Communications in Nonlinear Science and Numerical Simulation 126 (2023) 107497

Contents lists available at ScienceDirect

Communications in Nonlinear Science and Numerical Simulation

journal homepage: www.elsevier.com/locate/cnsns

The damped vibrating string equation on the positive half-line

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ARTICLE INFO

Article history:
Available online 11 August 2023

MSC:
34A60
34G25
35A01

Keywords:
Damped vibrating string equation
Second-order Cauchy problem
Banach spaces
Fundamental system
Approximation solvability method
Mild solution

ABSTRACT

In this paper, the existence of a solution to the problem describing the small vertical vibration of an elastic string on the positive half-line is investigated in the case when both viscous and material damping coefficients are present. The result is obtained by transforming the original partial differential equation into an appropriate abstract second-order ordinary differential equation in a suitable infinite dimensional space. The abstract problem is then studied using the combination of the Kakutani fixed point theorem together with the approximation solvability method and the weak topology. The applied procedure enables obtaining the existence result also for problems depending on the first derivative, without any strict compactness assumptions put on the right-hand side and on the fundamental system generated by the linear term. The paper ends by applying the obtained result to the studied mathematical model describing the small vertical vibration of an elastic string with a nonlinear Balakrishnan–Taylor-type damping term.

1. Introduction

Second-order evolution equations and inclusions have been arising in many areas of applied mathematics, mathematical physics, or biology and have been therefore attracting quite a lot of attention (see, e.g., [1–3], and the references therein). A variety of problems in mechanics, molecular dynamics, or quantum mechanics can be described by nonlinear partial differential equations or inclusions of second-order in time. In particular, in this paper, we investigate a mathematical model describing the small vertical vibration of an elastic string described by

$$\left.\begin{aligned} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial \xi^2} + b(t,\xi)\frac{\partial u}{\partial \xi} + a(t,\xi)u + f\left(t,\xi,\int_0^1 k(\xi,s)\frac{\partial u}{\partial t}(t,s)ds\right), \quad t \geq 0, \xi \in [0,1] \\ u(t,0) = u(t,1) = 0, \quad t \geq 0, \\ u(0,\xi) = x_0(\xi), \ \frac{\partial u}{\partial t}(0,\xi) = \bar{x}_0(\xi) \quad \xi \in (0,1)\,. \end{aligned}\right\} \tag{1}$$

In the studied problem, we consider both viscous and material damping coefficients like in [4]. The viscous damping term $f\left(t,\xi,\int_0^1 k(\xi,s)\frac{\partial u}{\partial t}(t,s)ds\right)$ is of nonlinear Balakrishnan–Taylor-type like in [5–7] and the material damping coefficient a is variable. Also, the coefficient b is variable, taking into account that the wave propagation speed can depend on time and space like, e.g., in [8].

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https://doi.org/10.1016/j.cnsns.2023.107497

Motivated by a variety of applications, many authors have been studying the existence of solutions to partial differential equations as well as to the corresponding Cauchy problems. For this reason, a new approach in literature has been appearing in the study of some types of partial differential equations recently. It consists in transforming the partial differential equation that governs the model into an abstract ordinary differential equation in a suitable infinite dimensional space. Following this approach, we consider in this paper $p \in (1, +\infty)$, denote $u(t, \cdot) = x(t)$, E as the reflexive Banach space $L^p([0, 1])$,

$$F(t, z, w) = a(t, \cdot)z + f\left(t, \cdot, \int_0^1 k(\cdot, s)w(s)ds\right),$$

for $z, w \in E$, $D(A)$ as the dense subspace $W^{2,p}([0, 1]) \cap W_0^{1,p}([0, 1])$ of E and, for every $t \geq 0$, $A(t) : D(A) \to E$ as the linear operator defined by

$$A(t)z(\xi) = \ddot{z} + b(t, \xi)\dot{z}. \tag{2}$$

Subsequently, problem (1) can be rewritten as

$$\left.\begin{aligned} \ddot{x}(t) = A(t)x(t) + F(t, x(t), \dot{x}(t)), \quad t \geq 0, \\ x(0) = x_0,\ x'(0) = \bar{x}_0. \end{aligned}\right\} \tag{3}$$

Furthermore, if the studied problem contains some discontinuities or uncertainty or if its controllability is considered, the original partial differential problem can lead to the Cauchy problem for a second-order inclusion like

$$\left.\begin{aligned} \ddot{x}(t) \in A(t)x(t) + F(t, x(t), \dot{x}(t)), \quad \text{for a.a. } t \in [0, +\infty) \\ x(0) = x_0,\ \dot{x}(0) = \bar{x}_0. \end{aligned}\right\} \tag{4}$$

In previous works, at first the easiest case of (3) – abstract linear second-order initial value problems on compact intervals

$$\ddot{x}(t) = A(t)x(t) + f(t), \quad \text{for a.a. } t \in [0, a]$$

– has been studied, see, e.g., [9,10], or [11].
Initial value problems for the following abstract semilinear second-order integro-differential equation on a compact interval in Banach spaces

$$\ddot{x}(t) = A(t)x(t) + \int_0^t P(t, s)x(s)\, ds + f(t), \quad \text{for a.a. } t \in [0, a],$$

have been studied in [12]; besides the existence of mild, also classical, solutions have been investigated there.
The Cauchy problem for an equation with a more general right-hand side (shortly r.h.s.) on a non-compact interval

$$\ddot{x}(t) = A(t)x(t) + f(t, x(t)), \quad \text{for a.a. } t \in [0, +\infty), \tag{5}$$

has been investigated, e.g., in [8], where the existence of pseudo S-asymptotically periodic mild solutions to (5) has been studied. Moreover the existence of a solution for the vibrating string Eq. (1) has been deduced in [8] in the absence of the viscous damping term and under quite strong conditions. The non-local problem for the multivalued version of (5) in a compact interval was considered in [13,14].

Let us note that only very few authors consider a Cauchy (or boundary value) problem for semilinear second-order differential equations or inclusions in Banach spaces whose r.h.s. depend also on the first derivative (sometimes solving the second-order problem by reducing it to a first-order one). See, e.g., [15–18], and the references therein. In [15,16], the operator A was considered to be bounded; on the other hand, boundary value problems were studied there instead of the Cauchy problem. In [17], the operator A was not considered bounded, but quite strict assumptions dealing with the local Lipschitz continuity of the r.h.s. played the key role in the proof of the main result. Furthermore, the result there was proven by reducing the second-order evolution equations to the first-order ones. In [18], the operator A could again be unbounded and the existence of a mild solution for the Cauchy problem has been studied in the case of a C^1 non-linear term.

The strong assumptions put on the r.h.s. (like the Lipschitz continuity or the use of the measure of non-compactness) were removed in [19], where the controllability of (5) has been taken into account. In the recent paper [20], the existence of a mild solution of the Cauchy problem for impulsive semilinear second-order differential inclusion in a Banach space has been investigated, without reducing it to the first-order problem, in the case when the non-linear term depends also on the first derivative, but when the operator A in (4) does not depend on t. Motivated by practical applications and results developed in the works mentioned previously, our objective in this paper is to study the existence of a mild solution for a general second-order semilinear inclusion in a Banach space depending also on the first derivative, namely studying the problem (4) above.

Throughout the paper, we assume that:

(i) E is a reflexive Banach space having a Schauder basis $\{e_n\}_n$;

(ii) $\{A(t)\}_t$, with $A(t) : D(A) \subset E \to E$, for every $t \geq 0$, is a family of closed, linear, and densely defined operators that generates a fundamental system $\{S(t,s)\}_{t,s}$;
(iii) $F : [0, +\infty) \times E \times E \multimap E$ is a multivalued mapping with bounded, closed and convex values;
(iv) $x_0, \bar{x}_0 \in E$.

After studying the Cauchy problem for the abstract second-order inclusion (4), we apply the obtained result for getting a solution $u : [0, +\infty) \times [0, 1] \to \mathbb{R}$ of (1) continuously differentiable with respect to t, such that at every value $t \in [0, +\infty)$ the function $u(t, \cdot)$ belongs to the Sobolev space $W^{1,p}([0, 1]) \cap C_0([0, 1])$. When (1) is reformulated in its abstract form as (3), the state variable $x(t)$ belongs to the Banach space $L^p([0, 1])$ which admits a Schauder basis. Thus, we are led to study (4) in a Banach space having a Schauder basis $\{e_n\}_n$. This allows to apply the approximation solvability method, which was introduced for the first time in [21] to study the existence of a classical solution for first-order fully non-linear inclusions and then extended to the first-order semilinear case in [22] and to the second order fully non-linear case in [23]. The approximation solvability method consists in introducing a sequence of approximating problems for (4) in the finite-dimensional space $E_n = \mathrm{span}\{e_1, \ldots, e_n\}$.

Since the conversion of the second-order problem into the first-order one may not yield desired results due to the behavior of the semigroup generated by the linear part of the converted first-order system, in this paper we will treat directly the second-order abstract differential problem (4). The results will be obtained under easily verifiable conditions and without assuming any compactness of the r.h.s., which becomes the main advantage of the paper.

When dealing with a r.h.s. that is not dependent on the first derivative, mild solutions of a semilinear inclusion are continuous functions satisfying a sort of variation constant formula. On the contrary, since the considered r.h.s. depends on the first derivative of the solution, we are looking for C^1-solutions. As shown in [24], when A is independent of t, denoted by $\{\tilde{C}(t)\}_t$ the cosine family generated by A, a necessary and sufficient condition guaranteeing such regularity of the solution requires to take $x_0 \in \{x \in E : \tilde{C}(\cdot)x \text{ is continuously differentiable}\} := \tilde{X}$. Since we consider $A = A(t)$, we will introduce the analogous set X of initial values (see the formula (7) below), extending the concept from the case of cosine and sine families to the case of the fundamental systems. We are also able to show that, if $A(t) = A + B(t)$, where A generates a cosine family and B is sufficiently regular, as for the abstract Eq. (3) associated to (1), then $X = \tilde{X}$. We stress that, to the best of our knowledge, the introduction of the set X for a class of second-order evolution differential systems in Banach spaces with the r.h.s. also depending on the first derivative and the operator A depending on t considered in this paper is an untreated topic in the literature, where in the few papers studying such general case the initial value x_0 is assumed to belong to the smaller set $D(A)$.

The paper is organized as follows. In Section 2, the necessary preliminaries about fundamental systems and multivalued analysis are introduced. This section also contains lemmas and propositions that are used in the proof of our main results. The main theorems for abstract problems on compact intervals are contained in Section 3.1. Subsequently, making use of the results for compact intervals, the abstract non-compact one is studied in Section 3.2. Finally, in Section 3.3 we apply the result obtained for the abstract Eq. (3) to get the existence of a solution of the equivalent model (1).

2. Materials and methods

Let E be an infinite-dimensional reflexive Banach space with norm $\|\cdot\|$, and let us denote the Banach space dual to E by E^*. The notation $\mathcal{L}(E)$ stands for the Banach space of linear and bounded operators from E into itself; by $\mathcal{L}(E, X)$ we will denote the Banach space of bounded linear operators from E into a Banach space X. In the paper we say that a sequence $\{e_n\}_n$ of vectors in E converges weakly to $e \in E$ if $\varphi(e_n) \to \varphi(e)$, for every $\varphi \in E^*$. The symbol $\rightharpoonup$ will denote the weak convergence. Throughout this paper, by E^ω, we denote the space E endowed with the weak topology.

For every $x \in E$ and $r > 0$, $B_r(x)$ is the open ball centered in x with radius r. Given $C \subset E$ and $\varepsilon > 0$, the symbol $B(C, \varepsilon)$ will denote, as usual, the set $C + \varepsilon B$, where B is the open unit ball in E, i.e., $B = B_1(0) = \{x \in E \mid \|x\| \leq 1\}$.

Given $b > a \geq 0$, we denote by $C([a, b], E)$ the Banach space of continuous functions defined in $[a, b]$ and taking values in E endowed with the maximum norm

$$\|x\|_C = \max_{t\in[a,b]} \|x(t)\|.$$

By $C^1([a, b], E)$, the Banach space of continuously differentiable functions defined in $[a, b]$ and taking values in E endowed with the norm

$$\|x\|_{C^1} = \max\{\|x\|_C, \|\dot{x}\|_C\}$$

will be denoted. Similarly, we define the Banach space $L^1([a, b], E)$, with the norm

$$\|x\|_{L^1} = \int_a^b \|x(t)\|\, dt,$$

and the Banach space $L^1_{loc}([0, +\infty), E) = \{x : [0, +\infty) \to E : x \in L^1([a, b], E)\, \forall\, b > a \geq 0\}$.

The technique used in the paper consists (besides others) in the approximation of problem (4) by means of a sequence of problems in finite-dimensional spaces. For this purpose, let us recall now the notion of Schauder basis and natural projection.

Definition 2.1. A sequence $\{e_n\}_n$ of vectors in E is a *Schauder basis* for E if for every $x \in E$, there exists a unique sequence of real numbers $\alpha_n = \alpha_n(x)$, $n \in \mathbb{N}$, such that

$$\left\| x - \sum_{i=1}^{n} \alpha_i e_i \right\| \to 0, \quad \text{as } n \to \infty.$$

Given a Schauder basis $\{e_n\}_n$ for E, let $E_n = \text{span}\{e_1, \dots, e_n\}$ denote the n-dimensional Euclidean space generated by the first n vectors of the basis, and let $\mathbb{P}_n : E \to E_n$ be the natural projection of E onto E_n, i.e.,

$$\mathbb{P}_n \left(\sum_{k=1}^{\infty} \alpha_k e_k \right) = \sum_{k=1}^{n} \alpha_k e_k.$$

It holds that $\alpha_n \in E^*$, for every $n \in \mathbb{N}$ (see [25, pp. 18–20]) and that the sequence $\{\|\mathbb{P}_n\|\}_n$ is bounded, i.e. there exists $K \geq 1$ such that

$$\|\mathbb{P}_n(x)\| \leq K\|x\| \quad \forall n \in \mathbb{N}, \ \forall x \in E \tag{6}$$

(see [26, Proposition 1.a.2]). The Schauder basis $\{e_n\}_n$ is said to be monotone if $K = 1$, i.e., if $\|\mathbb{P}_n\| = 1$, for every $n \in \mathbb{N}$.

Remark 2.1.

Trivially, if E is a separable Hilbert space, every orthonormal system of E is a monotone Schauder basis. Moreover, if a Banach space admits a Schauder basis, it is separable. On the other hand, it was proven in [27] that there exists a separable Banach space without a Schauder basis. However, for every $1 < p < \infty$ and for each bounded subset $\Omega \subset \mathbb{R}^n$, $L^p(\Omega, \mathbb{R})$ has a monotone Schauder basis (see, e.g., [28, Chap. 1.3 and 1.4]).

Some of the main properties of the projection $\mathbb{P}_n$ are contained in the following lemma (see [22, Lemma 2.2], [29, Lemma 6], and [30, Proposition 2.4]).

Lemma 2.1. *The projection $\mathbb{P}_n : E \to E_n$ satisfies the following properties:*

(a) $\mathbb{P}_n : E^\omega \to E_n$ is continuous;
(b) If $x_n \to x$, then $\mathbb{P}_n(x_n) \to x$;
(c) If $x_n \rightharpoonup x$, then $\mathbb{P}_n(x_n) \rightharpoonup x$;
(d) If $f_n \to f$ in $L^1([0,T], E)$, then $\mathbb{P}_n f_n \to f$ in $L^1([0,T], E)$;
(d) If $f_n \rightharpoonup f$ in $L^1([0,T], E)$, then $\mathbb{P}_n f_n \rightharpoonup f$ in $L^1([0,T], E)$;
(e) For every $x \in E$, $\|P_n(x) - x\| \to 0$.

We shall now introduce the definitions and notions from the multivalued analysis that we will need in the sequel. Let X, Y be two metric spaces. We say that H is a multivalued mapping from X to Y (written $H : X \multimap Y$) if, for every $x \in X$, a non-empty subset $H(x)$ of Y is given.
A multivalued mapping $H : X \multimap Y$ is called upper semicontinuous (shortly, u.s.c.) if, for each open subset $U \subset Y$, the set $\{x \in X \mid H(x) \subset U\}$ is open in X. It is called completely continuous if $H(C)$ is relatively compact for every bounded set $C \subset X$. If H is u.s.c. with convex values, then H has a closed graph (see [31, Theorem 1.1.4]). If H is u.s.c. and completely continuous with compact values, then it has a closed graph ([31, Theorem 1.1.5]). Conversely, if H is a completely continuous multivalued mapping with compact values and has a closed graph, then H is u.s.c. (see [31, Theorem 1.1.5]).
Let $J \subset \mathbb{R}$ be a compact interval. A mapping $H : J \multimap Y$ with closed values, where Y is a separable metric space, is called measurable if, for each open subset $U \subset Y$, the set $\{t \in J \mid H(t) \subset U\}$ belongs to the σ-algebra of subsets of J. If Y is separable, the measurability is indifferently strong and weak measurability (see [32, Chap. II]).
In this paper, we will consider problem (4) assuming that the family of operators $\{A(t)\}_t$ generates a fundamental system. This concept was introduced in [33,34]. We will assume that the domain $D(A)$ of $A(t)$ is a dense and closed subset of E which does not depend on t. In the following, we will discuss its definition and main properties.
A family of linear bounded operators $S(t,s) : E \to E$, with $t, s \geq 0$, is called a fundamental system if

(a) for each $x \in E$, the mapping $(t,s) \to S(t,s)x$ is of class C^1;
(b) for each $t \geq 0$, $S(t,t) = 0$;
(c) for all $t, s \geq 0$ and each $x \in E$,

$$\left.\frac{\partial}{\partial t} S(t,s)\right|_{t=s} x = x, \quad \left.\frac{\partial}{\partial s} S(t,s)\right|_{t=s} x = -x;$$

(d) for all $t, s \geq 0$, if $x \in D(A)$, then $S(t,s)x \in D(A)$, the mapping $(t,s) \to S(t,s)x$ is of class C^2 and

$$\frac{\partial^2}{\partial t^2} S(t,s)x = A(t)S(t,s)x,$$

$$\frac{\partial^2}{\partial s^2} S(t,s)x = S(t,s)A(s)x,$$

$$\left.\frac{\partial^2}{\partial s \partial t} S(t,s)\right|_{t=s} x = 0;$$

(e) for all $t, s \geq 0$, if $x \in D(A)$, then $\frac{\partial}{\partial s} S(t,s)x \in D(A)$, the mapping $(t,s) \to A(t)\frac{\partial}{\partial s} S(t,s)x$ is continuous, and

$$\frac{\partial^3}{\partial t^2 \partial s} S(t,s)x = A(t)\frac{\partial}{\partial s} S(t,s)x$$

$$\frac{\partial^3}{\partial s^2 \partial t} S(t,s)x = \frac{\partial}{\partial t} S(t,s)A(s)x.$$

The map $S : [0,+\infty) \times [0,+\infty) \to \mathcal{L}(E)$ is said to be a fundamental operator if $\{S(t,s)\}_{t,s}$ is a fundamental system. Since $S(t,s)$ is of class C^1, we introduce, for each $t, s \geq 0$, the linear and bounded operator

$$C(t,s) = -\frac{\partial}{\partial s} S(t,s).$$

By using the Banach–Steinhaus Theorem, it is possible to prove that a fundamental system satisfies properties mentioned in the following lemma.

Lemma 2.2. *For every $T > 0$ there exists a constant $K_T > 0$ such that, for every $t, s \in [0,T]$,*

(i) $\|C(t,s)\| \leq K_T$;
(ii) $\|S(t,s)\| \leq K_T$;
(iii) $\left\|\frac{\partial}{\partial t} S(t,s)\right\| \leq K_T$.

In what follows, we shall make use of the following set

$$X = \left\{ x \in E : C(\cdot, s)x \text{ is continuously differentiable } \forall\, s \geq 0 \right\}. \tag{7}$$

Several techniques have been used in the literature in order to ensure the existence of the fundamental operator (see, e.g. [18,34–37]).
In particular, a very frequent situation is when $A(t)$ is a perturbation of an operator A, that generates a cosine family $\{\tilde{C}(t)\}_t$. For this reason, we will below study also the simpler case when $A(t) = A$.

A one-parameter family $\{\tilde{C}(t)\}_t$ of bounded linear operators mapping the space E into itself is called a strongly continuous cosine family if:

- $\tilde{C}(t+s) + \tilde{C}(s-t) = 2\tilde{C}(s)\tilde{C}(t)$, for all $t,\ s \in \mathbb{R}$;
- $\tilde{C}(0) = I$;
- The map $t \to \tilde{C}(t)x$ is continuous in $\mathbb{R}$, for each fixed $x \in E$.

If $\{\tilde{C}(t)\}_t$ is a strongly continuous cosine family, then there exist $M \geq 1$ and $\omega \geq 0$ such that, for all $t \in \mathbb{R}$,

$$\|\tilde{C}(t)\| \leq Me^{\omega|t|} \tag{8}$$

and moreover, the set

$$D(A) = \left\{ x \in E : \exists \lim_{t \to 0^+} \frac{\tilde{C}(t)x - x}{t^2} \right\}$$

is dense in E.
The linear closed operator $A : D(A) \subset E \to E$ defined by

$$Ax = \frac{d^2}{dt^2}\left[\tilde{C}(t)x\right]_{t=0} = 2 \lim_{t \to 0^+} \frac{\tilde{C}(t)x - x}{t^2}$$

is called the infinitesimal generator of the cosine family.

Define

$$\tilde{X} = \left\{ x \in E \mid \tilde{C}(\cdot)x \text{ is continuously differentiable} \right\}.$$

It was proven in [24] that A is the generator of the cosine family $\{\tilde{C}(t)\}_t$ if and only if set $\tilde{X}$ endowed with the norm

$$\|x\|_{\tilde{X}} = \|x\|_E + \max_{t\in[0,1]} \|AS(t)x\|_E \tag{9}$$

is a Banach space.

The one-parameter family $\{\tilde{S}(t)\}_t$ of the bounded linear operators mapping the space E into itself defined, for all $t \in \mathbb{R}$ and $x \in E$, by

$$\tilde{S}(t)x = \int_0^t \tilde{C}(s)x\,ds \tag{10}$$

is called the strongly continuous sine family associated with the cosine family.

The families $\{\tilde{S}(t)\}_t$ and $\{\tilde{C}(t)\}_t$ possess several important properties; the most crucial are summarized in [38, Propositions 2.1, 2.2].

Let us now return to our original problem (4) and assume that $A(t) = A + B(t)$, where A is the generator of a strongly continuous cosine family $\{\tilde{C}(t)\}_t$ and $B : [0, \infty) \to \mathcal{L}(\tilde{X}, E)$.

Lemma 2.3 (*see [8]*). *If $B : [0, \infty) \to \mathcal{L}(\tilde{X}, E)$ is an operator-valued function such that $B(\cdot)x$ is continuously differentiable in E, for each $x \in \tilde{X}$ and A is the generator of a cosine family $\{\tilde{C}(t)\}_t$, then $A(t) = A + B(t)$ generates a fundamental system $\{S(t, s)\}_{t,s}$ satisfying*

$$S(t, s)z = \tilde{S}(t-s)z + \int_s^t \tilde{S}(t-\xi)B(\xi)S(\xi, s)z\,d\xi, \quad s, t \in [0, \infty). \tag{11}$$

The same conclusion holds if $B : [0, \infty) \to \mathcal{L}(E)$ is a strongly continuous operator-valued function; in this case, it is not necessary to require that $B(\cdot)x$ is continuously differentiable.

Lemma 2.4. *Assume that $A(t) = A + B(t)$, where A generates a cosine family and $B : [0, +\infty) \to \mathcal{L}(\tilde{X}, E)$. Then $X = \tilde{X}$.*

Proof. Consider the function

$$v(t, s) = \int_s^t \tilde{S}(t-\xi)B(\xi)S(\xi, s)x\,d\xi.$$

Notice that the conditions on $\tilde{S}, B$ and S imply that the function $\xi \to \tilde{S}(t-\xi)B(\xi)S(\xi, s)x$ is continuous, the function $s \to \tilde{S}(t-\xi)B(\xi)S(\xi, s)x$ is continuously differentiable and $S(s, s)x = 0$, for every $x \in E$. Thus, it holds that

$$\exists\, \frac{\partial}{\partial s} v(t, s) = -\int_s^t \tilde{S}(t-\xi)B(\xi)C(\xi, s)x\,d\xi$$

for every $x \in E$. Similarly, the function $\xi \to \tilde{S}(t-\xi)B(\xi)C(\xi, s)x$ is continuous, the function $t \to \tilde{S}(t-\xi)B(\xi)C(\xi, s)x$ is continuously differentiable and $\tilde{S}(0)x = 0$, for every $x \in E$, hence

$$\exists\, \frac{\partial^2}{\partial t \partial s} v(t, s) = -\int_s^t \tilde{C}(t-\xi)B(\xi)C(\xi, s)x\,d\xi,$$

for every $x \in E$.

Therefore, according to the definition of C and (11), it holds that $x \in X$ if and only if there exists $\frac{\partial^2}{\partial t\partial s}\tilde{S}(t-s)x$, i.e. if and only if $x \in \tilde{X}$. In this case,

$$\frac{\partial}{\partial t}C(t, s)x = -\frac{\partial^2}{\partial t \partial s}\left[\tilde{S}(t-s)x + \int_s^t \tilde{S}(t-\xi)B(\xi)S(\xi, s)x\,d\xi\right] = \frac{\partial}{\partial t}\tilde{C}(t-s)x + \int_s^t \tilde{C}(t-\xi)B(\xi)C(\xi, s)x\,d\xi.$$

Remark 2.2. In Lemma 2.3, we summarized conditions ensuring the existence of the fundamental operator in the case when $A(t) = A + B(t)$.
The more general case for the operator $A(t)$, $t \in [0, T]$, is exploited in [39,40] reducing the second-order equation to the first-order system and assuming that the corresponding linear operator generates an evolution operator.
More precisely, let $\{A(t)\}_t$, with $A(t) : D(A) \subset E \to E$, for every $t \in [0, T]$, be a family of closed, linear, and densely defined operators such that, for all $t \in [0, T]$, $A(t)$ is the infinitesimal generator of a strongly continuous cosine family of bounded linear operators.
Assume that, for each $t \in [0, T]$, 0 belongs to the resolvent set of $A(t)$, for each $x \in D(A)$ the mapping $t \to A(t)x$ is of class C^1 in $[0, T]$, and that the family of generators $\{A(t)\}_t$ is stable (see [39, Lemma 2] for the definition). Moreover, assume that, for each $t \in [0, T]$, there exists a linear operator $B(t) : D(B) \subset E \to E$ such that $B^2(t) = A(t)$, 0 belongs to the resolvent set of $B(t)$, and for each $x \in D(B)$ the mapping $t \to B(t)x$ is of class C^1 in $[0, T]$.
Then $A(t)$ generates a fundamental system $\{S(t, s)\}_{t,s}$.

The solution of problem (4) will be understood in a mild sense. More concretely, given $x_0 \in X$ and $\bar{x}_0 \in E$, by a mild solution of (4), we mean a continuously differentiable function $x\colon [0,\infty) \to E$ such that, for all $t \in [0,\infty)$,

$$x(t) = C(t,0)x_0 + S(t,0)\bar{x}_0 + \int_0^t S(t,s)f(s)\,ds, \tag{12}$$

where

$$f \in S^1_{F,x} = \left\{ f \in L^1_{loc}([0,\infty),E) : f(t) \in F(t,x(t),\dot{x}(t)),\ \text{ for a.a. } t \in [0,\infty) \right\}. \tag{13}$$

Remark 2.3. Since the function $(t,s) \to S(t,s)$ is of class C^1, $f \in L^1_{loc}([0,+\infty),E)$ and $S(t,t)=0$, for every $t \geq 0$, we get that the function $v : [0,+\infty) \to E$ defined as

$$v(t) = \int_0^t S(t,s)f(s)\,ds$$

is continuously differentiable and

$$v'(t) = \int_0^t \frac{\partial}{\partial t} S(t,s)f(s)\,ds$$

(see [41, page 6]). Therefore $x_0 \in X$ guarantees that the function x defined in (12) is continuously differentiable and

$$\dot{x}(t) = \frac{\partial}{\partial t} C(t,0)x_0 + \frac{\partial}{\partial t} S(t,0)\bar{x}_0 + \int_0^t \frac{\partial}{\partial t} S(t,s)f(s)\,ds.$$

Remark 2.4 (*see, e.g., [42]*)**.** Take $a \geq 0$ and consider the problem

$$\left.\begin{array}{r} \ddot{x}(t) \in A(t)x(t) + F(t,x(t),\dot{x}(t)), \quad \text{for a.a. } t \geq a, \\ x(a) = x_a,\ \dot{x}(a) = \bar{x}_a. \end{array}\right\} \tag{14}$$

Then x is a mild solution of (14) if there exists $f \in S^1_{F,x}$ such that, for all $t \geq a$,

$$x(t) = C(t,a)x_a + S(t,a)\bar{x}_a + \int_a^t S(t,s)f(s)\,ds. \tag{15}$$

Notice that, since $0 \in D(A)$, the homogeneous problem

$$\left.\begin{array}{r} \ddot{x}(t) = A(t)x(t), \quad \text{for a.a. } t \geq a, \\ x(a) = 0,\ \dot{x}(a) = 0 \end{array}\right\} \tag{16}$$

has only the trivial solution. This implies that, for every $x_a, \bar{x}_a \in E$ and $f \in L^1_{loc}([0,+\infty);E)$, the problem

$$\left.\begin{array}{r} \ddot{x}(t) \in A(t)x(t) + f(t), \quad \text{for a.a. } t \geq a, \\ x(a) = x_a,\ \dot{x}(a) = \bar{x}_a \end{array}\right\} \tag{17}$$

has a unique solution.

Now, given $x_0 \in X$, consider the solution x of problem (4). According to the definition, x satisfies expression (12), it is continuously differentiable,

$$x(a) = C(a,0)x_0 + S(a,0)\bar{x}_0 + \int_0^a S(a,s)f(s)ds$$

and

$$\dot{x}(a) = \frac{\partial}{\partial t} C(a,0)x_0 + \frac{\partial}{\partial t} S(a,0)\bar{x}_0 + \int_0^a \frac{\partial}{\partial t} S(a,s)f(s)\,ds,$$

for some $f \in S^1_{F,x}$. Since (17) has a unique solution, it follows that, for every $t \geq a$, x is also solution of (14), i.e. the expression given in (15) is equivalent to the expression given in (12) when taking

$$x_a = C(a,0)x_0 + S(a,0)\bar{x}_0 + \int_0^a S(a,s)f(s)ds$$

and

$$\bar{x}_a = \frac{\partial}{\partial t} C(a,0)x_0 + \frac{\partial}{\partial t} S(a,0)\bar{x}_0 + \int_0^a \frac{\partial}{\partial t} S(a,s)f(s)\,ds.$$

In particular, this implies that, whenever $x_0 \in X$, the expression in (15) is continuously differentiable, i.e. $x_a \in X$ as well and

$$\dot{x}(t) = \frac{\partial}{\partial t} C(t,a)x_a + \frac{\partial}{\partial t} S(t,a)\overline{x}_a + \int_a^t \frac{\partial}{\partial t} S(t,s)f(s)\,ds.$$

In order to ensure that the $S^1_{F,x}$ in (13) is non-empty, the following selection result can be employed.

Proposition 2.1 *(see, e.g., [29, Proposition 2.2]). Let $[a,b] \subset \mathbb{R}$ be a compact interval, E_1, E_2 be Banach spaces and $Q : [a,b] \times E_1 \multimap E_2$ be a multivalued mapping satisfying:*

(A1) $Q(t,x)$ is non-empty, convex, and weakly compact, for every $t \in [a,b]$ and $x \in E_1$;
(A2) For every $x \in E_1$, $Q(\cdot,x)$ has a measurable selection;
(A3) For a.a. $t \in [a,b]$, $Q(t,\cdot) : E_1^w \multimap E_2^w$ is weakly sequentially closed;
(A4) For each bounded $\Omega \subset E_1$, there exists $\eta_\Omega \in L^1([a,b],\mathbb{R})$ such that, for a.a. $t \in [a,b]$,

$$\sup_{x\in\Omega} \|Q(t,x)\| \le \eta_\Omega(t).$$

Then, for every $q \in C([a,b],E_1)$, there exists $f \in L^1([a,b],E_2)$ such that $f(t) \in Q(t,q(t))$, for a.a. $t \in [a,b]$.

The proofs of the existence results below are based on a fixed point theorem. Its application requires besides other that the fundamental Cauchy operators $G, H : L^1([a,b];E) \to C([a,b];E)$ defined by

$$Gf(t) = \int_a^t S(t,s)f(s)ds \tag{18}$$

and

$$Hf(t) = \int_a^t \frac{\partial}{\partial t} S(t,s)f(s)ds \tag{19}$$

satisfy proper regularity and compactness conditions. For this purpose, we recall now the concept of semicompact sequence and its properties.

A sequence $\{f_k\}_k \subset L^1([a,b],E)$ is called semicompact if it is integrably bounded and the set $\{f_k(t)\}_k$ is relatively compact, for a.a. $t \in [a,b]$. The following convergence result for semicompact sequences is useful in our discussion.

Theorem 2.1 *([31, Theorem 5.1.1]). Let $P : L^1([a,b],E) \to C([a,b],E)$ be an operator satisfying the following conditions*

(i) there exists $L > 0$ such that $\|Pf - Pg\|_C \le L\|f-g\|_1$, for all $f,g \in L^1([a,b],E)$;
(ii) for every compact $K \subset E$ and every sequence $\{f_k\}_k \subset L^1([a,b],E)$ such that $\{f_k(t)\}_k \subset K$, for a.a. $t \in [a,b]$, the weak convergence $f_k \rightharpoonup f$ implies $Pf_k \to Pf$.

Then, for every semicompact sequence $\{f_k\}_k \subset L^1([a,b],E)$, the sequence $\{Pf_k\}_k$ is relatively compact in $C([a,b],E)$ and, moreover, if $f_k \rightharpoonup f$, then $Pf_k \to Pf$.

Remark 2.5. Let us note that the fundamental Cauchy operators G and H defined by (18) and (19) satisfy all assumptions of Theorem 2.1. The result for the operator G was proved in [13, Theorem 3.2] and it is based on the facts that $x \to S(t,s)x$ is continuous and $\|S(t,s)\| \le K_T$, for every $s,t \in [0,T]$ and some $K_T > 0$. The proof related to H follows in a similar way, just recalling that, according to the definition of the fundamental system and Lemma 2.2, $\frac{\partial}{\partial t}S(t,s)$ enjoys the same properties stated for S.
In particular, we can conclude that, for every semicompact sequence $\{f_k\}_k$ weakly converging to f in $L^1([a,b],E)$, both the convergences $Gf_k \to Gf$ and $Hf_k \to Hf$ hold.

3. Results

3.1. Abstract inclusion: Existence of a mild solution on a compact interval

In this section, the existence of a mild solution to the Cauchy problem (14) in $[a,b] \subset [0,\infty)$ will be discussed. At first, the natural projections of the space E onto the finite-dimensional spaces generated by the first n vectors of the Schauder basis will be used in order to introduce a sequence of finite-dimensional approximating problems. Then the Kakutani fixed point theorem will be applied for getting the existence of a solution for each finite-dimensional problem. Finally, a limiting procedure based on the usage of the weak topology will be applied for getting a solution to the original problem in the abstract space.

Due to the used procedure, any requirements for compactness within the assumptions will be avoided. Furthermore, since the r.h.s. of the studied problem depends also on the first derivative, the initial value has to belong to the set X defined previously by the formula (7).

Theorem 3.1. *Consider the Cauchy problem (14) on the interval $[a, b]$, where x_a and $F : [a, b] \times E \times E \multimap E$ satisfy the following assumptions:*

$(F1^c)$ *$F(t, x, y)$ is non-empty, convex, closed, and bounded, for every $t \in [a, b]$ and $x, y \in E$,*
$(F2^c)$ *for every $(x, y) \in E \times E$, $F(\cdot, x, y)$ has a measurable selection,*
$(F3^c)$ *for a.a. $t \in [a, b]$, $F(t, \cdot, \cdot) : E^w \times E^w \multimap E^w$ is weakly u.s.c.,*
$(F4^c)$ *for every $n \in \mathbb{N}$, there exists $\varphi_n \in L^1([a, b], \mathbb{R})$ with*

$$\liminf_{n\to\infty} \frac{\|\varphi_n\|_{L^1}}{n} = 0$$

and such that

$$\|z\| \le \varphi_n(t),$$

for a.a. $t \in [a, b]$, every $(x, y) \in nB \times nB$ and every $z \in F(t, x, y)$,
$(F5^c)$ *$x_a \in X$.*

Then the Cauchy problem (14) has a solution defined on $[a, b]$.

Proof. For the sake of simplicity, we will assume all along with the proof that the space E has a monotone Schauder basis, i.e., that $\|\mathbb{P}_m\| \le 1$, for every $m \in \mathbb{N}$. Let us note that the proof also works in the general case with little changes. Since the proof consists of several parts, it will be split into the relevant steps from now on.

Step 1. *Introduction of a sequence of approximating operators*

To prove the existence of a solution to problem (14), we will use the approximation solvability method. Thus, for each $m \in \mathbb{N}$, consider the multimap $G_m : [a, b] \times E \times E \to E_m$ defined as $G_m = \mathbb{P}_m \circ F$ and the operator $\Sigma_m : C^1([a, b], E_m) \multimap C^1([a, b], E_m)$ defined as

$$\Sigma_m(q)(t) = \left\{ \mathbb{P}_m C(t, a) x_a + \mathbb{P}_m S(t, a)\bar{x}_a + \int_a^t \mathbb{P}_m S(t, s) f(s)\, ds : f \in S^1_{G_m, q} \right\}. \tag{20}$$

Let us note that, since $\mathbb{P}_m$ is a bounded and linear operator with $\|P_m\| \le 1$, the mapping G_m satisfies properties $(F1^c)$–$(F4^c)$ as well. Therefore, for every $q \in C^1([a, b], E_m)$, the existence of a selection $f \in S^1_{G_m, q}$ is guaranteed by Proposition 2.1 taking $\eta_\Omega = \varphi_n$ with $\Omega \subset nB \times nB$. We stress that, since $x_a \in X$, the function

$$y(t) = C(t, a) x_a + S(t, a)\bar{x}_a + \int_a^t S(t, s) f(s)\, ds$$

is continuously differentiable and

$$\dot{y}(t) = \frac{\partial}{\partial t} C(t, a) x_a + \frac{\partial}{\partial t} S(t, a)\bar{x}_a + \int_a^t \frac{\partial}{\partial t} S(t, s) f(s) ds.$$

Hence, since the natural projection is bounded, Σ_m is well defined and

$$\dot{h}(t) = \mathbb{P}_m \frac{\partial}{\partial t} C(t, a) x_a + \mathbb{P}_m \frac{\partial}{\partial t} S(t, a)\bar{x}_a + \int_a^t \mathbb{P}_m \frac{\partial}{\partial t} S(t, s) f(s) ds,$$

for every $q \in C^1([a, b], E_m)$ and every $h \in \Sigma_m(q)$.

In order to show that Σ_m has a fixed point, we will prove that it satisfies all assumptions of the Kakutani fixed point theorem ([43, Theorem 1]). For this purpose, given $n \in \mathbb{N}$, we use the following notation

$$nB_m = \left\{ q \in C^1([a, b]; E_m) : \|q(t)\|, \|\dot{q}(t)\| \le n, \quad \text{for every } t \in [a, b] \right\}.$$

(a) *Proving that the solution mapping Σ_m has convex values.*

Let $q \in C^1([a, b], E_m)$ and let $h_1, h_2 \in \Sigma_m(q)$. Then, there exist $f_1, f_2 \in S^1_{G_m, q}$ such that

$$h_i(t) = \mathbb{P}_m C(t, a) x_a + \mathbb{P}_m S(t, a)\bar{x}_a + \int_a^t \mathbb{P}_m S(t, s) f_i(s)\, ds, \ \ i = 1, 2.$$

Let $\alpha \in [0, 1]$. Then, for each $t \in [a, b]$, since $\mathbb{P}_m$ is linear, we obtain that

$$(\alpha h_1 + (1-\alpha) h_2)(t) = \mathbb{P}_m C(t, a) x_a + \mathbb{P}_m S(t, a)\bar{x}_a + \int_a^t \mathbb{P}_m S(t, s) \left(\alpha f_1(s) + (1-\alpha) f_2(s)\right)\, ds.$$

Since G_m has convex values, it holds that

$$\alpha h_1 + (1-\alpha) h_2 \in \Sigma_m(q).$$

(*b*) *Proving that* Σ_m *has a closed graph.*

Assume that $(q_k,\ h_k) \to (q,\ h)$ in $C^1([a,b],E_m)\times C^1([a,b],E_m)$, where $h_k \in \Sigma_m\,(q_k)$, for all $k\in\mathbb{N}$, and let us prove that $h\in\Sigma_m\,(q)$.

Since, for all $k\in\mathbb{N}$, $h_k\in\Sigma_m\,(q_k)$, there exists, for all $k\in\mathbb{N}$, $f_k\in S^1_{G_m,\,q_k}$ such that

$$h_k(t)=\mathbb{P}_mC(t,a)x_a+\mathbb{P}_mS(t,a)\overline{x}_a+\int_a^t \mathbb{P}_mS(t,s)f_k(s)\,ds,\qquad \text{for a.a. } t\in[a,b].$$

Since every converging sequence is bounded, there exists $n\in\mathbb{N}$ such that, for every $k\in\mathbb{N}$ and every $s\in[a,b]$, $\|q_k(s)\|\le n$, $\|\dot{q}_k(s)\|\le n$. Then, $(F4^c)$ and the monotonicity of the Schauder basis yield that the sequence $\{f_k\}_k\subset L^1([a,b],E_m)$ is bounded and uniformly integrable, and, for a.a. $s\in[a,b]$, the sequence $\{f_k(s)\}_k$ is bounded, thus relatively compact, in E_m, because E_m is finite-dimensional. According to the Dunford–Pettis Theorem (see [44, p. 294]), we have the existence of a subsequence, denoted as the sequence, and a function f such that $f_k\rightharpoonup f$ in $L^1([a,b],E_m)$.

Given $\phi\in E_m^*$ and $t\in[a,b]$, consider the operator $\Phi: L^1([a,t],E_m)\to\mathbb{R}$ defined by

$$\Phi(p):=\phi\left(\int_a^t S(t,s)p(s)\,ds\right).$$

According to Lemma 2.2 (ii), Φ is clearly linear and bounded. Moreover, $f_k\rightharpoonup f$ also in $L^1([a,t],E_m)$, and hence, we have that

$$\phi\left(\int_a^t S(t,s)f_k(s)\,ds\right)=\Phi(f_k)\to\Phi(f)=\phi\left(\int_a^t S(t,s)f(s)\,ds\right).$$

By the arbitrariness of ϕ, we conclude that

$$\int_a^t S(t,s)f_k(s)\,ds\rightharpoonup\int_a^t S(t,s)f(s)\,ds.$$

By applying Lemma 2.1 *(d)* and recalling that E_m is finite-dimensional, we obtain that

$$h_k(t)\to\mathbb{P}_mC(t,a)x_a+\mathbb{P}_mS(t,a)\overline{x}_a+\int_a^t\mathbb{P}_mS(t,s)f(s)\,ds,$$

for every $t\in[a,b]$. Since the uniform convergence implies the pointwise convergence, it follows that

$$h(t)=\mathbb{P}_mC(t,a)x_a+\mathbb{P}_mS(t,a)\overline{x}_a+\int_a^t\mathbb{P}_mS(t,s)f(s)\,ds.$$

It remains to be proven that $f\in S^1_{G_m,q}$. Due to Mazur's convexity theorem, for each $k\in\mathbb{N}$, there exist $p_k\in\mathbb{N}$ and positive numbers $\beta_{k,i}$, $i=0,\dots,p_k$, such that $\sum_{i=0}^{p_k}\beta_{k,i}=1$ and $r_k:=\sum_{i=0}^{p_k}\beta_{k,i}f_{k+i}\to f$ in $L^1([a,b],E_m)$. From the sequence $\{r_k\}_k$, we extract a subsequence, denoted as the sequence as usual, such that $r_k(t)\to f(t)$, for all $t\in[a,b]\setminus N_1$, with $\lambda(N_1)=0$. Moreover, for all $t\in[a,b]\setminus N_2$, with $\lambda(N_2)=0$, $G_m(t,\cdot,\cdot)$ is weakly u.s.c.

Put $N=N_1\cup N_2$ and consider $t_0\in[a,b]\setminus N$. Then, for every weak neighborhood V of $G_m(t_0,q(t_0),\dot{q}(t_0))$, there exists a weak neighborhood W of $(q(t_0),\dot{q}(t_0))$ such that $G_m(t,x,y)\subset V$ when $(x,y)\in W$. Since the uniform convergence implies the weak pointwise convergence, it follows that $q_k(t_0)\rightharpoonup q(t_0)$ and $\dot{q}_k(t_0)\rightharpoonup\dot{q}(t_0)$. Thus, there exists $\overline{k}$ such that, for all $k\ge\overline{k}$, $(q_k(t_0),\dot{q}_k(t_0))\in W$, yielding that $f_k(t_0)\in G_m(t_0,q_k(t_0),\dot{q}_k(t_0))\subset V$, i.e. that $r_k(t_0)\in V$, because G_m is convex valued. Since $r_k(t_0)\to f(t_0)$, it follows that $f(t_0)\in\overline{V}$, for every weak neighborhood V of $G_m(t_0,q(t_0),\dot{q}(t_0))$. Since G_m is closed valued, the proof is complete.

(*c*) *Showing that* Σ_m *maps bounded sets into bounded sets.*

Let $C\subset C^1([a,b],E_m)$ be bounded, $q\in C$, and $h\in\Sigma_m(q)$. Then, there exists $f\in S^1_{G_m,\,q}$ such that

$$h(t)=\mathbb{P}_mC(t,a)x_a+\mathbb{P}_mS(t,a)\overline{x}_a+\int_a^t\mathbb{P}_mS(t,s)f(s)\,ds \tag{21}$$

and

$$\dot{h}(t)=\mathbb{P}_m\frac{\partial}{\partial t}C(t,a)x_a+\mathbb{P}_m\frac{\partial}{\partial t}S(t,a)\overline{x}_a+\int_a^t\mathbb{P}_m\frac{\partial}{\partial t}S(t,s)f(s)ds. \tag{22}$$

Now, since C is bounded, there exists $n\in\mathbb{N}$ such that $C\subset nB_m$. Thus the monotonicity of the Schauder basis, $(F4^c)$ and Lemma 2.2, yield that

$$\|h(t)\|\le\|C(t,a)\|\,\|x_a\|+\|S(t,a)\|\,\|\overline{x}_a\|+\int_a^b\|S(t,s)\|\,\|f(s)\|\,ds\le K_b(\|x_a\|+\|\overline{x}_a\|+\|\varphi_n\|_{L^1}). \tag{23}$$

Moreover, since $x_a\in X$, there exists

$$M:=\max_{t\in[a,b]}\left\|\frac{\partial}{\partial t}C(t,a)x_a\right\|,$$

and therefore

$$\|\dot{h}(t)\| \le \left\|\frac{\partial}{\partial t}C(t,a)x_a\right\| + \left\|\frac{\partial}{\partial t}S(t,a)\right\|\|\overline{x}_a\| + \int_a^t \left\|\frac{\partial}{\partial t}S(t,s)\right\|\|f(s)\|ds \le M + K_b(\|\overline{x}_a\| + \|\varphi_n\|_{L^1}). \tag{24}$$

Thus, Σ_m maps bounded sets into bounded sets.
(d) *Showing that Σ_m maps bounded sets into relatively compact sets.*

Let $C \subset C^1([a,b],E_m)$ be bounded. Since $C^1([a,b],E_m)$ is a metric space, it is sufficient to prove that given any sequence $\{q_k\}_k \subset C$ and any sequence $\{h_k\}_k$, with $h_k \in \Sigma_m(q_k)$ for every k, there exists a subsequence $\{h_{k_p}\}_p$ and h such that $h_{k_p} \to h$ in $C^1([a,b],E_m)$. Now, let $f_k \in S^1_{G_m,q_k}$ such that

$$h_k(t) = \mathbb{P}_mC(t,a)x_a + \mathbb{P}_mS(t,a)\overline{x}_a + \int_a^t \mathbb{P}_mS(t,s)f_k(s)\,ds,$$

for a.a. $t \in [a,b]$. Since $\{q_k\}_k \subset C$ and C is bounded, ($F4^c$) and the monotonicity of the Schauder basis yield that the sequence $\{f_k\}_k \subset L^1([a,b],E_m)$ is bounded and uniformly integrable, and, for a.a. $s \in [a,b]$, the sequence $\{f_k(s)\}_k$ is bounded, hence relatively compact, in E_m, because E_m is finite-dimensional. Hence $\{f_k\}_k$ is semicompact and, reasoning like in (b), it is possible to prove that there exists a subsequence, denoted as the sequence, and a function $f \in L^1([a,b],E_m)$ such that $f_k \rightharpoonup f$ in $L^1([a,b],E_m)$.

Consider now the function

$$h(t) = \mathbb{P}_mC(t,a)x_a + \mathbb{P}_mS(t,a)\overline{x}_a + \int_a^t \mathbb{P}_mS(t,s)f(s)\,ds.$$

Since $x_a \in X$ and the natural projection is bounded, $h \in C^1([a,b],\mathbb{E}_m)$ and

$$\dot{h}(t) = \mathbb{P}_m\frac{\partial}{\partial t}C(t,a)x_a + \mathbb{P}_m\frac{\partial}{\partial t}S(t,a)\overline{x}_a + \int_a^t \mathbb{P}_m\frac{\partial}{\partial t}S(t,s)f(s)ds.$$

Therefore, recalling Theorem 2.1, Remark 2.5 and Lemma 2.1 *(d)*, we get that $h_k \to h$ in $C^1([a,b],E_m)$, which implies that $\Sigma_m(C)$ is relatively compact.

(e) *Showing that there exists $N \in \mathbb{N}$ independent of m such that $\Sigma_m(NB_m) \subset NB_m$*

By Eqs. (23) and (24), we have that, for every $n,m \in \mathbb{N}$, $q \in nB_m$, and every $h \in \Sigma_m(q)$,

$$\|h\|_{C^1} \le M + K_b(\|x_a\| + \|\overline{x}_a\| + \|\varphi_n\|_{L^1}). \tag{25}$$

According to ($F4^c$), there exists a subsequence, still denoted as the sequence, such that

$$\lim_{n\to\infty}\frac{M + K_b(\|x_a\| + \|\overline{x}_a\| + \|\varphi_n\|_{L^1})}{n} = 0.$$

Therefore, there exists $N > 0$ such that

$$\frac{M + K_b(\|x_a\| + \|\overline{x}_a\| + \|\varphi_N\|_{L^1})}{N} < 1,$$

which, combined with Eq. (25), implies that

$$\frac{1}{N}\|h\|_{C^1} < 1,$$

i.e., that $h \in NB_m$, for every $m \in \mathbb{N}$, $q \in NB_m$, $h \in \Sigma_m(q)$, and the claim is proven.

Since Σ_m is closed and completely continuous, it has compact values; hence, it is u.s.c. Thus, $\Sigma_m : NB_m \multimap NB_m$ is a u.s.c. compact map with convex and closed values. Applying the Kakutani fixed point theorem, we obtain that, for all $m \in \mathbb{N}$, the operator Σ_m has a fixed point q_m. Because of the technique used, we are also able to localize the fixed point in the set

$$NB = \left\{q \in C^1([a,b],E) : \|q(t)\|, \|\dot{q}(t)\| \le N, \text{ for every } t \in [a,b]\right\}.$$

Step 2. *Limiting procedure.* Let us now prove that the sequence $\{q_m\}_m$ found in Step 1 admits a subsequence pointwise weakly converging to a solution q of Problem (14).

The sequence $\{q_m\}_m$ satisfies, for all $m \in \mathbb{N}$ and $t \in [a,b]$,

$$q_m(t) = \mathbb{P}_mC(t,a)x_a + \mathbb{P}_mS(t,a)\overline{x}_a + \int_a^t \mathbb{P}_mS(t,s)f_m(s)\,ds,$$

where $f_m \in S^1_{G_m,q_m}$, for every $m \in \mathbb{N}$. Thus, there exists $g_m \in S^1_{F,q_m}$ such that $f_m = \mathbb{P}_m g_m$. Since $q_m \in NB$ for every m, we then obtain from ($F4^c$) that

$$\|g_m(s)\| \le \varphi_N(s),$$

for a.e. $s \in [a, b]$. Therefore, $\{g_m\}_m$ is bounded and uniformly integrable and $\{g_m(s)\}_m$ is bounded for a.a $s \in [a, b]$. Since E is reflexive, according the Dunford–Pettis Theorem, we obtain the existence of a subsequence, denoted as the sequence, and of a function f such that $g_m \rightharpoonup f$ in $L^1([a, b], E)$. From Lemma 2.1 *(d)*, we then also obtain that $f_m \rightharpoonup f$ in $L^1([a, b], E)$. Reasoning like in (b), it is possible to prove that

$$q_m(t) \rightharpoonup q(t) = C(t, a)x_a + S(t, a)\bar{x}_a + \int_a^t S(t, s)f(s)\,ds$$

and that $\dot{q}_m(t) \rightharpoonup \dot{q}(t)$, for every $t \in [a, b]$. Moreover, according to Mazur's convexity theorem, for each $m \in \mathbb{N}$, there exist $p_m \in \mathbb{N}$ and positive numbers $\beta_{m,i}$, $i = 0, \dots, p_m$, such that $\sum_{i=0}^{p_m} \beta_{m,i} = 1$ and $r_m := \sum_{i=0}^{p_m} \beta_{m,i} g_{m+i} \to f$ in $L^1([a, b], E)$. Therefore, reasoning like in (b), we get that $f \in S^1_{F,q}$ and the proof is complete.

The following theorem shows that it is possible to prove the result when assuming the growth condition $(F4')$ instead of $(F4)$ in Theorem 3.1. For a comparison between these conditions, we refer to [30]. The sketch of the proof is a generalization of the technique used in [13] for second-order inclusions, where the non-linear term does not depend on the first derivative.

Theorem 3.2. *Consider the Cauchy problem (14) on the interval $[a, b]$, where $x_a \in X$, and $F : [a, b] \times E \times E \multimap E$ satisfies conditions $(F1^c) - (F3^c)$. Moreover, let the following assumption hold:*

$(F4'^c)$ *There exist $\alpha, \beta \in L^1([a, b], \mathbb{R})$ such that, for a.a. $t \in [a, b]$ and all $x, y \in E$,*

$$\|F(t, x, y)\| \le \alpha(t)\max\{\|x\|, \|y\|\} + \beta(t). \tag{26}$$

Then the Cauchy problem (14) has a solution has a solution defined on $[a, b]$.

Proof. The result can be proven similarly to Theorem 3.1 when replacing $\varphi_n(t)$ by $n\alpha(t)+\beta(t)$ and consequently modifying the proof. More concretely, the most difficult point is showing that there exists a bounded and convex set H_m, such that $\Sigma_m(H_m) \subset H_m$, for all $m \in \mathbb{N}$.

For this purpose, for every fixed $j \in \mathbb{N}$, define

$$q_j = \max_{t\in[a,b]} \int_a^b e^{-j(t-s)}\chi_{[a,t]}(s)\alpha(s)\,ds,$$

whose existence is guaranteed by continuity. For every $j \in \mathbb{N}$, let t_j be the point where the maximum is achieved. Since $\{t_j\}_j \subset [a, b]$, there exists $\bar{t}$ such that (eventually passing to a subsequence) $t_j \to \bar{t}$. Thus, the sequence $\{\phi_j\}_j \subset L^1([a, b], E)$ defined as $\phi_j(s) = e^{-j(t_j-s)}\chi_{[a,t_j]}(s)\alpha(s)$ converges pointwise to 0. The convergence is dominated, which implies that $\phi_j \to 0$ in $L^1([a, b], E)$. In particular, there exists a subsequence, still denoted as the sequence, such that $q_j \to 0$. Take $\bar{j} \in \mathbb{N}$ and $R \in \mathbb{R}$ such that $1 - K_b q_{\bar{j}} > 0$, and

$$R > \frac{e^{-\bar{j}a}\left[\max\{M, K_b\|x_a\|\} + K_b(\|\bar{x}_a\| + \|\beta\|_1)\right]}{1 - K_b q_{\bar{j}}}.$$

Moreover, let us consider the bounded and convex set

$$H_m = \left\{x \in C^1([a, b], E_m) : \max_{t\in[a,b]}\left(e^{-\bar{j}t}\max\left\{\|x(t)\|, \|\dot{x}(t)\|\right\}\right) \le R\right\}.$$

Now, with reasoning like in Eqs. (23) and (24), we have that, for every $q \in H_m$, $h \in \Sigma_m(q)$, $t \in [a, b]$,

$$\begin{aligned}
e^{-\bar{j}t}\|h(t)\| &\le e^{-\bar{j}t}K_b(\|x_a\| + \|\bar{x}_a\|) + e^{-\bar{j}t}K_b \textstyle\int_a^t(\alpha(s)\max\{\|x(s)\|, \|\dot{x}(s)\|\} + \beta(s))ds \\
&\le e^{-\bar{j}t}K_b(\|x_a\| + \|\bar{x}_a\|) + e^{-\bar{j}t}K_b\|\beta\|_{L^1} + e^{-\bar{j}t}K_b \textstyle\int_a^t e^{\bar{j}s}\alpha(s)e^{-\bar{j}s}\max\{\|x(s)\|, \|\dot{x}(s)\|\}ds \\
&\le e^{-\bar{j}t}K_b(\|x_a\| + \|\bar{x}_a\| + \|\beta\|_{L^1}) + K_bR \textstyle\int_a^t e^{-\bar{j}(t-s)}\alpha(s)ds \\
&\le e^{-\bar{j}t}K_b(\|x_a\| + \|\bar{x}_a\| + \|\beta\|_{L^1}) + K_bR \textstyle\int_a^b e^{-\bar{j}(t-s)}\chi_{[a,t]}(s)\alpha(s)ds \\
&\le e^{-\bar{j}a}K_b(\|x_a\| + \|\bar{x}_a\| + \|\beta\|_{L^1}) + K_bRq_{\bar{j}} < R
\end{aligned}$$

and similarly

$$\begin{aligned}
e^{-\bar{j}t}\|\dot{h}(t)\| &\le e^{-\bar{j}t}M + e^{-\bar{j}t}K_b\|\bar{x}_a\| + e^{-\bar{j}t}K_b \textstyle\int_a^t(\alpha(s)\max\{\|x(s)\|, \|\dot{x}(s)\|\} + \beta(s))ds \\
&\le e^{-\bar{j}a}\left[M + K_b(\|\bar{x}_a\| + \|\beta\|_{L^1})\right] + K_bRq_{\bar{j}} < R.
\end{aligned}$$

Therefore, $h \in H_m$. Since H_m is a subset of the bounded set

$$H = \left\{ x \in C^1([a,b],E) : \max_{t\in[a,b]} \left(e^{-\bar{j}t} \max\left\{ \|x(t)\|, \|\dot{x}(t)\| \right\} \right) \le R \right\},$$

the conclusion then follows like in the proof of Theorem 3.1.

3.2. Abstract inclusion: Existence of a mild solution on the positive half line

In this section, previous results for problems on compact intervals will be applied in order to obtain the existence of a solution to the Cauchy problem for the abstract second-order inclusion (4) on a half-line.

Theorem 3.3. *Consider the Cauchy problem (4) and assume that $F : [0,\infty) \times E \times E \multimap E$ satisfies the following assumptions:*

(F1) *$F(t,x,y)$ is convex, closed, and bounded, for every $(t,x,y) \in [0,\infty) \times E \times E$;*
(F2) *For every $(x,y) \in E \times E$, $F(\cdot,x,y)$ has a measurable selection on every compact subinterval of $[0,\infty)$;*
(F3) *For a.a. $t \in [0,\infty)$, $F(t,\cdot,\cdot) : E^w \times E^w \multimap E^w$ is weakly u.s.c.;*
(F5) *$x_0 \in X$.*

Moreover, suppose that

(F4) *For every $n \in \mathbb{N}$, there exists $\varphi_n \in L^1_{loc}([0,\infty),\mathbb{R})$ with*

$$\liminf_{n\to\infty} \frac{\|\varphi_n\|_{L^1([0,b])}}{n} = 0$$

for every $b > 0$ and such that

$$\|z\| \le \varphi_n(t),$$

for a.a. $t \in [0,\infty)$, every $(x,y) \in nB \times nB$ and every $z \in F(t,x,y)$

or that

(F4′) *There exist $\alpha, \beta \in L^1_{loc}([0,+\infty),\mathbb{R})$ such that, for a.a. $t \in [0,+\infty)$ and all $x, y \in E$,*

$$\|F(t,x,y)\| \le \alpha(t) \max\left\{ \|x\|, \|y\| \right\} + \beta(t).$$

Then the Cauchy problem (4) has a solution defined on $[0,\infty)$.

Proof. The proof will be given in a few steps.

Step 1. Let us consider the Cauchy problem in the interval $[0,1]$:

$$\left.\begin{aligned} \ddot{x}(t) \in A(t)x(t) + F(t,x(t),\dot{x}(t)), \quad & \text{for a.a. } t \in [0,1], \\ x(0) = x_0,\ \dot{x}(0) = \bar{x}_0. & \end{aligned}\right\} \tag{27}$$

According to Theorem 3.1 or 3.2 with $[a,b] = [0,1]$, this problem has a mild solution $x_{[0]} \in C^1([0,1],E)$ verifying

$$x_{[0]}(t) = C(t,0)x_0 + S(t,0)\bar{x}_0 + \int_0^t S(t,s) f_{[0]}(s)\,ds,$$

where $f_{[0]} \in L^1([0,1],E)$ and $f_{[0]}(t) \in F(t, x_{[0]}(t), \dot{x}_{[0]}(t))$, for a.a. $t \in [0,1]$.

Step 2. Let us consider now the following problem in $[1,2]$:

$$\left.\begin{aligned} \ddot{x}(t) \in A(t)x(t) + F(t,x(t),\dot{x}(t)), \quad & \text{for a.a. } t \in [1,2], \\ x(1) = x_{[0]}(1), & \\ \dot{x}(1) = \dot{x}_{[0]}(1). & \end{aligned}\right\} \tag{28}$$

By $x_0 \in X$, we get that $x_{[0]}(t) \in X$, for every $t \in [0,1]$ (see Remark 2.4). In particular, $x_{[0]}(1) \in X$ and, analogously like in Step 1, it is possible to get the existence of a mild solution $x_{[1]} \in C^1([1,2],E)$ to (28) such that

$$x_{[1]}(t) = C(t,1)x_{[0]}(1) + S(t,1)\dot{x}_{[0]}(1) + \int_1^t S(t,s) f_{[1]}(s)\,ds,$$

where $f_{[1]} \in L^1([1,2],E)$ and $f_{[1]}(t) \in F(t, x_{[1]}(t), \dot{x}_{[1]}(t))$, for a.a. $t \in [1,2]$. Combining these two solutions together, we obtain that the function

$$x(t) = \begin{cases} x_{[0]}(t), & \text{for } t \in [0,1], \\ x_{[1]}(t), & \text{for } t \in (1,2] \end{cases}$$

is a continuously differentiable mild solution to the Cauchy problem (4) in $[0, 2]$ (see Remark 2.4).

Step 3. By such a "gluing" for particular solutions of the Cauchy problems, we are able to obtain a solution x defined in the interval $[0, m]$, for an arbitrary $m \in \mathbb{N}$. This solution is defined by

$$x(t) = \begin{cases} x_{[0]}(t), & \text{for } t \in [0, 1], \\ x_{[1]}(t), & \text{for } t \in (1, 2], \\ . & \\ . & \\ . & \\ x_{[m-1]}(t), & \text{for } t \in (m-1, m]. \end{cases}$$

Finally, in order to extend the result from Theorem 3.1 or 3.2 into the half-line, it is sufficient to realize that $[0, \infty) = \bigcup_{m \in \mathbb{N}} [m-1, m]$. Therefore, for every $t \in [0, \infty)$, there exists $m \in \mathbb{N}$ such that $t \in [m-1, m]$. The conclusion follows also observing that the function

$$f(t) = \begin{cases} f_{[0]}(t), & \text{for } t \in [0, t_1], \\ f_{[1]}(t), & \text{for } t \in (t_1, t_2], \\ . & \\ . & \\ . & \\ x_{[m-1]}(t), & \text{for } t \in (m-1, m] \\ . & \\ . & \\ . & \end{cases}$$

belongs to $L^1_{loc}([0, +\infty), E)$ and $f(t) \in F(t, x(t), \dot{x}(t))$, for a.a. $t \in [0, +\infty)$.

Remark 3.1. Existence results for abstract second-order equations and inclusions on non-compact intervals have been obtained in very few papers. In [45] and [46], respectively, a non-local problem and an impulsive Cauchy problem have been studied when A is the generator of a cosine family and F is independent of the first derivative. The non-local problem has been investigated in [47] and [48] for respectively functional and implicit equations under the dependence of the non-linear term on the first derivative, but again for A not dependent on t. Notice that, in all the quoted papers, there are very strict compactness conditions on the sine family generated by A, i.e. the request that the set

$$\{S(t)f(s, x, y) : s \in [0, t], x, y \in E, \|x\|, \|y\| \le r\}$$

is compact for every $t, r > 0$. Moreover, in all the quoted papers the proofs have been based on the usage of a fixed point theorem in the space of continuous functions with weight, which requires strong growth conditions on the non-linear term.

As far as we know, this is the first paper where a solution on a non-compact interval is obtained when $A = A(t)$ and $F = F(t, x(t), \dot{x}(t))$. Moreover, we stress that our result is obtained without requiring any compactness and, since it is based on a very different strategy, neither such strong growth conditions.

3.3. The vibrating string equation

In the previous section, we proved an existence result for (4), which is a generalization of the abstract reformulation (3) of the partial integro-differential problem (1). In this section we discuss the assumptions on $b, a, f, k, x_0, \overline{x}_0$ guaranteeing that (3) has a solution, which is therefore a solution $u \in C^1([0, +\infty), L^p([0, 1]))$ for (1).

In order to apply previous results to the problem (1), let us consider the following assumptions on functions $b, a : [0, +\infty) \times [0, 1] \to \mathbb{R}$, $f : [0, +\infty) \times [0, 1] \times \mathbb{R} \to \mathbb{R}$, $k : [0, 1] \times [0, 1] \to \mathbb{R}$ and $x_0, \overline{x}_0 : [0, 1] \to \mathbb{R}$:

- (b) $b(t, \cdot) \in L^\infty([0, 1])$, for every $t \ge 0$, and there exists $\dot{b}_t$ and it is continuous in $[0, +\infty) \times [0, 1]$;
- (a) a is measurable and there exists $\rho \in L^1_{loc}([0, +\infty))$ such that $|a(t, \xi)| \le \rho(t)$, for every $t \ge 0$ and a.a. $\xi \in [0, 1]$;
- (k) $k \in C([0, 1] \times [0, 1])$;
- (f) f is measurable, $f(t, \xi, \cdot)$ is continuous, for every $t \ge 0$, a.a. $\xi \in [0, 1]$ and there exist $\delta \in L^1_{loc}([0, +\infty)), \gamma \in L^p([0, 1])$ such that $|f(t, \xi, c)| \le \delta(t)\gamma(\xi)|c|$, for every $c \in \mathbb{R}, t \ge 0$ and a.a. $\xi \in [0, 1]$;
- (ic) $x_0 \in W^{1,p}([0, 1]) \cap C_0([0, 1]), \overline{x}_0 \in L^p([0, 1])$.

We recall that the Banach space E is defined as the space $L^p([0, 1])$ and the single-valued mapping $F : [0, +\infty) \times L^p([0, 1]) \times L^p([0, 1]) \to L^p([0, 1])$ is defined by

$$F(t, z, w)(\xi) := a(t, \xi)z(\xi) + f\left(t, \xi, \int_0^1 k(\xi, s)w(s)ds\right).$$

If we denote

$$\bar{k} := \max\left\{|k(\xi, s)| : \xi, s \in [0, 1]\right\},$$

then, according to (a), (k), (f) and the Holder inequality, it holds that

$$\begin{aligned} \left|a(t,\xi)z(\xi)+f\left(t,\xi,\int_0^1 k(\xi,s)w(s)ds\right)\right| &\le \rho(t)|z(\xi)|+\delta(t)\gamma(\xi)\left|\int_0^1 k(\xi,s)w(s)ds\right| \\ &\le \rho(t)|z(\xi)|+\delta(t)\gamma(\xi)\int_0^1 |k(\xi,s)w(s)|ds \\ &\le \rho(t)|z(\xi)|+\delta(t)\gamma(\xi)\bar{k}\|w\|_p. \end{aligned} \tag{29}$$

Thus, $F(t, z, w) \in L^p([0, 1])$, for every $t \ge 0$ and $z, w \in L^p([0, 1])$.

Moreover, by (a) and (f), we obtain that $F(\cdot, z, w)$ is measurable for every $z, w \in L^p([0, 1])$. Fixed now $t \ge 0$, consider a sequence $\{(z_n, w_n)\}_n$ weakly converging to (z, w) in $L^p([0, 1])\times L^p([0, 1])$ and denote $q = \frac{p}{p-1}$. From (a), we get that $a(t, \cdot) \in L^\infty([0, 1]) \subset L^q([0, 1])$, thus $a(t, \cdot)z_n$ converges weakly to $a(t, \cdot)z$. Moreover, for every $\xi \in [0, 1]$, $k(\xi, \cdot) \in L^q([0, 1])$. Thus,

$$\int_0^1 k(\xi, s)w_n(s)ds \to \int_0^1 k(\xi, s)w(s)ds.$$

Condition (f) then implies that

$$f\left(t, \xi, \int_0^1 k(\xi, s)w_n(s)ds\right) \to f\left(t, \xi, \int_0^1 k(\xi, s)w(s)ds\right).$$

Since every weakly converging sequence is bounded, we obtain the existence of a positive constant L such that, for every $n \in \mathbb{N}$, $\|w_n\|_p \le L$. Hence, recalling (29), it follows that

$$\left|f\left(t, \xi, \int_0^1 k(\xi, s)w_n(s)ds\right)\right| \le \delta(t)\gamma(\xi)\bar{k}L,$$

i.e. the convergence is also dominated, which implies that $f(t, \cdot, \int_0^1 k(\cdot, s)w_n(s)ds)$ converges strongly, thus also weakly, to $f(t, \cdot, \int_0^1 k(\cdot, s)w(s)ds)$ in $L^p([0, 1])$, i.e. that $F(t, \cdot, \cdot)$ is weakly continuous, for a.e. $t \ge 0$.

Recalling (29) again and using the estimate

$$(x + y)^p \le 2^p(x^p + y^p) \quad \text{for } x, y \ge 0, p > 1,$$

for every $z, w \in L^p([0, 1])$, we have that

$$\begin{aligned} \|F(t,z,w)\|_p^p &= \int_0^1 \left|a(t,\xi)z(\xi)+f\left(t,\xi,\int_0^1 k(\xi,s)w(s)ds\right)\right|^p d\xi \\ &\le \int_0^1 \left[\rho(t)|z(\xi)|+\delta(t)\gamma(\xi)\bar{k}\|w\|_p\right]^p d\xi \le 2^p[\rho(t)^p\|z\|_p^p+\delta(t)^p\bar{k}^p\|w\|_p^p\|\gamma\|_p^p]. \end{aligned}$$

Thus, (F_4') holds with $\beta(t) = 0$ and $\alpha(t) = 2\max\{\rho(t), \delta(t)\bar{k}\|\gamma\|_p\}$. Now, consider

$$A : W^{2,p}([0, 1]) \cap W_0^{1,p}([0, 1]) \to L^p([0, 1])$$

defined as

$$A(z) := \ddot{z}.$$

The operator A generates a strongly continuous cosine family (see, e.g., [49, Section IV.8]) and $\tilde{X} = W^{1,p}([0, 1])\cap C_0([0, 1])$. Moreover, consider

$$B : [0, +\infty) \to \mathcal{L}(\tilde{X}, E)$$

defined as

$$B(t)z(\xi) := b(t, \xi)\dot{z}(\xi).$$

Clearly, according to (b), B is well defined and, for every $z \in \tilde{X}$, $B(\cdot)z$ is continuously differentiable with

$$\left(\frac{d}{dt}B(t)z\right)(\xi) = \dot{b}_t(t, \xi)z(\xi)$$

(see [8, Section 4]). Then, according to Lemmas 2.3 and 2.4 and (ic), the operator $A(t) = A + B(t)$ generates a fundamental system and $x_0 \in \tilde{X} = X$.

Thus, the existence of a mild solution of problem (3), and hence of (1), can be obtained as the direct consequence of Theorem 3.3.

Remark 3.2. It is possible to generalize the previous result in the case when the condition that $f(t,\xi,\cdot)$ is continuous, for every $t \geq 0$, and $\xi \in [0,1]$, is replaced by the following one:

(f_2) there exist $r_1 < r_2 < \cdots < r_k$ such that $f(t,\xi,r)$ is continuous, for every $t \geq 0$, $\xi \in [0,1]$ and $r \neq r_i$, and $f(t,\xi,\cdot)$ has discontinuities at r_i, for $i = 1,\ldots,k$, with

$$f(t,\xi,r_i^{\mp}) := \lim_{r\to r_i^{\mp}} f(t,\xi,r) \in \mathbb{R}$$

and $|f(t,\xi,r)| \leq \delta(t)\gamma(\xi)|r|$, for every $r \in \mathbb{R}$, $r \neq r_i$, $t \geq 0$ and a.a. $\xi \in [0,1]$.

In this case, it is appropriate to define the multivalued mapping $\tilde{f} : [0,\infty) \times [0,1] \times \mathbb{R} \multimap \mathbb{R}$ by the formula

$$\tilde{f}(t,\xi,r) := \begin{cases} f(t,\xi,r) & \text{if } r \neq r_i, \\ \left[\min\left\{f(t,\xi,r_i), f(t,\xi,r_i^-), f(t,\xi,r_i^+)\right\},\ \max\left\{f(t,\xi,r_i), f(t,\xi,r_i^-), f(t,\xi,r_i^+)\right\}\right] & \\ & \text{if } r = r_i, \quad i = 1,2,\ldots,k. \end{cases}$$

In rewriting into the abstract form is then $F : [0,\infty) \times L^p([0,1]) \times L^p([0,1]) \multimap L^p([0,1])$ defined as

$$F(t,z,w)(\xi) := a(t,\xi)z(\xi) + \tilde{f}\left(t,\xi,\int_0^1 k(\xi,s)w(s)ds\right).$$

4. Conclusions

In this paper, the existence of a solution to the problem describing the small vertical vibration of an elastic string on the positive half-line in the case when both viscous and material damping coefficients are present was investigated. The result was obtained by transforming the original partial differential equation into an appropriate abstract second-order ordinary differential equation in an infinite-dimensional space. The abstract problem was studied using the combination of the Kakutani fixed point theorem together with the approximation solvability method and the weak topology. The applied method enabled us to deal with problems with r.h.s. depending on the first derivative, to obtain the conclusions without any requirement of the compactness of the r.h.s. and without the transformation of the studied second-order problem to the corresponding first-order one.

The conclusions of the paper generalize several previous results dealing with the Cauchy problem for semilinear second-order differential equations or inclusions on the half-line in Banach spaces since the considered r.h.s. depends also on the first derivative and since the conclusions are proved under less restrictive conditions.

We stress that, even if we investigated problem (1) in $C^1([0,+\infty), L^p([0,1]))$, with $p > 1$, the restriction on the exponent is just technical and can be removed, allowing p to assume also value 1. In fact, when dealing with topological methods in infinite-dimensional spaces, it is necessary to provide conditions ensuring compactness. In many papers, the latter is obtained by assuming the compactness of the fundamental system generated by the linear operator or by putting some conditions on the non-linear term, such as lipschitizianity; in both cases limiting the class of application models. We overcome this problem by assuming the reflexivity of the Banach space $E = L^p([0,1])$ in which the abstract form of the partial differential equation takes place and assuming the regularity of the non-linear term with respect to the weak topology. However, it is possible to apply topological methods in a non-reflexive Banach space, i.e. allowing $p = 1$ in the original partial differential equation. In this case, it is sufficient to make use of the weak measure of non-compactness like in [50].

Some directions for possible future research related to the studied topics are the following:

1. Like any other fixed point result, a difficulty in the application of the Kakutani fixed point theorem consists in finding a subset which is mapped into itself by the solution operator associated with the abstract equation. For this purpose, dealing with an initial value problem, we don't need very strict growth conditions on the non-linear term, since an (at most) linear growth like in condition (26) is allowed. Our purpose for future work is to investigate more general boundary conditions, such as, e.g., periodic, antiperiodic, Dirichlet, multipoint, integral, and so on. In this case, the application of a fixed point theorem usually requires a strictly sublinear growth condition. We aim to overcome this problem using a suitable continuation principle, which would allow relaxing the growth condition and enlarging the class of application models in future research.
2. It would be suitable to implement the technique used in the paper for the study of the controllability problems and corresponding real-life applications.

Funding

This work was supported by European Structural and Investment Funds (Operational Programme Research, Development and Education) and by Ministry of Education, Youth and Sports of the Czech Republic [grant number CZ.02.2.69/0.0/0.0/18 054/0014592 *The Advancement of Capacities for Research and Development at Moravian Business College Olomouc*] and by Italian Ministry of University and Research [grant number MIUR-PRIN 2020F3NCPX "Mathematics for industry 4.0 (Math4I4)"].

CRediT authorship contribution statement

Martina Pavlačková: Conceptualization, Methodology, Writing – original draft, Writing – review & editing, Supervision, Project administration. **Valentina Taddei:** Conceptualization, Methodology, Writing – original draft, Writing – review & editing, Supervision, Project administration.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgments

The second author is member of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

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