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Profinite groups and centralizers of coprime automorphisms whose elements are Engel

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Abstract. Let q be a prime, n a positive integer and A an elementary abelian group of order q^r with $r \geq 2$ acting on a finite q'-group G. We show that if all elements in $\gamma_{r-1}(C_G(a))$ are n-Engel in G for any $a \in A^{\#}$, then $\gamma_{r-1}(G)$ is k-Engel for some $\{n, q, r\}$-bounded number k, and if, for some integer d such that $2^d \leq r-1$, all elements in the dth derived group of $C_G(a)$ are n-Engel in G for any $a \in A^{\#}$, then the dth derived group $G^{(d)}$ is k-Engel for some $\{n, q, r\}$-bounded number k. Assuming $r \geq 3$, we prove that if all elements in $\gamma_{r-2}(C_G(a))$ are n-Engel in $C_G(a)$ for any $a \in A^{\#}$, then $\gamma_{r-2}(G)$ is k-Engel for some $\{n, q, r\}$-bounded number k, and if, for some integer d such that $2^d \leq r-2$, all elements in the dth derived group of $C_G(a)$ are n-Engel in $C_G(a)$ for any $a \in A^{\#}$, then the dth derived group $G^{(d)}$ is k-Engel for some $\{n, q, r\}$-bounded number k. Analogous (non-quantitative) results for profinite groups are also obtained.

1 Introduction

Let A be a finite group acting on a finite group G. Many well-known results show that the structure of the centralizer $C_G(A)$ (the fixed-point subgroup) of A has influence over the structure of G. The influence is especially strong if $(|A|, |G|) = 1$, that is, the action of A on G is coprime. By following the solution of the restricted Burnside problem, it was discovered that the exponent of $C_G(A)$ may have strong impact over the exponent of G. Recall that a group G is said to have exponent n if $x^n = 1$ for every $x \in G$ and n is the minimal positive integer with this property. The following theorem was obtained in [14].

Theorem 1.1. *Let q be a prime, n a positive integer and A an elementary abelian group of order q^2. Suppose that A acts coprimely on a finite group G and assume that $C_G(a)$ has exponent dividing n for each $a \in A^{\#}$. Then the exponent of G is $\{n, q\}$-bounded.*

Here and throughout the paper $A^{\#}$ denotes the set of nontrivial elements of A. Moreover, we will use the expression "$\{a, b, \dots\}$-bounded" to abbreviate

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"bounded from above in terms of $a, b, \dots$ only". The proof of the above result involves a number of deep ideas. In particular, Zelmanov's techniques that led to the solution of the restricted Burnside problem [26, 27] are combined with the Lubotzky–Mann theory of powerful p-groups [17, 18], and a theorem of Bahturin and Zaicev on Lie algebras admitting a solvable group of automorphisms whose fixed-point subalgebra is PI [5].

A profinite (non-quantitative) version of the above theorem was established in [22]. In the context of profinite groups all the usual concepts of group theory are interpreted topologically. In particular, by a subgroup of a profinite group we always mean a closed subgroup and a subgroup is said to be generated by a set S if it is topologically generated by S. By an automorphism of a profinite group we mean a continuous automorphism. We say that a group A acts on a profinite group G *coprimely* if A is finite while G is an inverse limit of finite groups whose orders are relatively prime to the order of A. The profinite (non-quantitative) version of Theorem 1.1 is as follows.

Theorem 1.2. *Let q be a prime and A an elementary abelian group of order q^2. Suppose that A acts coprimely on a profinite group G and assume that $C_G(a)$ is torsion for each $a \in A^\#$. Then G is locally finite.*

In [23] the situation where the centralizers $C_G(a)$ consist of Engel elements was dealt with. If x, y are elements of a (possibly infinite) group G, the commutators $[x,_n y]$ are defined inductively by the rule

$$[x,_0 y] = x, \qquad [x,_n y] = [[x,_{n-1} y], y] \quad \text{for all } n \geq 1.$$

An element x is called a (left) Engel element if for any $g \in G$ there exists n, depending on x and g, such that $[g,_n x] = 1$. A group G is called Engel if all elements of G are Engel. The element x is called a (left) n-Engel element if for any $g \in G$ we have $[g,_n x] = 1$. The group G is n-Engel if all elements of G are n-Engel. The following result was proved in [23].

Theorem 1.3. *Let q be a prime, n a positive integer and A an elementary abelian group of order q^2. Suppose that A acts coprimely on a finite group G and assume that for each $a \in A^\#$ every element of $C_G(a)$ is n-Engel in G. Then the group G is k-Engel for some $\{n, q\}$-bounded number k.*

A profinite (non-quantitative) version of the above theorem was established in the recent work [4].

Theorem 1.4. *Let q be a prime and A an elementary abelian group of order q^2. Suppose that A acts coprimely on a profinite group G and assume that all elements in $C_G(a)$ are Engel in G for each $a \in A^\#$. Then G is locally nilpotent.*

A very deep theorem of Wilson and Zelmanov [24, Theorem 5] tells us that a profinite group is locally nilpotent if and only if it is Engel. Thus, there is a clear relation between Theorem 1.3 and Theorem 1.4.

If, in Theorem 1.3, we relax the hypothesis that every element of $C_G(a)$ is n-Engel in G and require instead that every element of $C_G(a)$ is n-Engel in $C_G(a)$, we quickly see that the result is no longer true. An example of a finite non-nilpotent group G admitting an action of a noncyclic group A of order four such that $C_G(a)$ is abelian for each $a \in A^{\#}$ can be found for instance in [3]. On the other hand, another result, that was established in [4], is the following.

Theorem 1.5. *Let q be a prime, n a positive integer and A an elementary abelian group of order q^3. Suppose that A acts coprimely on a finite group G and assume that for each $a \in A^{\#}$ every element of $C_G(a)$ is n-Engel in $C_G(a)$. Then the group G is k-Engel for some $\{n, q\}$-bounded number k.*

In [3] a profinite (non-quantitative) version of Theorem 1.5 was obtained. The statement is as follows.

Theorem 1.6. *Let q be a prime and A an elementary abelian group of order q^3. Suppose that A acts coprimely on a profinite group G and assume that $C_G(a)$ is locally nilpotent for each $a \in A^{\#}$. Then G is locally nilpotent.*

The relation between Theorems 1.3 and 1.4 noted above naturally extends to Theorems 1.5 and 1.6.

Let us denote by $\gamma_i(H)$ the ith term of the lower central series of a group H and by $H^{(i)}$ the ith term of the derived series of H. It was shown in [9] and further in [1,2] that if the rank of the acting group A is big enough, then results of similar nature to that of Theorem 1.1 can be obtained while imposing conditions on elements of $\gamma_i(C_G(a))$ or $C_G(a)^{(i)}$ rather than on elements of $C_G(a)$. In the same spirit, one of the goals of the present article is to extend Theorems 1.3 and 1.5, respectively, as follows.

Theorem 1.7. *Let q be a prime, n a positive integer and A an elementary abelian group of order q^r with $r \geq 2$ acting on a finite q'-group G.*

(1) *If all elements in $\gamma_{r-1}(C_G(a))$ are n-Engel in G for any $a \in A^{\#}$, then $\gamma_{r-1}(G)$ is k-Engel for some $\{n, q, r\}$-bounded number k.*

(2) *If, for some integer d such that $2^d \leq r - 1$, all elements in the dth derived group of $C_G(a)$ are n-Engel in G for any $a \in A^{\#}$, then the dth derived group $G^{(d)}$ is k-Engel for some $\{n, q, r\}$-bounded number k.*

Theorem 1.8. *Let q be a prime, n a positive integer and A an elementary abelian group of order q^r with $r \geq 3$ acting on a finite q'-group G.*

1. *If all elements in $\gamma_{r-2}(C_G(a))$ are n-Engel in $C_G(a)$ for any $a \in A^{\#}$, then $\gamma_{r-2}(G)$ is k-Engel for some $\{n, q, r\}$-bounded number k.*
2. *If, for some integer d such that $2^d \leq r-2$, all elements in the dth derived group of $C_G(a)$ are n-Engel in $C_G(a)$ for any $a \in A^{\#}$, then the dth derived group $G^{(d)}$ is k-Engel for some $\{n, q, r\}$-bounded number k.*

Let H, K be subgroups of a profinite group G. We denote by $[H, K]$ the closed subgroup of G generated by all commutators of the form $[h, k]$, with $h \in H$ and $k \in K$. Thus one can consider inductively the following closed subgroups:

$$\gamma_1(G) = G, \qquad \gamma_k(G) = [\gamma_{k-1}(G), G] \qquad \text{for } k \geq 1,$$
$$G^{(0)} = G, \qquad G^{(k)} = [G^{(k-1)}, G^{(k-1)}] \quad \text{for } k \geq 1.$$

Finally, we formulate the (non-quantitative) analogues of Theorems 1.7 and 1.8, respectively.

Theorem 1.9. *Let q be a prime, n a positive integer and A an elementary abelian group of order q^r with $r \geq 2$ acting coprimely on a profinite group G.*

1. *If all elements in $\gamma_{r-1}(C_G(a))$ are Engel in G for any $a \in A^{\#}$, then $\gamma_{r-1}(G)$ is locally nilpotent.*
2. *If, for some integer d such that $2^d \leq r-1$, all elements in the dth derived group of $C_G(a)$ are Engel in G for any $a \in A^{\#}$, then the dth derived group $G^{(d)}$ is locally nilpotent.*

Theorem 1.10. *Let q be a prime, n a positive integer and A an elementary abelian group of order q^r with $r \geq 3$ acting coprimely on a profinite group G.*

1. *If all elements in $\gamma_{r-2}(C_G(a))$ are Engel in $C_G(a)$ for any $a \in A^{\#}$, then $\gamma_{r-2}(G)$ is locally nilpotent.*
2. *If, for some integer d such that $2^d \leq r-2$, all elements in the dth derived group of $C_G(a)$ are Engel in $C_G(a)$ for any $a \in A^{\#}$, then the dth derived group $G^{(d)}$ is locally nilpotent.*

Thus the purpose of the present article is to provide the proofs for Theorems 1.7, 1.8, 1.9 and 1.10. The paper is organized as follows. In Sections 2 and 3 we present the Lie-theoretic machinery that will be useful within the proofs. Later in Section 4 a technical tool, introduced in [1], is extended to the context of profinite groups. Sections 5 and 6 are devoted to proving Theorems 1.7 and 1.8. Finally, in Section 7 we give the details of the proofs of Theorems 1.9 and 1.10.

Throughout the paper we use, without special references, the well-known properties of coprime actions (see for example [20, Lemma 3.2]):

- If α is a coprime automorphism of a profinite group G, then
$$C_{G/N}(\alpha) = C_G(\alpha)N/N$$
for any α-invariant normal subgroup N.
- If A is a noncyclic abelian group acting coprimely on a profinite group G, then G is generated by the subgroups $C_G(B)$, where A/B is cyclic.

2 Results on Lie algebras and Lie rings

Let X be a subset of a Lie algebra L. By a commutator in elements of X we mean any element of L that can be obtained as a Lie product of elements of X with some system of brackets. If $x_1, \dots, x_k, x, y$ are elements of L, we define inductively

$$[x_1] = x_1, \quad [x_1, \dots, x_k] = [[x_1, \dots, x_{k-1}], x_k]$$

and

$$[x,_0 y] = x, \quad [x,_m y] = [[x,_{m-1} y], y]$$

for all positive integers k, m. As usual, we say that an element $a \in L$ is ad-nilpotent if there exists a positive integer n such that $[x,_n a] = 0$ for all $x \in L$. If n is the least integer with the above property, then we say that a is ad-nilpotent of index n.

The next theorem represents the most general form of the Lie-theoretical part of the solution of the restricted Burnside problem. It was announced by Zelmanov in [27]. A detailed proof was published in [28].

Theorem 2.1. *Let L be a Lie algebra over a field and suppose that L satisfies a polynomial identity. If L can be generated by a finite set X such that every commutator in elements of X is ad-nilpotent, then L is nilpotent.*

The next theorem, which was proved by Bahturin and Zaicev [5] for soluble groups A and later extended by Linchenko [16] to the general case, provides an important criterion for a Lie algebra to satisfy a polynomial identity.

Theorem 2.2. *Let L be a Lie algebra over a field K. Assume that a finite group A acts on L by automorphisms in such a manner that $C_L(A)$ satisfies a polynomial identity. Assume further that the characteristic of K is either 0 or prime to the order of A. Then L satisfies a polynomial identity.*

Both Theorems 2.1 and 2.2 admit the following respective quantitative versions (see for example [14] and [21]).

Theorem 2.3. *Let L be a Lie algebra over a field K generated by $a_1, \dots, a_m$. Suppose that L satisfies a polynomial identity $f \equiv 0$ and each commutator in $a_1, \dots, a_m$ is ad-nilpotent of index at most n. Then the Lie algebra L is nilpotent of $\{f, K, m, n\}$-bounded class.*

Theorem 2.4. *Let L be as in Theorem 2.2. Assume that $C_L(A)$ satisfies a polynomial identity $f \equiv 0$. Then L satisfies a polynomial identity of $\{|A|, f, K\}$-bounded degree.*

By combining Theorems 2.3 and 2.4, the following corollary can be obtained.

Corollary 2.5. *Let L be a Lie algebra over a field K and A a finite group of automorphisms of L such that $C_L(A)$ satisfies the polynomial identity $f \equiv 0$. Suppose that the characteristic of K is either 0 or prime to the order of A. Assume that L is generated by an A-invariant set of m elements in which every commutator is ad-nilpotent of index at most n. Then L is nilpotent of $\{|A|, f, K, m, n\}$-bounded class.*

For our purpose we will need to work with Lie rings, and not only with Lie algebras. As usual, $\gamma_i(L)$ denotes the ith term of the lower central series of L. In [23] the following result was established for Lie rings, similar to Corollary 2.5.

Theorem 2.6. *Let L be a Lie ring and A a finite group of automorphisms of L such that $C_L(A)$ satisfies the polynomial identity $f \equiv 0$. Further, assume that L is generated by an A-invariant set of m elements such that every commutator in the generators is ad-nilpotent of index at most n. Then there exist positive integers e and c, depending only on $|A|, f, m$ and n, such that $e\gamma_c(L) = 0$.*

We also require the following useful lemma whose proof can be found in [14].

Lemma 2.7. *Let L be a Lie ring and H a subring of L generated by m elements $h_1, \dots, h_m$ such that all commutators in h_i are ad-nilpotent in L of index at most n. If H is nilpotent of class c, then for some $\{c, m, n\}$-bounded number u we have*

$$[L, \underbrace{H, \dots, H}_{u}] = 0.$$

Recall that the identity

$$\sum_{\sigma \in S_n} [y, x_{\sigma(1)}, \dots, x_{\sigma(n)}] \equiv 0$$

is called the linearized n-Engel identity. In general, the statement of Theorem 2.1 cannot be extended to the case where L is just a Lie ring (rather than a Lie algebra over a field). However, such an extension does hold in the particular case where the polynomial identity $f \equiv 0$ is a linearized Engel identity. More precisely, by combining Theorems 2.6, 2.3 and Lemma 2.7, the following result can be obtained. See [23] for further details.

Theorem 2.8. *Let F be the free Lie ring and f an element of F (Lie polynomial) such that $f \notin pF$ for any prime p. Suppose that L is a Lie ring generated by finitely many elements $a_1, \dots, a_m$ such that all commutators in the generators are ad-nilpotent of index at most n. Assume that L satisfies the identity $f \equiv 0$. Then L is nilpotent with $\{f, m, n\}$-bounded class.*

3 On associated Lie rings

Given a group G, there are several well-known ways to associate a Lie ring to it (see [11, 13, 21]). For the reader's convenience we will briefly describe the construction that we are using in the present paper.

A series of subgroups for G

$$G = G_1 \geq G_2 \geq \cdots \tag{$*$}$$

is called an N-series if it satisfies $[G_i, G_j] \leq G_{i+j}$ for all i, j. Obviously any N-series is central, i.e. $G_i/G_{i+1} \leq Z(G/G_{i+1})$ for any i. Given an N-series $(*)$, let $L^*(G)$ be the direct sum of the abelian groups $L_i^* = G_i/G_{i+1}$, written additively. Commutation in G induces a binary operation $[\ ,\ ]$ in $L^*(G)$. For homogeneous elements $xG_{i+1} \in L_i^*$, $yG_{j+1} \in L_j^*$ the operation is defined by

$$[xG_{i+1}, yG_{j+1}] = [x, y]G_{i+j+1} \in L_{i+j}^*$$

and extended to arbitrary elements of $L^*(G)$ by linearity. It is easy to check that the operation is well-defined and that $L^*(G)$ with the operations $+$ and $[\ ,\ ]$ is a Lie ring.

An N-series $(*)$ is called an N_p-series if $G_i^p \leq G_{pi}$ for all i. An important example of an N_p-series is the case where the series $(*)$ is the p-dimension central series, also known under the name Zassenhaus–Jennings–Lazard series (see [11, p. 250] for details). Observe that if all quotients G_i/G_{i+1} of an N-series $(*)$ have prime exponent p, then $L^*(G)$ can be viewed as a Lie algebra over the field with p elements.

Any automorphism of G in the natural way induces an automorphism of $L^*(G)$. If G is profinite and α is a coprime automorphism of G, then the subring (subalgebra) of fixed points of α in $L^*(G)$ is isomorphic to the Lie ring (algebra) associated to the group $C_G(\alpha)$ via the series formed by intersections of $C_G(\alpha)$ with the series $(*)$ (see [21] for further details).

In the case where the series $(*)$ is just the lower central series of G we write $L(G)$ for the associated Lie ring. In the case where the series $(*)$ is the p-dimension central series of G we write $L_p(G)$ for the subalgebra generated by the first homogeneous component G_1/G_2 in the associated Lie algebra over the field with p elements.

Let H be a subgroup of G. For a series $(*)$ we write

$$L^*(G,H) = \bigoplus_{j\geq 1} \frac{(G_j \cap H)G_{j+1}}{G_{j+1}}$$

and, if the series $(*)$ is the p-dimension central series, we write

$$L_p(G,H) = L_p(G) \cap L^*(G,H).$$

In particular, if a group A acts coprimely on G, then we have

$$L^*(G, C_G(A)) = C_{L^*(G)}(A) \quad \text{and} \quad L_p(G, C_G(A)) = C_{L_p(G)}(A).$$

We will also require the following lemma that essentially is due to Wilson and Zelmanov (cf. [24, Lemma in Section 3]).

Lemma 3.1. *Let G be a profinite group and $g \in G$ an element such that for any $x \in G$ there exists a positive n with the property that $[x,_n g] = 1$. Let $L^*(G)$ be the Lie algebra associated with G using an N_p-series $(*)$ for some prime p. Then the image of g in $L^*(G)$ is ad-nilpotent.*

We close this section by quoting the following result whose proof can be found in [14].

Theorem 3.2. *Let P be a d-generated finite p-group and suppose that the Lie algebra $L_p(G)$ is nilpotent of class c. Then P has a powerful characteristic subgroup of $\{p,c,d\}$-bounded index.*

Recall that powerful p-groups were introduced by Lubotzky and Mann [17, 18]. A finite p-group P is said to be *powerful* if and only if $[P,P] \leq P^p$ for $p \neq 2$ (or $[P,P] \leq P^4$ for $p = 2$), where P^i denotes the subgroup of P generated by all ith powers. Powerful p-groups have some nice properties. In particular, if P is a powerful p-group, then the subgroups $\gamma_i(P)$, $P^{(i)}$ and P^i are also powerful. Moreover, for given positive integers $n_1,\dots,n_s$, it follows, by repeated applications of [17, Propositions 1.6 and 4.1.6], that $[P^{n_1},\dots,P^{n_s}] \leq \gamma_s(P)^{n_1\cdots n_s}$. Furthermore, if a powerful p-group P is generated by d elements, then any subgroup of P can be generated by at most d elements and P is a product of d cyclic subgroups, that is, P has cyclic subgroups $C_1,\dots,C_d$ with the property that for every element $x \in P$ there exist $x_1 \in C_1,\dots,x_d \in C_d$ such that $x = x_1\cdots x_d$.

4 On a technical tool: A-special subgroups

The main step in order to deal with the proof of part (2) of Theorems 1.7 and 1.8 is to consider the case where G is a p-group, which can be treated via Lie methods. Then the general case will follows from a reduction to the case of p-groups. We will deal with the case of p-groups by combining Lie methods with the use of the technical concept of A-special subgroups of a group G. This concept was introduced in [1]. In what follows, we are going to provide the reader with the most relevant information on that topic. Let us start by recalling the definition.

Definition 4.1. Suppose that A is an elementary abelian q-group acting on a finite q'-group G. Let $A_1, \dots, A_s$ be the maximal subgroups of A and H a subgroup of G.

(1) We say that H is an A-special subgroup of G of degree 0 if and only if $H = C_G(A_i)$ for suitable $i \leq s$.

(2) Suppose that $k \geq 1$ and the A-special subgroups of G of degree $k-1$ are defined. Then H is said to be an A-special subgroup of G of degree k if and only if there exist A-special subgroups J_1 and J_2 of G of degree $k-1$ such that $H = [J_1, J_2] \cap C_G(A_i)$ for suitable $i \leq s$.

Note that the A-special subgroups of G of any degree are A-invariant. If A has order q^r, then for a given integer k the number of A-special subgroups of G of degree k is bounded in terms of q, r and k. Moreover, the A-special subgroups have nice properties that are crucial for our purpose. We state here some of those properties whose proofs can be found in [1, Proposition 3.2, Theorem 4.1 and Corollary 4.4].

Proposition 4.2. *Let A be an elementary abelian q-group of order q^r with $r \geq 2$ acting on a finite q'-group G and let $A_1, \dots, A_s$ be the maximal subgroups of A. Let $k \geq 0$ be an integer.*

(1) *If $k \geq 1$, then every A-special subgroup of G of degree k is contained in some A-special subgroup of G of degree $k-1$.*

(2) *If $2^k \leq r-1$ and H is an A-special subgroup of G of degree k, then H is contained in the kth derived group of $C_G(B)$ for some subgroup $B \leq A$ such that $|A/B| \leq q^{2^k}$.*

(3) *Let R_k be the subgroup generated by all A-special subgroups of G of degree k. Then $R_k = G^{(k)}$.*

(4) *Let H be an A-special subgroup of G. If N is an A-invariant normal subgroup of G, then the image of H in G/N is an A-special subgroup of G/N.*

Theorem 4.3. *Let A be an elementary abelian q-group of order q^r with $r \geq 2$ acting on a finite q'-group G. Let p be a prime and P an A-invariant Sylow p-subgroup of $G^{(d)}$, for some integer $d \geq 0$. Let $P_1, \dots, P_t$ be the subgroups of the form $P \cap H$, where H ranges through A-special subgroups of G of degree d. Then $P = P_1 \cdots P_t$.*

In order to deal with the statements (1) of Theorems 1.7 and 1.8, we will need to use the concept of γ-A-special subgroups of a group G, whose definition was also given in [1]. They are analogues to the A-special subgroups defined above, and their definition is more suitable to treat situations involving the terms of the lower central series of a group. The definition is as follows.

Definition 4.4. Suppose that A is an elementary abelian q-group acting on a finite q'-group G. Let $A_1, \dots, A_s$ be the maximal subgroups of A and H a subgroup of G.

(1) We say that H is a γ-A-special subgroup of G of degree 1 if and only if $H = C_G(A_i)$ for suitable $i \leq s$.

(2) Suppose that $k \geq 2$ and the γ-A-special subgroups of G of degree $k-1$ are defined. Then H is said to be a γ-A-special subgroup of G of degree k if and only if there exist a γ-A-special subgroup J of G of degree $k-1$ such that $H = [J, C_G(A_i)] \cap C_G(A_j)$ for suitable $i, j \leq s$.

The next results are similar to Proposition 4.2 and Theorem 4.3, respectively, and their proofs can be found in [1].

Proposition 4.5. *Let A be an elementary abelian q-group of order q^r with $r \geq 2$ acting on a finite q'-group G and $A_1, \dots, A_s$ the maximal subgroups of A. Let $k \geq 1$ be an integer.*

(1) *If $k \geq 2$, then every γ-A-special subgroup of G of degree k is contained in some γ-A-special subgroup of G of degree $k-1$.*

(2) *If $k \leq r-1$ and H is a γ-A-special subgroup of G of degree k, then one has $H \leq \gamma_k(C_G(B))$ for some subgroup $B \leq A$ such that $|A/B| \leq q^k$.*

(3) *Let R_k be the subgroup generated by all γ-A-special subgroups of G of degree k. Then $R_k = \gamma_k(G)$.*

(4) *Let H be a γ-A-special subgroup of G. If N is an A-invariant normal subgroup of G, then the image of H in G/N is a γ-A-special subgroup of G/N.*

Theorem 4.6. *Let A be an elementary abelian q-group of order q^r with $r \geq 2$ acting on a finite q'-group G. Let p be a prime and let P be an A-invariant*

Sylow p-subgroup of $\gamma_{r-1}(G)$. Let $P_1, \dots, P_t$ be all the subgroups of the form $P \cap H$ where H ranges through γ-A-special subgroups of G of degree $r-1$. Then $P = P_1 \cdots P_t$.

Since Theorems 1.9 and 1.10 are (non-quantitative) profinite versions of Theorems 1.7 and 1.8, respectively, it is natural to expect that the main step of their proofs is to consider the case of pro-p groups, which will be treated by using Lie methods. For our purpose we also need to extend the concepts of A-special and γ-A-special subgroups to profinite groups.

Let H and K be subgroups of a profinite group G. Recall that we denote by $[H, K]$ the closed subgroup of G generated by all commutators of the form $[h, k]$, with $h \in H$ and $k \in K$. In the same spirit of what was done in [1] for the finite case, we can define the concept of A-special subgroups for a profinite group as follows.

Definition 4.7. Let A be an elementary abelian q-group acting coprimely on a profinite group G. Let $A_1, \dots, A_s$ be the maximal subgroups of A and H a subgroup of G.

(1) We say that H is an A-special subgroup of G of degree 0 if and only if $H = C_G(A_i)$ for suitable $i \le s$.

(2) Suppose that $k \ge 1$ and the A-special subgroups of G of degree $k-1$ are defined. Then H is said to be an A-special subgroup of G of degree k if and only if there exist A-special subgroups J_1 and J_2 of G of degree $k-1$ such that $H = [J_1, J_2] \cap C_G(A_i)$ for suitable $i \le s$.

Note that combining the definition above with a standard inverse limit argument and the results obtained in [1], it is easy to show that A-special subgroups of a profinite group satisfy properties analogous to those listed in Proposition 4.2. Moreover, a profinite version of Theorem 4.3 holds.

In order to deal with part (1) of Theorems 1.9 and 1.10, we need to introduce γ-A-special subgroups of a profinite group. This is done by slightly modifying Definition 4.7 in a similar way to what is stated in Definition 4.4 for the finite case. As a consequence we obtain that analogous profinite versions of Proposition 4.5 and Theorem 4.6 can be established. We omit further details.

5 Proof of Theorem 1.7

Our goal here is to prove Theorem 1.7. First of all we need to establish the following result about associated Lie rings.

Proposition 5.1. *Let G be a finite group satisfying the hypothesis of part* (2) *in Theorem* 1.7. *Suppose that there exists an A-invariant p-subgroup H of $G^{(d)}$, with p a prime divisor of the order of $G^{(d)}$, such that $H = H_1 \cdots H_t$, where each subgroup H_i is contained in some A-special subgroup of G of degree d and H is generated by a $\{q, r, t\}$-bounded number of elements. Then:*

(1) *$L_p(H)$ is nilpotent of $\{n, p, q, r, t\}$-bounded class.*

(2) *There exist positive integers e and c, depending only n, q, r and t, such that $e\gamma_c(L(H)) = 0$.*

We now deal with the proof of the second statement of Proposition 5.1.

Proof. Let $L = L(H)$ be the Lie ring associated with the p-subgroup H of $G^{(d)}$. Denote by $V_1, \dots, V_t$ the images of $H_1, \dots, H_t$ in $H/\gamma_2(H)$. It follows that the Lie ring L is generated by $V_1, \dots, V_t$.

Since H is A-invariant, the group A acts on L in the natural way. Let $A_1, \dots, A_s$ be the distinct maximal subgroups of A. Let W be an additive subgroup of L. We say that W is a *special subspace* of weight 1 if and only if $W = V_j$ for some $j \le t$ and say that W is a special subspace of weight $\alpha \ge 2$ if $W = [W_1, W_2] \cap C_L(A_l)$, where W_1 and W_2 are some special subspaces of L of weight α_1 and α_2 such that $\alpha_1 + \alpha_2 = \alpha$ and A_l is some maximal subgroup of A for a suitable $l \le s$.

We claim that every special subspace W of L corresponds to a subgroup of an A-special subgroup of G of degree d. We argue by induction on the weight α of W. If $\alpha = 1$, then $W = V_j$ and so W corresponds to H_j for some $j \le t$. Assume that $\alpha \ge 2$ and write $W = [W_1, W_2] \cap C_L(A_l)$. By induction we know that W_1 and W_2 correspond respectively to some J_1 and J_2 which are subgroups of some A-special subgroups of G degree d.

Note also that $[W_1, W_2]$ is contained in the image of $[J_1, J_2]$. This implies that the special subspace W corresponds to a subgroup of $[J_1, J_2] \cap C_G(A_l)$ which, by Proposition 4.2 (1), is contained in some A-special subgroup of G of degree d, as claimed. Moreover, it follows from Proposition 4.2 (2) that every element of W corresponds to some element of $C_G(a)^{(d)}$ for a suitable $a \in A^{\#}$. Therefore, since all elements of $C_G(a)^{(d)}$ are n-Engel in G, we have:

Claim 5.1. *Every element of W is ad-nilpotent of index at most n.*

From the previous argument we deduce that $L = \langle V_1, \dots, V_t \rangle$ is generated by ad-nilpotent elements of index at most n, but we do not know whether every Lie commutator in these generators is again in some special subspace of L and hence it is ad-nilpotent of bounded index. In order to overcome this problem, we take a qth primitive root of unity ω and put $\overline{L} = L \otimes \mathbb{Z}[\omega]$. We regard $\overline{L}$ as a Lie ring and remark that there is a natural embedding of the ring L into the ring $\overline{L}$. In what follows we write $\overline{V}$ to denote $V \otimes \mathbb{Z}[\omega]$, for some subspace V of L.

Let W be a special subspace of L. We claim the following:

Claim 5.2. *There exists an $\{n, q\}$-bounded number u such that every element w of $\overline{W}$ is ad-nilpotent of index at most u.*

Indeed, choose $w \in \overline{W}$ and write

$$w = l_0 + \omega l_1 + \cdots + \omega^{q-2} l_{q-2},$$

for suitable elements $l_0, \dots, l_{q-2}$ of W which in particular correspond to some elements $x_0, \dots, x_{q-2}$ of $C_G(a)^{(d)}$, for a suitable $a \in A^{\#}$. Denote by

$$K = \langle l_0, \omega l_1, \dots, \omega^{q-2} l_{q-2} \rangle$$

the subring of $\overline{L}$ generated by $l_0, \omega l_1, \dots, \omega^{q-2} l_{q-2}$; put $H_0 = \langle x_0, \dots, x_{q-2} \rangle$. We will show that K is nilpotent of $\{n, q\}$-bounded class. Note that $L(H, H_0)$ satisfies the linearized n-Engel identity and, so, the same identity is also satisfied in $\overline{L(H, H_0)}$ which contains K. Observe that a commutator in the elements $l_0, \omega l_1, \dots, \omega^{q-2} l_{q-2}$ is of the form $\omega^{\alpha} v$ for some $v \in W$, and so, by Claim 5.1, it is ad-nilpotent of index at most n. Hence, by Theorem 2.8, K is nilpotent of $\{n, q\}$-bounded class. Now Lemma 2.7 ensures that there exists an integer $u > 0$, depending only on n and q, such that $[\overline{L},_u K] = 0$. Since $w \in K$, we conclude that w is ad-nilpotent in $\overline{L}$ with $\{n, q\}$-bounded index, as claimed in Claim 5.2.

The group A acts on $\overline{L}$ in the natural way. An element $x \in \overline{L}$ will be called a common "eigenvector" for A if for any $a \in A^{\#}$ there exists a number λ such that $x^a = \omega^{\lambda} x$. Since $(|A|, |G|) = 1$ and H can be generated by a $\{q, r, t\}$-bounded number of elements, we can choose elements $v_1, \dots, v_\tau$ in $\overline{V_1} \cup \cdots \cup \overline{V_t}$, that generated the Lie ring $\overline{L}$, where τ is a $\{q, r, t\}$-bounded number, and each v_i is a common eigenvector for A (see for example [13, Lemma 4.1.1]).

Let v be any Lie commutator in $v_1, \dots, v_\tau$. We wish to show that v belongs to some $\overline{W}$, where W is a special subspace of L. We argue by induction on the weight of v. If v has weight 1, there is nothing to prove. Assume that v has weight at least 2. Write $v = [w_1, w_2]$ for some $w_1 \in \overline{W_1}$ and $w_2 \in \overline{W_2}$, where W_1 and W_2 are two special subspace of L of smaller weights. It is clear that v belongs to $[\overline{W_1}, \overline{W_2}]$. Note that any commutator in common eigenvectors is again a common eigenvector for A. Therefore v is a common eigenvector and it follows that there exists some maximal subgroup A_l of A such that $v \in C_{\overline{L}}(A_l)$. Thus $v \in [\overline{W_1}, \overline{W_2}] \cap C_{\overline{L}}(A_l)$. Hence v lies in $\overline{W}$, where W is the special subspace of L of the form $[W_1, W_2] \cap C_L(A_l)$ and so by Claim 5.2, v is ad-nilpotent of index at most u. This proves the following:

Claim 5.3. *Any commutator in $v_1, \dots, v_\tau$ is ad-nilpotent of index at most u.*

Note that for any $a \in A^{\#}$, the centralizer $C_L(a) = L(H, C_H(a))$ satisfies the linearized version of the identity

$$[y,_n \delta_{2^d}(y_1, \ldots, y_{2^d})] \equiv 0,$$

where $\delta_i(y_1, \ldots, y_{2^i})$ is given recursively by

$$\begin{aligned} \delta_0(y_1) &= y_1, \\ \delta_i(y_1, \ldots, y_{2^i}) &= [\delta_{i-1}(y_1, \ldots, y_{2^{i-1}}), \delta_{i-1}(y_{2^{i-1}+1}, \ldots, y_{2^i})] \end{aligned}$$

for any $i \geq 1$. The same identity also holds in $C_{\overline{L}}(a) = \overline{C_L(a)}$. Thus, by Theorem 2.6, there exist positive integers e and c depending only on n, q, r and t such that $e\gamma_c(\overline{L}) = 0$. Since L embeds into $\overline{L}$, we also have $e\gamma_c(L) = 0$, as desired. □

Note that the proofs of items (1) and (2) of the previous proposition are very similar. As for the proof of item (1), we only observe that it can be obtained, with obvious changes, simply by replacing every appeal to Theorem 2.6 in the proof of (2) by an appeal to Corollary 2.5.

We are now ready to embark on the proof of part (2) of Theorem 1.7.

Proof. By Proposition 4.2 (3) we know that $G^{(d)}$ is generated by A-special subgroups of G of degree d and Proposition 4.2 (2) tells us that any A-special subgroup of G of degree d is contained in $C_G(B)^{(d)}$ for some suitable nontrivial subgroup $B \leq A$ such that $|A/B| \leq q^{r-1}$. Thus each A-special subgroup of G of degree d is contained in $C_G(a)^{(d)}$ for some suitable $a \in A^{\#}$. This implies that $G^{(d)}$ is generated by n-Engel elements. Hence by Baer's Theorem [10, Theorem III 6.14] we get that $G^{(d)}$ is nilpotent. Then $G^{(d)}$ is a direct product of its Sylow subgroups.

Let $\pi(G^{(d)})$ be the set of prime divisors of $|G^{(d)}|$. Choose now $p \in \pi(G^{(d)})$ and let P be the Sylow p-subgroup of $G^{(d)}$. By Theorem 4.3, $P = P_1 \cdots P_t$, where each P_i is of the form $P \cap H$ for some A-special subgroup H of G of degree d. Combining this with Proposition 4.2 (2), we see that each P_i is contained in $C_G(a)^{(d)}$, for some $a \in A^{\#}$. Furthermore, t is a $\{q, r\}$-bounded number.

Choose arbitrary elements $x, y \in P$. Let us write $x = x_1 \cdots x_t$, $y = y_1 \cdots y_t$, where x_i and y_i belong to P_i. In what follows we will show that $\langle x, y\rangle$ is nilpotent of $\{n, q, r\}$-bounded class. Let Y be the subgroup generated by the orbits x_i^A and y_i^A for $i = 1, \ldots, t$. Note that Y is generated by a $\{q, r\}$-bounded number of elements. Since the subgroup $\langle x, y\rangle$ is contained in Y, it is enough to show that Y is nilpotent of $\{n, q, r\}$-bounded class.

Set $Y_i = P_i \cap Y$, $i = 1, \ldots, t$, and note that every Y_i is a subgroup of $C_G(a)^{(d)}$ for a suitable $a \in A^{\#}$. Since we have $Y = \langle x_i^A, y_i^A : i = 1, \ldots, t\rangle$ and every P_i

is an A-invariant subgroup, it follows that $Y = \langle Y_1, \dots, Y_t \rangle$. By [1, Lemma 2.1] we see that $Y = Y_1 \cdots Y_t$. Moreover, note that Y is generated by a $\{q, r\}$-bounded number of elements which are n-Engel.

Now by Proposition 5.1 (2) there exist integers e and c, that depend only on n, q and r, such that $e\gamma_c(L(Y)) = 0$. If p is not a divisor of e, then we have $\gamma_c(L(Y)) = 0$ and so Y is nilpotent of class at most $c - 1$. In that case Y is nilpotent of $\{n, q, r\}$-bounded class and, in particular, the same holds for $\langle x, y \rangle$. Assume now that p is a divisor of e. By Proposition 5.1 (1) we know that $L_p(Y)$ is nilpotent of $\{n, q, r\}$-bounded class. Now Theorem 3.2 tells us that Y has a powerful characteristic subgroup K of $\{n, q, r\}$-bounded index. It follows from [19, Theorem 6.1.8 (ii), p. 164] that K has an $\{n, q, r\}$-bounded rank.

Put $R = K^e$ and assume that $R \neq 1$. Note that, if $p \neq 2$, then we have

$$[R, R] \leq [K, K]^{e^2} \leq K^{pe^2} = R^{pe},$$

and if $p = 2$, then we have

$$[R, R] \leq R^{4e}.$$

Since $e\gamma_c(L(R)) = 0$, we get that

$$\gamma_c(R)^e \leq \gamma_{c+1}(R).$$

Taking into account that R is powerful, we obtain, if $p \neq 2$, that

$$\gamma_c(R)^e \leq \gamma_{c+1}(R) = [R',_{c-1} R] \leq [R^{pe},_{c-1} R] \leq \gamma_c(R)^{pe}$$

and, if $p = 2$, that

$$\gamma_c(R)^e \leq \gamma_c(R)^{4e}.$$

Hence we have $\gamma_c(R)^e = 1$. Since $\gamma_c(R)$ is also powerful and generated by an $\{n, q, r\}$-bounded number of elements, we infer that $\gamma_c(R)$ is of $\{n, q, r\}$-bounded order, since it is a product of an $\{n, q, r\}$-bounded number of cyclic subgroups. It follows that R has an $\{n, q, r\}$-bounded derived length. Recall that $R = K^e$ and K is a powerful p-group. Thus, K has $\{n, q, r\}$-bounded derived length and this implies that the derived length of Y is $\{n, q, r\}$-bounded, as well. Now [23, Lemma 4.1] tells us that Y has $\{n, q, r\}$-bounded nilpotency class, and the same holds for $\langle x, y \rangle$, as desired.

From the argument above we deduce that each Sylow p-subgroup of $G^{(d)}$ is k-Engel, for some $\{n, q, r\}$-bounded number k. The result follows. □

We conclude this section observing that the proof of part (1) of Theorem 1.7 has a very similar structure to that of part (2). We will omit details and describe only main steps that are somewhat different from those of part (2). More precisely, the first step consists in proving the following analogue of Proposition 5.1.

Proposition 5.2. *Let G be a finite group satisfying the hypothesis of part* (1) *in Theorem* 1.7. *Suppose that there exists an A-invariant p-subgroup H of $\gamma_{r-1}(G)$, with p a prime divisor of the order of $\gamma_{r-1}(G)$, such that $H = H_1 \cdots H_t$, where each subgroup H_i is contained in some γ-A-special subgroup of G of degree $r-1$ and H is generated by a $\{q, r, t\}$-bounded number of elements. Then:*

(1) *$L_p(H)$ is nilpotent of $\{n, p, q, r, t\}$-bounded class.*

(2) *There exist positive integers e and c, depending only n, q, r and t, such that $e\gamma_c(L(H)) = 0$.*

Next, one can establish part (1) of Theorem 1.7 by replacing every appeal to Theorem 4.3 and Proposition 4.2 in the proof of item (2) by an appeal to Theorem 4.6 and Proposition 4.5, respectively.

6 Proof of Theorem 1.8

In this section we are concerned with the proof of Theorem 1.8. In parallel to what we did in the previous section, we will focus our attention on the proof of statement (2) of Theorem 1.8.

First of all, we will require the following analogue of Proposition 5.1.

Proposition 6.1. *Let G be a finite group satisfying the hypothesis of part* (2) *in Theorem* 1.8. *Suppose that there exists an A-invariant p-subgroup H of $G^{(d)}$, with p a prime divisor of the order of $G^{(d)}$, such that $H = H_1 \cdots H_t$, where each subgroup H_i is contained in some A-special subgroup of G of degree d and H is generated by a $\{q, r, t\}$-bounded number of elements. Then:*

(1) *$L_p(H)$ is nilpotent of $\{n, p, q, r, t\}$-bounded class.*

(2) *There exist positive integers e and c, depending only n, q, r and t, such that $e\gamma_c(L(H)) = 0$.*

In what follows, we outline the proof of item (2) of Proposition 6.1. The proof of part (1) can be obtained with a similar argument.

Proof. By hypothesis we know that each subgroup H_i for $i = 1, \dots, t$, is contained in some A-special subgroup of G of degree d. Hence, Proposition 4.2 (2) implies that each H_i is contained in $C_G(B)^{(d)}$, for some subgroup B of A such that $|A/B| \leq q^{2^d}$. Let $A_1, \dots, A_s$ be the maximal subgroups of A. For any A_j the intersection $B \cap A_j$ is not trivial. Thus, there exists $a \in A^{\#}$ such that the centralizer $C_G(A_j)$ is contained in $C_G(a)$ and H_i is contained in $C_G(a)^{(d)}$. Since all elements of $C_G(a)^{(d)}$ are n-Engel in $C_G(a)$, we deduce:

Claim 6.1. *Each element $h \in H_i$ is n-Engel in $C_G(A_j)$, for any $j \leq s$.*

We now consider $L = L(H)$ the Lie ring associated to the p-subgroup H of $G^{(d)}$. In the same spirit of what we did in the proof of Proposition 5.1 we define special subspaces of L for any weight and observe that every element of a special subspace W of L corresponds to an element of a subgroup of an A-special subgroup of G of degree d. Since $2^d \leq r - 2$, it follows that $L = \sum_{j \leq s} C_L(A_j)$. Now taking into account that every special subspace W of L is contained in some $L(H, H_i)$ and that, by Claim 6.1, we have $[C_L(A_j),_n l] = 0$ for any $l \in L(H, H_i)$, we deduce that any element of a special subspace W is ad-nilpotent of index at most n in L.

The rest of the proof consists in mimicking the argument used in the proof of Proposition 5.1, with only obvious changes, so we omit the further details. □

Now we are ready to deal with the proof of part (2) of Theorem 1.8.

Proof. By the well-known Zorn's Theorem [10, Theorem III 6.3] each $C_G(a)^{(d)}$ is nilpotent. Furthermore, [2, Theorem 31] implies that $G^{(d)}$ is nilpotent. It follows that $G^{(d)}$ is a direct product of its Sylow subgroups.

Choose $p \in \pi(G^{(d)})$ and let P be a Sylow p-subgroup of $G^{(d)}$. By Theorem 4.3 we know that $P = P_1 \cdots P_t$, where t is a $\{q, r\}$-bounded number and each subgroup P_i is of the form $P \cap H$, where H is some A-special subgroup of G of degree d. Moreover, Proposition 4.2 (2) tells us that each $P_i \leq C_G(B)^{(d)}$, for some subgroup B of A such that $|A/B| \leq q^{2^d}$.

Let $A_1, \ldots, A_s$ be the maximal subgroups of A. For any A_j the intersection $B \cap A_j$ is not trivial. Thus, there exists $a \in A^{\#}$ such that the centralizer $C_G(A_j)$ is contained in $C_G(a)$ and P_i is contained in $C_G(a)^{(d)}$. Thus:

Claim 6.2. *Each element $l \in P_i$ is n-Engel in $C_G(A_j)$, for any $j \leq s$.*

Choose arbitrary elements $x, y \in P$. We will show that the subgroup $\langle x, y\rangle$ is nilpotent of $\{n, q, r\}$-bounded class. Following an argument similar to that used in the proof of Theorem 1.7 (2), we write $x = x_1 \cdots x_t$ and $y = y_1 \cdots y_t$, where each x_i and y_i belongs to P_i, for $i = 1, \ldots, t$, and want to show that the subgroup $Y = \langle x_i^A, y_i^A : i = 1, \ldots, t\rangle$ is nilpotent of $\{n, q, r\}$-bound class. Appealing to Proposition 6.1 and Theorem 3.2 we find that Y has a powerful characteristic subgroup N of $\{n, q, r\}$-bounded index and, by [19, Theorem 6.1.8 (ii), p. 164], of $\{n, q, r\}$-bounded rank as well.

It follows that each $C_N(a)^{(d)}$ is an n-Engel subgroup and can be generated by an $\{n, q, r\}$-bounded number of elements. Zelmanov noted in [25] that the nilpotency class of a finite n-Engel group is bounded in terms of n and the number of generators of that group. We conclude that each $C_N(a)^{(d)}$ is nilpotent of $\{n, q, r\}$-bounded class. Now [2, Theorem 31] tells us that $N^{(d)}$ is nilpotent of $\{n, q, r\}$-bounded class, and this implies that Y has $\{n, q, r\}$-bounded derived length l, say.

By [23, Lemma 4.1] it is enough to see that there exists an $\{n, q, r\}$-bounded number u such that each generator of Y is an u-Engel element in Y. Indeed, let $M = Y^{(l-1)}$ be the last nontrivial term of derived series of Y. By induction on l we see that Y/M is of $\{n, q, r\}$-bounded nilpotency class, say c_1. For any generator x of Y we have

$$[Y, \underbrace{x, \ldots, x}_{c_1}] \subseteq M.$$

Since M is abelian and A-invariant, we can write

$$M = \sum_{j \leq s} C_M(A_j).$$

By Claim 6.2 we have

$$[M, \underbrace{x, \ldots, x}_{n}] = 1.$$

Thus,

$$[Y, \underbrace{x, \ldots, x}_{c_1+n}] = 1.$$

This implies that $\langle x, y\rangle$ is nilpotent of $\{n, q, r\}$-bound class, as desired.

By the argument above we obtain that each Sylow p-subgroup P of $G^{(d)}$ is k-Engel for some $\{n, q, r\}$-bounded number k. The result follows. □

We finish by noting that the proof of item (1) of Theorem 1.8 can be obtained by replacing every appeal to [2, Theorem 31], Theorem 4.3, Proposition 4.2 and Proposition 6.1 in the proof of (2) by an appeal to [2, Theorem 41], Theorem 4.6, Proposition 4.5 and the following analogue of Proposition 6.1, respectively.

Proposition 6.2. *Let G be a finite group satisfying the hypothesis of part* (1) *in Theorem* 1.8. *Suppose that there exists an A-invariant p-subgroup H of $\gamma_{r-2}(G)$, with p a prime divisor of the order of $\gamma_{r-2}(G)$, such that $H = H_1 \cdots H_t$, where each subgroup H_i is contained in some γ-A-special subgroup of G of degree $r-2$ and H is generated by a $\{q, r, t\}$-bounded number of elements. Then:*

(1) *$L_p(H)$ is nilpotent of $\{n, p, q, r, t\}$-bounded class.*

(2) *There exist positive integers e and c, depending only n, q, r and t, such that $e\gamma_c(L(H)) = 0$.*

7 Results on profinite groups

In this section we deal with the proofs of Theorems 1.9 and 1.10 which are profinite non-quantitative analogues of Theorems 1.7 and 1.8, respectively.

Let $w = w(x_1, x_2, \dots, x_k)$ be a group-word. Let H be a subgroup of a group G and $g_1, g_2, \dots, g_k \in G$. We say that the law $w \equiv 1$ is satisfied on the cosets $g_1H, g_2H, \dots, g_kH$ if $w(g_1h_1, g_2h_2, \dots, g_kh_k) = 1$ for all $h_1, h_2, \dots, h_k \in H$. Let us start with a lemma.

Lemma 7.1. *Assume that a finite group A acts coprimely on a profinite group G. Then for each prime p the following holds:*

(1) *If, for some integer k, all elements in $\gamma_k(C_G(A))$ are Engel in $C_G(A)$, then $L_p(G)$ satisfies a multilinear Lie polynomial identity.*

(2) *If, for some integer k, all elements in the kth derived group of $C_G(A)$ are Engel in $C_G(A)$, then $L_p(G)$ satisfies a multilinear Lie polynomial identity.*

The proofs of items (1) and (2) of the lemma are similar, so we give a detailed proof of the second statement.

Proof. Let $L = L_p(G)$. In view of Theorem 2.2, it is sufficient to show that $C_L(A)$ satisfies a polynomial identity. We know that $C_L(A)$ is isomorphic to the Lie algebra associated with the central series of $C_G(A)$ obtained by intersecting $C_G(A)$ with the p-dimension central series of G.

Let
$$T = \underbrace{C_G(A) \times \cdots \times C_G(A)}_{2^k+1}.$$
For each integer i we define the set
$$S_i = \{(t, t_1, \dots, t_{2^k}) \in T : [t,_i \delta_k(t_1, \dots, t_{2^k})] = 1\}.$$
Since the sets S_i are closed in T and their union coincides with T, by the Baire category theorem [12, p. 200] at least one of these sets has a non-empty interior. Therefore, we can find an open subgroup H in $C_G(A)$, elements $g, g_1, \dots, g_{2^k}$ in $C_G(A)$ and an integer n such that the identity $[x,_n \delta_k(y_1, \dots, y_{2^k})] \equiv 1$ is satisfied on the cosets $gH, g_1H, \dots, g_{2^k}H$. Thus, the Wilson–Zelmanov result [24, Theorem 1] tells us that $C_L(A)$ satisfies a polynomial identity. □

We will also require the following profinite version of [1, Lemma 2.1]. The proof is straightforward so we do not give details.

Lemma 7.2. *Suppose that a pronilpotent group G is generated by subgroups $G_1, \dots, G_t$ such that $\gamma_i(G) = \langle \gamma_i(G) \cap G_j : 1 \le j \le t \rangle$ for all $i \ge 1$. Then one has $G = G_1 \cdots G_t$.*

As usual, for a profinite group G we denote by $\pi(G)$ the set of prime divisors of the orders of finite continuous homomorphic images of G. We say that G is a π-group if $\pi(G)$ is contained in π and G is a π'-group if $\pi(G) \cap \pi = \emptyset$. If π is a set of primes, we denote by $O_\pi(G)$ the maximal normal π-subgroup of G and by $O_{\pi'}(G)$ the maximal normal π'-subgroup.

We are ready to embark on the proof of Theorem 1.9.

Proof of Theorem 1.9 (2). Let $\mathcal{S}$ be the subset of all A-invariant open normal subgroups of G. For any $N \in \mathcal{S}$, set $Q = G/N$. Observe that, for any $a \in A^\#$, one has $C_Q(a)^{(d)} = (C_G(a))^{(d)}N/N$. By hypothesis each $x \in C_G(a)^{(d)}$ is Engel in G and this implies that all elements of $C_Q(a)^{(d)}$ are Engel in Q. Hence Theorem 1.7 (2) tells us that $Q^{(d)}$ is Engel. By Zorn's Theorem [10, Theorem III 6.3] $Q^{(d)}$ is nilpotent. Since $G^{(d)} \cong \varprojlim_{N\in\mathcal{S}} (G/N)^{(d)}$, we get that $G^{(d)}$ is a pronilpotent group. Thus, $G^{(d)}$ is the Cartesian product of its Sylow subgroups.

Choose $a \in A^\#$. For each positive integer i we set

$$S_i = \{(x, y) \in G^{(d)} \times C_G(a)^{(d)} : [x,_i y] = 1\}.$$

Since the sets S_i are closed in $G^{(d)} \times C_G(a)^{(d)}$ and their union coincides with $G^{(d)} \times C_G(a)^{(d)}$, by the Baire Category Theorem at least one S_i has a non-empty interior. Therefore we can find an integer n, an open subgroup K in $G^{(d)}$, elements $u \in G^{(d)}$ and $v \in C_G(a)^{(d)}$ such that

$$[ul,_n vk] = 1 \quad \text{for any } l \in K \text{ and any } k \in K \cap C_G(a)^{(d)}.$$

Let $[G^{(d)} : K] = m$ and let $\pi_1 = \pi(m)$ be the set of primes dividing m. Denote $O_{\pi_1'}(G^{(d)})$ by T. Since T is isomorphic to the image of K in $G^{(d)}/O_{\pi_1}(G^{(d)})$, it is easy to see that $[T,_n x] = 1$ for all $x \in C_T(a)^{(d)}$. Thus, each element of $C_T(a)^{(d)}$ is n-Engel in T.

The open subgroup K, the set π_1 and the integer n depend only on the choice of $a \in A^\#$, so strictly speaking they should be denoted by K_a, π_a and n_a, respectively. We choose such K_a, π_a and n_a for any element $a \in A^\#$. Set $\pi = \bigcup_{a\in A^\#} \pi_a$, $n = \max\{n_a : a \in A^\#\}$ and $R = O_{\pi'}(G^{(d)})$. The choice of the set π guarantees that for each $a \in A^\#$ every element of $C_R(a)^{(d)}$ is n-Engel in R. Using a routine inverse limit argument, we deduce from Theorem 1.7 (2) that R is n_1-Engel for some suitable integer n_1. By [24, Theorem 5] we have that R is locally nilpotent. Let $p_1, p_2, \dots, p_r$ be the finitely many primes in π and let $S_1, \dots, S_r$ be the corresponding Sylow subgroups of $G^{(d)}$. Then $G^{(d)} = S_1 \times \cdots \times S_r \times R$ and therefore it is enough to show that each subgroup S_i is locally nilpotent.

Let P be such a p-Sylow subgroup of $G^{(d)}$, for some $p \in \pi$. By the profinite version of Theorem 4.3 we have $P = P_1 \cdots P_t$, where any $P_j = P \cap H$ and H is an A-special subgroup of G of degree d. Since $2^d \leq r - 1$, by the profinite version of Theorem 4.2 (2) each subgroup $P_j \subseteq C_G(a)^{(d)}$, for some suitable $a \in A^\#$.

Choose arbitrary elements $x_1, \dots, x_m$ in P. For $i = 1, 2, \dots, m$ we write $x_i = x_{i1} \cdots x_{it}$, where x_{ij} belongs to P_j for $j = 1, 2, \dots, t$. Let Y be the subgroup generated by the orbits x_{ij}^A. Put $Y_j = Y \cap P_j$. Since Y is generated by the orbits x_{ij}^A and every P_j is an A-invariant subgroup, we deduce that Y is generated by $Y_1, \dots, Y_t$ and, by Lemma 7.2, we have $Y = Y_1 \cdots Y_t$.

For our purpose it is sufficient to show that Y is nilpotent. To this end, we denote by $D_j = D_j(Y)$ the terms of the p-dimension central series of Y. Let $L = L_p(Y)$ be the Lie algebra generated by D_1/D_2 associated with the pro-p group Y. Observe that D_2 coincides with $\Phi(Y)$ and denote by $V_1, \dots, V_t$ the images of $Y_1, \dots, Y_t$ in $Y/\Phi(Y)$. It follows that the Lie algebra L is generated by $V_1, \dots, V_t$.

Since Y is A-invariant, the group A acts on L in the natural way. Let $A_1, \dots, A_s$ be the distinct maximal subgroups of A. In the same spirit of what was done in the proof of Proposition 5.1 we can define special subspaces of L of any weight and show that every special subspace W of L corresponds to a subgroup of an A-special subgroup of G of degree d. Moreover, it follows from the profinite version of Proposition 4.2 (2) that every element of W corresponds to some element of $C_G(a)^{(d)}$ for a suitable $a \in A^{\#}$ and so, by Lemma 3.1, it is ad-nilpotent on L, since all elements of $C_G(a)^{(d)}$ are Engel in G.

From the previous argument we deduce that $L = \langle V_1, \dots, V_t \rangle$ is generated by ad-nilpotent elements. As in the proof of Proposition 5.1 we extend the ground field $\mathbb{F}_p$ by a primitive qth root of unity ω. Put $\overline{L} = L \otimes \mathbb{F}_p[\omega]$ and identify, as usual, L with the $\mathbb{F}_p$-subalgebra $L \otimes 1$ of $\overline{L}$. In what follows we write $\overline{X}$ to denote $X \otimes \mathbb{F}_p[\omega]$, for some subspace X of L. Let W be a special subspace of L. We claim:

Claim 7.1. *Every element w of $\overline{W}$ is ad-nilpotent in $\overline{L}$.*

Indeed, choose $w \in \overline{W}$ and write

$$w = l_0 + \omega l_1 + \cdots + \omega^{q-2} l_{q-2},$$

for suitable elements $l_0, \dots, l_{q-2}$ of W that, in particular, correspond to some elements $x_0, \dots, x_{q-2}$ of $C_G(a)^{(d)}$, for a suitable $a \in A^{\#}$. Let denote by

$$K_0 = \langle l_0, \omega l_1, \dots, \omega^{q-2} l_{q-2} \rangle$$

the subalgebra of $\overline{L}$ generated by $l_0, \omega l_1, \dots, \omega^{q-2} l_{q-2}$. Using now an argument analogous to that used in the proof of Proposition 5.1, we first apply Theorem 2.1 to show that K_0 is nilpotent and, later, by appealing to Lemma 2.7, we conclude that w is ad-nilpotent in $\overline{L}$, as claimed in Claim 7.1.

The group A acts on $\overline{L}$ in the natural way and now the ground field is a splitting field for A. Since Y is finitely generated, we can choose finitely many elements $v_1, v_2, \dots$ in $\overline{V_1} \cup \cdots \cup \overline{V_t}$, that generate the Lie algebra $\overline{L}$, and each v_i is a common eigenvector for A. Let v be any Lie commutator in generators $v_1, v_2 \dots$ of $\overline{L}$. Mimicking what we did in the proof of Proposition 5.1 and arguing by induction on the weight of v, we can show that v belongs to some $\overline{W}$, where W is a special subspace of L. Thus, by Claim 7.1, v is ad-nilpotent. This proves:

Claim 7.2. *Any commutator in $v_1, v_2 \dots$ is ad-nilpotent in $\overline{L}$.*

Furthermore, it follows from Lemma 7.1 (2) that L satisfies a multilinear Lie polynomial identity. The multilinear identity is also satisfied in $\overline{L}$ and from Theorem 2.1 we deduce that $\overline{L}$ is nilpotent. Since L embeds into $\overline{L}$, we get that L is nilpotent as well.

According to Lazard [15] the nilpotency of L is equivalent to Y being p-adic analytic (for details see [15, A.1 in Appendice and Sections 3.1 and 3.4 in Chapter III] or [7, 1.(k) and 1.(o) in Interlude A]). By [7, 7.19 Theorem] Y admits a faithful linear representation over the field of p-adic numbers. A result of Gruenberg [8, Theorem 0] says that in a linear group the Hirsch–Plotkin radical coincides with the set of Engel elements. Since $Y = Y_1 \cdots Y_t$ is finitely generated and each Y_j is contained in $C_G(a)^{(d)}$ for some suitable $a \in A^{\#}$, we deduce that Y is nilpotent. The proof is complete. □

The proof of part (1) of Theorem 1.9 is analogous to that of item (2) and can be obtained by replacing every appeal to Theorem 1.7 (2), Theorem 4.3, Proposition 4.2 and Lemma 7.1 (2) in the proof of (2) by an appeal to Theorem 1.7 (1), Theorem 4.6, Proposition 4.5 and Lemma 7.1 (1), respectively. Therefore we omit the further details.

In what follows we give an outline of the proof of Theorem 1.10.

Proof of Theorem 1.10 (2). With an argument similar to that used in the proof of Theorem 1.9 (2) and appealing to Theorem 1.8 (2) it easy to show that $G^{(d)}$ is pronilpotent and so it is the Cartesian product of its Sylow subgroups.

Choose $a \in A^{\#}$. Since $C_G(a)^{(d)}$ is Engel, it follows from [24, Theorem 5] that $C_G(a)^{(d)}$ is locally nilpotent. By [3, Lemma 2.5] there exist a positive integer n, elements $u, v \in C_G(a)^{(d)}$ and an open subgroup $H \leq C_G(a)^{(d)}$ such that the law $[x,_n y] \equiv 1$ is satisfied on the cosets uH, vH. Let $[C_G(a)^{(d)} : H] = m$ and $\pi_1 = \pi(m)$. Denote $O_{\pi_1'}(C_G(a)^{(d)})$ by T. Thus, T satisfies the law $[x,_n y] \equiv 1$, that is, T is n-Engel. By the result of Burns and Medvedev [6] the subgroup T has a nilpotent normal subgroup U such that T/U has finite exponent, say e. Set $\pi_2 = \pi(e)$. The sets π_1 and π_2 depend on the choice of the element $a \in A^{\#}$, so strictly speaking they should be denoted by $\pi_1(a)$ and $\pi_2(a)$. For each such choice let $\pi_a = \pi_1(a) \cup \pi_2(a)$. We repeat this argument for every element $a \in A^{\#}$. Set $\pi = \bigcup_{a \in A^{\#}} \pi_a$ and $R = O_{\pi'}(G^{(d)})$. Since all sets $\pi_1(a)$ and $\pi_2(a)$ are finite, so is π. Let $p_1, \dots, p_r$ be the finitely many primes in π and let $S_1, \dots, S_r$ be the corresponding Sylow subgroups of $G^{(d)}$. Then $G^{(d)} = S_1 \times \cdots \times S_r \times R$.

The choice of the set π guarantees that $C_R(a)^{(d)}$ is nilpotent for every $a \in A^{\#}$. Using the routine inverse limit argument, we deduce from [2, Theorem 31] that $R^{(d)}$ is nilpotent. Thus R is solvable. We claim that R is an Engel group. Indeed, combining the profinite version of Proposition 4.2 (3) with Lemma 7.2, we obtain $R = R_1 \cdots R_t$, where $R_i = R \cap H$ and H is some A-special subgroup of G of degree d. Choose arbitrary elements $x, y \in R$. It suffices to prove that $\langle x, y\rangle$ is nilpotent. Note that we can write $x = x_1 \cdots x_t$ and $y = y_1 \cdots y_t$, where x_i and y_i belong to R_i, $i = 1, \dots, t$. Consider $Y = \langle x_i^A, y_i^A : i \leq t\rangle$ and set K to be the abstract subgroup generated by the elements x_i^A, y_i^A. Since K is a dense subgroup of Y, in order to prove that Y is nilpotent it is enough to prove that K is nilpotent.

By construction and since $2^d \leq r - 2$, there exists an element $a \in A^{\#}$ such that the centralizer $C_G(A_j)$ is contained in $C_G(a)$ and each subgroup R_i is contained in $C_G(a)^{(d)}$. Thus:

Claim 7.3. *Each element $x \in R_i$ is Engel in $C_G(A_j)$, for any $j \leq s$.*

In the same spirit of what was done in the proof of Theorem 1.8 (2), it is possible to show, by induction on the derived length of K, that all generators of K are Engel elements in K. Now by a well-known result of Gruenberg [19, Theorem 12.3.3] we conclude that K is nilpotent and, so, Y is nilpotent, as well. In particular, we deduce that R is Engel, as desired. Thus, by [24, Theorem 5] we have that R is locally nilpotent. Since $G^{(d)} = S_1 \times \cdots \times S_r \times R$, for our purpose it is sufficient to prove that each subgroup S_i is locally nilpotent, for $i = 1, \dots, r$.

Let P be such a p-Sylow subgroup of $G^{(d)}$, for some $p \in \pi$. By the profinite version of Theorem 4.3 we have $P = P_1 \cdots P_t$, where any $P_j = P \cap H$ and H is an A-special subgroup of G with degree d.

Choose arbitrary elements $x_1, \dots, x_m$ in P. Let us write $x_i = x_{i1} \cdots x_{it}$ for $i = 1, \dots, m$, where each x_{ij} belongs to P_j and so to $C_G(a)^{(d)}$ for a suitable $a \in A^{\#}$. Let X be the subgroup generated by the orbits x_{ij}^A and let $L = L_p(X)$. By the assumptions we have $L = \sum_{j \leq s} C_L(A_j)$ and by using the Lie theoretical machinery it is possible to prove that L is nilpotent.

According to Lazard [15] the nilpotency of L is equivalent to X being p-adic analytic. The Lubotzky and Mann theory [17, 18] now tells us that X is of finite rank, that is, all closed subgroups of X are finitely generated. In particular, we conclude that $C_X(a)^{(d)}$ is finitely generated for every $a \in A^{\#}$. From [24, Theorem 5] it follows that $C_X(a)^{(d)}$ is nilpotent, and by the profinite quantitative version of [2, Theorem 31], $X^{(d)}$ is nilpotent. Then X is soluble. Finally, by mimicking what we did above for Y, we can prove that X is nilpotent. This concludes the proof. □

The proofs of part (1) and (2) of Theorem 1.10 are very similar. We conclude noting that the proof of item (1) of Theorem 1.10 can be obtained by replacing every appeal to Theorem 1.8 (2), Theorem 4.3, Proposition 4.2 and Lemma 7.1 (2) in the proof of (2) by an appeal to Theorem 1.8 (1) Theorem 4.6, Proposition 4.5 and Lemma 7.1 (1), respectively. Therefore we will omit further details.

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