

A sharp upper bound for the harmonious total chromatic number of graphs and multigraphs*

Marién Abreu † 

Università degli Studi della Basilicata, Via dell'Ateneo Lucano 10, 85100 Potenza, Italy

John Baptist Gauci ‡ 

Department of Mathematics, Faculty of Science, University of Malta, Msida, Malta

Davide Mattiolo § 

Department of Computer Science, KU Leuven Kulak, 8500 Kortrijk, Belgium

Giuseppe Mazzuoccolo 


*Dipartimento di Scienze Fisiche, Informatiche e Matematiche,
Università di Modena e Reggio Emilia, Via Campi 213/b 41125 Modena*

Federico Romaniello 

*Dipartimento di Matematica "Giuseppe Peano", Università di Torino,
via Carlo Alberto 10, 10123 Torino, Italy*

Christian Rubio-Montiel 

*División de Matemáticas e Ingeniería, FES Acatlán,
Universidad Nacional Autónoma de México, Naucalpan, Mexico*

Tommaso Traetta 

*DICATAM - Sez. Matematica, Università degli Studi di Brescia,
Via Valotti 9, I-25123 Brescia, Italy*

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Abstract

A proper total colouring of a graph G is called harmonious if it has the further property that when replacing each unordered pair of incident vertices and edges with their colours,

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‡Corresponding author.

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then no pair of colours appears twice. The smallest number of colours for it to exist is called the harmonious total chromatic number of G , denoted by $h_t(G)$. Here, we give a general upper bound for $h_t(G)$ in terms of the order n of G . Our two main results are obvious consequences of the computation of the harmonious total chromatic number of the complete graph K_n and of the complete multigraph λK_n , where λ is the number of edges joining each pair of vertices of K_n . In particular, Araujo-Pardo et al. have recently shown that $\frac{3}{2}n \leq h_t(K_n) \leq \frac{5}{3}n + \theta(1)$. In this paper, we prove that $h_t(K_n) = \lceil \frac{3}{2}n \rceil$ except for $h_t(K_1) = 1$ and $h_t(K_4) = 7$; therefore, $h_t(G) \leq \lceil \frac{3}{2}n \rceil$, for every graph G on $n > 4$ vertices. Finally, we extend such a result to the harmonious total chromatic number of the complete multigraph λK_n and as a consequence show that $h_t(\mathcal{G}) \leq (\lambda - 1)(2 \lceil \frac{n}{2} \rceil - 1) + \lceil \frac{3n}{2} \rceil$ for $n > 4$, where \mathcal{G} is a multigraph such that λ is the maximum number of edges between any two vertices.

Keywords: Total colouring, harmonious colouring, complete graphs, complete multigraphs, Levi graph.

Math. Subj. Class.: 05C15, 05C70

1 Introduction

A harmonious colouring of a graph G is a proper vertex colouring such that every pair of colours appears on the end vertices of at most one edge of G . The harmonious chromatic number, denoted by $h(G)$, is the smallest number of colours required for a harmonious colouring of G . This parameter was introduced by Miller and Pritikin in [9], as a small variation of the parameter defined independently by Hopcroft and Krishnamoorthy in [7] and by Frank, Harary, and Plantholt in [6], where vertex colourings are not required to be proper.

Note that a vertex colouring where all pairs of vertices have different colours is harmonious. Thus, for every graph G , $h(G) \leq |V(G)|$. If G has diameter 2, then equality holds. Indeed, note that any two vertices with a common neighbour must be coloured differently in a harmonious colouring. On the other hand, the problem of finding $h(G)$ for graphs with diameter at least three is quite difficult. It is proved in [7] that determining $h(G)$ is NP-complete (in particular, in the appendix of [7], a short and elegant proof by David S. Johnson is presented). Results on $h(G)$ can nevertheless be achieved by considering families of graphs with specific structural properties, see for example [1, 2, 4, 5, 10, 11], and interesting upper bounds for general graphs can be found, for example, in [8, 13].

In [3], the authors give a lower bound and an upper bound for the harmonious chromatic number of the Levi graph of the complete graph K_n on n vertices, stated here in Theorem 1.1. The *Levi graph* of a graph G is the incidence graph of G . Recall that a total colouring of a graph is a colouring of both vertices and edges, where pairs of adjacent vertices, pairs of adjacent edges, and pairs of incident vertices and edges receive different colours. A harmonious colouring of the Levi graph of G is then a proper total colouring of G with the additional property that if we write the colours occurring on a vertex and each

E-mail addresses: marien.abreu@unibas.it (Marién Abreu), john-baptist.gauci@um.edu.mt (John Baptist Gauci), davide.mattiolo@kuleuven.be (Davide Mattiolo), giuseppe.mazzuocollo@unimore.it (Giuseppe Mazzuocollo), federico.romaniello@unibas.it (Federico Romaniello), christian.rubio@acatlan.unam.mx (Christian Rubio-Montiel), tommaso.traetta@unibs.it (Tommaso Traetta)

edge incident to it as unordered pairs of colours, then each pair would occur at most once. We refer to such a total colouring as a *harmonious total colouring*. Hence, we can refer to the harmonious chromatic number of the Levi graph of a graph G as the harmonious total chromatic number of G , denoted by $h_t(G)$.

In this paper, we first furnish a general upper bound for $h_t(G)$ in terms of the order n of the graph G , and we prove it to be optimal in all cases. It is straightforward that $h_t(H) \leq h_t(G)$ for every subgraph H of a graph G . Hence, the computation of $h_t(K_n)$ proves a general upper bound for the harmonious total chromatic number of a graph of order n .

In this direction, the following result was proved in [3].

Theorem 1.1 ([3]). *Let K_n be the complete graph on $n \geq 2$ vertices. Then, the following holds*

$$\frac{3}{2}n \leq h_t(K_n) \leq \frac{5}{3}n + \theta(1).$$

Here, we improve such a result by giving the exact value for $h_t(K_n)$ in the following.

Theorem 1.2. *The harmonious total chromatic number $h_t(K_n)$ of the complete graph K_n equals $\lceil \frac{3}{2}n \rceil$ for $n \neq 1, 4$, while $h_t(K_1) = 1$ and $h_t(K_4) = 7$.*

An illustration of a harmonious total colouring of K_8 obtained as described in Section 2.2 is shown in Figure 1.

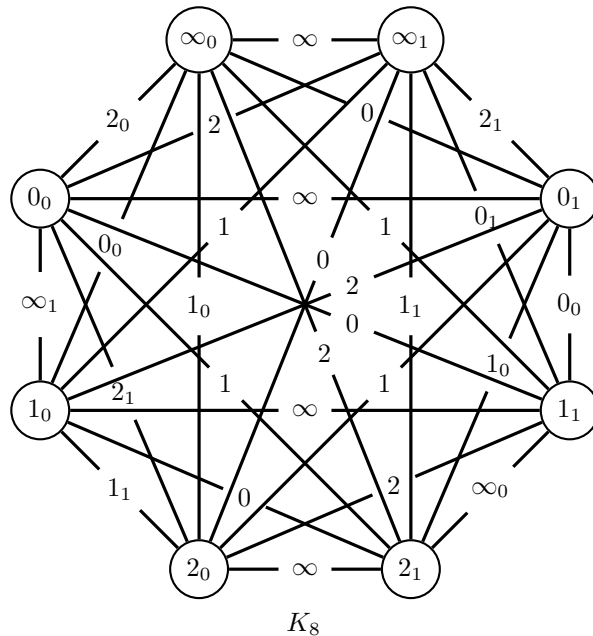


Figure 1: Harmonious total colourings of K_8 with 12 colours.

As already remarked, our first main result is a natural consequence of Theorem 1.2.

Corollary 1.3. *Let G be a graph of order n different from K_1 and K_4 . Then,*

$$h_t(G) \leq \left\lceil \frac{3}{2}n \right\rceil.$$

The next section is devoted to the proof of Theorem 1.2.

Finally, in Section 3, we consider the *complete multigraph* λK_n defined as the multigraph with n vertices and λ edges joining each pair of vertices. We prove the following result.

Theorem 1.4. *The harmonious total chromatic number $h_t(\lambda K_n)$ of the complete multigraph λK_n equals $(\lambda - 1)(2 \lceil \frac{n}{2} \rceil - 1) + \lceil \frac{3n}{2} \rceil$ for $2 \leq n \neq 4$ and $h_t(\lambda K_4) = 3\lambda + 4$.*

Our second main result is thus a direct consequence of Theorem 1.4.

Corollary 1.5. *Let \mathcal{G} be a multigraph of order $n \geq 2$, $n \neq 4$, such that λ is the maximum number of edges between any two vertices. Then,*

$$h_t(\mathcal{G}) \leq (\lambda - 1)(2 \lceil \frac{n}{2} \rceil - 1) + \lceil \frac{3n}{2} \rceil.$$

2 Proof of Theorem 1.2

Figure 2 displays equality for $n \leq 3$, and, for $n = 4$, it shows that $h_t(K_4) \leq 7$. It is easy to check that 6 colours do not suffice for a harmonious total colouring of K_4 , so $h_t(K_4) = 7$. Together with the case $n = 1$, these instances are the only exceptions in which $h_t(K_n) > \lceil \frac{3}{2}n \rceil$.

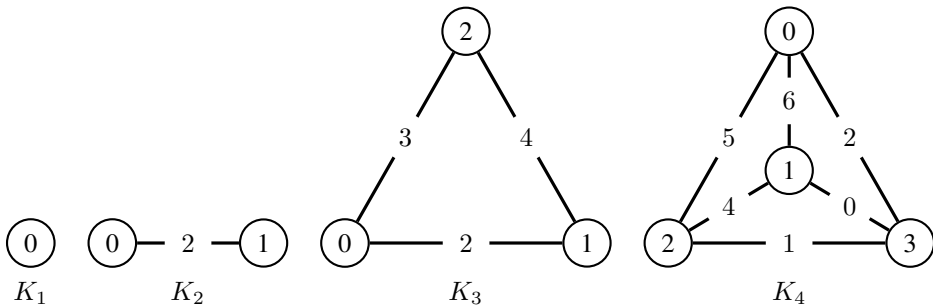


Figure 2: Harmonious total colourings of some small complete graphs.

For the remainder of this proof, we need to distinguish cases according to the congruence of $n > 5$ modulo 4 (or 8).

2.1 The case $n \neq 4k$

In this subsection we consider the case $n \not\equiv 0 \pmod 4$ and construct a harmonious total colouring of K_n which uses $\lceil \frac{3}{2}n \rceil$ colours. More precisely, we prove for all positive values of k that:

- $h_t(K_{4k+1}) = 6k + 2;$
- $h_t(K_{4k+2}) = 6k + 3;$
- $h_t(K_{4k+3}) = 6k + 5.$

The general idea behind our constructions is to first colour each of the vertices using a different colour and decompose K_n into factors. The edges of one of the factors are cleverly coloured using the colours of the vertices, and the remaining edges are then coloured using the unused colours.

The proposed construction is divided into two main steps, with the initial step being the same for all three cases. We begin by identifying the vertex-set of $K_{4k+\ell}$, for $\ell \in \{1, 2, 3\}$, with the elements of the additive group $(\mathbb{Z}_{4k+\ell}, +)$ and we denote the edges of the complete graph by the unordered pairs $\{i, j\}$ where $i, j \in \mathbb{Z}_{4k+\ell}$ such that $i \neq j$. We make use of a set of colours $\mathcal{C} = \mathbb{Z}_{4k+\ell} \cup \mathcal{C}'$, where \mathcal{C}' is a set disjoint from $\mathbb{Z}_{4k+\ell}$. In other words, we are using the elements of the additive group both for the vertices of the graph and for some of the colours. Set $c(i) = i$ for all $i \in \mathbb{Z}_{4k+\ell}$, that is, to every vertex we assign a colour equal to its label. For $h \in \{0, 1, \dots, 4k + \ell - 1\}$, we set $c(\{i + h, j + h\}) = h$ (addition modulo $4k + \ell$), for all $i \neq j \in \{1, \dots, 2k\}$ such that $i + j = 2k + 1$. In other words, we assign colour 0 to the k edges $\{i, j\}$ with $i \neq j \in \{1, \dots, 2k\}$ such that $i + j = 2k + 1$, and then colour h to an edge obtained by adding h to both the ends of an edge with colour 0. Up to now, we have assigned a colour to $(4k + \ell)k$ edges of $K_{4k+\ell}$. More precisely, each colour is used for exactly k edges and we assigned colours to exactly $2k$ edges incident to each vertex.

The second step of our construction aims to colour the remaining edges, and we must distinguish between three different cases, as follows:

- K_{4k+1} : The set of the uncoloured edges is a $2k$ -regular spanning subgraph H_1 on an odd number of vertices. We complete the colouring c by choosing an arbitrary proper $(2k + 1)$ -edge-colouring of H_1 with $2k + 1$ colours in \mathcal{C}' ;
- K_{4k+2} : The set of the uncoloured edges is a $(2k + 1)$ -regular spanning subgraph H_2 on an even number of vertices. Observe that H_2 is the Cayley graph of the group \mathbb{Z}_{4k+2} with generating set S consisting of $\{2, \dots, 4k\}$ and the involution $2k + 1$. Therefore, by Corollary 2.3.1 of [12], we get that H_2 is $(2k + 1)$ -edge-colourable. Hence, we complete the colouring c by choosing an arbitrary proper $(2k + 1)$ -edge-colouring of H_2 with $2k + 1$ colours in \mathcal{C}' .
- K_{4k+3} : The set of the uncoloured edges is a $(2k + 2)$ -regular spanning subgraph H_3 on an odd number of vertices. We complete the colouring c by choosing an arbitrary proper $(2k + 3)$ -edge-colouring of H_3 with $2k + 3$ colours in \mathcal{C}' .

It is worth remarking that Vizing’s Theorem assures the existence of such edge-colourings in the first and third case.

Claim 1. The total colouring c of $K_{4k+\ell}$ is harmonious.

Proof of the Claim 1. We need to prove that the list of all unordered pairs $\{c(i), c(\{i, j\})\}$ has no repetition. Since no two vertices share the same colour, we only need to prove that there are no $i \neq j$ and $h \neq k$ such that

$$c(i) = c(\{h, k\}) \text{ and } c(h) = c(\{i, j\}). \tag{2.1}$$

Note that no vertex receives a colour in C' , then relation (2.1) cannot hold for edges with a colour in C' .

By the symmetry of c , it suffices to prove that the edges incident to the vertex 0 and the vertices incident to edges having colour 0 are a partition of $\{1, \dots, 4k + \ell\}$. Indeed, this is the case since, by definition of c , if $c(\{i, j\}) = 0$ then $i, j \in \{1, \dots, 2k\}$. On the other hand, the edges $\{i + h, j + h\}$ are coloured h , and, without loss of generality, setting $i + h = 0$ we get $h = -i$. Now, for each $j \in \{1, \dots, 2k\}$, $i = 2k + 1 - j$, implying that the $2k$ edges $\{0, j - i\}$ get distinct colours $-i \in \{2k + \ell, \dots, 4k + \ell - 1\}$. \square

2.2 The case $n = 4k$

In this case, we subdivide the graph K_n into two equal complete graphs and a bipartite graph that joins them. We use three sets of $\frac{1}{2}n$ colours that will be used both for the vertices and for the edges under an algebraic rule. The original assignment of colours almost works, except for one edge, so we will have to modify the colours of several edges at the end in order to attain the chromatic total colouring with exactly $\frac{3}{2}n$ colours.

2.2.1 The case k even

We let $k = 2m$, and show that K_{8m} admits a harmonious total colouring with $12m$ colours.

Set $C' = \mathbb{Z}_{4m-1} \cup \{\infty\}$, and for $u \in \mathbb{Z}_2$, let $C_u = \{x_u \mid x \in C'\}$ be a copy of C' . Also, set $V(K_{8m}) = C_0 \cup C_1$. We proceed by constructing a proper total $12m$ -colouring c of K_{8m} which turns out to be harmonious when $m = 1$; otherwise, c is “almost” harmonious, in the sense that there are only two pairs $(x, \{x, y\}), (z, \{z, t\}) \in V(K_{8m}) \times E(K_{8m})$ with $x \neq z$ such that $\{c(x), c(\{x, y\})\} = \{c(z), c(\{z, t\})\}$. We then modify c to obtain the desired harmonious colouring for every $m \geq 2$. We use as a set of colours $C = C_0 \cup C_1 \cup C'$.

Set $c(x_u) = x_u$ for every vertex x_u ($u = 0, 1$) of K_{8m} . To colour its edges, we start by decomposing K_{8m} into the edge disjoint union of the following graphs:

- (1) $K_{4m,4m}$ is the complete bipartite graph whose parts are C_0 and C_1 ;
- (2) Γ_0 and Γ_1 are copies of K_{4m} with vertex-sets C_0 and C_1 , respectively.

Consider a 1-factorization $\mathcal{F} = \{F_y \mid y \in C'\}$ of $K_{4m,4m}$ and set $c(e) = y$ for every edge $e \in F_y$ and every $y \in C'$. It is left to colour the edges of Γ_0 and Γ_1 .

If $m = 1$, set $c(\{\infty_u, (j + 1)_u\}) = j_u$ for every $j \in \mathbb{Z}_3$ and $u \in \{0, 1\}$, and let

$$c(\{0_0, 1_0\}) = \infty_1, c(\{1_0, 2_0\}) = 1_1, c(\{2_0, 0_0\}) = 2_1,$$

$$c(\{0_1, 1_1\}) = 0_0, c(\{1_1, 2_1\}) = \infty_0, c(\{2_1, 0_1\}) = 1_0.$$

One can easily check that c is harmonious (refer to Figure 1).

For $m \geq 2$, we first colour the edges incident with ∞_u and those of the form $\{x_u, y_u\}$ where $x, y \in \mathbb{Z}_{4m-1}$ and $y - x \neq \pm 1$; for $u = 0, 1, i = 0, \dots, m - 2$ and $j \in \mathbb{Z}_{4m-1}$, set

$$c(\{\infty_u, (2m - 1 + j)_u\}) = j_u,$$

$$c(\{j_u, (4i + j + 3)_u\}) = (2i + j + 1)_u, \text{ and}$$

$$c(\{j_u, (4i + j + 5)_u\}) = (2i + j + 3)_{u+1}.$$

The remaining edges, those of the form $\{x_u, (x + 1)_u\}$ will be coloured as follows:

$$c(\{j_0, (j + 1)_0\}) = \begin{cases} \infty_1 & \text{if } j = 0, 2, 4, \dots, 4m - 4, \\ j_1 & \text{if } j = 1, 3, 5, \dots, 4m - 3, 4m - 2; \end{cases}$$

$$c(\{j_1, (j + 1)_1\}) = \begin{cases} \infty_0 & \text{if } j = 1, 3, 5, \dots, 4m - 3, \\ j_0 & \text{if } j = 0, 2, 4, \dots, 4m - 2. \end{cases}$$

One can check that c is almost harmonious; more precisely,

- (1) c is a proper total colouring of K_{8m} ,
- (2) $c(\{(4m - 2)_0, 0_0\}) = (4m - 2)_1$, and $c(\{(4m - 2)_1, 0_1\}) = (4m - 2)_0$, and
- (3) c is harmonious on K_{8m} minus the edge $\{(4m - 2)_0, 0_0\}$.

By recolouring the edges $\{(4m - 2)_0, 0_0\}$, $\{0_0, (4m - 3)_0\}$, $\{(2m - 1)_1, (6m - 4)_1\}$ and $\{(4m - 3)_0, (8m - 6)_0\}$ in the following way, we obtain the desired harmonious colouring of K_{8m} :

$$c(\{(4m - 2)_0, 0_0\}) = (2m - 1)_1, \quad c(\{0_0, (4m - 3)_0\}) = (4m - 2)_1,$$

$$c(\{(2m - 1)_1, (6m - 4)_1\}) = (4m - 3)_0 \quad c(\{(4m - 3)_0, (8m - 6)_0\}) = (4m - 4)_1.$$

2.2.2 The case k odd

We let $k = 2m + 1$, and show that K_{8m+4} admits a harmonious total colouring with $12m + 6$ colours.

Set $\mathcal{C}' = \mathbb{Z}_{4m+1} \cup \{\infty\}$, and for $u \in \mathbb{Z}_2$, let $\mathcal{C}_u = \{x_u \mid x \in \mathcal{C}'\}$ be a copy of \mathcal{C}' . Also, set $V(K_{8m+4}) = \mathcal{C}_0 \cup \mathcal{C}_1$. We proceed by constructing a proper total $(12m + 6)$ -colouring c of K_{8m+4} which turns out to be “almost” harmonious, in the sense that there are only two pairs $(x, \{x, y\}), (z, \{z, t\}) \in V(K_{8m+4}) \times E(K_{8m+4})$ with $x \neq z$ such that $\{c(x), c(\{x, y\})\} = \{c(z), c(\{z, t\})\}$. We then modify c to obtain the desired harmonious colouring for every $m \geq 1$. We use as a set of colours $\mathcal{C} = \mathcal{C}_0 \cup \mathcal{C}_1 \cup \mathcal{C}'$.

Set $c(x_u) = x_u$ for every vertex x_u ($u = 0, 1$) of K_{8m+4} . To colour its edges, we start by decomposing K_{8m+4} into the edge disjoint union of the following graphs:

- (1) $K_{4m+2, 4m+2}$ is the complete bipartite graph whose parts are \mathcal{C}_0 and \mathcal{C}_1 ;
- (2) Γ_0 and Γ_1 are copies of K_{4m+2} with vertex-sets \mathcal{C}_0 and \mathcal{C}_1 , respectively.

Consider a 1-factorization $\mathcal{F} = \{F_y \mid y \in \mathcal{C}'\}$ of $K_{4m+2, 4m+2}$ and set $c(e) = y$ for every edge $e \in F_y$ and every $y \in \mathcal{C}'$. It is left to colour the edges of Γ_0 and Γ_1 .

We first colour the edges incident with ∞_u and those of the form $\{x_u, y_u\}$ where $x, y \in \mathbb{Z}_{4m+1}$ and $y - x \neq \pm 1$; for $u = 0, 1$, $0 \leq i, i' \leq m - 1$ with $i' \neq m - 1$, and $j \in \mathbb{Z}_{4m+1}$, set

$$c(\{\infty_u, j_u\}) = (j + 2m)_{u+1},$$

$$c(\{j_u, (4i + j + 3)_u\}) = (2i + j + 2)_u, \text{ and}$$

$$c(\{j_u, (4i' + j + 5)_u\}) = (2i' + j + 3)_{u+1}.$$

The remaining edges, those of the form $\{x_u, (x + 1)_u\}$ will be coloured as follows:

$$c(\{j_0, (j + 1)_0\}) = \begin{cases} \infty_0 & \text{if } j = 0, 2, 4, \dots, 4m - 2, \\ j_1 & \text{if } j = 1, 3, 5, \dots, 4m - 1, 4m; \end{cases}$$

$$c(\{j_1, (j + 1)_1\}) = \begin{cases} \infty_1 & \text{if } j = 1, 3, 5, \dots, 4m - 1, \\ j_0 & \text{if } j = 0, 2, 4, \dots, 4m. \end{cases}$$

One can check that c is almost harmonious; more precisely,

- (1) c is a proper total colouring of K_{8m+4} ,
- (2) $c(\{4m_0, 0_0\}) = 4m_1$, and $c(\{4m_1, 0_1\}) = 4m_0$, and
- (3) c is harmonious on K_{8m+4} minus the edge $\{4m_0, 0_0\}$ (or $\{4m_1, 0_1\}$).

By recolouring, in the following way, the edges $\{0_1, 4_1\}$, $\{\infty_0, 3_0\}$, $\{4_0, 1_0\}$ when $m = 1$, and the edges $\{4m_0, 0_0\}$, $\{(4m - 1)_0, 0_0\}$, $\{(2m)_0, (2m - 2)_0\}$ and $\{(4m - 1)_0, (4m - 3)_0\}$ when $m \geq 2$, we obtain the desired harmonious total colouring of K_{8m+4} : when $m = 1$, set $c(\{0_1, 4_1\}) = 3_0$, $c(\{\infty_0, 3_0\}) = 4_0$ and $c(\{4_0, 1_0\}) = 0_1$, otherwise set

$$c(\{4m_0, 0_0\}) = (2m)_0, \quad c(\{(4m - 1)_0, 0_0\}) = (4m)_1,$$

$$c(\{(2m)_0, (2m - 2)_0\}) = (4m - 1)_0 \quad c(\{(4m - 1)_0, (4m - 3)_0\}) = (4m - 2)_1. \quad \square$$

3 Proof of Theorem 1.4

The computation of the harmonious total chromatic number $h_t(\lambda K_n)$ is divided into two parts. The lower bound is a generalization of the lower bound of Theorem 1.1, and the upper bound is a consequence of Theorem 1.2.

Proof. To achieve the lower bound, let $c: V \cup E \rightarrow \{c_1, \dots, c_{h_t(G)}\}$ be a harmonious total colouring of $G := \lambda K_n$, where $V = \{v_1, \dots, v_n\}$ and for each $v_i \in V$, let $c_i = c(v_i)$, since vertices are pairwise adjacent.

For each $i \in \{1, \dots, h_t(G)\}$, let $E_i = \{e \in E : c(e) = c_i\}$. Let G^* be the partial multigraph of G such that its vertices are V and its edges are $E^* = \bigcup_{i=1}^n E_i$. On the one

hand, $|E^*| = \sum_{i=1}^n |E_i| = \frac{1}{2} \sum_{i=1}^n \text{deg}_{G^*}(v_i)$ where $\text{deg}_{G^*}(v_i)$ is the number of edges of G^* incident to v_i . On the other hand, for each $v_i \in V$, if there is an edge of G^* , incident to v_i , coloured c_j , then v_j is not incident to any edge of E_i . Thus, $|E_i| \leq \frac{n-1-\text{deg}_{G^*}(v_i)}{2}$.

Therefore, $\sum_{i=1}^n |E_i| \leq \sum_{i=1}^n \frac{n-1-\text{deg}_{G^*}(v_i)}{2}$, which implies that $\sum_{i=1}^n |E_i| \leq \frac{n(n-1)}{2} - \sum_{i=1}^n |E_i|$. Hence, $\sum_{i=1}^n |E_i| \leq \lfloor \frac{n(n-1)}{4} \rfloor$ and therefore

$$\lambda \binom{n}{2} = \sum_{i=1}^{h_t(G)} |E_i| = \sum_{i=1}^n |E_i| + \sum_{i=n+1}^{h_t(G)} |E_i| \leq \left\lfloor \frac{n(n-1)}{4} \right\rfloor + \sum_{i=n+1}^{h_t(G)} |E_i|.$$

Since, for each $i \in \{1, \dots, h_t(G)\}$, $|E_i| \leq \lfloor \frac{n}{2} \rfloor$ it follows that

$$\lambda \binom{n}{2} \leq \left\lfloor \frac{n(n-1)}{4} \right\rfloor + (h_t(G) - n) \left\lfloor \frac{n}{2} \right\rfloor,$$

$$\left\lceil \frac{\lambda \binom{n}{2} - \left\lfloor \frac{n(n-1)}{4} \right\rfloor}{\left\lfloor \frac{n}{2} \right\rfloor} \right\rceil + n \leq h_t(G).$$

After some simple calculations, we get

$$h_t(G) \geq \begin{cases} (\lambda - 1)(n - 1) + \frac{3n}{2} & \text{if } n \text{ is even,} \\ (\lambda - 1)n + \frac{3n+1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Thus, we conclude that $(\lambda - 1)(2 \lceil \frac{n}{2} \rceil - 1) + \lceil \frac{3n}{2} \rceil \leq h_t(\lambda K_n)$.

For the upper bound, let us consider $\lambda K_n = (1 + t)K_n = K_n \cup tK_n$. We colour K_n as in Theorem 1.2 with $\lceil \frac{3n}{2} \rceil$ colours. If n is even, tK_n accepts a colouring by perfect matchings with $t(n - 1)$ classes of new colours, while if n is odd, tK_n accepts a colouring by maximal matchings with tn classes of new colours. For the exceptional case of $n = 4$, it is not hard to see that G^* is a subgraph of $K_{1,3}$ or the multigraph $K_1 \cup K_1 \cup 2K_2$, and in any case, its complement contains a triangle that belongs to three different classes, not two, which are considered in the count since each remaining class is a matching, therefore $h_t(\lambda K_4) \geq 3\lambda + 4$, and the result follows. \square

ORCID iDs

Marien Abreu  <https://orcid.org/0000-0003-3992-1029>

John Baptist Gauci  <https://orcid.org/0000-0001-6584-8473>

Davide Mattiolo  <https://orcid.org/0000-0001-5876-0887>

Giuseppe Mazzuoccolo  <https://orcid.org/0000-0001-7775-065X>

Federico Romaniello  <https://orcid.org/0000-0003-1166-3179>

Christian Rubio-Montiel  <https://orcid.org/0000-0003-1474-8362>

Tommaso Traetta  <https://orcid.org/0000-0001-8141-0535>

References

- [1] A. Aflaki, S. Akbari, K. Edwards, D. Eskandani, M. Jamaali and M. Ravanbod, On harmonious colourings of trees, *Electron. J. Comb.* **19** (2012), 3, doi:10.37236/9, <https://doi.org/10.37236/9>.
- [2] S. Akbari, J. Kim and A. Kostochka, Harmonious coloring of trees with large maximum degree, *Discret Math.* **312** (2012), 1633–1637, doi:10.1016/j.disc.2012.02.009, <https://doi.org/10.1016/j.disc.2012.02.009>.
- [3] G. Araujo-Pardo, J. J. Montellano-Ballesteros, M. Olsen and C. Rubio-Montiel, On the harmonious chromatic number of graphs, *Bol. Soc. Mat. Mex. (3)* **30(2)** (2024), Paper No. 59, 16pp., doi:10.1007/s40590-024-00632-5, <https://doi.org/10.1007/s40590-024-00632-5>.
- [4] M. Buratti, F. Merola, A. Nakić and C. Rubio-Montiel, Banff designs: difference methods for coloring incidence graphs, *Art. Discrete Appl. Math.* **8**, Id/No P3.01, 11 pp., doi:10.26493/2590-9770.1739.91c, <https://doi.org/10.26493/2590-9770.1739.91c>.

- [5] K. Edwards and C. McDiarmid, New upper bounds on harmonious colorings, *J. Graph Theory* **18** (1994), 257–267, doi:10.1002/jgt.3190180305, <https://doi.org/10.1002/jgt.3190180305>.
- [6] O. Frank, F. Harary and M. Plantholt, The line-distinguishing chromatic number of a graph, *Ars Comb.* **14** (1982), 241–252.
- [7] J. Hopcroft and M. Krishnamoorthy, On the harmonious coloring of graphs, *SIAM J. Algebraic Discrete Methods* **4** (1983), 306–311, doi:10.1137/0604032, <https://doi.org/10.1137/0604032>.
- [8] Z. Lu, On an upper bound for the harmonious chromatic number of a graph, *J. Graph Theory* **15** (1991), 345–347, doi:10.1002/jgt.3190150402, <https://doi.org/10.1002/jgt.3190150402>.
- [9] Z. Miller and D. Pritikin, The harmonious coloring number of a graph, *Discrete Math.* **93** (1991), 211–228, doi:10.1016/0012-365X(91)90257-3, [https://doi.org/10.1016/0012-365X\(91\)90257-3](https://doi.org/10.1016/0012-365X(91)90257-3).
- [10] X. Ming, T. Feng and M. Zhang, The asymptotic existence of BIBDs having a nesting, *Des. Codes Cryptogr.* (2025), doi:10.1007/s10623-025-01583-2, <https://doi.org/10.1007/s10623-025-01583-2>.
- [11] J. Mitchem, On the harmonious chromatic number of a graph, *Discrete Math.* **74** (1989), 151–157.
- [12] R. Stong, On 1-factorizability of cayley graphs, *J. Comb. Theory Ser. B* **39** (1985), 298–307, doi:10.1016/0095-8956(85)90056-5, [https://doi.org/10.1016/0095-8956\(85\)90056-5](https://doi.org/10.1016/0095-8956(85)90056-5).
- [13] M. Venkatachalam, J. Vivin and K. Kaliraj, Harmonious coloring of double star graph families, *Tamkang J. Math.* **43** (2012), 153–158, doi:10.5556/j.tkjm.43.2012.675, <https://doi.org/10.5556/j.tkjm.43.2012.675>.