is regulated by an order parameter and driven by several biological mechanisms such as proliferation via nutrient consumption, apoptosis [15, 25], chemotaxis and active transport of specific chemical species [12]. Given the medical nature of the problem, models taking into account the effects of therapies have also been studied e.g. for brain tumors [6] and prostate cancer [4, 5].
In spite of the abundance of mathematical models in tumor growth, still an aspect has so far been neglected, namely, thermal effects on cancers. Nonetheless, it is experimentally verified that both high and low temperatures can lead to partial or complete destruction of tumor cells. In some cases, hyperthermia or hypothermia

2020 *Mathematics Subject Classification.* Primary: 35D30, 35Q92; Secondary: 92C50.

Key words and phrases. Allen-Cahn, phase-field, tumor growth, partial differential equations, diffuse interface.

S.G. and E.I. were partially supported by the project NOBILI_PRIN2020 "Mathematics for industry 4.0 (Math4I4) 2020F3NCPX". In particular, E.I. was financially supported by LIA-LYSM-AMU-CNRS-ECM-INdAM. This work was also supported by the DFG within CRC 1114 "Scaling Cascades in Complex Systems" Project Number 235221301, Project C02 "Interface dynamics: Bridging stochastic and hydrodynamic descriptions".

*Corresponding author: Alain Miranville.

are an adjuvant to other therapies and, in others, they are a last resort for untreatable deseases (see the reviews [22] for the former and [18, 21, 29] for the latter). It should be noted that they are quite difficult to administer without damaging the surrounding healthy tissues and the overall mechanism behind their efficiency is not well understood.

In this paper, we present and study the well-posedness of a non-isothermal Allen-Cahn model for tumor growth coupled with nutrient dynamics. Here, we suppose the evolution of the tumor to be governed by proliferation via nutrient consumption, apoptosis and thermal effects. Although we do not include thermal control, the temperature contribution to the tumor equation is a first attempt to render the fact that low temperatures slow down tumor growth. We also underline that the tumor cells are assumed to die only by apoptosis, so that we do not consider e.g. necrosis [11, 31]. Our thermodynamically consistent model allows to include temperature effects on tumor growth, achieving at the same time the global existence and uniqueness of *not too weak* solutions. Indeed, up to our knowledge, the only other non-isothermal phase-field model for tumor growth in the literature is a Cahn-Hilliard model recently studied in [15] (see also [7, 8] for similar non isothermal phase-field equations for fluids mixtures). In this previous paper, only the existence of weak entropy solutions was proven, due to the chemical potential term in the right-hand side of the temperature equation and therefore to the impossibility of gaining enough regularity to pass to the limit. As a consequence, also the uniqueness of the solution could not be achieved. Besides, the Cahn-Hilliard framework excludes any maximum principles for the order parameter. Here, switching to an Allen-Cahn system, we are able to overcome analogous difficulties by exploiting a completely different approach, based on the decoupling of the system, as proposed in [27, 10]. This strategy will be made clear in Section 3.

We note that for Allen-Cahn models for tumor growth, the literature is not as extensive as for Cahn-Hilliard ones. In fact, we refer to the works [4, 32, 33, 34] for models and well-posedness and to [1, 5] for optimal control studies.

Our PDE system consists of three evolution equations, governing the phase-field (or order parameter), the absolute temperature and the nutrient concentration, respectively. Namely, given a bounded regular domain $\Omega \subset \mathbb{R}^3$, we consider the following initial and boundary value problem:

$$\beta\varphi_t - \Delta\varphi + F'(\varphi) - \theta = (\mathcal{P}\sigma - \mathcal{A})h(\varphi), \quad \text{in} \quad \Omega \times (0,T), \tag{1.1}$$

$$c_V\theta_t - \operatorname{div}[\kappa(\theta)\nabla\theta] - \beta(\varphi_t)^2 + \theta\varphi_t = 0, \quad \text{in} \quad \Omega \times (0,T), \tag{1.2}$$

$$\sigma_t - \Delta\sigma = -\mathcal{C}\sigma h(\varphi) + \mathcal{B}(\sigma_B - \sigma), \quad \text{in} \quad \Omega \times (0,T), \tag{1.3}$$

$$\partial_{\boldsymbol{n}}\varphi = \partial_{\boldsymbol{n}}\theta = \partial_{\boldsymbol{n}}\sigma = 0, \quad \text{on} \quad \partial\Omega \times (0,T), \tag{1.4}$$

$$\varphi(0) = \varphi_0, \quad \theta(0) = \theta_0, \quad \sigma(0) = \sigma_0, \quad \text{in} \quad \Omega. \tag{1.5}$$

The order parameter φ denotes the local concentration of tumor cells. Therefore, to be biologically relevant, φ should vary in a bounded interval, say $[0,1]$. This means that $\varphi = +1$ in the regions exclusively occupied by tumor cells, $\varphi = 0$ in the healthy ones, while $\varphi \in (0,1)$ in the so-called diffuse interface, that is, a thin layer where the phases coexist. Although we are able to analytically prove that $\varphi \geq 0$ is bounded on any finite time interval, the general source term prevents us to show that $\varphi \leq 1$. The absolute temperature is denoted by θ and σ represents the concentration of a nutrient, such as oxygen and glucose, for tumor cells; both variables should be nonnegative. The parameters $\mathcal{P}, \mathcal{A}, \mathcal{C}$, $\mathcal{B}$ and σ_B are positive

constants indicating the tumor proliferation rate, the apoptosis rate, the nutrient consumption rate, the blood-tissue transfer rate and the nutrient concentration in the pre-existing vasculature, respectively. The function h, whose choice is detailed in Section 3.1, is a bounded, monotone increasing function assumed to be nonnegative on $[0,1]$. Therefore, assuming that $\varphi \geq 0$, the tumor grows when $(\mathcal{P}\sigma - \mathcal{A}) > 0$ and decreases otherwise (in particular, the higher the concentration of tumor cells is, the faster they die). The term $\mathcal{C}\sigma h(\varphi)$ represents the consumption of nutrient by the tumor cells. We consider here the case when the tumor has its own vasculature, which leads to the presence of the term $\mathcal{B}(\sigma_B - \sigma)$ in the nutrient equation. In particular, if $\sigma_B > \sigma$, the term $\mathcal{B}(\sigma_B - \sigma)$ models the supply of nutrient from the blood vessels and, otherwise, the transport of nutrient away from the domain.
Eventually, $\kappa(\theta)$ stands for the heat conductivity, c_V is the specific heat and the nonnegative constant β is a constitutive modulus. Our last remark concerns the function $F(\varphi)$, assumed to be a polynomial double-well potential.

The paper is organized as follows. Since we present a new phase-field model, we first derive it in Section 2, following a microforces approach. Then, in Section 3, we investigate the well-posedness of a simplified version of the model. Namely, having introduced our assumptions and notation in Section 3.1, we prove the existence and uniqueness of a local-in-time solution in Theorem 3.1 in Section 3.2. Heuristically, the main idea behind the proof of this result is to decouple our problem, first fixing θ in a suitable regularity space and then the pair (φ, σ), in the spirit of [27]. This yields two intermediate results for the frozen problems. Each of these results, i.e. Lemmas 3.2 and 3.4, relies on a Galerkin scheme. The proof is concluded by composition of the two frozen problems and using Schauder's fixed point theorem. Finally, in Section 3.3, we show that we are able to extend the local solutions and thus gaining a global-in-time result (Theorem 3.6).

2. **Derivation of the model.** We suppose that a two-components mixture consisting of healthy cells and tumor cells occupies an open spatial domain $\Omega \subset \mathbb{R}^3$. We denote by $\varphi(x,t)$ the tumor phase concentration, $\theta(x,t)$ the absolute temperature and $\sigma(x,t)$ the concentration of a nutrient for the tumor cells.
We follow Gurtin's approach based on microforces (proposed in [14] and largely used later on, e.g. in [10, 15, 23, 26]) in order to derive our model (1.1)–(1.3). As a matter of fact, we treat separately the balance laws and the constitutive relations. In particular, we postulate the following balance law for internal microforces

$$\operatorname{div} \boldsymbol{\xi} + \pi + \gamma = 0, \tag{2.1}$$

where $\boldsymbol{\xi}$ is a vector representing the microstress, π and γ are scalars corresponding to the internal microforces and to the external microforces, respectively. Microforces describe the forces associated with microscopic configurations of atoms, differently from standard forces that are associated with macroscopic length scales. These different length scales are the reason why a separate balance law for microforces is needed. Such interatomic forces may be mirrored on the macroscopic level by fields that perform work when the order parameter undergoes changes. Therefore, this work can be expressed in terms of φ_t, which explains why the microforces are scalar rather than vector quantities.
For the sake of completeness, we also refer to the pioneering paper [2] after which models of phase transition with microscopic movements developed by Frémond have been studied extensively. In particular, we cite [19, 20], where the nonlinearities are similar to the ones appearing in our model.

2.1. **Order parameter equation.** The internal energy density of the system is given by Gibbs' relation

$$e = \psi + \theta s, \tag{2.2}$$

where s represents the entropy of the system, given by

$$s = -\frac{\partial \psi}{\partial \theta}. \tag{2.3}$$

The term $\psi = \psi(\theta, \varphi, \nabla\varphi)$ denotes the free energy, whose form will be chosen later in order to make this first part of the derivation more general.
Following [14], we write the first law of thermodynamics in the form

$$\frac{d}{dt}\int_R e \,\mathrm{d}x = -\int_{\partial R} \boldsymbol{q}\cdot\boldsymbol{\nu}\,\mathrm{d}\Sigma + \mathcal{W}(R) + \mathcal{M}(R), \tag{2.4}$$

where $\boldsymbol{q}$ is the heat flux, R is the control volume and $\boldsymbol{\nu}$ is the outward unit normal to ∂R. Besides,

$$\mathcal{W}(R) = \int_{\partial R}(\boldsymbol{\xi}\cdot\boldsymbol{\nu})\frac{\partial\varphi}{\partial t}\,\mathrm{d}\Sigma + \int_R \gamma\frac{\partial\varphi}{\partial t}\,\mathrm{d}x, \tag{2.5}$$

$$\mathcal{M}(R) = 0 \tag{2.6}$$

are the rate of work and the rate at which free energy is added to R, respectively, assuming no heat supply. We recall that (2.6) holds since we consider here the case of the Ginzburg-Landau equation.
By Green's formula, since the control volume R is arbitrary, (2.4) reads

$$\frac{\partial e}{\partial t} = -\operatorname{div}\boldsymbol{q} + \frac{\partial\varphi}{\partial t}\operatorname{div}\boldsymbol{\xi} + \boldsymbol{\xi}\cdot\nabla\frac{\partial\varphi}{\partial t} + \frac{\partial\varphi}{\partial t}\gamma. \tag{2.7}$$

Hence, from the microforce balance (2.1), we infer

$$\frac{\partial e}{\partial t} = -\operatorname{div}\boldsymbol{q} - \pi\frac{\partial\varphi}{\partial t} + \boldsymbol{\xi}\cdot\nabla\frac{\partial\varphi}{\partial t}. \tag{2.8}$$

We impose the validity of the second law of thermodynamics in the form of the Clausius-Duhem inequality, namely,

$$\theta\left(\frac{\partial s}{\partial t} + \operatorname{div}\boldsymbol{Q}\right) \geq 0, \tag{2.9}$$

where the entropy flux $\boldsymbol{Q}$ is given by the relation

$$\boldsymbol{Q} = \frac{\boldsymbol{q}}{\theta}. \tag{2.10}$$

Working on the left-hand side of (2.9), we have

$$\begin{aligned}
\theta\left(\frac{\partial s}{\partial t} + \operatorname{div}\boldsymbol{Q}\right) &\overset{(2.2)}{=} \frac{\partial e}{\partial t} - \frac{\partial\psi}{\partial t} - s\frac{\partial\theta}{\partial t} + \theta\operatorname{div}\boldsymbol{Q} \\
&\overset{(2.10)}{=} \frac{\partial e}{\partial t} - \frac{\partial\psi}{\partial t} - s\frac{\partial\theta}{\partial t} + \operatorname{div}\boldsymbol{q} - \boldsymbol{Q}\cdot\nabla\theta \\
&\overset{(2.8)}{=} -\pi\frac{\partial\varphi}{\partial t} + \boldsymbol{\xi}\cdot\nabla\frac{\partial\varphi}{\partial t} - \frac{\partial\psi}{\partial t} - s\frac{\partial\theta}{\partial t} - \boldsymbol{Q}\cdot\nabla\theta.
\end{aligned} \tag{2.11}$$

The third and fourth terms in the last equation, computed by the chain rule and (2.3), read

$$\frac{\partial\psi}{\partial t} + s\frac{\partial\theta}{\partial t} = \frac{\partial\psi}{\partial\varphi}\frac{\partial\varphi}{\partial t} + \frac{\partial\psi}{\partial\nabla\varphi}\frac{\partial\nabla\varphi}{\partial t}. \tag{2.12}$$

Then, inserting (2.12) into (2.11) yields

$$\theta\left(\frac{\partial s}{\partial t}+\operatorname{div}\boldsymbol{Q}\right)=-\pi\frac{\partial\varphi}{\partial t}+\boldsymbol{\xi}\cdot\nabla\frac{\partial\varphi}{\partial t}-\frac{\partial\psi}{\partial\varphi}\frac{\partial\varphi}{\partial t}-\frac{\partial\psi}{\partial\nabla\varphi}\frac{\partial\nabla\varphi}{\partial t}-\boldsymbol{Q}\cdot\nabla\theta$$
$$=\left(-\pi-\frac{\partial\psi}{\partial\varphi}\right)\frac{\partial\varphi}{\partial t}+\left(\boldsymbol{\xi}-\frac{\partial\psi}{\partial\nabla\varphi}\right)\frac{\partial\nabla\varphi}{\partial t}-\boldsymbol{Q}\cdot\nabla\theta.$$

Hence, in order for (2.9) to hold, we take

$$\boldsymbol{\xi}=\frac{\partial\psi}{\partial\nabla\varphi}, \tag{2.13}$$

$$\left(-\pi-\frac{\partial\psi}{\partial\varphi}\right)\frac{\partial\varphi}{\partial t}-\boldsymbol{Q}\cdot\nabla\theta\geq 0. \tag{2.14}$$

Dividing (2.14) by θ (assumed to be positive), relation (2.10) gives

$$\frac{1}{\theta}\left(\pi+\frac{\partial\psi}{\partial\varphi}\right)\frac{\partial\varphi}{\partial t}+\frac{\boldsymbol{q}}{\theta^2}\cdot\nabla\theta=\frac{1}{\theta}\left(\pi+\frac{\partial\psi}{\partial\varphi}\right)\frac{\partial\varphi}{\partial t}-\boldsymbol{q}\cdot\nabla\frac{1}{\theta}\leq 0. \tag{2.15}$$

Arguing as in [26], we can take

$$\boldsymbol{q}=\boldsymbol{b}\frac{\partial\varphi}{\partial t}+\mathbb{B}\nabla\frac{1}{\theta}, \tag{2.16}$$

where $\boldsymbol{b}$ is a vector and $\mathbb{B}$ is a matrix possibly depending on the constitutive variables $\varphi,\nabla\varphi,\theta,\nabla\theta,\sigma$. According to [14], we set

$$\frac{1}{\theta}\left(\pi+\frac{\partial\psi}{\partial\varphi}\right)=-\beta\frac{\partial\varphi}{\partial t}-\boldsymbol{a}\cdot\nabla\frac{1}{\theta}, \tag{2.17}$$

where β is a positive constant and $\boldsymbol{a}$ is a vector depending on the constitutive variables. In particular, we set

$$\boldsymbol{a}=0,$$
$$\boldsymbol{b}=0,$$
$$\mathbb{B}=\mathbb{I}_d\kappa(\theta)\theta^2,$$

with $\kappa(\theta)=1+\kappa_0\theta^q$, for some $q\geq 2$ and some $\kappa_0\geq 0$. These constitutive choices ensure that (2.15) holds. Combining (2.17) with the microforce balance (2.1), it follows that

$$\beta\frac{\partial\varphi}{\partial t}=\frac{1}{\theta}\operatorname{div}\boldsymbol{\xi}+\frac{1}{\theta}\gamma-\frac{1}{\theta}\frac{\partial\psi}{\partial\varphi}. \tag{2.18}$$

We now postulate that the free energy density ψ takes the form

$$\psi=\frac{\varepsilon}{2}|\nabla\varphi|^2+\frac{1}{\varepsilon}F(\varphi)+f(\theta)-\theta\varphi+N(\varphi), \tag{2.19}$$

where ε is a positive constant related to the thickness of the interface and the function F denotes a polynomial double-well potential having at least a cubic growth at infinity. Since we assume 0 and 1 to be the pure phases, we take $F(\varphi)=\varphi^2(1-\varphi)^2$ (cf. [32]).

The term f in (2.19) describes the part of free energy that is purely caloric and is related to the specific heat $c_V(\theta)=Q'(\theta)$ through the relation $Q(\theta)=f(\theta)-\theta f'(\theta)$. Finally, N represents both the chemical energy of the nutrient and the energy contributions given by the interactions between the tumor tissues and the nutrient.

Since ψ is now supposed to take the form (2.19), according to (2.13), we infer

$$\boldsymbol{\xi}=\varepsilon\nabla\varphi. \tag{2.20}$$

Moreover,

$$\frac{\partial \psi}{\partial \varphi} = \frac{1}{\varepsilon}F'(\varphi) - \theta + \frac{\partial N}{\partial \varphi}.$$

The term $\frac{\partial N}{\partial \varphi}$ is known in the literature to describe chemotaxis effects (see e.g. [12]) that we neglec here, thus taking $\frac{\partial N}{\partial \varphi} = 0$. Inserting these last relations into equation (2.18), we obtain

$$\beta \frac{\partial \varphi}{\partial t} = \frac{\varepsilon}{\theta}\Delta\varphi + \frac{1}{\theta}\gamma - \frac{1}{\theta\varepsilon}F'(\varphi) + 1.$$

Assuming that $\beta\theta$ is a constant still denoted by β, we get

$$\beta\varphi_t - \varepsilon\Delta\varphi + \frac{1}{\varepsilon}F'(\varphi) - \theta = \gamma. \tag{2.21}$$

Postulating the source term γ to be

$$\gamma = (\mathcal{P}\sigma - \mathcal{A})h(\varphi),$$

we recover equation (1.1). In particular, $\mathcal{P} > 0$ is the tumor proliferation rate and $\mathcal{A} > 0$ is the apoptosis rate, while the function h models how the tumor regulates several phenomena, proliferation and apoptosis in our case. Therefore, $h(0) = 0$ and h is monotone increasing with an upper threshold.

Note that the constitutive relation regarding the source term γ complies with the fact that tumor growth is proportional to the nutrient supply in the tumoral region. In particular, the tumor grows if $\mathcal{P}\sigma - \mathcal{A} > 0$, otherwise it decreases, due to apoptosis of tumor cells.

2.2. **Temperature equation.** According to (2.19), Gibbs' relation (2.2) reads

$$e = \frac{\varepsilon}{2}|\nabla\varphi|^2 + \frac{1}{\varepsilon}F(\varphi) + f(\theta) - \theta f'(\theta) + N(\varphi).$$

Therefore,

$$\frac{\partial e}{\partial t} = \varepsilon\nabla\varphi \cdot \frac{\partial \nabla\varphi}{\partial t} + \frac{1}{\varepsilon}F'(\varphi)\varphi_t + Q'(\theta)\theta_t, \tag{2.22}$$

recalling that $\frac{\partial N}{\partial \varphi} = 0$. Moreover, combining the energy balance (2.8) with (2.1), (2.13) and (2.16), we get

$$\frac{\partial e}{\partial t} = -\operatorname{div}\left(-\kappa(\theta)\nabla\theta - \frac{\partial \psi}{\partial \nabla\varphi}\frac{\partial \varphi}{\partial t}\right) + \gamma\frac{\partial \varphi}{\partial t}. \tag{2.23}$$

Putting together (2.22) and (2.23) and recalling (2.20), it follows that

$$Q'(\theta)\theta_t - \operatorname{div}(\kappa(\theta)\nabla\theta) = \varepsilon\Delta\varphi\varphi_t - \frac{1}{\varepsilon}F'(\varphi)\varphi_t + \gamma\varphi_t. \tag{2.24}$$

Substituting the term $\varepsilon\Delta\varphi$ in (2.24) through relation (2.21) and assuming a constant specific heat c_V, that is, $Q'(\theta) = c_V$ is constant, we obtain the temperature equation (1.2).

2.3. **Nutrient equation.** The nutrient balance equation is postulated to take the form

$$\sigma_t = -\operatorname{div}\mathbf{J}_\sigma - \mathcal{S}, \tag{2.25}$$

where J_σ is the nutrient flux and $\mathcal{S}$ denotes a source/sink term for the nutrient. Making the following constitutive choices:

$$\mathbf{J}_\sigma = -\nabla\sigma, \tag{2.26}$$

$$\mathcal{S} = \mathcal{C}\sigma h(\varphi) - \mathcal{B}(\sigma_B - \sigma), \tag{2.27}$$

we recover (1.3). Here, we recall that $\mathcal{C} > 0$ corresponds to the nutrient consumption rate and $\mathcal{B} > 0$ is the nutrient supply rate, while σ_B is the vascular nutrient concentration. For the sake of simplicity, the nutrient diffusion constant has been normalized.
Hence, the sink/source term in the nutrient equation is assumed to be regulated by both consumption of the nutrient and nutrient supply via the capillary network. In particular, we consider here the case where the nutrient supply is linked to the fact that the tumor has its own vasculature, according to the latter term in (2.27). Therefore, the term σ_B acts as a threshold, indicating whether the nutrient is drawn to the tumor or is transported in the other direction.

3. **Well-posedness.** Having derived our model, we can now make precise our assumptions and subsequently our notation.

3.1. **Assumptions and notation.** We consider the double-well potential $F(\varphi) = \varphi^2(1-\varphi)^2$, so that $F'(\varphi) = 4\varphi^3 - 6\varphi^2 + 2\varphi$. Concerning h, we assume $h \in C^1(\mathbb{R})$ to be a monotone increasing function. We further assume that $h(0) = 0$ and

$$|h(r)| + |h'(r)| \le C, \quad r \in \mathbb{R}, \tag{3.1}$$

for some positive constant C. Notice that $h(0) = 0$ complies with the biological meaning of the corresponding term in the tumor and nutrient equations, in which it accounts for the tumor regulation of proliferation and apoptosis, and the nutrient uptake by the tumor, respectively.

We assume that the heat conductivity takes the form

$$\kappa(r) = 1 + r^q, \quad r \ge 0, \quad \text{for some} \quad q \ge 2. \tag{3.2}$$

This choice of q mainly depends on mathematical reasons and is related to a better integrability for θ. Since in principle the sign of the variable θ is unknown, we extend κ to the whole real line as $\kappa(r) = 1 + |r|^q$. In the end, the modulus would be useless since the solution to the corresponding problem is nonnegative (actually $\theta \ge \underline{\theta} > 0$, provided that we properly modify the assumption on the initial datum, as explained in Remark 3.5). Besides, we assume that the constant $\sigma_B \in [0,1]$ and recall that $\mathcal{P}, \mathcal{A}, \mathcal{C}, \mathcal{B} > 0$ are constant as well. Finally, for the sake of simplicity, we set $c_V = 1$ and $\beta = \varepsilon = 1$ in system (1.1)–(1.3). Then the model reads

$$\begin{cases} \varphi_t - \Delta\varphi + F'(\varphi) - \theta = (\mathcal{P}\sigma - \mathcal{A})h(\varphi), & \text{in} \quad \Omega \times (0,T), \\ \theta_t - \operatorname{div}[\kappa(\theta)\nabla\theta] - \varphi_t^2 + \theta\varphi_t = 0, & \text{in} \quad \Omega \times (0,T), \\ \sigma_t - \Delta\sigma = -\mathcal{C}\sigma h(\varphi) + \mathcal{B}(\sigma_B - \sigma), & \text{in} \quad \Omega \times (0,T), \end{cases} \tag{3.3}$$

supplemented with (1.4) and (1.5). We recall that Ω is a smooth and bounded domain of $\mathbb{R}^3$ with boundary Γ, while $(0,T)$ represents an assigned but otherwise arbitrary time interval. Then, we define the parabolic domain $Q_T := \Omega \times (0,T)$.
We set $H := L^2(\Omega)$ and $V := H^1(\Omega)$, using these symbols also when referring to vector valued functions. In general, given a Banach space X, $\|\cdot\|_X$ represents its norm, using for brevity $\|\cdot\|$ in place of $\|\cdot\|_H$, and $\|\cdot\|_*$ for the norm of V', dual space to V. The symbol $\langle\cdot,\cdot\rangle$ will stand for the duality pairing between V' and V. We also omit the integration variables when no misinterpretation occurs.
Finally, c stands for a positive constant, possibly changing from line to line; it is in general structural, further dependencies being pointed out on occurrence.

In order to make precise the assumptions on the initial data, we define

$$K(r) := \int_0^r \kappa(s)ds = r + \frac{r|r|^q}{q+1}, \quad r \in \mathbb{R}, \tag{3.4}$$

where $\kappa(r) = 1 + |r|^q$, for some $q \geq 2$, $r \in \mathbb{R}$ (cf. (3.2)). Indeed, we assume that

$$\theta_0 \in L^{3q}(\Omega) \cap V \quad \text{s.t.} \quad K(\theta_0) \in V \quad \text{and} \quad \theta_0 \geq 0 \quad \text{a.e. in} \quad \Omega, \tag{3.5}$$

$$\varphi_0 \in W^{\frac{5}{3},6}(\Omega), \quad \varphi_0 \geq 0 \quad \text{a.e. in } \Omega, \quad \text{and} \quad \partial_{\boldsymbol{n}}\varphi_0 = 0 \quad \text{on} \quad \partial\Omega, \tag{3.6}$$

$$\sigma_0 \in L^\infty(\Omega), \quad 0 \leq \sigma_0 \leq 1 \quad \text{a.e. in} \quad \Omega. \tag{3.7}$$

For further reference, we notice in particular that

$$\int_\Omega \kappa(\theta)^2 |\nabla\theta|^2 \, \mathrm{dx} = \|\nabla K(\theta)\|^2 \, , \tag{3.8}$$

whenever it makes sense. We also recall the nonlinear Poincaré inequality (see [13])

$$\||v|^{\frac{p}{2}}\|_V^2 \leq c_p \left(\|v\|_{L^1}^p + \|\nabla |v|^{\frac{p}{2}}\|^2 \right), \quad \forall v \in L^1(\Omega) \quad \text{such that} \quad \nabla |v|^{\frac{p}{2}} \in L^2(\Omega), \tag{3.9}$$

holding for any finite $p \geq 2$. Besides, without mentioning it, we will repeatedly use the following relation:

$$\int_\Omega |v|^r |\nabla v|^2 = \frac{4}{(r+2)^2} \int_\Omega |\nabla (|v|^{\frac{r}{2}+1})|^2. \tag{3.10}$$

3.2. **Local in time existence of a solution.** We can now state the existence and uniqueness of a local solution that we will prove via the Schauder fixed point theorem in several steps. Namely,

Theorem 3.1. *For any $R > 0$ there exists $T = T(R) > 0$ such that if $(\theta_0, \varphi_0, \sigma_0)$ satisfies assumptions (3.5)–(3.7) and*

$$\|\theta_0\|_{V \cap L^{3q}(\Omega)} + \|K(\theta_0)\|_V + \|\varphi_0\|_{W^{\frac{5}{3},6}(\Omega)} \leq R, \tag{3.11}$$

then there exists a unique local solution $(\theta, \varphi, \sigma)$ to system (3.3) in $(0,T)$, namely,

$$\theta \in C([0,T]; H) \cap L^\infty(0,T; L^{3q}(\Omega) \cap V) \cap L^{4q}(0,T; L^{12q}(\Omega)) \cap H^1(0,T;H),$$
$$K(\theta) \in L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega)) \cap H^1(0,T; L^{\frac{3}{2}}(\Omega)),$$
$$\varphi \in C([0,T];V) \cap W^{1,4}(0,T;L^6(\Omega)),$$
$$\sigma \in L^\infty(0,T;L^\infty(\Omega)) \cap L^2(0,T;V) \cap H^1(0,T;V'),$$

and

$$\theta_t - \Delta K(\theta) = \varphi_t^2 - \varphi_t\theta, \quad \text{a.e. in} \quad Q_T,$$
$$\varphi_t - \Delta\varphi + F'(\varphi) = \theta + (\mathcal{P}\sigma - \mathcal{A})h(\varphi), \quad \text{a.e. in} \quad Q_T,$$
$$\langle \sigma_t, v\rangle + \int_\Omega \nabla\sigma\cdot\nabla v + \mathcal{B}\int_\Omega \sigma v = \int_\Omega (-\mathcal{C}\sigma h(\varphi) + \mathcal{B}\sigma_B)v, \quad \forall v \in V, \text{ a.e. in} \quad (0,T),$$
$$\partial_{\boldsymbol{n}}\varphi = 0 \quad \text{and} \quad \partial_{\boldsymbol{n}}K(\theta) = 0, \quad \text{a.e. in} \quad \partial\Omega\times(0,T),$$
$$\theta(0) = \theta_0, \quad \varphi(0) = \varphi_0 \quad \sigma(0) = \sigma_0, \quad \text{a.e. in } \Omega.$$

Besides,

$$\theta \geq 0, \quad 0 \leq \sigma \leq 1 \quad \text{and} \quad \varphi \geq 0 \quad \text{a.e. in} \quad Q_T.$$

The strategy used to prove the above result is to decouple the problem, first freezing the temperature, then the order parameter and the nutrient concentration, with suitable regularities. Hence, we obtain the well-posedness of the fixed point map.
For any fixed $R, T > 0$, we introduce the space for the fixed point argument as

$$\Theta_R(T) := \{\tilde{\theta} \in L^4(0,T;L^6(\Omega)) \text{ s.t. } \tilde{\theta} \geq 0 \text{ a.e. in } Q_T \text{ and } \|\tilde{\theta}\|_{L^4(0,T;L^6(\Omega))} \leq R\}.$$

As a first step to prove Theorem 3.1, we have the following.

Lemma 3.2. *Let assumptions (3.6) and (3.7) hold true. Then, given $R, T > 0$, fix $\tilde{\theta} \in \Theta_R(T)$, there exists a unique pair (φ, σ) such that*

$$\varphi \in C([0,T];V) \cap W^{1,4}(0,T;L^6(\Omega)), \quad \sigma \in L^\infty(0,T;L^\infty(\Omega)) \cap L^2(0,T;V) \cap H^1(0,T;V'),$$

$$\varphi_t - \Delta\varphi + F'(\varphi) = \tilde{\theta} + (\mathcal{P}\sigma - \mathcal{A})h(\varphi), \quad \textit{a.e. in} \quad Q_T, \tag{3.12}$$

$$\langle \sigma_t, v\rangle + \int_\Omega \nabla\sigma\cdot\nabla v + \mathcal{B}\int_\Omega \sigma v = \int_\Omega (-\mathcal{C}\sigma h(\varphi) + \mathcal{B}\sigma_B)v, \ \forall v \in V, \ \textit{a.e. in} (0,T), \tag{3.13}$$

as well as

$$\partial_{\boldsymbol{n}}\varphi = 0, \quad \textit{a.e. in} \quad \partial\Omega \times (0,T),$$
$$\varphi(0) = \varphi_0 \quad \sigma(0) = \sigma_0, \quad \textit{a.e. in } \Omega.$$

Moreover,

$$0 \leq \sigma \leq 1 \quad \textit{and} \quad \varphi \geq 0 \quad \textit{a.e. in} \quad Q_T.$$

In particular,

$$\|\varphi\|_{L^\infty(0,T;L^\infty(\Omega))} + \|\varphi_t\|_{L^4(0,T;L^6(\Omega))} \leq m_0(R)m_1(T),$$

where $m_i > 0$ are monotone increasing functions.

Proof. Assuming the existence of a pair (φ, σ) meeting the above requirements, its uniqueness is straightforward. We will argue formally, although a Galerkin scheme allows to make rigorous the next computations (see e.g. [6]). Indeed, testing (3.12) by φ and $-\Delta\varphi$ in H and (3.13) by σ in H, owing to (3.1), we easily see that

$$\|\varphi\|^2_{L^\infty(0,T;V)\cap L^2(0,T;H^2(\Omega))} + \|\sigma\|^2_{L^\infty(0,T;H)\cap L^2(0,T;V)} \leq m_0(R)m_1(T),$$

where, for $i = 0, 1$, m_i is a positive and increasing function that may change from line to line in this proof. Besides, by comparison in the equations, we further infer

$$\|\varphi_t\|^2_{L^2(0,T;H)} + \|\sigma_t\|^2_{L^2(0,T;V')} \leq m_0(R)m_1(T).$$

Then, the local (and actually global) existence of a pair $(\varphi, \sigma) \in C([0,T],V) \times C([0,T],H)$ can be deduced by the Galerkin method.

Moreover, notice that σ is a weak solution to

$$\sigma_t - \Delta\sigma = \mathcal{B}(\sigma_B - \sigma) - \mathcal{C}\sigma h(\varphi) \in L^\infty(0,T;H),$$

with $\sigma_0 \in L^\infty(\Omega)$. Applying classical regularity results [16, Theorem 7.1 p. 181], we can see that $\sigma \in L^\infty(Q_T)$. At this point, we can also prove that $\sigma \geq 0$ a.e. in Q_T. Indeed, testing (3.13) by $-\sigma_-$, where $\sigma_- \geq 0$ is the negative part of σ, we infer

$$\frac{1}{2}\frac{d}{dt}\|\sigma_-\|^2 + \|\nabla\sigma_-\|^2 \leq c\|\sigma_-\|^2. \tag{3.14}$$

By Gronwall's Lemma, the assumption $\sigma_0 \geq 0$ entails $\sigma(x,t) \geq 0$. Recalling that $\sigma \in L^\infty(Q_T)$, we can see that

$$0 \leq \sigma \leq \overline{\sigma}_T, \quad \text{a.e. in} \quad Q_T,$$

for some $\overline{\sigma}_T \geq 1$.

We now turn to the lower bound for the order parameter. Testing the corresponding equation (3.12) by $-\varphi_-$, where $\varphi_- \geq 0$ is the negative part of φ, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\varphi_-\|^2 + \|\nabla\varphi_-\|^2 + 4\|\varphi_-\|^4_{L^4(\Omega)} + 6\|\varphi_-\|^3_{L^3(\Omega)} + 2\|\varphi_-\|^2 \\ &= -\int_\Omega \tilde{\theta}\varphi_- - \int_\Omega (\mathcal{P}\sigma - \mathcal{A})h(\varphi)\varphi_-. \end{aligned} \tag{3.15}$$

Since $h(0) = 0$ and owing to the Lipschitz continuity in (3.1), we estimate the last term on the right-hand side as

$$-\int_\Omega (\mathcal{P}\sigma - \mathcal{A})h(\varphi)\varphi_- \leq \|\mathcal{P}\sigma - \mathcal{A}\|_{L^\infty(\Omega)}\int_\Omega |h(\varphi)|\varphi_- \leq c\int_\Omega |\varphi|\varphi_- = c\|\varphi_-\|^2,$$

for some $c = c_T$, noting that $0 \leq \sigma \leq \bar{\sigma}_T$ a.e. in Q_T. Then, in view of the assumption $\tilde{\theta} \geq 0$, the differential inequality reduces to

$$\frac{1}{2}\frac{d}{dt}\|\varphi_-\|^2 + \|\nabla\varphi_-\|^2 + 4\|\varphi_-\|^4_{L^4(\Omega)} + 6\|\varphi_-\|^3_{L^3(\Omega)} + 2\|\varphi_-\|^2 \leq c\|\varphi_-\|^2.$$

Therefore, by Gronwall's Lemma and since $\varphi_0 \geq 0$ a.e. in Ω, it follow that $\varphi \geq 0$ a.e. in Q_T.

Going back to the nutrient's equation, we can now show that $\sigma \leq 1$ a.e. in Q_T. Indeed, having the sign of the order parameter, testing (3.13) by $(\sigma - 1)_+$, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(\sigma-1)_+\|^2 + \|\nabla(\sigma-1)_+\|^2 + \mathcal{B}\|(\sigma-1)_+\|^2 \\ &= \mathcal{B}(\sigma_B - 1)\int_\Omega (\sigma-1)_+ - \mathcal{C}\int_\Omega h(\varphi)\sigma(\sigma-1)_+ \leq 0, \end{aligned}$$

since $\sigma_B \leq 1$. On account of the initial condition, we deduce that $\sigma \leq 1$ a.e. in Q_T.

An upper bound for the L^∞-norm of the order parameter φ is also available, employing a Moser iterations technique.
Indeed, assumption (3.6), in particular, provides

$$\varphi_0 \in L^p(\Omega), \qquad p \in [1, +\infty].$$

Then, according to (3.12), φ solves

$$\varphi_t - \Delta\varphi + 4\varphi^3 - 6\varphi^2 + 2\varphi = g, \tag{3.16}$$

where $g := \tilde{\theta} + (\mathcal{P}\sigma - \mathcal{A})h(\varphi) \in L^4(0,T;L^6(\Omega))$. Testing (3.16) by φ^p, we obtain

$$\begin{aligned} &\frac{1}{p+1}\frac{d}{dt}\int_\Omega \varphi^{p+1} - \int_\Omega (\Delta\varphi)\varphi^p + 4\int_\Omega \varphi^{p+3} + 2\int_\Omega \varphi^{p+1} \\ &\qquad\qquad = 6\int_\Omega \varphi^{p+2} + \int_\Omega g\varphi^p. \end{aligned} \tag{3.17}$$

Since

$$-\int_\Omega (\Delta\varphi)\varphi^p = \frac{4p}{(p+1)^2}\int_\Omega \left|\nabla\left(\varphi^{\frac{p+1}{2}}\right)\right|^2,$$

multiplying (3.17) by $(p+1)$, we infer

$$\begin{aligned} &\frac{d}{dt}\int_\Omega \varphi^{p+1} + \frac{4p}{p+1}\int_\Omega \left|\nabla\left(\varphi^{\frac{p+1}{2}}\right)\right|^2 + 4(p+1)\int_\Omega \varphi^{p+3} + 2(p+1)\int_\Omega \varphi^{p+1} \\ &= 6(p+1)\int_\Omega \varphi^{p+2} + (p+1)\int_\Omega g\varphi^p. \end{aligned}$$

The second term on the right-hand side reads

$$(p+1)\int_\Omega g\varphi^p = (p+1)\int_\Omega g|\varphi|^{\frac{p+1}{2}}|\varphi|^{\frac{p-1}{2}}.$$

Then, using Hölder's inequality with exponents $6, 6$ and $\frac{3}{2}$ and the Sobolev embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$, we have

$$\begin{aligned}
&(p+1)\int_\Omega g\varphi^p \\
&\leq c(p+1)\|g\|_{L^6(\Omega)}\|\varphi^{\frac{p+1}{2}}\|_V \left(\int_\Omega |\varphi|^{\frac{3}{4}(p-1)}\right)^{\frac{2}{3}} \\
&\leq \frac{2p}{p+1}\|\varphi^{\frac{p+1}{2}}\|_V^2 + c(p+1)^2\|g\|_{L^6(\Omega)}^2 \left(\int_\Omega |\varphi|^{\frac{3}{4}(p-1)}\right)^{\frac{4}{3}} \\
&= \frac{2p}{p+1}\|\nabla(\varphi^{\frac{p+1}{2}})\|^2 + \frac{2p}{p+1}\int_\Omega \varphi^{p+1} + c(p+1)^2\|g\|_{L^6(\Omega)}^2 \left(\int_\Omega |\varphi|^{\frac{3}{4}(p-1)}\right)^{\frac{4}{3}},
\end{aligned}$$

where in the last line we have used once again Young's inequality. Here and throughout this proof, we emphasize that the positive constant c is independent of p. Injecting these computations in the differential inequality, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \varphi^{p+1} + \frac{2p}{p+1}\int_\Omega \left|\nabla\left(\varphi^{\frac{p+1}{2}}\right)\right|^2 + 4(p+1)\int_\Omega \varphi^{p+3} + \frac{2}{p+1}\int_\Omega \varphi^{p+1} \\
&\quad \leq 6(p+1)\int_\Omega \varphi^{p+2} + c(p+1)^2\|g\|_{L^6(\Omega)}^2 \left(\int_\Omega |\varphi|^{\frac{3}{4}(p-1)}\right)^{\frac{4}{3}}. \qquad (3.18)
\end{aligned}$$

Since the first term on the right-hand side of (3.18) can be estimated by the generalized Young's inequality as

$$6(p+1)\int_\Omega \varphi^{p+2} \leq (p+1)\int_\Omega \varphi^{p+3} + 6^{p+3}(p+1)|\Omega|,$$

we infer

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \varphi^{p+1} + \frac{2p}{p+1}\int_\Omega \left|\nabla\left(\varphi^{\frac{p+1}{2}}\right)\right|^2 + 3(p+1)\int_\Omega \varphi^{p+3} \\
&\quad \leq c(p+1)6^{p+3} + c(p+1)^2\|g\|_{L^6(\Omega)}^2 \left(\int_\Omega |\varphi|^{\frac{3}{4}(p+1)}\right)^{\frac{4}{3}}.
\end{aligned}$$

Thus, integrating in time, it holds

$$\begin{aligned}
&\int_\Omega \varphi^{p+1}(t) \leq \int_\Omega \varphi_0^{p+1} + Tc(p+1)6^{p+3} \\
&\qquad + c(p+1)^2\int_0^T \|g(s)\|_{L^6(\Omega)}^2 \left(\int_\Omega |\varphi(s)|^{\frac{3}{4}(p+1)}\right)^{\frac{4}{3}} ds. \qquad (3.19)
\end{aligned}$$

In order to apply Moser's iteration scheme, let us consider the sequence $(p_k)_k$ of real numbers defined by

$$p_0 = 3, \qquad p_{k+1} = \frac{4}{3}p_k, \qquad k \in \mathbb{N}.$$

Choosing $p = p_{k+1} - 1$ in (3.19), we obtain

$$\int_\Omega \varphi^{p_{k+1}}(t) \leq c\left(M^{p_{k+1}} + p_{k+1}^2 + p_{k+1}^2 \sup_{[0,T]}\left(\int_\Omega \varphi^{p_k}\right)^{\frac{4}{3}}\right),$$

for some constant M depending on T, $\|\varphi_0\|_{V\cap L^\infty(\Omega)}$ and $\|g\|_{L^2(0,T;L^6(\Omega))}$, but independent of k. Hence, it follows that

$$\sup_{[0,T]}\int_\Omega \varphi^{p_{k+1}}(t) \le cp_{k+1}^2 \max\left\{M^{p_{k+1}}, \sup_{[0,T]}\left(\int_\Omega \varphi^{p_k}\right)^{\frac{4}{3}}\right\},$$

and we can conclude that $\varphi \in L^\infty(0,T;L^\infty(\Omega))$ (see, e.g., [17]), with

$$\|\varphi\|_{L^\infty(0,T;L^\infty(\Omega))} \le m_0(R)m_1(T).$$

Thus, having written (3.12) as

$$\varphi_t - \Delta\varphi = \tilde f$$

where, owing to the regularity of the initial datum,

$$\tilde f := -F'(\varphi) + \tilde\theta + (\mathcal{P}\sigma - \mathcal{A})h(\varphi) \in L^4(0,T;L^6(\Omega)),$$

we also deduce that $\varphi_t, \Delta\varphi \in L^4(0,T;L^6(\Omega))$ (see the subsequent Remark). In particular,

$$\|\varphi_t\|_{L^4(0,T;L^6(\Omega))} \le c(\|\varphi_0\|_{W^{\frac{5}{3},6}(\Omega)} + \|\tilde f\|_{L^4(0,T;L^6(\Omega))}) \le m_0(R)m_1(T).$$

□

Remark 3.3. *Note that the regularity of φ_t and its estimate stated in Lemma 3.2, both crucial in the fixed point argument, rely upon assumption (3.6) on φ_0. Indeed, assume that (φ,σ) is a solution to the problem corresponding to a fixed $\tilde\theta$. Then, in particular, φ solves*

$$\begin{cases} \varphi_t - \Delta\varphi = \tilde f, & \text{in} \quad \Omega\times(0,T),\\ \partial_{\boldsymbol{n}}\varphi = 0, & \text{on} \quad \Gamma\times(0,T),\\ \varphi(0) = \varphi_0, & \text{in} \quad \Omega, \end{cases}$$

where $\tilde f = -F'(\varphi) + \tilde\theta + (\mathcal{P}\sigma - \mathcal{A})h(\varphi)$. On account of [30, Theorem III.1]*, we observe that*

$$\varphi_t \in L^4(0,T;L^6(\Omega)) \quad \text{and} \quad \|\varphi_t - \psi_t\|_{L^4(0,T;L^6(\Omega))} \le c(\|\varphi_0\|_{L^6(\Omega)} + \|\tilde f\|_{L^4(0,T;L^6(\Omega))}),$$

where ψ is the solution to

$$\begin{cases} \psi_t - \Delta\psi = 0, & \text{in} \quad \Omega\times(0,T),\\ \partial_{\boldsymbol{n}}\psi = 0, & \text{on} \quad \Gamma\times(0,T),\\ \psi(0) = \varphi_0, & \text{in} \quad \Omega. \end{cases}$$

Applying [16, Theorem 9.1] *to the last system, we deduce that $\psi \in W_6^{2,1}(Q_T)$, so that, in particular, $\psi_t \in L^6(Q_T)$ with*

$$\|\psi_t\|_{L^6(Q_T)} \le c\|\varphi_0\|_{W^{\frac{5}{3},6}(\Omega)}.$$

Thus, we obtain the desired estimate

$$\|\varphi_t\|_{L^4(0,T;L^6(\Omega))} \le c(\|\varphi_0\|_{W^{\frac{5}{3},6}(\Omega)} + \|\tilde f\|_{L^4(0,T;L^6(\Omega))}).$$

As a second step, we work on the temperature equation, freezing the order parameter and the nutrient concentration variables.
Therefore, given $R, T > 0$, let us define the set

$$\mathcal{X}_R(T) := \big\{(\varphi, \sigma) \in [C([0,T]; V) \cap W^{1,4}(0,T; L^6(\Omega))] \times L^\infty(0,T; L^\infty(\Omega)), \\ \varphi \geq 0 \,\text{and}\, 0 \leq \sigma \leq 1 \;\text{ in }\; Q_T, \quad \|\varphi_t\|_{L^4(0,T;L^6(\Omega))} \leq m_0(R) m_1(T)\big\}.$$

Notice in particular that for any $(\tilde\varphi, \tilde\sigma) \in \mathcal{X}_R(T)$, it holds $\tilde\varphi_t \in \mathcal{H}_\rho(T)$, where

$$\mathcal{H}_\rho(T) := \{u \in L^4(0,T; L^6(\Omega)) : \quad \|u\|_{L^4(0,T;L^6(\Omega))} \leq \rho\},$$

since $\rho = m_0(R) m_1(T)$.

Given $(\tilde\varphi, \tilde\sigma) \in \mathcal{X}_R(T)$, let $m := \tilde\varphi_t$ and consider the problem

$$\theta_t - \operatorname{div}[\kappa(\theta)\nabla\theta] = m^2 - m\theta, \tag{3.20}$$

complemented with homogeneous Neumann boundary conditions and the initial condition $\theta(0) = \theta_0$.

Lemma 3.4. *Assume that (3.5) holds true. For fixed $\rho, T > 0$, let $m \in \mathcal{H}_\rho(T)$. Then there exists a unique θ such that*

$$\begin{aligned} &\theta \in C([0,T]; H) \cap L^\infty(0,T; L^{3q}(\Omega) \cap V) \cap L^{4q}(0,T; L^{12q}(\Omega)) \\ &\theta_t \in L^2(0,T; H) \\ &K(\theta) \in L^\infty(0,T; V) \cap L^2(0,T; H^2(\Omega)) \cap H^1(0,T; L^{\frac{3}{2}}(\Omega)) \\ &\theta \geq 0, \quad \textit{a.e. in} \quad Q_T \end{aligned}$$

that solves

$$\theta_t - \Delta K(\theta) = m^2 - m\theta, \quad \textit{a.e. in} \quad Q_T, \tag{3.21}$$

with $\partial_{\boldsymbol{n}} K(\theta) = 0$ a.e. on $\partial\Omega \times (0,T)$ and $\theta(0) = \theta_0$, a.e. in Ω. Moreover,

$$\|\theta\|_{L^4(0,T;L^6(\Omega))} \leq \mu_0(\rho)\mu_1(T), \tag{3.22}$$

for some monotone increasing functions $\mu_i \geq 0$.

Proof. We first derive formal a priori estimates that can be made rigorous in a Faedo-Galerkin scheme.
Testing (3.21) by θ, we infer

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2 + \int_\Omega \kappa(\theta)|\nabla\theta|^2 = \int_\Omega m^2\theta - \int_\Omega m\theta^2. \tag{3.23}$$

Exploiting the definition of $\kappa(\theta)$ in (3.2), the second term on the left-hand side reads

$$\begin{aligned} \int_\Omega \kappa(\theta)|\nabla\theta|^2 &= \|\nabla\theta\|^2 + \int_\Omega |\theta|^q |\nabla\theta|^2 \\ &= \|\nabla\theta\|^2 + \frac{4}{(q+2)^2}\left\|\nabla\left(|\theta|^{\frac{q}{2}+1}\right)\right\|^2. \end{aligned} \tag{3.24}$$

The first term on the right-hand side of (3.23) can be controlled by Hölder's and Young's inequalities:

$$\int_\Omega m^2\theta \leq \|m\|^2_{L^6(\Omega)}\|\theta\|_{L^{\frac{3}{2}}(\Omega)} \leq c\|m\|^2_{L^6(\Omega)}\|\theta\| \leq c\|m\|^2_{L^6(\Omega)}(1 + \|\theta\|^2). \tag{3.25}$$

On the other hand, similarly, we can take care of the second term on the right-hand side of (3.23). Employing the interpolation inequality

$$\|\theta\|_{L^{\frac{12}{5}}(\Omega)} \leq c\|\theta\|^{\alpha}_{L^2(\Omega)}\|\theta\|^{1-\alpha}_{L^6(\Omega)},$$

with $\frac{5}{12} = \frac{\alpha}{2} + \frac{1-\alpha}{6}$ $\left(\alpha = \frac{3}{4}\right)$, and the Sobolev embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$, it holds

$$\begin{aligned}\int_\Omega m\theta^2 &\leq c\|m\|_{L^6(\Omega)}\|\theta\|^{\frac{3}{2}}_{L^2(\Omega)}\|\theta\|^{\frac{1}{2}}_{L^6(\Omega)} \leq c\|m\|_{L^6(\Omega)}\|\theta\|^{\frac{3}{2}}\|\theta\|^{\frac{1}{2}}_V \\ &\leq \frac{1}{2}\|\theta\|^2_V + c\|m\|^{\frac{4}{3}}_{L^6(\Omega)}\|\theta\|^2 \leq \frac{1}{2}\|\nabla\theta\|^2 + c(1+\|m\|^{\frac{4}{3}}_{L^6(\Omega)})\|\theta\|^2, \qquad (3.26)\end{aligned}$$

where we have also exploited Young's inequality. Combining (3.24), (3.25), (3.26) in (3.23), we infer

$$\frac{d}{dt}\|\theta\|^2 + \|\nabla\theta\|^2 + \frac{2}{(q+2)^2}\left\|\nabla\left(|\theta|^{\frac{q}{2}+1}\right)\right\|^2 \leq c\left(1+\|m\|^2_{L^6(\Omega)}\right)\|\theta\|^2 + c\|m\|^2_{L^6(\Omega)},$$

hence

$$\|\theta\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} + \left\|\nabla\left(|\theta|^{\frac{q}{2}+1}\right)\right\|_{L^2(0,T;H)} \leq c(\rho,T), \qquad (3.27)$$

where, in this proof, $c \geq 0$ stands for monotone increasing functions in both arguments and may possibly change from line to line. The bound on the second norm above yields

$$|\theta|^{\frac{q}{2}+1} \in L^2(0,T;V), \qquad (3.28)$$

since

$$\||\theta|^{\frac{q}{2}+1}\|^2_V \leq c(\|\theta\|^{q+2}_{L^1(\Omega)} + \|\nabla(|\theta|^{\frac{q}{2}+1})\|^2) \leq c(\|\theta\|^{q+2} + \|\nabla(|\theta|^{\frac{q}{2}+1})\|^2)$$

according to the nonlinear Poincaré inequality (3.9) and the continuous inclusion $L^2(\Omega) \subset L^1(\Omega)$.

Testing the temperature equation (3.20) by $\theta|\theta|^{3q-2}$ leads to

$$\int_\Omega \theta_t\theta|\theta|^{3q-2} + \int_\Omega \kappa(\theta)\nabla\theta\cdot\nabla(\theta|\theta|^{3q-2}) = \int_\Omega m^2\theta|\theta|^{3q-2} - \int_\Omega m|\theta|^{3q}. \qquad (3.29)$$

We first focus on the left-hand side of (3.29). Since

$$\frac{d}{dt}\int_\Omega |\theta|^{3q} = 3q\int_\Omega \theta|\theta|^{3q-2}\theta_t,$$

the first term on the left-hand side can be equivalently written as

$$\int_\Omega \theta_t\theta|\theta|^{3q-2} = \frac{1}{3q}\frac{d}{dt}\int_\Omega |\theta|^{3q}. \qquad (3.30)$$

On the other hand,

$$\nabla(\theta|\theta|^{3q-2}) = (3q-1)|\theta|^{3q-2}\nabla\theta$$

entails, using the definition of $\kappa(\vartheta)$ in (3.2),

$$\begin{aligned}\int_\Omega \kappa(\theta)\nabla\theta\cdot\nabla(\theta|\theta|^q) &= (3q-1)\int_\Omega(|\theta|^{3q-2}+|\theta|^{4q-2})|\nabla\theta|^2 \\ &= \frac{4(3q-1)}{(3q)^2}\|\nabla\left(|\theta|^{\frac{3q}{2}}\right)\|^2 + \frac{3q-1}{4q^2}\|\nabla(|\theta|^{2q})\|^2. \qquad (3.31)\end{aligned}$$

Inserting (3.30) and (3.31) into (3.29), we see that

$$\begin{aligned}\frac{1}{3q}\frac{d}{dt}\int_\Omega |\theta|^{3q} + \frac{4(3q-1)}{(3q)^2}\|\nabla\left(|\theta|^{\frac{3q}{2}}\right)\|^2 + \frac{3q-1}{4q^2}\|\nabla(|\theta|^{2q})\|^2 \\ = \int_\Omega m^2\theta|\theta|^{3q-2} - \int_\Omega m|\theta|^{3q} =: I_A + I_B. \qquad (3.32)\end{aligned}$$

We now take care of the right-hand side. For simplicity, we start by studying the second term I_B. We preliminarily observe that, by interpolation,

$$\|\theta\|_{L^{\frac{3q}{2}}(\Omega)} \le \|\theta\|_{L^2(\Omega)}^{\frac{2}{3q-2}}\|\theta\|_{L^{3q}(\Omega)}^{\frac{3q-4}{3q-2}}. \qquad (3.33)$$

By Hölder's inequality with exponents $6, 3/2$ and 6, the Sobolev embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$, (3.33) and then Young's inequality together with (3.9), we have

$$\begin{aligned}I_B &:= -\int_\Omega m|\theta|^{3q} = \int_\Omega |m||\theta|^q|\theta|^{2q} \le c\|m\|_{L^6(\Omega)}\|\theta\|^q_{L^{\frac{3q}{2}}(\Omega)}\||\theta|^{2q}\|_V \\ &\le \frac{\delta}{2}\||\theta|^{2q}\|_V^2 + c\|m\|^2_{L^6(\Omega)}\|\theta\|^{2q}_{L^{\frac{3q}{2}}(\Omega)} \\ &\le \frac{3q-1}{16q^2}\|\nabla|\theta|^{2q}\|^2 + c\|\theta\|^{4q}_{L^1(\Omega)} + c\|m\|^2_{L^6(\Omega)}\|\theta\|^{\frac{4q}{3q-2}}_{L^2(\Omega)}\|\theta\|^{2q\frac{3q-4}{3q-2}}_{L^{3q}(\Omega)} \\ &\le \frac{3q-1}{16q^2}\|\nabla|\theta|^{2q}\|^2 + c\|\theta\|^{4q} + c\|m\|^2_{L^6(\Omega)}\|\theta\|^{3q}_{L^{3q}(\Omega)} + c\|m\|^2_{L^6(\Omega)},\end{aligned}$$

for a suitable choice of $\delta > 0$ depending on q. On the other hand, arguing as above, with exponents $3, 6$ and 2 we obtain

$$\begin{aligned}I_A &:= \int_\Omega m^2\theta|\theta|^{3q-2} \le \int_\Omega m^2|\theta|^{2q}|\theta|^{q-1} \le \|m\|^2_{L^6(\Omega)}\||\theta|^{2q}\|_V\|\theta\|^{q-1}_{L^{2q-2}(\Omega)} \\ &\le \frac{\delta}{2}\||\theta|^{2q}\|_V^2 + c\|m\|^4_{L^6(\Omega)}\|\theta\|^{2(q-1)}_{L^{2q-2}(\Omega)} \\ &\le \frac{3q-1}{16q^2}\|\nabla|\theta|^{2q}\|^2 + c\|\theta\|^{4q}_{L^1(\Omega)} + c\|m\|^4_{L^6(\Omega)}\|\theta\|^{3q}_{L^{3q}(\Omega)} + c\|m\|^4_{L^6(\Omega)} \\ &\le \frac{3q-1}{16q^2}\|\nabla|\theta|^{2q}\|^2 + c\|\theta\|^{4q} + c\|m\|^4_{L^6(\Omega)}\|\theta\|^{3q}_{L^{3q}(\Omega)} + c\|m\|^4_{L^6(\Omega)},\end{aligned}$$

having also exploited (3.9) and a properly chosen $\delta > 0$. Replacing the above estimates in (3.32), on account of (3.27), we have

$$\begin{aligned}\frac{1}{3q}\frac{d}{dt}\|\theta\|^{3q}_{L^{3q}(\Omega)} + \frac{4(3q-1)}{(3q)^2}\|\nabla\left(|\theta|^{\frac{3q}{2}}\right)\|^2 + \frac{3q-1}{8q^2}\|\nabla(|\theta|^{2q})\|^2 \\ \le c(\|m\|^4_{L^6(\Omega)} + 1)\|\theta\|^{3q}_{L^{3q}(\Omega)} + c(\|m\|^4_{L^6(\Omega)} + 1), \qquad (3.34)\end{aligned}$$

for some $c = c(\rho, T)$. Therefore, $\theta \in L^\infty(0,T;L^{3q}(\Omega))$, with

$$\|\theta\|_{L^\infty(0,T;L^{3q}(\Omega))} + \||\theta|^{2q}\|_{L^2(0,T;V)} \le \mu_0(\rho)\mu_1(T), \qquad (3.35)$$

for some monotone increasing functions $\mu_i \ge 0$, possibly changing from line to line. In particular, the previous inequality yields

$$\|K(\theta)\|_{L^\infty(0,T;H)} \le \mu_0(\rho)\mu_1(T), \qquad (3.36)$$

since $2q + 2 \le 3q$.

Our next task is an estimate for the time derivative of θ. We observe that, according to (3.4), the temperature equation (3.20) reads

$$\theta_t - \Delta K(\theta) = m^2 - m\theta$$

that we formally multiply by $[K(\theta)]_t = \kappa(\theta)\theta_t$. Then, we infer

$$\|\sqrt{\kappa(\theta)}\theta_t\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla K(\theta)\|^2 = \int_\Omega m^2\kappa(\theta)\theta_t - \int_\Omega m\theta\kappa(\theta)\theta_t =: J_1 + J_2. \tag{3.37}$$

The first term on the right-hand side is managed by Hölder's inequality with exponents $3, 6$ and 2, Young's inequality, (3.2) and (3.35), so that

$$\begin{aligned} J_1 := \int_\Omega m^2\sqrt{\kappa(\theta)}\sqrt{\kappa(\theta)}\theta_t &\le \|m\|^2_{L^6(\Omega)}\|\sqrt{\kappa(\theta)}\|_{L^6(\Omega)}\|\sqrt{\kappa(\theta)}\theta_t\| \\ &\le \frac{1}{4}\|\sqrt{\kappa(\theta)}\theta_t\|^2 + c\|m\|^4_{L^6(\Omega)}\left(\int_\Omega [\kappa(\theta)]^3\right)^{\frac{1}{3}} \\ &\le \frac{1}{4}\|\sqrt{\kappa(\theta)}\theta_t\|^2 + c\|m\|^4_{L^6(\Omega)}\left(\int_\Omega \left(1+|\theta|^{3q}\right)\right)^{\frac{1}{3}} \\ &\le \frac{1}{4}\|\sqrt{\kappa(\theta)}\theta_t\|^2 + c\|m\|^4_{L^6(\Omega)}. \end{aligned} \tag{3.38}$$

Arguing similarly, we can estimate J_2 as follows:

$$\begin{aligned} J_2 &= \int_\Omega m\theta\kappa(\theta)\theta_t = \int_\Omega m(\sqrt{\kappa(\theta)}\theta)\sqrt{\kappa(\theta)}\theta_t \\ &\le \|m\|_{L^6(\Omega)}\|\theta\sqrt{\kappa(\theta)}\|_{L^3(\Omega)}\|\|\sqrt{\kappa(\theta)}\theta_t\| \\ &\le \frac{1}{4}\|\sqrt{\kappa(\theta)}\theta_t\|^2 + c\|m\|^2_{L^6(\Omega)}\|\theta\sqrt{\kappa(\theta)}\|^2_{L^3(\Omega)} \\ &\le \frac{1}{4}\|\sqrt{\kappa(\theta)}\theta_t\|^2 + c\|m\|^2_{L^6(\Omega)}\left(\int_\Omega (|\theta|^3 + |\theta|^{\frac{3}{2}q+3}\right)^{\frac{2}{3}} \\ &\le \frac{1}{4}\|\sqrt{\kappa(\theta)}\theta_t\|^2 + c\|m\|^2_{L^6(\Omega)}\left(\int_\Omega (1+|\theta|^{3q})\right)^{\frac{2}{3}} \\ &\le \frac{1}{4}\|\sqrt{\kappa(\theta)}\theta_t\|^2 + c\|m\|^2_{L^6(\Omega)}, \end{aligned} \tag{3.39}$$

having applied the Hölder inequality with exponents $6, 3$ and 2. Replacing (3.38) and (3.39) in (3.37), we are lead to

$$\|\sqrt{\kappa(\theta)}\theta_t\|^2 + \frac{d}{dt}\|\nabla K(\theta)\|^2 \le c(\|m\|^4_{L^6(\Omega)} + 1).$$

An integration in time on account of (3.36) provides

$$\|\sqrt{\kappa(\theta)}\theta_t\|_{L^2(0,T;H)} + \|K(\theta)\|_{L^\infty(0,T;V)} \le c(\rho, T). \tag{3.40}$$

Thus, it follows in particular that

$$\|\theta\|_{L^\infty(0,T;V)} \le c(\rho, T), \tag{3.41}$$

$$\|\theta_t\|_{L^2(0,T;H)} \le c(\rho, T), \tag{3.42}$$

$$\|\Delta K(\theta)\|_{L^2(0,T;H)} \le c(\rho, T). \tag{3.43}$$

Besides, by interpolation, we eventually infer

$$K(\theta) \in L^4(0, T; W^{1,3}(\Omega)), \tag{3.44}$$

so that, by definition of K,

$$\|\nabla\theta\|_{L^4(0,T;L^3(\Omega))} \leq c(\rho,T). \tag{3.45}$$

We can also prove that $\partial_t[K(\theta)] = \kappa(\theta)\theta_t \in L^2(0,T;L^{\frac{3}{2}}(\Omega))$. Indeed, by (3.35), we learn that

$$\|\kappa(\theta)\|_{L^\infty(0,T;L^3(\Omega))} \leq c(\rho,T). \tag{3.46}$$

Therefore, owing also to (3.40),

$$\int_0^T \|\partial_t[K(\theta(s)]\|^2_{L^{\frac{3}{2}}(\Omega)} ds \leq \|\kappa(\theta)\|_{L^\infty(0,T;L^3(\Omega))}\|\sqrt{\kappa(\theta)}\theta_t\|_{L^2(Q_T)} \leq c.$$

Since we already know that $K(\theta) \in L^2(0,T;H^2(\Omega))$ by (3.40) and (3.43), it follows that

$$K(\theta) \in C([0,T];V). \tag{3.47}$$

The previous computations hold for each approximating solution θ_n in a Faedo-Galerkin scheme. Therefore,

$$\|\partial_t\theta_n\|_{L^2(Q_T)} + \|\theta_n\|_{L^\infty(0,T,V)} + \|\theta_n\|_{L^\infty(0,T;L^{3q}(\Omega))} + \|\theta_n\|_{L^4(0,T;W^{1,3}(\Omega))} \leq c(\rho,T)$$

$$\|K(\theta_n)\|_{L^\infty(0,T;V)\cap L^2(0,T;H^2(\Omega))} + \|\partial_t K(\theta_n)\|_{L^2(0,T;L^{\frac{3}{2}}(\Omega))} \leq c(\rho,T),$$

as well as

$$\||\theta_n|^{2q}\|_{L^2(0,T,V)} \leq c(\rho,T), \tag{3.48}$$

where $c(\rho,T)$ does not depend on n. Besides,

$$\|\theta_n\|_{L^{4q}(0,T;L^{12q}(\Omega))} \leq c(\rho,T), \tag{3.49}$$

on account (3.35) and of the inclusion $V \subset L^6(\Omega)$. Indeed,

$$\begin{aligned}\|\theta_n\|^{4q}_{L^{4q}(0,T;L^{12q}(\Omega))} &= \int_0^T \left(\int_\Omega |\theta_n|^{12q}\right)^{\frac{1}{3}} = \int_0^T \left(\int_\Omega (|\theta_n|^{2q})^6\right)^{\frac{1}{3}} \\ &\leq c\int_0^T \left(\||\theta_n|^{2q}\|^2_V\right) \leq c(\rho,T).\end{aligned}$$

Since

$$W^{1,3}(\Omega) \Subset L^s(\Omega), \quad \text{for any finite} \quad s \geq 3$$

and applying Aubin-Simon compactness results, the inclusion of $H^1(0,T;L^2(\Omega)) \cap L^4(0,T;W^{1,3}(\Omega))$ in $L^4(0,T;L^s(\Omega))$ is compact for any $s \geq 3$ (in particular, when $s = \frac{12q}{q+1}$). Then, up to a subsequence,

$$\theta_n \to \theta \quad \text{strongly in} \quad L^4(0,T;L^s(\Omega)), \tag{3.50}$$

for any finite $s \geq 3$, as well as

$$\begin{aligned}\partial_t\theta_n \to \partial_t\theta \quad &\text{weakly in} \quad L^2(0,T;L^2(\Omega)),\\ \theta_n \to \theta \quad \text{weakly in} \quad &L^4(0,T;W^{1,3}(\Omega)) \cap L^{4q}(0,T;L^{12q}(\Omega)),\\ \theta_n \to \theta \quad &\text{weakly-* in} \quad L^\infty(0,T;L^{3q}(\Omega)),\\ \theta_n \to \theta \quad &\text{a.e. in} \quad \Omega \times (0,T).\end{aligned}$$

Thus, we also have

$$\begin{aligned}K(\theta_n) \to K(\theta) \quad &\text{strongly in} \quad C([0,T];H) \quad \text{and} \quad L^2(0,T;V),\\ K(\theta_n) \to K(\theta) \quad &\text{weakly in} \quad H^1(0,T;L^{\frac{3}{2}}(\Omega)) \quad \text{and} \quad L^2(0,T;H^2(\Omega)),\end{aligned}$$

$$K(\theta_n) \to K(\theta) \quad \text{weakly-* in} \quad L^\infty(0,T;V).$$

In view of these convergences, θ is indeed a solution to (3.20).
Actually, $\theta \geq 0$ a.e. in $\Omega \times (0,T)$ and, thus, there was no loss of generality in taking $\kappa(s) = 1 + |s|^q$.

Indeed, using Young's inequality on the term $m\theta$ in the temperature equation (3.20), it is easy to see that

$$\begin{aligned}\theta_t - \operatorname{div}[\kappa(\theta)\nabla\theta] &= m^2 - m\theta \\ &\geq m^2 - \frac{\theta^2}{2} - \frac{m^2}{2} \geq -\frac{\theta^2}{2}.\end{aligned}$$

Hence, $v \equiv 0$ is a subsolution to (3.20) and thus $\theta(\cdot,t) \geq 0, \quad \forall t \in [0,T]$.

Once the existence of a solution to (3.20) is proved, we also show its uniqueness.

Having fixed two solutions θ_1, θ_2 departing from the same initial datum θ_0, notice that $\Theta = \theta_1 - \theta_2$ solves

$$\Theta_t + A[K(\theta_1) - K(\theta_2)] = K(\theta_1) - K(\theta_2) - m\Theta, \tag{3.51}$$

where $A = -\Delta + \mathbb{I} : V \to V'$ is the invertible operator with inverse $\mathcal{N} = A^{-1}$. Multiplying the above equation by $\mathcal{N}\Theta$, we are led to

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Theta\|_*^2 + (K(\theta_1) - K(\theta_2), \Theta) = (K(\theta_1) - K(\theta_2), \mathcal{N}\Theta) - (m\Theta, \mathcal{N}\Theta).$$

Since $(K(\theta_1) - K(\theta_2), \Theta) \geq \|\Theta\|^2$, it follows that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Theta\|_*^2 + \|\Theta\|^2 \leq \|K(\theta_1) - K(\theta_2)\|_{L^{\frac{6}{5}}(\Omega)}\|\mathcal{N}\Theta\|_{L^6(\Omega)} + \|m\|_{L^3(\Omega)}\|\Theta\|\|\mathcal{N}\Theta\|_{L^6(\Omega)}.$$

We now introduce the function $\ell(r) = r^{q+1}$ for $r \geq 0$, whose derivative is $\ell'(r) = (q+1)r^q$. Taking advantage of this notation, we have

$$\begin{aligned}|K(\theta_1) - K(\theta_2)| &= \left|\Theta + \frac{1}{q+1}[\ell(\theta_1) - \ell(\theta_2)]\right| \\ &= \left|\Theta + \frac{1}{q+1}\int_0^1 \ell'(s\theta_1 + (1-s)\theta_2)ds\,\Theta\right| \\ &= |\Theta|\left|1 + \int_0^1 |s\theta_1 + (1-s)\theta_2|^q ds\right| \\ &\leq |\Theta|(1 + |\theta_1|^q + |\theta_2|^q).\end{aligned}$$

Therefore,

$$\begin{aligned}\|K(\theta_1) - K(\theta_2)\|_{L^{\frac{6}{5}}(\Omega)} &\leq c\left(\int_\Omega |\Theta|^{\frac{6}{5}}(1 + |\theta_1|^{\frac{6}{5}q} + |\theta_2|^{\frac{6}{5}q})\right)^{\frac{5}{6}} \\ &\leq \|\Theta\|\left(\int_\Omega (1 + |\theta_1|^{\frac{6}{5}q} + |\theta_2|^{\frac{6}{5}q})^{\frac{5}{2}}\right)^{\frac{1}{3}} \\ &\leq c\|\Theta\|\left(\int_\Omega (1 + |\theta_1|^{3q} + |\theta_2|^{3q})\right)^{\frac{1}{3}} \\ &\leq c\|\Theta\|(1 + \|\theta_1\|_{L^{3q}(\Omega)}^q + \|\theta_2\|_{L^{3q}(\Omega)}^q),\end{aligned}$$

by Hölder's inequality with exponents $5/3$ and $5/2$. Thus, by the continuous inclusion $W^{1,2}(\Omega) \subset L^6(\Omega)$ and the continuity of $\mathcal{N}$, we deduce that

$$\begin{aligned}
&\|K(\theta_1) - K(\theta_2)\|_{L^{\frac{6}{5}}(\Omega)} \|\mathcal{N}\Theta\|_{L^6(\Omega)} \\
&\quad \le c\|\Theta\|_* \|\Theta\| (1 + \|\theta_1\|^q_{L^{3q}(\Omega)} + \|\theta_2\|^q_{L^{3q}(\Omega)}) \\
&\quad \le \frac{1}{4}\|\Theta\|^2 + c\|\Theta\|_*^2 (1 + \|\theta_1\|^q_{L^{3q}(\Omega)} + \|\theta_2\|^q_{L^{3q}(\Omega)})^2. \qquad (3.52)
\end{aligned}$$

Recalling that

$$\|\theta_1\|_{L^\infty(0,T;L^{3q}(\Omega))} + \|\theta_2\|_{L^\infty(0,T;L^{3q}(\Omega))} \le c,$$

we are lead to

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Theta\|_*^2 + \|\Theta\|^2 \le c(1 + \|m\|^2_{L^3(\Omega)})\|\Theta\|_*^2.$$

Therefore, Gronwall's Lemma allows us to conclude. □

Proof of Theorem 3.1. Let us define the operator $\mathcal{T} := \mathcal{T}_2 \circ \mathcal{T}_1$, where $(\varphi, \sigma) =: \mathcal{T}_1(\tilde{\theta}) : Q_T \to \mathbb{R}^2$, with $\tilde{\theta} \in \Theta_R(T)$, and $\theta =: \mathcal{T}_2(\varphi, \sigma) : Q_T \to \mathbb{R}$, with $(\varphi, \sigma) \in \mathcal{X}_R(T)$.
The local (in time) existence of the solution is obtained by showing that, for any fixed $R > 0$, the Schauder fixed point theorem applies to the operator

$$\begin{aligned}
\mathcal{T} : \Theta_R(T) &\to L^4(0,T;L^6(\Omega)) \\
\tilde{\theta} &\mapsto \theta,
\end{aligned}$$

provided that T is small enough.
First, we see that $\mathcal{T}$ is well defined for any $T, R > 0$. Indeed, having fixed $(\theta_0, \varphi_0, \sigma_0)$ satisfying (3.5)–(3.7) and (3.11), given any $\tilde{\theta} \in \tilde{\Theta}_R(T)$, by Lemma 3.2, there exists a unique solution $(\varphi, \sigma) \in \mathcal{X}_R(T)$ to (3.12)-(3.13) corresponding to $\tilde{\theta}$, complemented with homogeneous Neumann boundary conditions and meeting the initial data. Indeed, $\varphi_t \in L^4(0,T;L^6(\Omega))$ with

$$\|\varphi_t\|_{L^4(0,T;L^6(\Omega))} \le m_0(R) m_1(T).$$

Then, we can plug $m = \varphi_t \in \mathcal{H}_\rho(T)$ with $\rho = m_0(R)m_1(T)$ in (3.21) and, by Lemma 3.4, see that there exists a unique solution $\theta \ge 0$ such that

$$\|\theta\|_{L^\infty(0,T;L^{3q}(\Omega))} \le \mu_0(R)\mu_1(T).$$

Therefore the operator $\mathcal{T}(\tilde{\theta}) = \theta$ is well defined. We now prove the following.

- **The operator $\mathcal{T} : \Theta_R(T) \to \Theta_R(T)$ with a suitable choice of T.** Since

$$\|\theta\|_{L^4(0,T;L^6(\Omega))} \le T^{\frac{1}{4}} \|\theta\|_{L^\infty(0,T;L^{3q}(\Omega))} \le T^{\frac{1}{4}} c(R,T) \le R,$$

  provided that T is small enough.
- **The operator $\mathcal{T}$ is continuous with respect to the $L^4(0,T;L^6(\Omega))$-norm.** This can be easily seen along the same lines as in the uniqueness' proof.
- **The operator $\mathcal{T}$ is compact.** Since

$$H^1(0,T;H) \cap L^4(0,T;W^{1,3}(\Omega)) \Subset L^4(0,T;L^6(\Omega)),$$

  $\mathcal{T}(\Theta_R(T))$ is compact in $\Theta_R(T)$.

Since the assumptions of the Schauder fixed point theorem are satisfied for $T = T(R) > 0$ small enough, we deduce the local existence of a solution to our problem. Moreover, since the estimates needed to prove uniqueness hold true on any time interval $(0,T)$, the solution is also unique. Indeed, given two solutions $(\theta_i, \varphi_i, \sigma_i)$ originating from the same initial datum $(\theta_0, \varphi_0, \sigma_0)$, let $(\theta, \varphi, \sigma) = (\theta_1 - \theta_2, \varphi_1 - \varphi_2, \sigma_1 - \sigma_2)$. This triplet solves

$$\begin{cases} \theta_t - \Delta[K(\theta_1) - K(\theta_2)] + K(\theta_1) - K(\theta_2) \\ \quad = K(\theta_1) - K(\theta_2) + \partial_t \varphi_1^2 - \partial_t \varphi_2^2 - \theta_1 \partial_t \varphi_1 + \theta_2 \partial_t \varphi_2, \quad \text{a.e. in } Q_T \\ \varphi_t - \Delta \varphi \\ \quad = -[F'(\varphi_1) - F'(\varphi_2)] + \theta + (\mathcal{P}\sigma_1 - \mathcal{A})h(\varphi_1) - (\mathcal{P}\sigma_2 - \mathcal{A})h(\varphi_2), \text{ a.e. in } Q_T \\ \sigma_t - \Delta\sigma + \mathcal{B}\sigma = -\mathcal{C}\sigma_1 h(\varphi_1) + \mathcal{C}\sigma_2 h(\varphi_2), \quad \text{in} \quad V', \text{ a.e. in } (0,T), \end{cases}$$

complemented with no flux boundary conditions and null initial conditions. We introduce the functional

$$\mathcal{E}(t) = \|\theta(t)\|_*^2 + \|\varphi(t)\|^2 + \frac{1}{2}\|\nabla\varphi(t)\|^2 + \|\sigma(t)\|^2,$$

for which a suitable differential inequality holds, as we see exploiting in particular the continuous operator $\mathcal{N} : V \to V'$ as above. Indeed, summing the product of the first equation above by $\mathcal{N}\theta$, the second one by $\varphi + \frac{1}{2}\varphi_t$ and the third one by σ, we see that

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} + \|\nabla\varphi\|^2 + \|\nabla\sigma\|^2 + \mathcal{B}\|\sigma\|^2 + \frac{1}{2}\|\varphi_t\|^2 + \|\theta\|^2 \\ &\le c\|\varphi\|^2 + \|\theta\|\|\varphi\| + c\|\sigma\|\|\varphi\| - \frac{1}{2}\int_\Omega [F'(\varphi_1) - F'(\varphi_2)]\varphi_t + \frac{1}{2}\|\theta\|\|\varphi_t\| + c\|\sigma\|\|\varphi_t\| \\ &\quad + \int_\Omega [K(\theta_1) - K(\theta_2)]\mathcal{N}\theta + \int_\Omega \partial_t\varphi(\partial_t\varphi_1 + \partial_t\varphi_2)\mathcal{N}\theta - \int_\Omega \theta\partial_t\varphi_1\mathcal{N}\theta - \int_\Omega \theta_2\partial_t\varphi\mathcal{N}\theta. \end{aligned}$$

We focus on the five non trivial terms on the right-hand side. First, notice that

$$-\frac{1}{2}\int_\Omega [F'(\varphi_1) - F'(\varphi_2)]\varphi_t \le (1 + \|\varphi_1\|_{L^6(\Omega)}^2 + \|\varphi_1\|_{L^6(\Omega)}^2)\|\varphi\|_V\|\varphi_t\| \le c\|\varphi\|_V\|\varphi_t\|.$$

Arguing exactly as in (3.52), we find

$$\begin{aligned} \int_\Omega [K(\theta_1) - K(\theta_2)]\mathcal{N}\theta &\le \|K(\theta_1) - K(\theta_2)\|_{L^{\frac{6}{5}}(\Omega)}\|\mathcal{N}\theta\|_{L^6(\Omega)} \\ &\le c\|\theta\|\|\theta\|_*(1 + \|\theta_1\|_{L^{3q}(\Omega)}^q + \|\theta_2\|_{L^{3q}(\Omega)}^q) \le c\|\theta\|\|\theta\|_*. \end{aligned}$$

Applying Hölder's inequality, we have

$$\begin{aligned} \int_\Omega \partial_t\varphi(\partial_t\varphi_1 + \partial_t\varphi_2)\mathcal{N}\theta &\le \|\varphi_t\|(\|\partial_t\varphi_1\|_{L^3(\Omega)} + \|\partial_t\varphi_2\|_{L^3(\Omega)})\|\mathcal{N}\theta\|_{L^6(\Omega)} \\ &\le c\|\varphi_t\|(\|\partial_t\varphi_1\|_{L^3(\Omega)} + \|\partial_t\varphi_2\|_{L^3(\Omega)})\|\theta\|_*. \end{aligned}$$

Finally,

$$\begin{aligned} -\int_\Omega \theta\partial_t\varphi_1\mathcal{N}\theta - \int_\Omega \theta_2\partial_t\varphi\mathcal{N}\theta &\le \|\theta\|\|\mathcal{N}\theta\|_{L^6(\Omega)}\|\partial_t\varphi_1\|_{L^3(\Omega)} + \|\theta_2\|_{L^3(\Omega)}\|\varphi_t\|\|\mathcal{N}\theta\|_{L^6(\Omega)} \\ &\le c\|\partial_t\varphi_1\|_{L^3(\Omega)}\|\theta\|\|\theta\|_* + c + \|\theta_2\|_{L^3(\Omega)}\|\varphi_t\|\|\theta\|_*. \end{aligned}$$

Collecting the above estimates and applying Young's inequality, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} \leq c(\|\partial_t\varphi_1\|^2_{L^3(\Omega)} + \|\partial_t\varphi_2\|^2_{L^3(\Omega)} + \|\theta_2\|^2_{L^3(\Omega)})\mathcal{E}.$$

Gronwall's lemma allows us to conclude. □

Remark 3.5. *Provided that initial datum $\theta_0 \geq \underline{\theta} > 0$ a.e. in Ω, where $\underline{\theta}$ is a constant, the same argument used in Lemma 3.4 to prove that $\theta \geq 0$ a.e. in Q_T, allows us to show that $\theta \geq \underline{\theta}$ a.e. in Q_T.*

3.3. **Global in time existence of the solution.** We finally observe that the local solution is in fact global.

Theorem 3.6. *Let $(\theta_0, \varphi_0, \sigma_0)$ satisfy assumptions (3.5)–(3.7) and let $T > 0$ be an arbitrary final time. Then, there exists a unique solution $(\theta, \varphi, \sigma)$ to system (3.3) on the whole time interval $(0, T)$. In particular,*

$$\theta \geq 0, \quad 0 \leq \sigma \leq 1 \quad \textit{and} \quad \varphi \geq 0 \quad \textit{a.e. in} \quad Q_T.$$

Proof. The local existence proved in Theorem 3.1 is actually global, thanks to an additional regularity of the solution holding for positive times. Indeed, assume now that $[0, T)$ is the maximal interval of existence of the solution and let

$$(\hat{\theta}, \hat{\varphi}, \hat{\sigma}) = \lim_{t\to T^-}(\theta(t), \varphi(t), \sigma(t)).$$

We first observe that $\hat{\sigma} \in L^\infty(\Omega)$, with $0 \leq \hat{\sigma} \leq 1$. Indeed, $\sigma \in C([0,T]; L^2(\Omega))$, with $0 \leq \sigma \leq 1$ a.e. in Q_T, and then $\hat{\sigma} \in L^2(\Omega)$, with $0 \leq \hat{\sigma} \leq 1$, meaning, in particular, that $\hat{\sigma} \in L^\infty(\Omega)$.

The condition $\hat{\varphi} \in W^{\frac{5}{3},6}(\Omega)$ follows from $\varphi \in C([\delta, T]; W^{\frac{5}{3},6}(\Omega))$, for any $\delta > 0$. Indeed, observe that φ solves

$$\varphi_t - \Delta\varphi = \tilde{f},$$

where now $\tilde{f} \in L^6(\delta, T; L^6(\Omega))$ for any $\delta \in (0, T)$. Then, [16, Theorem 9.1] implies that

$$\varphi \in L^6(\delta, T; W^{2,6}(\Omega) \cap W^{1,6}(\delta, T; L^6(\Omega)),$$

so that $\varphi \in C([\delta, T]; W^{\frac{5}{3},6}(\Omega))$, yielding $\hat{\varphi} \in W^{\frac{5}{3},6}(\Omega)$ and $\hat{\varphi} \geq 0$.

Finally, we see that $\hat{\theta} \in L^{3q}(\Omega) \cap V$, with $\hat{\theta} \geq 0$ a.e. in Ω and $K(\hat{\theta}) \in V$. Lemma 3.4 yields $\theta \in C([0,T]; H) \cap L^\infty(0,T; L^{3q}(\Omega) \cap V)$, with $\theta \geq 0$ a.e. in Q_T. A subsequent application of Strauss' Lemma (see, e.g., [24]) provides $\theta \in C_w([0,T]; L^{3q}(\Omega) \cap V)$, so that $\hat{\theta} \in L^{3q}(\Omega) \cap V$ and $\hat{\theta} \geq 0$ exactly as in [27]. Besides, $K(\hat{\theta}) \in V$, owing to (3.47).

We then conclude that $(\hat{\theta}, \hat{\varphi}, \hat{\sigma})$ is an admissible initial datum, so that we can extend further the solution, covering any time interval $[0, T]$ in a finite number of steps (see [27]). □

Acknowledgments. S. G. is a member of G.N.A.M.P.A. of Istituto Nazionale di Alta Matematica I.N.d.A.M. Part of this work was done while S.G. and E.I. were visiting the Laboratoire de Mathématiques et Applications at the Université de Poitiers, whose hospitality is gratefully acknowledged.

REFERENCES

[1] H. Alsayed, H. Fakih, A. Miranville and A. Wehbe, Optimal control of an Allen-Cahn model for tumor growth through supply of cytotoxic drugs, *Discrete Contin. Dyn. Syst. Ser. S*, **15** (2022), 3481-3515.

[2] G. Bonfanti, M. Frémond and F. Luterotti, Global solution to a nonlinear system forirreversible phase changes, *Adv. Math. Sci. Appl.*, **10** (2000), 1-24.

[3] H. Byrne, The role of mathematics in solid tumour growth, *Math. Today (Southend-on-Sea)*, **35** (1999), 48-53.

[4] P. Colli, H. Gomez, G. Lorenzo, G. Marinoschi, A. Reali and E. Rocca, Mathematical analysis and simulation study of a phase-field model of prostate cancer growth with chemotherapy and antiangiogenic therapy effects, *Math. Models Methods Appl. Sci.*, **30** (2020), 1253-1295.

[5] P. Colli, H. Gomez, G. Lorenzo, G. Marinoschi, A. Reali and E. Rocca, Optimal control of cytotoxic and antiangiogenic therapies on prostate cancer growth, *Math. Models Methods Appl. Sci.*, **31** (2021), 1419-1468.

[6] M. Conti, S. Gatti and A. Miranville, Mathematical analysis of a phase-field model of brain cancers with chemotherapy and antiangiogenic therapy effects, *AIMS Math.*, **7** (2022), 1536-1561.

[7] M. Eleuteri, S. Gatti and G. Schimperna, Regularity and long-time behavior for a thermodynamically consistent model for complex fluids in two space dimensions, *Indiana Univ. Math. J.*, **68** (2019), 1465-1518.

[8] M. Eleuteri, E. Rocca and G. Schimperna, Existence of solutions to a two-dimensional model for nonisothermal two-phase flows of incompressible fluids, *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, **33** (2016), 1431-1454.

[9] M. Fritz, Tumor Evolution Models of Phase-Field Type with Nonlocal Effects and Angiogenesis, *Bull. Math. Biol.*, **85** (2023), Paper No. 44, 34 pp.

[10] N. Fterich, Global solution to a generalized nonisothermal Ginzburg-Landau system, *Appl. Math.*, **55** (2010), 1-46 .

[11] H. Garcke, K.F. Lam, R. Nürnberg and E. Sitka, A multiphase Cahn-Hilliard-Darcy model for tumour growth with necrosis, *Math. Models Methods Appl. Sci.*, **28** (2018), 525-577.

[12] H. Garcke, K.F. Lam, E. Sitka and V. Styles, A Cahn-Hilliard-Darcy model for tumour growth with chemotaxis and active transport, *Math. Models Methods Appl. Sci.*, **26** (2016), 1095-1148.

[13] M. Grasselli, A. Miranville, R. Rossi and G. Schimperna, Analysis of the Cahn-Hilliard equation with a chemical potential dependent mobility, *Comm. Partial Differential Equations*, **36** (2011) 1193-1238.

[14] M.E. Gurtin, Generalized Ginzburg-Landau and Cahn-Hilliard equations based on a microforce balance, *Phys. D*, **92** (1996), 178-192.

[15] E. Ipocoana, On a non-isothermal Cahn-Hilliard model for tumor growth, *J. Math. Anal. Appl.*, **506** (2022), Paper No. 125665, 18 pp.

[16] O.A. Ladyzhenskaja, V.A. Solonnikov and N. N. Ural'ceva, *Linear and Quasilinear Equations of Parabolic Type*, Translations of Mathematical Monographs, 23, American Mathematical Society, Providence, RI, 1968.

[17] Ph. Laurençot, Solutions to a Penrose-Fife model of phase-field type, *J. Math. Anal. Appl.*, **185** (1994), 262-274.

[18] J. Lu, L. Chen, Z. Song, M. Das and J. Chen, Hypothermia effectively treats tumors with temperature-sensitive p53 mutations, *Cancer Res.*, **81** (2021), 3905-3915.

[19] F. Luterotti, G.Schimperna and U. Stefanelli, Existence Result for a nonlinear model related to irriversible phase changes, *Math. Models Methods Appl. Sci.*, **11** (2001), 809-825.

[20] F. Luterotti, G.Schimperna and U. Stefanelli, Global solution to a phase field model with irreversible and constrained phase evolution, *Quart. Appl. Math.*, **60** (2002), 301-306.

[21] S. F. Enam et al., Cytostatic hypothermia and its impact on glioblastoma and survival, *Sci. Adv.*, **8** (2022), 13 pp.

[22] M. Mallory et al., Therapeutic hyperthermia: The old, the new, and the upcoming, *Critical Reviews in Oncology/Hematology*, **97** (2016), 56-64.

[23] A. Marveggio and G. Schimperna, On a non-isothermal Cahn-Hilliard model based on a microforce balance, *J. Differ. Equ.*, **274** (2021), 924-970.

[24] A. Miranville, The Cahn-Hilliard equation: recent advances and applications, CBMS-NSF Regional Conference Series in Applied Mathematics 95, Society for Industrial and Applied Mathematics (SIAM) Philadelphia, PA, 2019, 6 pp.

[25] A. Miranville, E. Rocca and G. Schimperna, On the long time behavior of a tumor growth model, *J. Differ. Equ.*, **267** (2019), 2616-2642.

[26] A. Miranville and G. Schimperna, Nonisothermal phase separation based on a microforce balance, *Discrete Contin. Dyn. Syst. Ser.B*, **5** (2005), 753-768.

[27] A. Miranville and G. Schimperna, Global solution to a phase transition model based on a microforce balance, *J. Evol. Equ.*, **5** (2005), 253-276.

[28] A. Rivaz, M. Azizian and M. Soltani, Various Mathematical Models of Tumor Growth with Reference to Cancer Stem Cells: A Review, *Iran. J. Sci. Technol. Trans. Sci.*, **43** (2019), 687–700.

[29] T. Seki et al., Brown-fat-mediated tumour suppression by cold-altered global metabolism, *Nature*, **608** (2022), 421–428.

[30] W. Von Wahl, The equation $u' + A(t)u = f$ in a Hilbert space and L^p-estimates for parabolic equations, *London Math. Soc.*, **25** (1982), 483-497.

[31] S.M. Wise, J.S. Lowengrub, H.B. Frieboes and V. Cristini, Three-dimensional multispecies nonlinear tumor growth—I: Model and numerical method, *J. Theoret. Biol.,* **253** (2008), 524-543.

[32] J. Xu, G. Vilanova and H. Gomez, A Mathematical Model Coupling Tumor Growth and Angiogenesis, *PLOS ONE*, **11** (2016), 20 pp.

[33] J. Xu, G. Vilanova and H. Gomez, Full-scale, three-dimensional simulation of early-stage tumor growth: the onset of malignancy, *Comput. Meth. Appl. Mech. Eng.*, **314** (2016), 126-146.

[34] J. Xu, G. Vilanova and H. Gomez, Phase-field model of vascular tumor growth: three-dimensional geometry of the vascular network and integration with imaging data, *Comput. Methods Appl. Mech. Eng.*, **359** (2020), 20 pp.

[35] J. Xu, Y. Wang, H. Gomez and X. Feng, Biomechanical modelling of tumor growth with chemotherapeutic treatment: a review, *Smart Mater. Struct.*, **32** (2023), 23 pp.

Received September 6, 2025; revised February 10, 2026; early access March 10, 2026.