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Computation of robust control invariant sets with predefined complexity for uncertain systems

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Abstract

This paper presents an algorithm that computes polytopic robust control-invariant (RCI) sets for rationally parameter-dependent systems with additive disturbances. By means of novel linear matrix inequalities (LMI) feasibility conditions for invariance along with a newly developed method for volume maximization, an iterative algorithm is proposed for the computation of RCI sets with maximized volumes. The obtained RCI sets are symmetric around the origin by construction and have a user-defined level of complexity. Unlike many similar approaches, the proposed algorithm directly computes the RCI sets without requiring control inputs to be in a specific feedback form. In fact, a specific control input is obtained from the LMI problem for each extreme point of the RCI set. The outcomes of the proposed algorithm can be used to construct a piecewise-affine controller based on offline computations.

KEYWORDS

invariant set, linear fractional transformation, linear matrix inequalities, linear systems, semi-definite program

1 | INTRODUCTION

Robust control-invariant (RCI) sets have been an important area of research within the controls community for the past six decades.[1,2] For a controlled dynamical system, these sets are bounded regions of the state-space, wherein the state trajectories can always be restricted by the application of some admissible control inputs.[1] Invariant sets can be also used for controller synthesis (e.g., model predictive control [MPC[3]]) and stability analysis of closed-loop systems.[1] Consequently, a lot of research has been done specifically focusing on the computation of these sets.

The algorithms to compute an RCI set for a linear system with polytopic constraints are well established in the literature.[1,2,4-6] A widely used algorithm to compute an RCI set for a linear system belongs to the class of the so-called geometric approach, in which a recursive method based on one-step backward reach operator is employed until some termination condition is matched. In each backward reach operation, numerical procedures like Minkowski sum, projection and finding minimal set representation are performed on polytopes, which can be computationally very demanding.[1,7] Depending upon the choice of the initial set, the result of such a recursive method is the arbitrarily close outer/inner approximation of the maximal RCI set.[8,9] Although effective, the geometric approach does not guarantee finite time termination of the procedure, and also the obtained set may have a very high representational complexity.[2,4,8-11] For polytopic systems, the computational complexity grows exponentially with each additional vertex and system dimension.[1]

Recently, many algorithms have been proposed for the efficient computation of RCI sets with restricted representational complexity. In these algorithms, the candidate RCI sets are assumed to be symmetric ellipsoids or polytopes, and the computation thereof generally involves solving an optimization problem to find a set with maximum or minimum volume. For example, algorithms are proposed by References 12,13 to compute ellipsoidal RCI sets of maximum volumes for uncertain systems. For linear systems with polytopic constraints, it is well known from the literature that the maximal RCI set is also polytopic.[14] Therefore, using a polytopic candidate RCI set would qualify to be an obvious choice. As a result, References 11,15-18 proposed different approaches for the computation of restricted complexity or low complexity polytopic RCI sets. Algorithms developed by References 15-17 are applicable to polytopic linear systems, whereas those in References 11,18 can be applied to affinely parameter-dependent systems. It is important to mention here that all these algorithms assume the invariance inducing controller to be based on linear state feedback, which is computed by the algorithm while optimizing the volume of the set. It was proved in[19] that, even for simple LTI systems, assigning a linear state feedback controller to an already known polytopic RCI set is not always feasible. In fact, it was shown that a piece-wise affine controller can always be assigned. Thus, if the main goal is to compute large volume RCI sets, in general, a linear state feedback law is not a suitable candidate for inducing invariance. In References 20,21, methods are proposed to compute RCI sets using a more general feedback structure, which is though only applicable to LTI systems.

In this paper, we present a novel algorithm to compute an RCI set with a predefined complexity for an uncertain system with additive disturbances. Similar to References 10,22, we consider rationally parameter-dependent systems, which are more general than other linear system descriptions discussed above in that they can be used to approximately describe many nonlinear engineering systems.[23] We start by proposing a new necessary and sufficient condition for invariance, which is a slight extension of the result in Reference 24 for the considered system. By then using this condition along with the full block S-procedure,[25] a sufficient linear matrix inequality (LMI) feasibility condition for invariance is derived. Also, an approach to directly maximize the volume of the RCI sets is presented, which is less conservative than using an indirect approach used in References 10,11. In contrast to References 10,11,15,17,18,22, we directly compute the RCI sets without imposing any structural restrictions on the control input so that possibly larger RCI sets can be obtained. The algorithm generates RCI sets of monotonically increasing volume at each iteration until convergence. Hence, it can be terminated at any iteration, unlike the geometric approach which needs a termination condition to be satisfied to eventually obtain a maximal RCI set. Further, by applying the procedure in References 24,26,27, we show that solutions obtained from the proposed algorithm can be used to construct a piece-wise affine controller for the simple constrained control of the system. We demonstrate the applicability of the proposed algorithm by using numerical examples, where considerably larger RCI sets were obtained in comparison with other similar algorithms.

2 | PRELIMINARIES

For completeness, we recall some basic definitions and well-established results from,[1,6] which will be useful in the later parts.

Definition 1. A set is a C-set if it is convex, compact and contains the origin in its interior.

Definition 2. A C-set $\mathcal{S}$ is 0-symmetric if

$$x \in \mathcal{S} \Rightarrow -x \in \mathcal{S}. \tag{1}$$

Definition 3. The convex hull of a set $\boldsymbol{S} \subset \mathbb{R}^n$ is the smallest convex set containing $\boldsymbol{S}$. For any finite set $\boldsymbol{S} = \{s^1, s^2, \dots, s^r\}$, where $s^i \in \mathbb{R}^n$, $i = 1, 2, \dots, r$, the convex hull of $\boldsymbol{S}$ is given by

$$\mathcal{S} = \text{conv}(\boldsymbol{S}) \triangleq \left\{ \sum_{i=1}^{r} \alpha_i s^i \, : \, \sum_{i=1}^{r} \alpha_i = 1, \alpha_i \in [0, 1] \right\}. \tag{2}$$

Definition 4. An n-simplex $\mathcal{S} \subset \mathbb{R}^n$ is an n-dimensional polytope, which is the convex hull of $n + 1$ vertices.

Consider a dynamical system

$$x(k+1) = f(x(k), u(k), w(k)), \tag{3}$$

where $x(k) \in \mathcal{X}$, $u(k) \in \mathcal{U}$, $w(k) \in \mathcal{W}$ are the state, input and disturbance (or uncertainty) vectors.

Definition 5. A set $\boldsymbol{\Omega} \subseteq \mathcal{X}$ is an RCI set for the system (3) if

$$x(k) \in \boldsymbol{\Omega} \Rightarrow \exists u(k) \in \mathcal{U} : x(k+1) \in \boldsymbol{\Omega}, \forall w(k) \in \mathcal{W}. \tag{4}$$

Definition 6. A set $\boldsymbol{\Omega}_\infty \subseteq \mathcal{X}$ is said to be the maximal RCI set for the system (3) if it is RCI and contains all other RCI sets.

Definition 7. Given λ, $0 < \lambda \leq 1$, a C-set $\mathcal{S} \subseteq \mathcal{X}$ is a λ-contractive set for the system (3) if

$$x(k) \in \mathcal{S} \Rightarrow \exists u(k) \in \mathcal{U} : x(k+1) \in \lambda\mathcal{S}, \forall w(k) \in \mathcal{W}. \tag{5}$$

For $\lambda = 1$, we say $\mathcal{S}$ is an RCI set.

Lemma 1. *A set $\mathcal{S}$ is λ-contractive for the system (3) if and only if it is controlled-invariant for the modified system*

$$x(k+1) = \frac{f(x(k), u(k), w(k))}{\lambda}. \tag{6}$$

3 | PROBLEM STATEMENT

We consider a discrete-time uncertain system described by

$$x^+ = \mathcal{A}(\Delta)x + \mathcal{B}(\Delta)u + \mathcal{E}(\Delta)w, \tag{7}$$

where $x \in \mathbb{R}^{n_x}$ is the current state vector, x^+ is the successor state vector, and $u \in \mathbb{R}^{n_u}$ and $w \in \mathbb{R}^{n_w}$ are the control input and the (additive) disturbance vectors,respectively. Furthermore, $\mathcal{A}(\Delta)$, $\mathcal{B}(\Delta)$ and $\mathcal{E}(\Delta)$ are the rationally parameter dependent system matrices expressed as an LFT of the form

$$\left[\begin{array}{c|c|c} \mathcal{A}(\Delta) & \mathcal{B}(\Delta) & \mathcal{E}(\Delta) \end{array}\right] = \left[\begin{array}{c|c|c} A & B & E \end{array}\right] + B_p\Delta(I - D_p\Delta)^{-1}\left[\begin{array}{c|c|c} A_d & B_d & E_d \end{array}\right], \tag{8}$$

where A, B, E, A_d, B_d, E_d, B_p, and D_p are known matrices of appropriate dimensions. Δ is an uncertain (and possibly) time-varying parameter matrix that satisfies

$$\Delta(k) \in \boldsymbol{\Delta}, \ \forall k \geq 0, \tag{9}$$

where $\boldsymbol{\Delta}$ is a known compact polytopic set expressed as the convex-hull of finitely many given matrices $\boldsymbol{\Delta}_v = \{\Delta^1, \dots, \Delta^\eta\}$:

$$\boldsymbol{\Delta} = \mathrm{conv}(\boldsymbol{\Delta}_v). \tag{10}$$

We assume that $(I - D_p\Delta)$ is invertible $\forall \Delta \in \boldsymbol{\Delta}$, which guarantees that the LFT representation is well-posed.

The system is subject to the following polytopic state and input constraints, respectively:

$$\begin{aligned} \mathcal{X} &= \{x : Hx \leq \mathbf{1}\}, \\ \mathcal{U} &= \{u : Gu \leq \mathbf{1}\}. \end{aligned} \tag{11}$$

Here $H \in \mathbb{R}^{n_h \times n_x}$ and $G \in \mathbb{R}^{n_g \times n_u}$ are given matrices and $\mathbf{1}$ represents the vector of ones of compatible dimension. We assume $w \in \mathcal{W}$, with $\mathcal{W}$ as a C-set and represented by the convex-hull of a set of finitely many known vertices $\mathcal{W}_v = \{w^1, w^2, \dots, w^\gamma\}$:

$$\mathcal{W} = \mathrm{conv}(\mathcal{W}_v). \tag{12}$$

We allow the sets $\mathcal{X}$, $\mathcal{U}$ and $\mathcal{W}$ to be non-symmetric, unlike References 10,11,17,18 where some or all the sets are assumed to be symmetric. The affine parameter dependence of the system, as considered in for example, References 11,18, is just a special case that corresponds to $D_p = 0$ in (8). Moreover, the algorithm proposed in this paper does not impose any structural restriction on the matrix Δ.

The goal in this paper is to compute a 0-symmetric RCI set with a predefined complexity n_p described as

$$\boldsymbol{C} = \{x \in \mathbb{R}^{n_x} \, : \, -\mathbf{1} \leq PW^{-1}x \leq \mathbf{1}\}, \tag{13}$$

where $P \in \mathbb{R}^{n_p \times n_x}$ and $W \in \mathbb{R}^{n_x \times n_x}$. We tacitly assume that W is invertible, which would be later guaranteed by the LMI conditions for invariance.

From Definition 5, a set $\boldsymbol{C} \subseteq \boldsymbol{\mathcal{X}}$ is RCI if for each x

$$x \in \boldsymbol{C} \;\Rightarrow\; \exists u \in \boldsymbol{\mathcal{U}} \, : \, x^+ \in \boldsymbol{C}, \; \forall (w, \Delta) \in (\boldsymbol{\mathcal{W}}, \boldsymbol{\Delta}). \tag{14}$$

Notice that we do not impose any structure on the control input. The problem of finding an RCI set can now be formulated as follows:

Problem 1. Given a matrix P and the discrete-time system (7) subject to constraints (10), (11), and (12) with given $\boldsymbol{\Delta}_v$, H and G, find a matrix W such that

1. the controlled system in (7) and (8) satisfies (14);
2. the set $\boldsymbol{C}$ in (13) must satisfy $\boldsymbol{C} \subseteq \boldsymbol{\mathcal{X}}$.

The representational complexity of the set $\boldsymbol{C}$ (i.e., n_p) is decided by the choice of the matrix P. For $P = I$, the candidate RCI set $\boldsymbol{C}$ has a complexity similar to the one considered in References 10,17,18. Systematic choices of P with increasing degrees of complexity will be exemplified briefly in the sequel.

Thus, in the next section, we derive tractable feasibility conditions for the solvability of Problem 1. The obtained feasibility conditions will be then utilized to develop an iterative algorithm to compute RCI sets with potentially increased volume at each step.

4 | SUFFICIENT CONDITIONS FOR INVARIANCE

Let us first introduce a state transformation

$$\theta \triangleq W^{-1}x \Leftrightarrow x = W\theta. \tag{15}$$

Using (15), the controlled system (7) can be expressed in the transformed space as

$$\theta^+ = \underbrace{W^{-1}\mathcal{A}(\Delta)W}_{\overline{\mathcal{A}}(W,\Delta)}\theta + \underbrace{W^{-1}\mathcal{B}(\Delta)}_{\overline{\mathcal{B}}(W,\Delta)}u + \underbrace{W^{-1}\mathcal{E}(\Delta)}_{\overline{\mathcal{E}}(W,\Delta)}w. \tag{16}$$

For notational simplicity, we will suppress the arguments of the matrices $\mathcal{A}(\Delta)$, $\mathcal{B}(\Delta)$, $\mathcal{E}(\Delta)$, $\overline{\mathcal{A}}(W, \Delta)$, $\overline{\mathcal{B}}(W, \Delta)$, and $\overline{\mathcal{E}}(W, \Delta)$ in the later parts. Using (15), we can also express the set $\boldsymbol{C}$ as

$$\boldsymbol{C} = \{W\theta \in \mathbb{R}^{n_x} \, : \, \theta \in \boldsymbol{\Theta}\}, \tag{17}$$

where $\boldsymbol{\Theta}$ is a symmetric set defined as follows:

$$\boldsymbol{\Theta} \triangleq \{\theta \in \mathbb{R}^{n_x} \, : \, -\mathbf{1} \leq P\theta \leq \mathbf{1}\}. \tag{18}$$

We have thus introduced a θ-state-space in which the candidate RCI set is a known 0-symmetric set identified as in (18). The RCI set $\boldsymbol{C}$ in the x-state-space will be determined as in (17) based on the choice of W.

According to the Definition 5, for invariance of $\boldsymbol{\Theta}$, we have to show that for each $\theta \in \boldsymbol{\Theta}$, there exists a $u \in \boldsymbol{\mathcal{U}}$ for which $\theta^+ \in \boldsymbol{\Theta}$, $\forall (w, \Delta) \in (\boldsymbol{\mathcal{W}}, \boldsymbol{\Delta})$. Verifying the existence of u for each $\theta \in \boldsymbol{\Theta}$ for invariance is an intractable problem. To obtain a tractable formulation, we first note that the set $\boldsymbol{\Theta}$ can always be expressed as the convex hull of finitely many (known) vertices. The vertices (and thus the number of them) are determined by the choice of the matrix P. With the set of vertices represented as $\boldsymbol{\Theta}_v = \{\theta^1, \ldots, \theta^{2\sigma}\}$, we can write

$$\boldsymbol{\Theta} = \mathrm{conv}(\boldsymbol{\Theta}_v). \tag{19}$$

For now, we ignore the state constraints on θ obtained using (11) and (15), and propose a necessary and sufficient condition for the invariance of the set $\boldsymbol{\Theta}$.

Lemma 2. *Consider following statements:*

i. *$\boldsymbol{\Theta}$ in (18) is an RCI set for the system (16), i.e. for each $\theta \in \boldsymbol{\Theta}$,*

$$\exists u \in \boldsymbol{\mathcal{U}} : \overline{\mathcal{A}}\theta + \overline{\mathcal{B}}u + \overline{\mathcal{E}}w \in \boldsymbol{\Theta}, \forall (w, \Delta) \in (\boldsymbol{\mathcal{W}}, \boldsymbol{\Delta}). \tag{20}$$

ii. *for each $\theta^j \in \boldsymbol{\Theta}_v$,*

$$\exists u^j \in \boldsymbol{\mathcal{U}} : \overline{\mathcal{A}}\theta^j + \overline{\mathcal{B}}u^j + \overline{\mathcal{E}}w \in \boldsymbol{\Theta}, \forall (w, \Delta) \in (\boldsymbol{\mathcal{W}}, \boldsymbol{\Delta}). \tag{21}$$

The above two statements are equivalent.

Proof. It can be clearly seen that ($i \Rightarrow ii$) since $\theta^j \in \boldsymbol{\Theta}$. To prove the converse statement (i.e., $ii \Rightarrow i$), we first assume the existence of u^j as in (21) and aim to construct u for any given $\theta \in \boldsymbol{\Theta}$ such that the invariance condition expressed in (20) is satisfied. As implied by the notation, the idea would be to compute the control input as a convex combination of u^js. This convex combination is identified by first expressing θ in terms of the extreme points θ^j as

$$\theta = \sum_{j=1}^{2\sigma} \alpha_j \theta^j, \quad \text{where} \quad \sum_{j=1}^{2\sigma} \alpha_j = 1, \alpha_j \in [0, 1]. \tag{22}$$

By using the α^js identified from this decomposition, the control input is then constructed as

$$u = \sum_{j=1}^{2\sigma} \alpha_j u^j \in \boldsymbol{\mathcal{U}}. \tag{23}$$

With this input, the transformed state vector in the next step would be obtained based on (16) as

$$\theta^+ = \overline{\mathcal{A}}\underbrace{\left(\sum_{j=1}^{2\sigma} \alpha_j \theta^j\right)}_{\theta} + \overline{\mathcal{B}}\underbrace{\left(\sum_{j=1}^{2\sigma} \alpha_j u^j\right)}_{u} + \overline{\mathcal{E}}\underbrace{\left(\sum_{j=1}^{2\sigma} \alpha_j\right)}_{1} w = \sum_{j=1}^{2\sigma} \alpha_j \underbrace{(\overline{\mathcal{A}}\theta^j + \overline{\mathcal{B}}u^j + \overline{\mathcal{E}}w)}_{y(w,\Delta)}. \tag{24}$$

We know from (21) that $y(w, \Delta) \in \boldsymbol{\Theta}, \forall (w, \Delta) \in (\boldsymbol{\mathcal{W}}, \boldsymbol{\Delta})$. Since θ^+ is obtained as a convex combination of $y(w, \Delta) \in \boldsymbol{\Theta}$ and as $\boldsymbol{\Theta}$ is a convex set, it then necessarily follows that $\theta^+ \in \boldsymbol{\Theta}, \forall (w, \Delta) \in (\boldsymbol{\mathcal{W}}, \boldsymbol{\Delta})$. ■

Similar results were originally proved[24] and extended to uncertain systems in Reference 26. Lemma 2 shows that for invariance of the set $\boldsymbol{\Theta}$, existence of $u \in \boldsymbol{\mathcal{U}}$ needs to be verified only for the finite set of points $\theta^j \in \boldsymbol{\Theta}_v$. This is a crucial observation that paves the way toward a tractable solution for Problem 1.

We next derive matrix inequality conditions for the invariance of the set $\boldsymbol{\Theta}$ based on (21). For each $\theta^j \in \boldsymbol{\Theta}_v$, we can see that condition (21) can be rewritten as

$$\exists u^j \in \boldsymbol{\mathcal{U}} : -\mathbf{1} \leq P(\overline{\mathcal{A}}\theta^j + \overline{\mathcal{B}}u^j + \overline{\mathcal{E}}w) \leq \mathbf{1}, \forall (w, \Delta) \in (\boldsymbol{\mathcal{W}}, \boldsymbol{\Delta}). \tag{25}$$

Condition (25) must be satisfied element-wise. Using (16), we hence express (25) equivalently as

$$\exists u^j \in \boldsymbol{\mathcal{U}} \text{and } \phi_{j,i} \in \mathbb{R}_+ : \phi_{j,i}(1 - (e_i^T P W^{-1}(\mathcal{A}W\theta^j + \mathcal{B}u^j + \mathcal{E}w))^2) \geq 0, \forall (w, \Delta) \in (\boldsymbol{\mathcal{W}}, \boldsymbol{\Delta});\ i = 1, \ldots, n_p, \tag{26}$$

where e_i represents the ith column of the identity matrix of size $n_p \times n_p$ and $\phi_{j,i}$s are introduced for the convenience of our derivations in the sequel. Condition (26) is a nonlinear parameter-dependent necessary and sufficient condition for the invariance of the set $\boldsymbol{\Theta}$, which has to be satisfied for each $\theta^j \in \boldsymbol{\Theta}_v$.

Having obtained the condition for invariance of the set $\boldsymbol{\Theta}$, we now proceed to present the main result of this paper, which gives feasibility conditions for Problem 1. In order to deal with uncertain parameter dependence, we employ a specific version of the full-block S-procedure from,[10] which is cited in Appendix A1 for ease of reference. As ingredients that emerge through this procedure, we introduce a set of multiplier matrices (associated with $\boldsymbol{\Delta}$) and an inner approximation thereof respectively as

$$\boldsymbol{\mathcal{M}} \triangleq \{M : (A2)\} \text{ and } \boldsymbol{\mathcal{M}}_{pol} \triangleq \{M : (A4)\}. \tag{27}$$

Note that $\boldsymbol{\mathcal{M}}_{pol} \subseteq \boldsymbol{\mathcal{M}}$ is characterized by LMI conditions (as inherited from Reference 28) and is hence used to express the theorem statement.

Theorem 1. *Problem 1 is feasible if there exist $u^j \in \mathbb{R}^{n_u}$, $V_{j,i} = V_{j,i}^T \in \mathbb{R}^{n_x \times n_x}$, $\phi_{j,i} \in \mathbb{R}_+$, $M_{j,i} \in \boldsymbol{\mathcal{M}}_{pol}$; $i = 1, \dots, n_p$; $j = 1, \dots, 2\sigma$ and $W \in \mathbb{R}^{n_x \times n_x}$ with which (28), (29), and (30) are satisfied for $i = 1, \dots, n_p$, $j = 1, \dots, 2\sigma$ and $k = 1, \dots, \gamma$. A robust control invariant set can then be obtained as in (13).*

$$HW\theta^j \leq \mathbf{1},\ Gu^j \leq \mathbf{1}, \tag{28}$$

$$\begin{bmatrix} W^T V_{j,i}^{-1} W & * \\ \phi_{j,i} e_i^T P & \phi_{j,i} \end{bmatrix} \succ 0, \tag{29}$$

$$\begin{bmatrix} \phi_{j,i} & * & * \\ AW\theta^j + Bu^j + Ew^k & V_{j,i} + B_p R_{j,i} B_p^T & * \\ A_d W\theta^j + B_d u^j + E_d w^k & S_{j,i} B_p^T + D_p R_{j,i}^T B_p^T & Q_{j,i} + S_{j,i} D_p^T + D_p S_{j,i}^T + D_p R_{j,i} D_p^T \end{bmatrix} \succcurlyeq 0. \tag{30}$$

Proof. Recalling Lemma 2, we aim for finding $u^j \in \boldsymbol{\mathcal{U}}; j = 1, \dots, 2\sigma$ that satisfy (21). In this fashion we will have established that $\boldsymbol{\Theta}$ in (18) and thus $\boldsymbol{C}$ in (17) are RCI sets. We hence assume that the control input is formed as in (23) and observe in reference to (15) that the state and control input constraints in (11) read as (28).

With the intention to resolve the nonlinearity (26), we now introduce a positive-definite matrix variable $V_{j,i}$ that satisfies

$$V_{j,i}^{-1} - \phi_{j,i} W^{-T} P^T e_i e_i^T P W^{-1} \succ 0. \tag{31}$$

Note that this condition can be expressed equivalently as in (29) by applying first a congruence transformation with W and then the Schur complement lemma. With $V_{j,i}$ satisfying (31), a sufficient condition for (26) can be formulated as

$$\phi_{j,i} - (\mathcal{A}W\theta^j + \mathcal{B}u^j + \mathcal{E}w)^T V_{j,i}^{-1} (\mathcal{A}W\theta^j + \mathcal{B}u^j + \mathcal{E}w) \geq 0. \tag{32}$$

This reads after an application of the Schur complement lemma as the parameter-dependent LMI

$$\begin{bmatrix} \phi_{j,i} & * \\ \mathcal{A}(\Delta)W\theta^j + \mathcal{B}(\Delta)u^j + \mathcal{E}(\Delta)w & V_{j,i} \end{bmatrix} \succ 0. \tag{33}$$

By now applying Lemma 4 with

$$\mathcal{Y} = \left[\begin{array}{c|cc} D_p & A_d W\theta^j + B_d u^j + E_d w & 0 \\ \hline 0 & \frac{1}{2}\phi_{j,i} & 0 \\ B_p & AW\theta^j + Bu^j + Ew & \frac{1}{2}V_{j,i} \end{array}\right], \tag{34}$$

we arrive at an LMI condition in terms of $M_{j,i} \in \mathcal{M}$, which needs to be satisfied for all $w \in \mathcal{W}$. Thanks to the fact that dependence on w is affine and $\mathcal{W}$ in (12) is polytopic, we arrive at the condition in (30) expressed in terms of the vertices v^j. ■

It should be noted at this point that $(1,1)$ block of (29) has nonlinear dependence on $(W, V_{j,i})$. This issue was also faced in[10] and resolved by adapting a successive linearization approach from Reference 18. We follow similar lines to develop an iterative scheme for volume optimization in the next section.

5 | ITERATIVE RCI SET COMPUTATION

In this section, we first briefly recall the linearization approach from References 10,18. We then formulate a cost based on which successive volume optimization can be performed.

5.1 | Linearization

Consider the following inequality

$$W^T V_{j,i}^{-1} W = \underbrace{(W - V_{j,i}Y_{j,i})^T}_{\mathcal{V}_{j,i}^T} V_{j,i}^{-1} \underbrace{(W - V_{j,i}Y_{j,i})}_{\mathcal{V}_{j,i}} + Y_{j,i}^T W + W^T Y_{j,i} - Y_{j,i}^T V_{j,i} Y_{j,i} \succcurlyeq \underbrace{Y_{j,i}^T W + W^T Y_{j,i} - Y_{j,i}^T V_{j,i} Y_{j,i}}_{\mathcal{N}_{j,i}}, \tag{35}$$

where $Y_{j,i}$ is an arbitrary matrix of compatible dimension. A sufficient condition for (29) can be obtained by replacing its $(1,1)$ block with $\mathcal{N}_{j,i}$ in (35). Nevertheless, the resulting condition is an LMI only if $Y_{j,i}$ fixed. Assuming that an iterative scheme can be started with an initial choice, the approach developed in Reference 10 is based on fixing the choice at the $n+1$st step as

$$Y_{j,i} = Y_{j,i}^n \triangleq (V_{j,i}^n)^{-1} W^n, \tag{36}$$

where $(W^n, V_{j,i}^n)$ are the values of $(W, V_{j,i})$ obtained at the nth step. Hence, the sufficient LMI condition for (29) to be used at the $n+1$st step of our iterative scheme is

$$\begin{bmatrix} (Y_{j,i}^n)^T W + W^T Y_{j,i}^n - (Y_{j,i}^n)^T V_{j,i} Y_{j,i}^n & e_i P^T \phi_{j,i} \\ \phi_{j,i} e_i^T P & \phi_{j,i} \end{bmatrix} \succ 0. \tag{37}$$

In Reference 10, it has been shown that such an iterative scheme reduces the conservatism introduced by (35). Note that (37) still holds if $W = W^n$ and $V_{j,i} = V_{j,i}^n$, which confirms that the solutions obtained at the nth step are also feasible at the $n+1$st step. We also recall that the invertibility of matrix W is implied from the $(1,1)$ block of (37).

5.2 | Iterative volume maximization

We now develop an iterative scheme based on Theorem 1 for the computation of RCI sets of potentially large volumes at each step. Some recent works like References 11,22 adopt an indirect approach to maximize the volume of the RCI set by iteratively maximizing the volume of the ellipsoidal set enclosed therein.

In contrast, we present in this paper a direct approach in which the volume of the actual RCI set is aimed to be maximized.

5.3 | Volume computation

As the first major step toward developing our algorithm, we first show that the volume of the set $\mathcal{C}$ is proportional to $|\det(W)|$ once P is fixed. For this, we use the fact that any 0-symmetric polytopic set $\mathcal{C}$ can be decomposed into n_x-simplices

$\boldsymbol{C}_m,\ m = 1, \ldots, 2\mu$, which are identified in terms of their vertex points x_m^i as

$$\boldsymbol{C}_m = \left\{ \sum_{i=0}^{n_x} \alpha_i x_m^i \ :\ \sum_{i=0}^{n_x} \alpha_i = 1, \alpha_i \in [0, 1] \right\}. \tag{38}$$

These simplices would have the following properties:

i. $\boldsymbol{C}_m$ is nonempty;
ii. $\mathrm{int}(\boldsymbol{C}_{m_1} \cap \boldsymbol{C}_{m_2}) = \emptyset$, if $m_1 \neq m_2$;
iii. $\bigcup_{m=1}^{2\mu} \boldsymbol{C}_m = \boldsymbol{C}$.

Each $\boldsymbol{C}_m$ would have origin as a common vertex point (i.e., $x_m^0 = 0, \forall m = 1, \ldots 2\mu$). The remaining n_x vertices of $\boldsymbol{C}_m$ can be represented as the columns of a matrix

$$X_m = \begin{bmatrix} x_m^1 & x_m^2 & \ldots & x_m^{n_x} \end{bmatrix}. \tag{39}$$

From Reference 29, its known that the volume of the simplex $\boldsymbol{C}_m$ is given by the Cayley–Menger determinant:

$$\mathrm{Volume}(\boldsymbol{C}_m) = \frac{1}{n_x!} |\mathrm{det}(X_m)|. \tag{40}$$

Let the matrix Θ_m be defined as

$$\Theta_m = W^{-1} X_m \Leftrightarrow X_m = W\Theta_m. \tag{41}$$

From (15), we know that the columns of the matrix Θ_m represent n_x independent vertices from the set $\boldsymbol{\Theta}_v$.

Lemma 3. *The volume of the set $\boldsymbol{C}$ identified as in (13) with a fixed P is proportional to $|\mathrm{det}(W)|$.*

Proof. Since the simplices $\boldsymbol{C}_m$ have disjoint interiors, the volume of the set $\boldsymbol{C}$ can be written as

$$\mathrm{Volume}(\boldsymbol{C}) = \sum_{m=1}^{2\mu} \mathrm{Volume}(\boldsymbol{C}_m). \tag{42}$$

It then follows from (40) and (41) that

$$\mathrm{Volume}(\boldsymbol{C}) = \frac{1}{n_x!} \sum_{m=1}^{2\mu} |\mathrm{det}(X_m)| = \frac{1}{n_x!} \sum_{m=1}^{2\mu} |\mathrm{det}(W\Theta_m)|. \tag{43}$$

Since the set $\boldsymbol{C}$ is symmetric, we can always order Θ_ms as

$$\Theta_{m+\mu} = -\Theta_m, m = 1, \ldots, \mu. \tag{44}$$

With $|\mathrm{det}(\Theta_m)| = |\mathrm{det}(-\Theta_m)|$, the summation in (43) can be rewritten as

$$\mathrm{Volume}(\boldsymbol{C}) = \frac{2}{n_x!} |\mathrm{det}(W)| \sum_{m=1}^{\mu} |\mathrm{det}(\Theta_m)|. \tag{45}$$

For a fixed P matrix, Θ_ms will be fixed too, which implies that

$$\mathrm{Volume}(\boldsymbol{C}) \propto |\mathrm{det}(W)|. \tag{46}$$

■

Note that the simplicial decomposition of the set $\boldsymbol{C}$ is not unique, which is though not a problem for our purposes. Once P is fixed, Lemma 3 basically justifies to maximize $|\mathrm{det}(W)|$ to obtain a set $\boldsymbol{C}$ of desirably large volume. Simplicial decomposition will also be useful when we consider controller design in the next section.

5.3.1 | Iterative volume maximization

In order to obtain an RCI set with a desirably large volume, we hence need to solve a determinant maximization problem under LMI conditions presented in Theorem 1. Such a problem reads as a generalized semi-definite program only when W is required to be symmetric,[30] which would necessarily introduce potential conservatism. An iterative volume maximization approached was developed in Reference 10 without enforcing symmetry on W, which we will also use in this paper.

The basic idea is to maximize the determinant of a different matrix $\overline{Z}$, which is required to be symmetric and positive-definite. We relate $\overline{Z}$ to W by a condition as

$$W^T W \succcurlyeq \overline{Z} \succ 0, \tag{47}$$

which would ensure that $\det(\overline{Z}) \leq |\det(W)|^2$. Since (47) is not an LMI, it needs to be replaced with a sufficient condition. This is done within an iterative scheme in which the solution of W at the nth step is represented as W^n. A sufficient condition for (47) is formulated by Reference 10 in terms of W^n as

$$W^T W^n + (W^n)^T W - (W^n)^T W^n \succcurlyeq \overline{Z} \succ 0. \tag{48}$$

Note that this condition is necessarily satisfied with $W = W^n$. As a result, maximization of $\det(\overline{Z})$ under (48) would lead to a solution W^{n+1} that satisfies

$$|\det(W^{n+1})| \geq |\det(W^n)|. \tag{49}$$

This allows us to develop the following iterative algorithm to compute RCI sets of increased volume at each step for a priori chosen matrix P:

Algorithm 1: [$n+1$st step]

$$\begin{array}{ll} \max\limits_{W, \overline{Z}, V_{j,i}, M_{j,i}, \phi_{j,i}, u^j} & \log\det(\overline{Z}) \\ \text{subject to:} & (28), (30), (37), \text{ and } (48) \end{array}$$

Initial optimization to compute W^0: Condition (48) is removed and $\log\det(\overline{Z})$ is changed to $\log\det(W + W^T)$; (37) is imposed with $Y^n_{j,i} \to \psi I$, where ψ is selected by a line search to find an initial feasible solution.

The invariance conditions (30) and (37) are only sufficient and hence might be conservative. By updating the matrix W at each iteration, we are actually scaling and rotating the initial polyhedron to a desirably large volume. It has been already shown before that the solution from step n is feasible at step $n+1$ in (37) and (48), which ensures that Algorithm 1 is recursively feasible. Hence the volume of set $\boldsymbol{C}$ increases iteratively until it converges to some stationary point. The algorithm should be terminated in practice when the change in $|\det(W)|$ is below a certain value or when infeasibility occurs due to numerical reasons.

Algorithm 1 requires a user-defined matrix P, which determines the representational complexity of the set $\boldsymbol{C}$. In general, it is difficult to devise a guideline for the selection of the matrix P following which an RCI set is guaranteed to exist. In fact, this is the common drawback of many similar approaches (e.g., References 11,17,22,31-33), which adopt the optimization-based strategy to compute RCI sets with restricted or predefined complexity. We explain some procedures to select the initial matrix P based on the heuristic in the following remark.

Remark 1. We can always choose matrix P in (13), while assuming $W = I$, such that it defines an initial candidate polytope of desired shape and complexity within the state constraints. We emphasize that the candidate polytope need not be RCI. A simple approach to select the candidate set could be to distribute the hyperplanes on an arbitrarily small ball of radius ϵ enclosed within the state constraints. For example, in the two-dimensional case, the ith row could be selected as

$$\epsilon > 0,\ e_i^T P = \left[\tfrac{1}{\epsilon}\cos\left(\tfrac{\pi(i-1)}{n_p}\right) \quad \tfrac{1}{\epsilon}\sin\left(\tfrac{\pi(i-1)}{n_p}\right)\right],\ i = 1, \ldots, n_p. \tag{50}$$

With this choice of the initial candidate set, the state and input constraints are most likely to be satisfied. Thus, at the first iteration, the algorithm now has to guarantee the satisfaction of invariance conditions, which can be achieved by generating a suitable matrix W. Alternatively, having an approximate knowledge of the shape of the RCI set can be very helpful while selecting the initial polytope. Thus one could select the initial set by performing a few iterations of the geometric approach[6] (convergence not needed) to get an initial estimate of the RCI set. Note that this could be still beneficial in the case of a higher-dimensional system where the geometric approach is known to be computationally inefficient and may not even converge.

One might obtain different RCI sets from Algorithm 1 for different P choices (as well as different initial solutions) and these can in fact be combined to construct larger RCI sets.

As is known from References 1,34, the convex hull and union of different RCI sets is a larger RCI set with higher complexity.

Remark 2. In Algorithm 1, the SDP consists of $\left(2\sigma \times n_p \times \left(\gamma + 1 + \frac{\eta(\eta+1)}{2}\right) + 1\right)$ LMIs due to (30), (37), (48); and $(2\sigma \times (n_h + n_g))$ scalar inequalities representing the system constraints in (28). Furthermore, these inequalities are in terms of $(2 + 2\sigma \times (1 + 4n_p))$ matrix variables and $(2\sigma \times n_p)$ scalar variables. The complexity due to system and invariance constraints can in fact be reduced significantly in the case when the sets $\boldsymbol{\mathcal{X}}$, $\boldsymbol{\mathcal{U}}$, and $\boldsymbol{\mathcal{W}}$ are 0-symmetric. Because of the symmetry, the invariance condition needs to be verified only on nonsymmetric elements of $\boldsymbol{\Theta}_v$. In this way the number of LMIs and scalar inequalities reduce to $\left(\sigma \times n_p \times \left(\gamma + 1 + \frac{\eta(\eta+1)}{2}\right) + 1\right)$ and $\sigma \times (n_h + n_g)$, respectively. We then also have less number of matrix and scalar variables as $(2 + \sigma(1 + 4n_p))$ and $(\sigma \times n_p)$, respectively. The computational complexity can be reduced by also considering alternative relaxation schemes or structured multipliers at the cost of potential conservatism. For instance, Pólya relaxation might be replaced with the so-called convex hull relaxation (e.g., Reference 28) to reduce the number of LMIs. The number of variables can be reduced by using D-scales, DG scales, or block-diagonal subblocks in the multiplier matrix M.[23]

Remark 3. An algorithm to obtain desirably small RCI sets can be formulated by minimizing $|\det(W)|$ iteratively in a similar fashion as done in Algorithm 1. This can be achieved by minimizing trace($\underline{Z}$) at each iteration and replacing (48) by $\begin{bmatrix} \underline{Z} & W^T \\ W & I \end{bmatrix} \succcurlyeq 0$ in Algorithm 1. For further details, we refer the reader to Reference 18.

Algorithm 1 would provide RCI sets for which control inputs that ensure invariance are known to exist. But to realize constrained control, a method needs to be devised to compute a feasible input for any given $x \in \boldsymbol{C}$. In the next section, we present an offline approach to constrained controller design based on an existing result.

6 | CONTROLLER DESIGN

The purpose of this section is to design a piecewise affine controller based on the approach presented in References 24,26 for the constrained control of the system in (7). The approach requires that at each vertex of the set $\boldsymbol{C}$, there should exist an admissible control action that brings the state to the interior of the set $\boldsymbol{C}$ within finite time. To be able to design such a controller,

we consider using Lemma 1 together with the RCI set computation algorithm developed in the previous section. For a specific $\lambda \in (0, 1)$ choice, we would then obtain an RCI set $\boldsymbol{C}$ for the modified system in (6), which would serve as a λ-contractive set for the original system of (3). A controller designed for the modified system of (6) would hence serve as a stabilizing piecewise-affine controller for the original system (3), thus ensuring asymptotic convergence to origin in the absence of any disturbance input. This controller will also keep the trajectory of (3) in $\boldsymbol{C}$ for any disturbance input within the considered disturbance set.

To this end, we first recall the decomposition of the RCI set $\boldsymbol{C}$ into n_x-simplicies $\boldsymbol{C}_m$ as in (38). The control law is then formulated as a domain-dependent state feedback of the form

$$x(k) \in \boldsymbol{C}_m \Rightarrow u(k) = K_m x(k). \tag{51}$$

If $x(k)$ happens to be at a point that is common to $\boldsymbol{C}_i$ and $\boldsymbol{C}_j$ (e.g., a point on a common edge), then the control input can be computed with either K_i or K_j. To identify the state-feedback gain matrices K_m, we first use the control inputs associated with the vertices x_m^j to form a matrix as

$$U_m = \begin{bmatrix} u_m^1 & u_m^2 & \dots & u_m^{n_x} \end{bmatrix}. \tag{52}$$

The associated state feedback gain matrix is then computed as

$$K_m = U_m X_m^{-1},\ m = 1, \dots, 2\mu. \tag{53}$$

In References 24,26, it is shown that the piece-wise affine control law obtained in this fashion is stabilizing and is recursively feasible, which indeed imply that it is an invariance-inducing controller. Indeed, if $x(k)$ is expressed as a convex combination of the vertices of $\boldsymbol{C}_m$, $u(k)$ of (51) can then also be expressed as a convex combination of the associated vertices u_m^j, which basically corresponds to the control law proposed in (23):

$$x(k) = X_m\alpha \Rightarrow u(k) = K_m X_m \alpha = U_m \alpha. \tag{54}$$

Assuming that the system starts from an initial state within the computed RCI set, it is also possible to develop a control algorithm based on online optimization by using the associated W matrix and the vertices $\theta^j,\ j = 1, \dots, 2\sigma$.[27] At each time step, a linear program would then be solved to obtain a set of weights, with which the control input would be constructed. With kth time step indicated with a superscript k, the weights are obtained and used to construct the control input as follows:

$$\beta^k = \arg\min_{\beta\in\mathbb{R}^{2\sigma}} \left\{ \sum_{j=1}^{2\sigma} \beta_j : \sum_{j=1}^{2\sigma} \beta_j W\theta^j = x(k),\ 0 \le \beta_j \le 1\ , j = 1, \dots, 2\sigma \right\} \Rightarrow u(k) = \sum_{j=1}^{2\sigma} \beta_j^k u^j. \tag{55}$$

A controller implemented in this fashion would ensure asymptotic stability, as established in detail by Reference 27. An online approach might be preferable especially when working with high dimensional systems, for which obtaining a simplicial decomposition of the set $\boldsymbol{C}$ would be difficult.

The closed-loop transient performance of the system can (to some extent) be shaped by the choice of the contraction factor λ, which also affects the size of the set $\boldsymbol{C}$. However, a known weakness of such a control structure is that a full range of control inputs are utilized only at the boundary of the set $\boldsymbol{C}$. In the interior of the set $\boldsymbol{C}$, progressively smaller control inputs are applied as the system trajectories approach origin. Thus it may take a long time for a controller to stabilize the system. It follows from Reference 27 that this weakness could easily be overcome by modifying the proposed controller into a so-called interpolation-based controller, which suboptimally minimizes a predefined cost function.

7 | ILLUSTRATIVE EXAMPLES

In this section, we provide some examples to illustrate the potential of the approach proposed in this paper. The algorithm was implemented in Matlab on a 3.1 GHz Intel Core i7-555U macOS computer with 8 GB RAM with YALMIP[35] and the solver SeDuMi.[36] The computation of the volume, vertices, and projections of the polytope was done using MPT.[37]

7.1 | Double integrator

In this section, the proposed algorithm is demonstrated with a discrete-time double integrator that has rational parameter dependence. The system dynamics are described as in (7) with

$$\mathcal{A} = \begin{bmatrix} 1+\delta_1 & 1+\delta_1 \\ 0 & 1+\frac{\delta_2}{1+\delta_1} \end{bmatrix},\ \mathcal{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ \mathcal{E} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{56}$$

where $|\delta_{1,2}| \le 0.25$ represent the uncertain parameters. The state and control input constraints and the input disturbance bound are expressed as

$$|x| \le \begin{bmatrix} 5 \\ 5 \end{bmatrix},\quad |u| \le 3,\quad |w| \le 0.6. \tag{57}$$

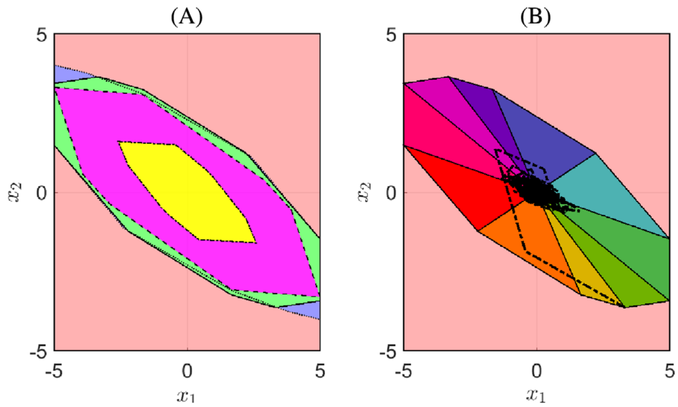

FIGURE 1 (A) Admissible states $\mathcal{X}$ (red) and maximal RCI set Ω_∞ geometric approach (blue; dotted), and maximum volume RCI set C_5 (green; solid) and C_4 (magenta; dashed) from Algorithm 1 with complexity $n_p = 5$ and $n_p = 4$, respectively, and minimum volume RCI set from Remark 3 with $n_p = 5$ (yellow; dot-dashed) and (B) Admissible states $\mathcal{X}$ (red) and Simplex decomposition of the maximum volume RCI set (with $\lambda = 0.99$) from Algorithm 1 with $n_p = 5$ (colored) [Colour figure can be viewed at wileyonlinelibrary.com]

In order to make comparisons with the existing algorithms[11,18] and the geometric approach, we first introduce a new parameter as $\delta_3 = \delta_2/(1 + \delta_1)$ and thus view (56) as a system that has affine dependence on (δ_1, δ_3). We illustrate the proposed algorithm by computing the sets C_4 and C_5 associated with $n_p = 4$ and $n_p = 5$, respectively. These sets are compared with the maximal RCI set Ω_∞ obtained by using an existing geometric approach, in which the control inputs are also assumed to be free. Figure 1(A) shows the generated RCI sets. We were unable to compute any RCI set with the methods from References 10,11,17,18,22, in which the control input is assumed to have a linear structure and/or a conservative system description is used to account for parametric uncertainty. It can be observed that the maximal RCI set from the geometric approach (Volume = 40.2445) is described by 10 hyperplanes and has a slightly larger volume than the one obtained using the proposed approach with the same complexity (Volume = 39.9720). The nonsymmetric vertices of the set C_5 and the corresponding inputs are given by

$$X = \begin{pmatrix} x^1 & x^2 & x^3 & x^4 & x^5 \\ 3.3198 & 5.0000 & 5.0000 & 2.2185 & -1.6636 \\ -3.6425 & -3.4323 & -1.4801 & 1.2336 & 3.2409 \end{pmatrix}$$
$$U = \begin{pmatrix} u^1 & u^2 & u^3 & u^4 & u^5 \\ 2.9922 & 2.6635 & -1.1764 & -2.9825 & -3.0000 \end{pmatrix}. \qquad (58)$$

Notice that the vertices $\{x^5, -x^5\} \notin \Omega_\infty$, this is due to the approximation of (56) with an affine parameter dependent system while computing Ω_∞. Lastly, in Figure 1(B), we demonstrate the controller proposed in the Section 6 in closed-loop with the system (56). Also, the proposed decomposition of the set C into simplices is shown. The closed-loop trajectory, when starting from one of the vertices, can be seen in black (dot-dashed) is produced by randomly varying disturbances and uncertainties within their bounds.

7.2 | 4D Vehicle lateral dynamics

We now compare the proposed algorithm with a recent work[10] using a four-dimensional bicycle model for vehicle lateral dynamics.[10] We assume the longitudinal velocity of the vehicle to be an uncertain parameter as $V_x \in [50, 70]$ km/h, which enters the system dynamics rationally. For a fair comparison, we keep the complexity of the RCI set same as Reference 10 by selecting $P = I$. Using Algorithm 1, the nonsymmetric vertices of the set C and corresponding control inputs were found to be

$$X = \begin{pmatrix} x^1 & x^2 & x^3 & x^4 & x^5 & x^6 & x^7 & x^8 \\ 0.3714 & 0.4000 & -0.2800 & -0.2514 & 0.2514 & 0.2800 & -0.4000 & -0.3714 \\ -3.0000 & -2.9754 & -2.3478 & -2.3233 & -1.3182 & -1.2937 & -0.6661 & -0.6415 \\ 0.0485 & 0.0035 & 0.1136 & 0.0687 & 0.0681 & 0.0232 & 0.1332 & 0.0883 \\ -0.2040 & -0.1502 & -0.0704 & -0.0165 & -1.3353 & -1.2815 & -1.2017 & -1.1479 \end{pmatrix}$$

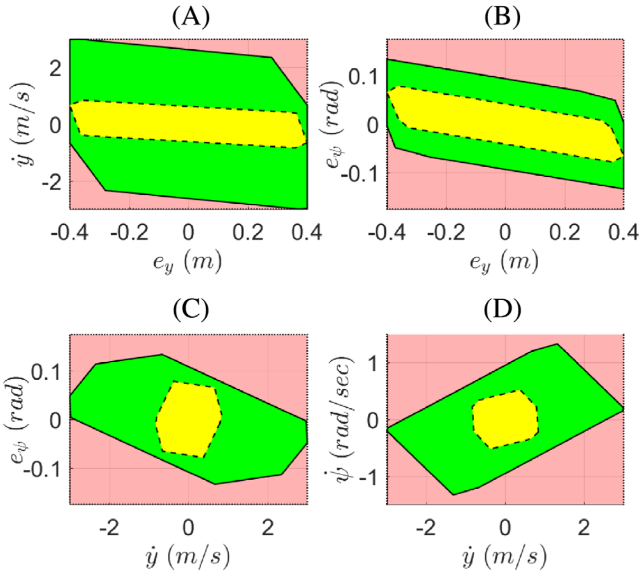

FIGURE 2 (A) Projection of admissible set $\mathcal{X}$ (red; dotted), maximum volume RCI set $\boldsymbol{C}$ from Algorithm 1(green; solid) and using Reference 10 (yellow; dashed) [Colour figure can be viewed at wileyonlinelibrary.com]

$$U = \begin{pmatrix} u^1 & u^2 & u^3 & u^4 & u^5 & u^6 & u^7 & u^8 \\ -0.0791 & 0.0776 & -0.0873 & 0.0873 & 0.0517 & 0.0873 & -0.0426 & 0.0873 \\ -0.1235 & 0.1407 & -1.0000 & -0.9999 & 1.0000 & 0.9991 & 0.3675 & -0.9999 \end{pmatrix}. \tag{59}$$

The plots in Figure 2(A)–(D) show the projections of the admissible states $\mathcal{X}$, and the maximum volume RCI set $\boldsymbol{C}$ obtained from Algorithm 1 after 3000 iterations, and the algorithm in Reference 10, respectively. The RCI set obtained from the proposed algorithm is much larger than the compared approach. However, larger set comes at the cost of a modestly complex controller. Furthermore, even though the computational complexity is high (see Remark 2), each iteration of the proposed algorithm took 24 s on average compared to 14 s taken by Reference 10. We also tried to compute the RCI set by using the approximate polytopic model, but were unable to compute any RCI set with the methods from Reference 11,18. On the other hand, the geometric approach did not converge even after more than 24 h.

8 | CONCLUSION

We have developed an iterative algorithm to compute a desirably large 0-symmetric RCI set for a rationally parameter-dependent system with additive disturbances. Sufficient LMI conditions guaranteeing invariance have been derived by favor of a state transformation, the full block S-procedure and a Newton update technique. These LMI conditions and the reformulated system constraints are then integrated into an algorithm where an SDP is solved to obtain RCI sets of successively larger volumes at every iteration. A relatively less-conservative formulation has been achieved by rendering the control inputs independent of any predefined feedback structure along with the employment of a direct approach for volume maximization. These claims are supported by including relevant numerical examples where the proposed algorithm was able to compute considerably larger RCI sets in comparison with other similar approaches.

It has been observed in our numerical exercises and is also emphasized in Remark 1 that the choice of initial polytope affects the final outcome, thereby encouraging investigations to identify an appropriate approach to select this initial set. Moreover, a future research direction could be to extend the proposed formulation to compute asymmetric polytopic RCI sets. It requires some further investigations to develop parameter-dependent set computations along with control algorithms that achieve certain performance objectives.

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APPENDIX A. A FULL-BLOCK S-PROCEDURE

The following Lemma is a special case of the full block S-procedure[25] as presented in Reference 10. We first introduce the LFT

$$\Delta \star \underbrace{\left[\begin{array}{c|c} \mathcal{Y}_{11} & \mathcal{Y}_{12} \\ \hline \mathcal{Y}_{21} & \mathcal{Y}_{22} \end{array}\right]}_{\mathcal{Y}} \triangleq \mathcal{Y}_{22} + \mathcal{Y}_{21}\Delta(I - \mathcal{Y}_{11}\Delta)^{-1}\mathcal{Y}_{12},$$

which is said to be well-posed when $I - \mathcal{Y}_{11}\Delta$ is nonsingular $\forall \Delta \in \mathbf{\Delta}$.

Lemma 4. *The LFT $\Delta \star \mathcal{Y}$ is well-posed and*

$$\mathrm{He}\{\Delta \star \mathcal{Y}\} \triangleq \Delta \star \mathcal{Y} + (\Delta \star \mathcal{Y})^T \succ 0, \ \forall \Delta \in \mathbf{\Delta}, \tag{A1}$$

holds if there exists a scaling matrix M that satisfies

$$\begin{bmatrix} \Delta^T \\ I \end{bmatrix}^T \underbrace{\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix}}_{M} \begin{bmatrix} \Delta^T \\ I \end{bmatrix} \preccurlyeq 0, \quad \forall \Delta \in \mathbf{\Delta}, \tag{A2}$$

$$\begin{bmatrix} \mathcal{Y}_{21}R\mathcal{Y}_{21}^T + \mathrm{He}\{\mathcal{Y}_{22}\} & \mathcal{Y}_{21}R\mathcal{Y}_{11}^T + \mathcal{Y}_{21}S^T + \mathcal{Y}_{12}^T \\ * & Q + \mathcal{Y}_{11}R\mathcal{Y}_{11}^T + \mathrm{He}\{\mathcal{Y}_{11}S^T\} \end{bmatrix} \succ 0. \tag{A3}$$

The condition (A2) is a semi-infinite LMI condition. Since we consider a polytopic uncertainty region, one can use Pólya's method to relax (A2). Hence using zeroth order Pólya relaxation in References 10,28, we replace (A2) by a sufficient condition given by

$$\left.\begin{array}{lll} \Omega_{jj}(M) & \preccurlyeq & 0, \quad j = 1, \dots, \eta \\ \mathrm{He}\{\Omega_{ji}(M)\} & \preccurlyeq & 0, \quad j = 1, \dots, \eta; \ i = j+1, \dots, \eta \end{array}\right\}, \tag{A4}$$

where

$$\Omega_{ij}(M) \triangleq \begin{bmatrix} (\Delta^i)^T \\ I \end{bmatrix}^T M \begin{bmatrix} (\Delta^j)^T \\ I \end{bmatrix}, i,j = 1, \dots, \eta. \tag{A5}$$

Note that, it is possible to obtain potentially less conservative conditions at the cost of a combinatorial increase in the number of LMIs by employing higher-order Pólya relaxations; see Reference 10. Lastly, for the uncertainty regions described by polynomial inequalities, the sum-of-squares (SOS) approach can be used; see References 28,38,39.