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Differential equations with maximal monotone operators

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ABSTRACT

The paper deals with multivalued differential equations in abstract spaces. Nonlocal conditions are assumed. The model includes an m-dissipative multioperator which generates an equicontinuous, not necessarily compact, semigroup. The regularity of the nonlinear term also depends on the Hausdorff measure of noncompactness. The existence of integral solutions is discussed, with a topological index argument. A transversality condition is required. The results are applied to a partial differential inclusion in a bounded domain in $\mathbb{R}^n$ with nonlocal integral conditions. The model also includes an m-dissipative but not necessarily compact semigroup generated by a suitable subdifferential operator.

1. Introduction

In this paper we prove the existence of integral solutions of the following differential equation

$$\begin{cases} u'(t) \in Au(t) + F(t, u(t)), \text{ for a.e. } t \in [0, T], \\ u(0) = g(u), \end{cases} \tag{1.1}$$

with nonlocal initial conditions in a Banach space $(X, \|\cdot\|)$. Here $A : D(A) \subset X \multimap X$ is an m-dissipative multioperator while $g : C([0, T]; X) \to \overline{D(A)}$ and $F : [0, T] \times X \multimap X$ are a given map and a multimap, respectively.

Starting from the seminal paper by Byszewski, [16], problem (1.1) has been widely investigated and the literature on this topic can be divided in two main classes. In the first one, about semilinear problems, A is linear and it generates a C_0 semigroup. In the second, about fully nonlinear problems, A is a nonlinear

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m-dissipative operator and it generates a nonlinear semigroup according to the Crandall-Lidget definition [19] (see also Definition 3.7). Among the results for semilinear problems involving a multivalued map F we cite [7], [8], [9], [17], [30]. In particular, the multipoint condition appears in [14]. To our knowledge the case when A is nonlinear and F multivalued was first discussed in [2, Theorem 3.8]; there F is closed-valued and lower semicontinuous in its second variable and A generates a compact semigroup. Subsequently, under the assumption of the compactness of the semigroup generated by A, the existence of the solutions for (1.1) with multivalued perturbations F has been extensively studied. See for instance [29], [35] and also [3], [18], where the topological structure of the solution is also studied. Furthermore, the case when $A = A(t)$ is a family of maximal monotone operator was recently investigated in [6].

When A is a m-dissipative operator, the nonlinearity F frequently satisfies some dissipative type conditions, such as $F(t, \cdot)$ Lipschitz (see, e.g., [1], [34]), one sided Perron as in [12], or one sided Lipschitz, see [13] and the recent result [27]. Furthermore, the assumption of uniformly convex dual for the Banach space X, as in our paper, is a common practice. However, it's worth noting that under Lipschitz conditions on F ([1]), or additional requirements on the operator A such as compactness of the generated semigroup ([15] and [29]), or complete continuity ([27]), Banach spaces without uniformly convex dual can be considered. For instance, in Theorem 7.2.1 in [15] A is assumed to be a m-ω-dissipative operator (a stronger assumption than m-dissipativity), generator of a compact semigroup and F is an almost strong-weak upper semicontinuous multivalued map.

We decided to assume the uniform convexity of the dual of X, in order to analyze more general operators A and nonlinearities F. More precisely, compared to Theorem 7.2.1 in [15] we can consider a non necessarily m-ω-dissipative operator that generates a non-compact semigroup and with weaker regularity assumptions on F. Indeed, we consider that F is strong-weak upper Carathéodory (see assumptions $(H_F^1), (H_F^2)$), a weaker assumption than the one assumed in the cited theorem. For an example of a multivalued map that is upper Carathéodory, but not almost upper semicontinuous see Example 4 in [5]. Moreover, we point out that in most applications the natural framework is the space $L^p(\Omega)$ with $1 < p < \infty$ that has a uniformly convex dual.

The motivation for these studies is that nonlocal Cauchy problems may have better effects in applications than the classical initial value problem $u(0) = u_0$. For example, it is used to represent mathematical models for evolution of various phenomena, such as nonlocal neural networks, nonlocal pharmacokinetics, nonlocal pollution and nonlocal combustion, see the monograph [28].

In this paper we assume that A is a m-dissipative operator generating an equicontinuous, but not necessarily compact, semigroup. Our main goal is to prove the existence of solutions of (1.1) under very general growth conditions on the nonlinear term F and on the nonlocal initial condition g. In particular, we have $F(t, \cdot)$ Lipschitz with respect to a measure of non compactness (see condition (H_F^4)). The growth condition on F allows us to handle a class of nonlinear terms F which could exhibit superlinear growth, see Example 4.8. Our main result is presented in Theorem 4.10. We note that in the case of F having compact values, the Lipschitz continuity assumed, for instance, in [1], constitutes a stronger condition than our Lipschitz condition (H_F^4), as detailed in Remark 4.5.

We solve (1.1) by introducing a suitable operator $G : [0, 1] \times C([0, T]; \overline{D(A)}) \multimap C([0, T]; \overline{D(A)})$, such that the fixed points of $G(1, \cdot)$ are integral solutions of (1.1). We construct an homotopy between the map G at level 1, that is $G(1, \cdot)$, and the map G at level zero, i.e. $G(0, \cdot)$, that coincides with the zero function; thus, by exploiting the properties of the index for condensing operators, we get the existence of at least one integral solution of (1.1).

The Lipschizianity with respect to a measure of non compactness of the nonlinear term F, are used also in [24] for single valued nonlinearities and in [34] for upper Carathéodory multimaps. In particular, in [24] the equicontinuity of the semigroup is not required, but the nonlinearity is a single valued map $F : L^1([0, T], X) \to L^1([0, T], X)$. The main novelty of our paper with respect to the cited results consists in the fact that we assume more general growth conditions on the nonlocal function g and on the nonlinear term F. Indeed, in both papers the map g and the nonlinear term F are required to have a sublinear growth condition with coefficients strictly smaller than one. Here, we are able to relax these growth hypotheses by requiring the map g only to send a ball into itself and, by imposing the so called *transversality condition*, see condition (H_F^5) below, on the multimap F. We would like to point out the fact that we require the transversality condition only for one element $z \in F(t,x)$ and not for every $z \in F(t,x)$ as is usually done in literature, for instance see again Theorem 7.2.1 in [15]. On the other side, to consider these general growths, we are compelled to introduce the assumption of compactness of the values for the nonlinear term.

We will apply the existence result in Theorem 4.10 to the following class of partial parabolic differential inclusions in a domain $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$

$$\begin{cases} u_t \in a\,\Delta u(t,x) - \partial\varphi(u(t,x)) - bu(t,x) + [f_1(t,u(t,x)), f_2(t,u(t,x))] & \text{in } \Omega_T \\ -\dfrac{\partial u}{\partial \mu}(t,x) \in \partial j(u(t,x)) & \text{in } \partial\Omega_T \\ u(0,x) = \displaystyle\int_0^T \int_\Omega h(s,x,\xi,u(s,\xi))\,d\xi\,ds & \text{in } \Omega \end{cases} \tag{1.2}$$

where $\Omega_T = \Omega \times]0,T[$, $\partial\Omega_T = \partial\Omega \times [0,T]$, $\partial\varphi$ is the subdifferential of φ, $\Delta u(t,x)$ is the Laplacian of $u(t,\cdot)$, $a \geq 0$ and $\dfrac{\partial u}{\partial \mu}(t,x)$ is the normal derivative of $u(t,\cdot)$ at $x \in \partial\Omega$, see Theorem 6.1. Being A in a subdifferential form, it is possible to prove that it is the generator of a nonlinear semigroup. We point out the fact that, under the hypotheses required on (1.2), such semigroup it is not necessarily compact. We provide, in particular, a simple example of map $\varphi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$, whose subdifferential generates a non compact semigroup, see Example 6.3.

The paper is divided into six sections. In Section 2 we recall the definition and the properties of the degree for condensing operator. In Section 3 we give the notion of integral solution and we recall the definition and the properties of m-dissipative operators and nonlinear semigroups. In Section 4 we formulate the problem and we state the main result, i.e. Theorem 4.10. In Section 5 we prove the main result, while in the last Section 6 we analyze the example presented above.

In the whole paper, we denote with $\|\cdot\|_p$ the norm in $L^p(\Omega;\mathbb{R})$, $1 \leq p \leq \infty$, where Ω is a domain in $\mathbb{R}^n$, $n \geq 1$, with $\|\cdot\|_0$ the norm in $C([0,T];X)$ and with $B_r(x)$ the ball in X of radius $r > 0$ and center $x \in X$.

2. Index for condensing operators

First we briefly recall some useful properties of the multivalued operators (see e.g. [26]).

Let X and Y be two topological spaces.

Definition 2.1. A multimap $F : D \subseteq X \multimap Y$ is said to be upper semicontinuous at $x \in D$ if for every open set $W \subseteq Y$ such that $F(x) \subset W$, there exists a neighbourhood $V(x)$ of x with the property that $F(V(x)) \subset W$. It is said to be upper semicontinuous if it is upper semicontinuous at every point $x \in X$.

Definition 2.2. A multimap $F : D \subseteq X \multimap Y$ is said to be

(a) sequentially closed: if for any sequences $\{x_n\} \subset D$, $\{z_n\} \subset X$, if $x_n \to x_0$ and $z_n \in F(x_n)$, $z_n \to z_0$, then $z_0 \in F(x_0)$;
(b) closed: if its graph is a closed subset of $X \times Y$.

If X and Y are metric spaces (a) and (b) in the above definition are equivalent.

Definition 2.3. A multimap $F : D \subseteq X \multimap Y$ is said to be

1. compact: if $\overline{F(D)}$ is compact in Y;
2. quasicompact: if its restriction to any compact set $K \subset X$ is compact.

Proposition 2.4. *A closed quasicompact multimap with compact values $F : D \subseteq X \multimap Y$ is upper semicontinuous.*

Proposition 2.5. *If Y is a Hausdorff topological space, an upper semicontinuous multimap with closed values $F : D \subseteq X \multimap Y$ is closed.*

Let now X be a real Banach space. We recall the notion of measure of non compactness.

Definition 2.6. Given a partially ordered set N, a function $\beta : P(X) \to N$ is said to be a measure of non-compactness (m.n.c.) in X if $\beta(\overline{\mathrm{co}}(\Omega)) = \beta(\Omega)$ for all $\Omega \subset X$, where $\overline{\mathrm{co}}(\Omega)$ denotes the closed convex hull of Ω.

A m.n.c. β is called:

(i) *monotone*: if $\beta(\Omega_0) \leq \beta(\Omega_1)$ for every $\Omega_0 \subset \Omega_1 \subset X$;
(ii) *nonsingular*: if $\beta(\{x\} \cup \Omega) = \beta(\Omega)$ for every $x \in X$ and $\Omega \subset X$;
(iii) *regular*: when $\beta(\Omega) = 0$ if and only if $\Omega \subset X$ is relatively compact.

The Hausdorff measure of non-compactness, defined as

$$\chi(\Omega) = \inf\{\varepsilon : \exists\, x_1, \dots, x_n \in X,\ \Omega \subset \bigcup_{i=1}^{n} B_\varepsilon(x_i)\},$$

is a typical example of monotone, nonsingular and regular m.n.c. Moreover by its definition

$$\chi\left(\bigcup_{\lambda\in[0,1]} \lambda\Omega\right) = \chi(\Omega) \tag{2.1}$$

for every $\Omega \subset X$.

In the space of continuous functions we consider the following measure of noncompactness (see Example 2.1.4 in [26])

$$\nu(\Omega) := \max_{\{y_n\}_{n=1}^\infty \subset \Omega} \left(\gamma(\{y_n\}_{n=1}^\infty), \mathrm{mod}_C(\{y_n\}_{n=1}^\infty)\right), \tag{2.2}$$

where

$$\gamma(\{y_n\}_{n=1}^\infty) = \sup_{t\in[0,T]} e^{-Lt}\chi(\{y_n(t)\}_{n=1}^\infty),$$

with $L > 0$ a suitable constant and

$$\mathrm{mod}_C(\{y_n\}_{n=1}^\infty) = \lim_{\delta\to 0} \sup_{n\in\mathbb{N}} \max_{|t_1-t_2|\leq\delta} \|y_n(t_1) - y_n(t_2)\|.$$

The ordering is induced by the positive cone in $\mathbb{R}^2$ and ν is a regular measure of non compactness.

We base our study of problem (1.1) on arguments related to the topological degree for upper-semicontinuous and condensing multimaps with R_δ values. We introduce in the following the definition of R_δ set, of condensing multioperator and the notion of the topological degree we are going to use. We refer to [4].

We say that a nonempty subset C of a metric space is contractible, provided there exist $x_0 \in C$ and a continuous map $h : [0,1] \times C \to C$ such that $h(0,x) = x_0$ and $h(1,x) = x$ for every $x \in C$; C is called an R_δ-set, provided there exists a decreasing sequence $\{C_n\}_{n=1}^{\infty}$ of compact contractible sets such that $C = \cap_{n=1}^{\infty} C_n$.

Let Λ be a topological space.

Definition 2.7. A multimap $F : D \subseteq X \multimap X$, or a family of multimaps $G : \Lambda \times D \multimap X$, is said to be condensing with respect to a measure of noncompactness β (β-condensing) if for every bounded set $\Omega \subseteq D$ the inequality

$$\beta(F(\Omega)) \geq \beta(\Omega), \quad \text{or} \quad \beta(G(\Lambda \times \Omega)) \geq \beta(\Omega),$$

implies Ω relatively compact.

Let $D \subseteq X$ be closed and convex. Let $F : D \multimap D$ be an R_δ valued, upper semicontinuous and β-condensing multimap with respect to a monotone, nonsingular m.n.c. β. Let $U \subset D$ be open and the boundary ∂U be fixed point free for F. In this case, one can associate to F an integer defined as the index of F with respect to U, $\operatorname{ind}(F, D, U)$, with the following properties:

(i) *Normalization property.* If $F \equiv u_0 \notin \partial U$, then

$$\operatorname{ind}(F, D, U) = \begin{cases} 1 & u_0 \in U \\ 0 & u_0 \notin U \end{cases}$$

(ii) *Homotopy invariance.* Assume that two upper semicontinuous β-condensing operators $F_1 : D \multimap X$ and $F_2 : D \multimap X$ are homotopic, i.e. there exists an upper semicontinuous, condensing operator $G : [0,1] \times D \multimap X$ with R_δ-values such that
(a) $u \notin G(\lambda, u)$ for every $\lambda \in [0,1]$ and $u \in \partial U$;
(b) $G(0, \cdot) = F_1, \quad G(1, \cdot) = F_2$.
Then

$$\operatorname{ind}(F_1, D, U) = \operatorname{ind}(F_2, D, U).$$

(iii) *Existence.* If $\operatorname{ind}(F, D, U) \neq 0$, then F admits a fixed point, that is there exists $u \in U$ such that $u \in F(u)$.

3. Definition of integral solutions

In this section we list some definitions, results and examples related to dissipative operators, duality maps, and integral solutions for linear problems. We refer to [32] for more details.

Let $(X, \|\cdot\|)$ be a real Banach space and X^* its dual, let $x, y \in X$, $h \in \mathbb{R} \setminus \{0\}$, define

$$(x, y)_h := \frac{1}{2h} \left(\|x + hy\|^2 - \|x\|^2 \right).$$

The limits $(x, y)_+ = \lim_{h \downarrow 0} (x, y)_h$ and $(x, y)_- = \lim_{h \uparrow 0} (x, y)_h$ exist and are finite; the function $(\cdot, \cdot)_+$ is called the upper semi-inner product on X and $(\cdot, \cdot)_-$ is called the lower semi-inner product on X. Denoting with $J : X \multimap X^*$ the duality map, i.e.

$$J(x) = \{x^* \in X^* \colon \|x^*\|_{X^*} = \|x\| \text{ and } \langle x^*, x\rangle = \|x\|^2\}$$

we have that for every $x, y \in X$, $x \neq 0$

$$(x, y)_+ = \sup\{\langle x^*, y\rangle \colon x^* \in J(x)\},$$

and

$$(x, y)_- = \inf\{\langle x^*, y\rangle \colon x^* \in J(x)\},$$

see Lemma 1.4.3 and 1.4.2 in [32]. In particular, if X^* is strictly convex then J is a single valued map, thus, we have that

$$(x, y)_+ = (x, y)_- = \langle J(x), y\rangle.$$

Furthermore, if X is a Hilbert space then,

$$(x, y)_+ = (x, y)_- = \langle x, y\rangle.$$

If moreover X^* is uniformly convex, as in this framework, then J is also continuous (see [20, Proposition 8.10]).

If Ω, is a bounded measurable subset of $\mathbb{R}^n$, $n \geq 1$ and $p \in]1, +\infty[$, then $X = L^p(\Omega; \mathbb{R})$ endowed with the usual norm is uniformly convex. So, since for every $p \in]1, +\infty[$ the dual of X is $X^* = L^q(\Omega; \mathbb{R})$ with $\frac{1}{p} + \frac{1}{q} = 1$, we get that $X = L^p(\Omega; \mathbb{R})$, $p \in]1, +\infty[$ has a uniformly convex dual.

Definition 3.1. A multioperator $A : D(A) \subset X \multimap X$ is called dissipative if

$$(x_1 - x_2, y_1 - y_2)_- \leq 0$$

for any $x_1, x_2 \in D(A)$ and $y_1 \in Ax_1$, $y_2 \in Ax_2$, and m-dissipative if it is dissipative and for every $\lambda > 0$ the range of the operator $I - \lambda A$ is equal to X.

Example 3.2. Given an Hilbert space H and a proper lower semicontinuous convex function $\phi : H \to \mathbb{R} \cup \{+\infty\}$, we recall that the subdifferential of ϕ at x is defined as

$$\partial\phi(x) := \{z \in H \ : \ \phi(x) \leq \phi(y) + \langle x - y, z\rangle \text{ for every } y \in H\}.$$

Now, the multioperator $A : D(A) \subset H \multimap H$ defined as

$$\begin{aligned} &D(A) = \{x \in H \ : \ \partial\phi(x) \neq \emptyset\} \\ &Ax = -\partial\phi(x) \end{aligned}$$

is an m-dissipative multioperator, see Theorem 1.6.2 in [32].

Example 3.3. Let Ω be a nonempty, bounded and open subset in $\mathbb{R}^d$, $d \geq 1$, with C^2 boundary Σ, let $p \in [2, +\infty)$, let $\lambda > 0$, and let $\beta : D(\beta) \subseteq \mathbb{R} \multimap \mathbb{R}$ such that $-\beta$ is an m-dissipative operator, with $0 \in D(\beta)$ and $0 \in \beta(0)$. The p-Laplace operator $\Delta_p^\lambda : D(\Delta_p^\lambda) \subseteq L^2(\Omega) \to L^2(\Omega)$ is defined as

$$\Delta_p^\lambda u = \sum_{i=1}^{d} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\right) - \lambda|u|^{p-2}u,$$

$$D(\Delta_p^\lambda) = \left\{ u \in W^{1,p}(\Omega); \Delta_p^\lambda u \in L^2(\Omega), -\frac{\partial u}{\partial \nu_p}(x) \in \beta(u(x)) \text{ for a.a. } x \in \Sigma \right\},$$

the p-conormal derivative of u, u_{ν_p}, being defined as

$$u_{\nu_p} = \frac{\partial u}{\partial \nu_p} = \sum_{i=1}^{d} \left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} cos(n, e_i),$$

in the above formula n is the unitary exterior normal to Σ and $\{e_1, e_2, \dots, e_d\}$ is the canonical base in $\mathbb{R}^d$.

It is possible to prove that the p-Laplace operator is an m-dissipative operator on $L^2(\Omega)$, see Example 1.5.4 in [32].

Notice that depending on the choice of the function β the condition

$$-\frac{\partial u}{\partial \nu_p}(x) \in \beta(u(x)) \text{ for a.a. } x \in \Sigma$$

incorporates the Dirichlet boundary conditions and the Neumann boundary conditions, see Remark 1.5.3 in [32].

Now consider the following quasi-autonomous differential inclusion

$$u'(t) \in Au(t) + f(t), \text{ for a.e. } t \in [0, T], \tag{3.1}$$

where $A\colon D(A) \subset X \multimap X$ is an m-dissipative multioperator and $f\colon [0, T] \to X$ is a given map. Several notions of solution for (3.1) were introduced. We assume $f \in L^1([0, T]; X)$ and consider the following one.

Definition 3.4. A function $u : [0, T] \to X$ is called an integral solution of (3.1) on $[0, T]$ if $u \in C([0, T]; X)$, $u(t) \in \overline{D(A)}$ for every $t \in [0, T]$ and

$$\|u(t) - x\|^2 \le \|u(s) - x\|^2 + 2 \int_s^t (u(\tau) - x, f(\tau) + y)_+ \, d\tau$$

for each $x \in D(A)$, $y \in Ax$ and $0 \le s \le t \le T$.

In the whole paper a function $u \in C([0, T]; X)$, with $u(t) \in \overline{D(A)}$, will be denoted by $u \in C([0, T]; \overline{D(A)})$. In this framework the integral solution is unique.

Theorem 3.5. *Let $A : D(A) \subset X \multimap X$ be an m-dissipative operator, let $f \in L^1([0, T]; X)$ and $x \in \overline{D(A)}$. There exists a unique integral solution $u : [0, T] \to X$ of (3.1) on $[0, T]$ satisfying $u(0) = x$.*

In this framework also the concept of limit solution has been introduced. Under the assumptions of the previous theorem the two definitions of solution are equivalent.

Lemma 3.6. *Let $A : D(A) \subseteq X \multimap X$ be an m-dissipative operator, let $f, g \in L^1([0, T], X)$ and u, v be two integral solutions of the equation in (3.1) corresponding to f and g respectively. Then*

$$\|u(t) - v(t)\| \le \|u(s) - v(s)\| + \int_s^t \|f(\tau) - g(\tau)\| \, d\tau,$$

and

$$\|u(t)-v(t)\|^2 \leq \|u(s)-v(s)\|^2 + 2\int_s^t (u(\tau)-v(\tau), f(\tau)-g(\tau))_+ \, d\tau,$$

for each $0 \leq s \leq t \leq T$.

Let $C \subseteq X$ be a nonempty subset of X.

Definition 3.7. A family of functions $\{S(t); S(t) : C \to C, t \geq 0\}$ is called a semigroup of nonexpansive mappings on C if

(1) $S(0) = I$;
(2) $S(t+s) = S(t)S(s)$;
(3) $\lim_{t\downarrow 0} S(t)x = x$ for every $x \in C$;
(4) $\|S(t)x - S(t)y\| \leq \|x-y\|$ for each $x, y \in C$ and $t \geq 0$.

Theorem 3.8. *Let* $A : D(A) \subset X \multimap X$ *be an* m*-dissipative operator, then*

$$S(t)x = \lim_{n\to\infty} \left(I + \frac{t}{n}A\right)^{-n} x \tag{3.2}$$

exists for each $x \in \overline{D(A)}$ *and uniformly for* t *in any compact subset of* $\mathbb{R}_+$. *In addition,* $\{S(t)\}_{t\geq 0}$ *is a semigroup of non expansive mappings on* $\overline{D(A)}$ *and for each* $x \in D(A)$ *we have*

$$\|S(t)x - x\| \leq t|Ax|,$$

where $|Ax| := \inf\{\|y\| \ : \ y \in Ax\}$.

Definition 3.9. The semigroup $\{S(t)\}_{t\geq 0}$ is said to be equicontinuous if $\{S(t)x, \ x \in D\}$ is equicontinuous at any $t > 0$ for any bounded subset $D \subset X$, i.e. for every $\varepsilon > 0$ there exists $\delta(\varepsilon)$ such that for every $t, s \in [0, T]$, with $|t-s| < \delta(\varepsilon)$ it holds

$$\|S(t)x - S(s)x\| < \varepsilon \quad \forall\, x \in D.$$

We recall that for each $x \in \overline{D(A)}$ the function $u : [a, +\infty) \to \overline{D(A)}$ defined by $u(t) := S(t-a)x$, for each $t \in [a, +\infty)$ is the unique integral solution of the Cauchy problem

$$\begin{cases} u'(t) \in Au(t), & a \leq t < +\infty \\ u(a) = x \end{cases}$$

Definition 3.10. Let $S \subseteq \mathbb{R}$ be a measurable subset. A subset $D \subset L^1(S, X)$ is called uniformly integrable if for every $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ such that $\Omega \subset S$ and $\mu(\Omega) < \delta(\varepsilon)$ implies

$$\int_\Omega \|f(s)\|\, ds < \varepsilon \qquad \text{for all } f \in D,$$

where μ is the Lebesgue measure on $\mathbb{R}$.

Notice that, when S is compact, an uniformly integrable set $D \subset L^1(S, X)$ is also bounded. Moreover, the following weak compactness criterion holds.

Theorem 3.11. *Let $D \subset L^1([0,T]; X)$ be an uniformly integrable subset such that $D(t) = \{f(t), f \in D\}$ is weakly relatively compact for a.e. $t \in [0,T]$. Then D is weakly relatively compact in $L^1([0,T]; X)$.*

Let $x \in \overline{D(A)}$. We denote by $K_x : L^1([0,T]; X) \to C([0,T]; \overline{D(A)})$ the solution operator of (3.1), i.e. for $f \in L^1([0,T]; X)$,

$$K_x f = u, \tag{3.3}$$

where u is the integral solution of (3.1) with $u(0) = x$.

Lemma 3.12 *(Lemma 3.3 in [34]). If X^* is uniformly convex and A is an m-dissipative multioperator generating an equicontinuous semigroup, then for any uniformly integrable sequence $\{w_k\}_{k=1}^{\infty} \subset L^1([0,T]; X)$ and relatively compact subset $\{x_k\}_{k=1}^{\infty} \subset \overline{D(A)}$, we have*

$$\chi(\{K_{x_k} w_k)(t) \; : \; k \geq 1\}) \leq \int_0^t \chi(\{w_k(s) : \; k \geq 1\})\, ds, \; t \in [0,T],$$

where χ is the Hausdorff measure of noncompactness.

Now we show an equicontinuity result for the solutions of (3.1). The result appeared in [25] in a special case (see also [33]).

Lemma 3.13 *(Lemma 3.5 in [34]). If A generates an equicontinuous semigroup $\{S(t)\}_{t \geq 0}$, $B \subset L^1([0,T]; X)$ is uniformly integrable and $C \subset \overline{D(A)}$ is compact, then the set*

$$\Pi = \{u \; : \; u \textit{ is the integral solution of } (3.1) \textit{ with } u(0) = x, \textit{ for some } f \in B \textit{ and } x \in C\}$$

is bounded and equicontinuous in $C([0,T]; X)$.

Proof. At first we prove that Π is bounded.

Let $u \in \Pi$, so there exists $f \in B$ such that $u = K_{u(0)} f$ with $u(0) \in C$. Let now $\overline{f} \in B$ and $\overline{u} = K_{u(0)} \overline{f}$. By Lemma 3.6, we have that

$$\begin{aligned} \|u(t)\| &\leq \|u(t) - \overline{u}(t)\| + \|\overline{u}(t)\| \leq \|u(0) - \overline{u}(0)\| + \int_0^t \|f(s) - \overline{f}(s)\|\, ds + \|\overline{u}(t)\| \\ &\leq \|f\|_1 + \|\overline{f}\|_1 + \|\overline{u}(t)\| \quad \text{for every } t \in [0,T] \end{aligned}$$

and then Π is bounded since B is uniformly integrable and hence bounded.

Now we prove the equicontinuity of Π, i.e. we show that for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for every $0 \leq s \leq t \leq T$, $|t - s| \leq \delta(\varepsilon)$ it holds

$$\|u(t) - u(s)\| \leq \varepsilon \quad \text{for every } u \in \Pi.$$

Let $x \in C$. First we prove the equicontinuity of the set

$$\Pi' = \{u = K_x f, \; f \in B\},$$

where K_x is defined in (3.3).

Fix $\varepsilon > 0$. By the uniform integrability of B, there exists $\gamma(\varepsilon) > 0$ such that for every measurable subset $E \subset [0,T]$ with Lebesgue measure $\mu(E) \leq \gamma$ it holds

$$\int_E \|f(r)\| \, dr \leq \varepsilon \quad \text{for every } f \in B.$$

Consider $s = 0$. By the continuity of $\{S(t)\}_{t\geq 0}$ there exists $\beta(\varepsilon) > 0$ such that for every $t \in [0, \beta(\varepsilon)]$

$$\|S(t)x - x\| \leq \varepsilon.$$

Moreover, the unique integral solution of the following problem

$$\begin{cases} z'(t) \in Az(t), \text{ for a.e. } t \geq 0, \\ z(0) = x, \end{cases} \tag{3.4}$$

is given by $z(t) = S(t)x$. By Lemma 3.6, we have that for every $t \in [0, \gamma(\varepsilon)]$

$$\|u(t) - z(t)\| \leq \|u(0) - z(0)\| + \int_0^t \|f(s)\| \, ds \leq \varepsilon,$$

uniformly with respect to $f \in B$. Therefore, denoting with $\delta(\varepsilon) = \min\{\gamma(\varepsilon), \beta(\varepsilon)\}$, for $t \in [0, \delta(\varepsilon)]$ we get

$$\begin{aligned} \|u(t) - u(0)\| &\leq \|u(t) - z(t)\| + \|z(t) - u(0)\| \\ &= \|u(t) - z(t)\| + \|S(t)x - x\| \leq 2\varepsilon, \end{aligned}$$

uniformly with respect to $u \in \Pi'$.

Now let $s > 0$. Assume $\gamma = \gamma(\varepsilon) < s$. Let $u \in \Pi'$ and consider $t, s \in [0,T]$ such that $|t - s| \leq \gamma$. Assume, without loss of generality, that $t > s$. The unique integral solution $w \in C([s-\gamma, T]; \overline{D(A)})$ of the problem

$$\begin{cases} w'(t) \in Aw(t), \quad \text{for a.e } t \geq s - \gamma \\ w(s-\gamma) = u(s-\gamma), \end{cases} \tag{3.5}$$

is given by $w(t) = S(t - (s - \gamma))u(s - \gamma)$. By Lemma 3.6, we have that

$$\|u(s) - w(s)\| \leq \|u(s-\gamma) - w(s-\gamma)\| + \int_{s-\gamma}^{s} \|f(r)\| \, dr \leq \varepsilon,$$

and also

$$\|u(t) - w(t)\| \leq \|u(s-\gamma) - w(s-\gamma)\| + \int_{s-\gamma}^{t} \|f(r)\| \, dr \leq \int_{s-\gamma}^{s} \|f(r)\| \, dr + \int_{s}^{t} \|f(r)\| \, dr \leq 2\varepsilon.$$

Moreover, by the equicontinuity of the semigroup $\{S(t)\}_{t\geq 0}$, applied to the bounded set

$$D = \{z(t), \ z \in \Pi', t \in [0,T]\},$$

there exists $\beta(\varepsilon) > 0$ such that for every $t, s \in [0,T]$, with $|t - s| < \beta(\varepsilon)$ we have that

$$\|S(t)\eta - S(s)\eta\| \leq \varepsilon \quad \forall\, \eta \in D.$$

Notice that $u(s-\gamma) \in D$. Thus, considering as above $\delta(\varepsilon) = \min\{\gamma(\varepsilon), \beta(\varepsilon)\}$, for every $t, s \in [0,T]$ such that $|t-s| \leq \delta(\varepsilon)$ we get

$$\begin{aligned}\|u(t) - u(s)\| &\leq \|u(t) - w(t)\| + \|w(t) - w(s)\| + \|w(s) - u(s)\| \\ &\leq 3\varepsilon + \|S(t-s+\gamma)u(s-\gamma) - S(\gamma)u(s-\gamma)\| \leq 4\varepsilon,\end{aligned}$$

uniformly with respect to $u \in \Pi'$.

Now we prove the equicontinuity of the set Π. Let $\varepsilon > 0$. By the compactness of C, there exists $\{x_i\}_{i=1}^n \subset C$ such that $C \subset \cup_{i=1}^n B(x_i, \varepsilon)$. Moreover by Lemma 3.6,

$$\|K_x f(t) - K_y f(t)\| \leq \|x - y\| \quad \text{for every } x, y \in \overline{D(A)}.$$

By the equicontinuity of the set Π' we have that there exists $\delta_x > 0$ such that for every $0 \leq s \leq t \leq T$, $|t-s| \leq \delta_x$

$$\|K_x f(t) - K_x f(s)\| \leq \varepsilon.$$

Thus, denoting with $\delta = \min\{\delta_{x_i},\ i = 1, \ldots, n\}$ and letting $x \in C$, $f \in B$ and $u = K_x f$, we obtain

$$\begin{aligned}\|u(t) - u(s)\| &= \|K_x f(t) - K_x f(s)\| \\ &\leq \|K_x f(t) - K_{x_i} f(t)\| + \|K_{x_i} f(t) - K_{x_i} f(s)\| + \|K_{x_i} f(s) - K_x f(s)\| \\ &\leq \|x - x_i\| + \varepsilon + \|x_i - x\| \leq 3\varepsilon,\end{aligned}$$

uniformly with respect to $u \in \Pi$. □

4. Problem statement

We consider the problem (1.1) in a Banach space $(X, \|\cdot\|)$ with uniformly convex dual X^*. We recall that in this setting

$$(x, y)_+ = (x, y)_- = \langle J(x), y \rangle,$$

where $J : X \to X^*$ is the duality map and we assume the following hypotheses on the problem (1.1).

(H_A) $A : D(A) \subset X \multimap X$ is an m-dissipative operator satisfying the following assumptions:
- (H_A^0) A generates an equicontinuous semigroup;
- (H_A^1) $\overline{D(A)}$ is a convex subset of X.

(H_F) $F : [0,T] \times X \multimap X$ satisfies the following assumptions:
- (H_F^0) for every $x \in X$ and every $t \in [0,T]$, $F(t,x)$ is a nonempty, convex, compact set;
- (H_F^1) for every $x \in X$ the map $F(\cdot, x) : [0,T] \to X$ has a measurable selection;
- (H_F^2) for a.e. $t \in [0,T]$ the map $F(t, \cdot) : X \to X$ is $X - X^w$ upper semicontinuous;
- (H_F^3) for every $\ell > 0$, there exists a map $\alpha_\ell \in L^1([0,T]; \mathbb{R}_+)$ such that

$$\|F(t,x)\| \leq \alpha_\ell(t), \quad \text{for a.e. } t \in [0,T] \text{ and every } x \in B_\ell(0);$$

- (H_F^4) there exists $\beta \in L^1([0,T]; \mathbb{R}_+)$ such that

$$\chi(F(t,D)) \leq \beta(t)\chi(D), \quad \text{for every bounded set } D \subset X,$$

where χ is the Hausdorff measure of non compactness defined in Section 2;

(H_F^5) *transversality condition:* there exist $\overline{x} \in D(A)$, $\overline{y} \in A(\overline{x})$, $\varepsilon > 0$ and

$$r > \max\left\{\varepsilon, \sup_{t\in[0,T]} \|S(t)\overline{x} - \overline{x}\|, \|\overline{x}\|\right\}$$

such that for every $x \in X$, $r - \varepsilon < \|x - \overline{x}\| < r$ and for a.e. $t \in [0,T]$ there exists $z \in F(t,x)$ (possibly depending on t) such that

$$\langle J(x - \overline{x}), \lambda z + \overline{y}\rangle \leq 0\,, \quad \text{for every } \lambda \in (0,1]. \tag{4.1}$$

(H_g) $g : C([0,T];X) \to \overline{D(A)}$ is a compact and continuous map such that

$$\|g(u) - \overline{x}\| \leq r \quad \text{for every} \quad u \in C([0,T];X),\ \|u - \overline{x}\|_0 \leq r,$$

where r and $\overline{x}$ are from (H_F^5).

Remark 4.1. If $0 \in D(A)$ and $0 \in A(0)$, in hypothesis (H_F^5) we can consider $\overline{x} = \overline{y} = 0$ and we can assume (4.1) only for $\lambda = 1$.

The uniform convexity of the dual X^* implies that X is reflexive. Thus by Proposition 2.2 of [10] we get the following result.

Proposition 4.2. *Under the conditions $(H_F^1) - (H_F^3)$, for every $q \in C([0,T];X)$ the set*

$$S_q = \{f \in L^1([0,T];X),\ f(s) \in F(s,q(s)),\ \textit{for a.e. } s \in [0,T]\}$$

is a nonempty, closed and convex subset of $L^1([0,T];X)$.

For $A \subset X$ we denote by $W_\varepsilon(A)$ the ϵ neighbourhood of A, i.e.

$$W_\varepsilon(A) = \{x \in X : d(x,A) < \varepsilon\}$$

where $d(x,A) = \inf_{y\in A} d(x,y)$. Now, denoting the Hausdorff metric by d_H, i.e. for $A, B \subset X$

$$d_H(A,B) = \inf\{\varepsilon : A \subset W_\varepsilon(B),\ B \subset W_\varepsilon(A)\},$$

we will show some easy to check properties on the space X and on the multivalued map $F : [0,T] \times X \multimap X$ implying assumptions (H_A^1) (H_F^0) and (H_F^4).

Remark 4.3. It is well known that if X is a uniformly convex Banach space, or if it is a reflexive Banach space with a Frechét differentiable dual X^*, then $\overline{D(A)}$ is convex (see [23, Remark 3.6]).

Remark 4.4. If $F : [0,T] \times X \multimap X$ has weakly compact values, $F(\cdot, x)$ is measurable for every $x \in X$, $F(t,\cdot)$ is upper semicontinuous with respect to the weak topology for a.e. $t \in [0,T]$, then has a restriction $F_0 : [0,T] \times X \multimap X$, defined as

$$F_0(t,x) = \{z_\lambda \in X,\ z_\lambda = \lambda y_1 + (1-\lambda)y_2,\ \lambda \in [0,1], y_1, y_2 \in F(t,x)\}$$

that is measurable in the first variable, strong-weak upper semicontinuous in the second variable and that has strongly compact values. Notice that $F_0(t,x) \subset F(t,x)$ for every $t \in [0,T]$ and $x \in X$, so it is possible to obtain a solution of problem (1.1) considering F replaced by F_0.

Remark 4.5. Trivially, if a multimap $F : [0,T] \times X \multimap X$ is compact, satisfies (H_F^4) with $\beta(t) \equiv 0$.

Moreover, if F has compact values and there exists a map $\beta \in L^1([0,T];X)$ such that

$$d_H(F(t,x), F(t,y)) \leq \beta(t)\|x-y\| \tag{4.2}$$

for a.e. $t \in [0,T]$ and every $x, y \in X$, where d_H is the Hausdorff distance, then the multimap F satisfies (H_F^4).

Indeed, let $t \in [0,T]$, $D \subset X$ be a bounded set, $\varepsilon > 0$ and $\mathcal{S}$ be a finite $\chi(D)+\varepsilon$-net of D, i.e. for every $x \in D$ there exists $\overline{y} \in \mathcal{S}$ such that

$$\|x - \overline{y}\| \leq \chi(D) + \varepsilon.$$

Thus, for every $x \in D$ and $z \in F(t,x)$ there exists $z_1 \in F(t,\overline{y})$ such that

$$\|z - z_1\| \leq \beta(t)\|x-\overline{y}\| \leq \beta(t)(\chi(D)+\varepsilon).$$

Hence

$$F(t,D) \subseteq \bigcup_{y\in\mathcal{S}} F(t,y) + \beta(t)(\chi(D)+\varepsilon)B_1(0).$$

Now, since $\mathcal{S}$ is a finite set and $F(t,y)$ is compact, it follows that the set $\bigcup_{y\in\mathcal{S}} F(t,y)$ is a compact set. Thus, by the subadditivity, regularity and semi-homogeneity of the Hausdorff measure of non compactness we get

$$\chi(F(t,D) \leq \chi\left(\bigcup_{y\in\mathcal{S}} F(t,y)\right) + \chi\left(\beta(t)(\chi(D)+\varepsilon)B_1(0)\right) \leq \beta(t)(\chi(D)+\varepsilon),$$

and we obtain the claimed result by the arbitrariness of ε.

Finally, by the additive properties of the Hausdorff measure of non compactness, a multimap $F : [0,T] \times X \multimap X$, $F = F_1 + F_2$, with F_1 a compact multimap and F_2 satisfying (4.2) verifies (H_F^4).

Here is an example of a map that is neither compact nor Lipschitz, but satisfies the hypothesis (H_F^4).

Example 4.6. Consider in ℓ_2 the map $f : B_1(0) \to B_1(0)$ defined as $f(x_1,x_2,x_3,\dots) = (\sqrt{1-\|x\|_{\ell^2}^2}, x_1, x_2, \dots)$, where ℓ_2 is the space of square-summable sequences with values in $\mathbb{R}$. The map f is the sum of the one dimensional mapping $f_1 : B_1(0) \to B_1(0)$, $f_1(x) = (\sqrt{1-\|x\|_{\ell^2}}, 0, \dots, 0, \dots)$ and the isometry $f_2 : B_1(0) \to B_1(0)$, $f_2(x_1,x_2,x_3,\dots) = (0, x_1, x_2, \dots)$. Thus f_1 is a compact map and f_2 is Lipschitz, so the sum satisfies (H_F^4), but does not satisfy (4.2) in $x \in \ell_2$ with $\|x\|_{\ell^2} = 1$.

Furthermore, below we show an example of a compact map, thus satisfying (H_F^4), but not necessarily Lipschitz, which arises from applications, precisely in population dynamics, representing the mortality rate of a population.

Example 4.7. Let $\Omega \subset \mathbb{R}^n$ be an open domain with regular boundary, $X = L^p(\Omega)$, $p > 1$ and $h : [0,T] \times \Omega \times \mathbb{R} \to \mathbb{R}$ be a map satisfying

(h_1) for every $u \in \mathbb{R}$, $h(\cdot, \cdot, u) : [0, T] \times \Omega \to \mathbb{R}$ is measurable;
(h_2) for every $(t, x) \in [0, T] \times \Omega$, $h(t, x, \cdot) : \mathbb{R} \to \mathbb{R}$ is continuous;
(h_3) there exists $a > 0$ and $b \in \mathbb{R}$ such that

$$|h(t, x, u)| \leq a + b|u| \quad (t, x, u) \in [0, T] \times \Omega \times \mathbb{R}.$$

Then the operator $f : [0, T] \times L^p(\Omega) \to L^p(\Omega)$

$$f(t, v)(x) = h\left(t, x, \int_\Omega v(\xi)\, d\xi\right)$$

is a Carathéodory map, sending bounded sets into relatively compact ones and so satisfying assumptions (H_F^1), (H_F^2) and also (H_F^4) with $\beta(t) \equiv 0$.

We will show now an example of a superlinear map satisfying the assumptions $(H_F^0) - (H_F^4)$.

Example 4.8. Let $\Omega \subset \mathbb{R}^n$ be an open domain with regular boundary, $X = L^2(\Omega)$ and $f : L^2(\Omega) \to L^2(\Omega)$ defined as $f(u) = \|u\|_2^2 - bu$ with $b > 0$. Being the assumptions $(H_F^0) - (H_F^3)$ trivially satisfied, we will check only condition (H_F^4). The map f is the sum of the maps $f_1 : L^2(\Omega) \to L^2(\Omega)$, $f_1(u) = \|u\|_2^2$ and $f_2 : L^2(\Omega) \to L^2(\Omega)$, $f_2(u) = -bu$. Thus, f is the sum of the compact map f_1 and the Lipschitz map f_2, so satisfies (H_F^4). Considering f the nonlinearity in (1.1), under condition H_A we have that f may also satisfy (H_F^5). For instance, we can choose $\overline{x} = \overline{y} = 0$, and $\lambda = 1$, as in problem (6.2). Since $L^2(\Omega)$ is a Hilbert space, we have that condition (4.1) reads as

$$\langle J(u), f(u)\rangle = \langle u, f(u)\rangle \leq 0$$

and we have that

$$\begin{aligned}\langle u, f(u)\rangle &= \int_\Omega u(\xi) f(u(\xi))\, d\xi = \int_\Omega u(\xi)\|u\|_2^2\, d\xi + \int_\Omega u(\xi)(-bu(\xi))\, d\xi \\ &\leq \|u\|_2^2\|u\|_2\sqrt{|\Omega|} - b\|u\|_2^2 = \|u\|_2^2(\|u\|_2\sqrt{|\Omega|} - b) \leq r^2(r\sqrt{|\Omega|} - b) \leq 0\end{aligned}$$

provided $r \leq \dfrac{b}{\sqrt{|\Omega|}}$.

We want to prove the existence of at least one integral solution for the problem (1.1).

Definition 4.9. A function $u \in C([0, T]; \overline{D(A)})$ is said to be an integral solution to (1.1) if there exists $f \in L^1([0, T]; X)$, $f(\tau) \in F(\tau, u(\tau))$ for a.e. $\tau \in [0, T]$ such that u is an integral solution of (3.1) in the sense of Definition 3.4 with $u(0) = g(u)$.

The main result of this paper is the following.

Theorem 4.10. *Under the assumptions $(H_F), (H_A)$ and (H_g) there exists at least one integral solution of the problem* (1.1).

5. Existence results

In this section we prove the main result of the paper, i.e. the existence of at least one integral solution for problem (1.1).

First of all, to overcome the fact that for every $x \in X, r-\varepsilon < \|x-\overline{x}\| < r$, and a.e. $t \in [0,T]$, we assumed (H_F^5) for at least one element of $F(t,x)$ and not for all elements of $F(t,x)$, following a technique developed in [11], we introduce an auxiliary multimap. More precisely, denoting by

$$B_r^{\varepsilon} = \{x \in E : r-\varepsilon < \|x-\overline{x}\| < r\},$$

let $B\colon X \multimap X$ defined as

$$B(x) = \begin{cases} \{z \in X \ : \ \langle J(x-\overline{x}), \lambda z + \overline{y}\rangle \leq 0\,, \forall\, \lambda \in (0,1]\} & \text{if} \ \ x \in B_r^{\varepsilon}(\overline{x}), \\ X \ \ \text{otherwise} \end{cases}$$

where $\varepsilon > 0$ and $r > 0$ are from (H_F^5). Set $F_B\colon [0,T] \times X \multimap X$, $F_B(t,x) = F(t,x) \cap B(x)$. Notice that, by (H_F^5), $F(t,x) \cap B(x) \neq \emptyset$ for every $t \in [0,T]$ and $x \in X$, thus F_B is well defined and it is clear that the multimap F_B satisfies (H_F^5) for every $z \in F_B(t,x)$. Moreover, F_B satisfies also $(H_F^0)-(H_F^4)$. We prove this assertion in the next proposition.

Proposition 5.1. *The map $F_B\colon [0,T] \times X \multimap X$ satisfies assumptions $(H_F^0)-(H_F^4)$.*

Proof. It is easy to see that the set $B(x)$ is closed and convex for every $x \in X$, therefore the set $F_B(t,x)$ is compact and convex for every $t \in [0,T]$ and $x \in X$. Notice that

$$F_B(t,x) = \begin{cases} \{z \in F(t,x) \ : \ \langle J(x-\overline{x}), \lambda z + \overline{y}\rangle \leq 0\,, \forall\, \lambda \in (0,1]\} & \text{if} \ \ x \in B_r^{\varepsilon}(\overline{x}), \\ F(t,x) \ \ \text{otherwise} \end{cases}$$

Thus, for every $x \in X$ the map $F_B(\cdot,x) : [0,T] \multimap X$ has a measurable selection. Indeed for $\|x\| \leq r-\varepsilon$, or $\|x\| \geq r$, it is true by (H_F^1) and for $r-\varepsilon < \|x-\overline{x}\| < r$ it is enough to define $f : [0,T] \to X$ as $f(t) \equiv z$, with $z \in F(t,x)$ from (H_F^5).

Moreover for a.e $t \in [0,T]$ the multimap $F_B : X \multimap X$ is $X - X^w$ upper semicontinuous. First of all we will show that the multimap B is a sequentially closed multimap from X to X^w. Indeed let $x_n \to x_0$ and $z_n \rightharpoonup z_0$ with $z_n \in B(x_n)$ for any $n \in \mathbb{N}$. We have the following cases:

1. there exists $\overline{n} > 0$ such that $x_n \notin B_r^{\varepsilon}$ for every $n > \overline{n}$.
   In this case, we have also that $x_0 \notin B_r^{\varepsilon}$, so $B(x_n) = B(x_0) = X$.
2. there exists $\overline{n} > 0$ such that $x_n \in B_r^{\varepsilon}$ for every $n > \overline{n}$.
   In this case, we have that $r-\varepsilon \leq \|x_0-\overline{x}\| \leq r$. So, we have the following two subcases:
   2.a $\|x_0-\overline{x}\| = r-\varepsilon$ or $\|x_0-\overline{x}\| = r$.
      In this case $B(x_0) = X$ and there is nothing to prove.
   2.b $r-\varepsilon < \|x_0-\overline{x}\| < r$.
      Recalling that in our setting, the duality map $J : X \multimap X^*$ is single valued and continuous we get

$$0 \geq \langle J(x_n-\overline{x}), \lambda z_n + \overline{y}\rangle \to \langle J(x_0-\overline{x}), \lambda z_0 + \overline{y}\rangle\,, \quad \forall\, \lambda \in (0,1].$$

It follows

$$\langle J(x_0-\overline{x}), \lambda z_0 + \overline{y}\rangle \leq 0\,, \quad \forall\, \lambda \in (0,1].$$

Hence $z_0 \in B(x_0)$.

Moreover, by Proposition 2.5, for a.e. $t \in [0,T]$ the multimap $F(t,\cdot) : X \multimap X$ is $X - X^w$-closed. Therefore, for a.e. $t \in [0,T]$, $F_B(t,\cdot) : X \multimap X$ is $X - X^w$-sequentially closed as intersection of a $X - X^w$-closed multimap and a $X - X^w$-sequentially closed multimap.

Clearly, for every $x \in X$, $F_B(t,x) \subset F(t,x)$, thus, by the monotonicity of the norm and of the Hausdorff measure of non compactness, for every $t \in [0,T]$, $x \in X$ and every bounded set $D \subset X$ it follows that

$$\|F_B(t,x)\| \leq \|F(t,x)\|,$$

and

$$\chi(F_B(t,D)) \leq \chi(F(t,D)),$$

thus, assumptions (H_F^3) and (H_F^4) are trivially satisfied. By (H_F^3) and the reflexivity of the space X, F_B is also $X - X^\omega$ quasi compact. Since, in a Banach space, the weak closure of a weakly relatively compact set coincides with its weak sequential closure (see [21, Theorem 8.12.1, p. 549]), by Proposition 2.4, we have that for a.e. $t \in [0,T]$, the multimap $F_B(t,\cdot) : X \multimap X$ is $X - X^w$ upper semicontinuous. □

Consider the following auxiliary problem

$$\begin{cases} u'(t) \in Au(t) + F_B(t,u(t)), \text{ for a.e. } t \in [0,T], \\ u(0) = g(u). \end{cases} \tag{5.1}$$

Clearly, all solutions of (5.1) are solutions of (1.1).

Moreover, for every $\lambda \in [0,1]$, we consider the following family of problems

$$\begin{cases} u'(t) \in Au(t) + \lambda F_B(t,v(t)), \text{ for a.e. } t \in [0,T], \\ u(0) = \lambda g(v) + (1-\lambda)\overline{x} \end{cases} \tag{5.2}$$

where $v \in C([0,T];X)$ is a given map.

By Propositions 5.1 and 4.2 the set

$$S_v^B = \{f \in L^1([0,T];X),\ f(s) \in F_B(s,v(s)), \text{ for a.e. } s \in [0,T]\}$$

is nonempty, convex and closed for every $v \in C([0,T],X)$. Thus, by Theorem 3.5, the operator $G : [0,1] \times C([0,T];\overline{D(A)}) \multimap C([0,T];\overline{D(A)})$ defined by

$$G(\lambda,v) = \{u \in C([0,T];\overline{D(A)}) \quad \text{integral solution of (5.2)}\},$$

is well defined. Recalling the definition of the operator K given in Section 3 and defining $\overline{g}(\lambda,v) : [0,1] \times C([0,T];X) \to X$, $\overline{g}(\lambda,v) = \lambda g(v) + (1-\lambda)\overline{x}$, we have that

$$G(\lambda,v) = \{K_{\overline{g}(\lambda,v)}(\lambda f), f \in S_v^B\}.$$

Clearly, a fixed point of the operator $G(1,\cdot)$ is an integral solution of (5.1) and so an integral solution of (1.1).

Proposition 5.2. *The operator* $G : [0,1] \times C([0,T];\overline{D(A)}) \multimap C([0,T];\overline{D(A)})$ *is closed.*

Proof. Let $\{v_n\}_{n=1}^{\infty} \subset C([0,T];\overline{D(A)})$, $\lambda_n \in [0,1]$ and $\{u_n\}_{n=1}^{\infty} \subset C([0,T];\overline{D(A)})$ with $u_n \in G(\lambda_n, v_n)$ such that $v_n \to v$, $u_n \to u$ in $C([0,T];\overline{D(A)})$ and $\lambda_n \to \lambda$ we want to prove that $u \in G(\lambda, v)$.

First of all, by the convergence of $\{v_n\}_{n=1}^{\infty}$ in $C([0,T];\overline{D(A)})$ we have the existence of a constant $M > 0$ such that

$$\|v_n(t)\| \leq M \quad \text{for every} \quad t \in [0,T] \quad \text{and for every } n \in \mathbb{N}.$$

Denote with $\{h_n\}_{n=1}^{\infty}$ the sequence of functions $h_n : [0,T] \to X$ such that $h_n \in S_{v_n}^B$, $u_n = K_{\overline{g}(\lambda_n, v_n)}(\lambda_n h_n)$. By assumption (H_F^3) we have that the sequence $\{h_n\}_{n=1}^{\infty}$ is uniformly integrable, moreover by (H_F^4)

$$\chi(\{h_n(t)\}_{n=1}^{\infty}) \leq \beta(t)\chi(\{v_n(t)\}_{n=1}^{\infty}) = 0.$$

Thus by Theorem 3.11, the sequence $\{h_n\}_{n=1}^{\infty}$ is weakly compact in $L^1([0,T];X)$, i.e. we can assume without loss of generality that there exists $h_0 \in L^1([0,T];X)$ such that $h_n \rightharpoonup h_0$. Moreover, by the continuity of the map $\overline{g}$ we get that $\overline{g}(\lambda_n, v_n) \to \overline{g}(\lambda, v)$ in $\overline{D(A)}$. Denoting with w the solution corresponding to λh_0 with $w(0) = \overline{g}(\lambda, v)$, i.e. $w = K_{\overline{g}(\lambda,v)}\lambda h_0$, by Lemma 3.6 we have that

$$\begin{aligned}
\|u_n(t) - w(t)\|^2 &\leq \|u_n(0) - w(0)\|^2 + 2\int_0^t \langle J(u_n(\tau) - w(\tau)), \lambda_n h_n(\tau) - \lambda h_0(\tau)\rangle \, d\tau \\
&= \|\overline{g}(\lambda_n, v_n) - \overline{g}(\lambda, v)\|^2 + 2\int_0^t \langle J(u(\tau) - w(\tau)), \lambda_n h_n(\tau) - \lambda h_0(\tau)\rangle \, d\tau \\
&\quad + 2\int_0^t \langle J(u_n(\tau) - w(\tau)) - J(u(\tau) - w(\tau)), \lambda_n h_n(\tau) - \lambda h_0(\tau)\rangle \, d\tau.
\end{aligned}$$

Notice that in our hypotheses $J : X \to X^*$ is a uniformly continuous operator on bounded subsets in X and, by the strong convergence of u_n to u, we have

$$\lim_{n\to\infty} J(u_n(\tau) - w(\tau)) = J(u(\tau) - w(\tau)).$$

Thus, by the boundedness of $\{\lambda_n\}$ and $\{h_n\}$ we have

$$\lim_{n\to\infty} 2\int_0^t \langle J(u_n(\tau) - w(\tau)) - J(u(\tau) - w(\tau)), \lambda_n h_n(\tau) - \lambda h_0(\tau)\rangle \, d\tau = 0, \quad \text{for every } t \in [0,T].$$

Moreover,

$$\|J(u(t) - w(t))\|_{X^*} = \|u(t) - w(t)\| \leq \|u\|_0 + \|w\|_0 \quad \text{for every } t \in [0,T].$$

Thus, $J(u(\cdot) - w(\cdot)) \in L^\infty([0,T], X^*)$. So, by the weak convergence of h_n to h_0 in $L^1([0,T], X)$ and the strong convergence of λ_n to λ in $\mathbb{R}$, we get

$$\lim_{n\to\infty} \int_0^t \langle J(u(\tau) - w(\tau)), \lambda_n h_n(\tau) - \lambda h_0(\tau)\rangle \, d\tau = 0, \quad \text{for every } t \in [0,T].$$

Hence, we obtain that for every $t \in [0,T]$

$$\lim_{n\to\infty} \|u_n(t) - w(t)\|^2 = 0.$$

Thus, by the uniqueness of the limit, we have $w(t) \equiv u(t)$. To conclude, we have only to prove that $h_0(t) \in F_B(t, v(t))$ for a.a. $t \in [0, T]$. By Mazur's convexity Theorem (see e.g. [22]) we have a sequence

$$\tilde{h}_m = \sum_{i=0}^{k_m} \lambda_{mi} h_{m+i}, \qquad \lambda_{mi} \geq 0, \; \sum_{i=0}^{k_m} \lambda_{mi} = 1$$

satisfying $\tilde{h}_m \to h_0$ in $L^1([0, T]; X)$. Hence, up to subsequence, there is $N_0 \subset [0, T]$ with Lebesgue measure zero such that $\tilde{h}_m(t) \to h_0(t)$ and $F_B(t, \cdot)$ is u.s.c for all $t \in [0, T] \setminus N_0$. Fix $t_0 \notin N_0$ and assume, by contradiction, that $h_0(t_0) \notin F_B(t_0, v(t_0))$. Since $F_B(t_0, v(t_0))$ is closed and convex, from the Hahn-Banach Theorem there is a weakly open convex set $V \supset F_B(t_0, v(t_0))$ satisfying $h_0(t_0) \notin \overline{V}^w$. Since $F_B(t_0, \cdot)$ is $X - X^w$ u.s.c., we can also find a neighbourhood V_1 of $h_0(t_0)$ such that $F(t_0, x) \subset V$ for all $x \in V_1$. The convergence $v_m(t_0) \to v(t_0)$ as $m \to \infty$ then implies the existence of $m_0 \in \mathbb{N}$ such that $v_m(t_0) \in V_1$ for all $m > m_0$. Therefore $h_m(t_0) \in F_B(t_0, v_m(t_0)) \subset V$ for all $m > m_0$. The convexity of V implies that $\tilde{h}_m(t_0) \in V$ for all $m > m_0$ and, by the convergence, we arrive to the contradictory conclusion that $h_0(t_0) \in \overline{V}^w$. We obtain that $h_0(t) \in F_B(t, v(t))$ for a.a. $t \in [0, T]$. □

Proposition 5.3. *The operator* $G : [0, 1] \times C([0, T]; \overline{D(A)}) \multimap C([0, T]; \overline{D(A)})$ *is upper semicontinuous.*

Proof. By Proposition 5.2, the operator G is closed. Thus, to get the upper semicontinuity of G, we will prove that the operator G is quasi compact and we will apply Proposition 2.4. To this aim, let $\{v_n\}_{n=1}^{\infty} \subset C([0, T]; \overline{D(A)})$, $\{\lambda_n\}_{n=1}^{\infty} \subset [0, 1]$ be two convergent sequences and consider $\{u_n\}_{n=1}^{\infty} \subset C([0, T]; \overline{D(A)})$ with $u_n \in G(\lambda_n, v_n)$. As before, denote by $\{h_n\}_{n=1}^{\infty}$ the sequence of functions $h_n : [0, T] \to X$, $h_n \in S^B_{v_n}$ and by $\{g_n\}_{n=1}^{\infty} \subset \overline{D(A)}$ the sequence of functions defined as $g_n = \overline{g}(\lambda_n, v_n)$. We observe that $\{g_n\}$ is a convergent sequence, hence, applying Lemma 3.12, (H^4_F) and (2.1), we have

$$\begin{aligned}\chi(\{u_n(t)\}_{n=1}^{\infty}) &= \chi(\{K_{g_n}(\lambda_n h_n)(t)\}_{n=1}^{\infty}) \leq \int_0^t \chi(\{\lambda_n h_n(s)\}_{n=1}^{\infty})\, ds \\ &\leq \int_0^t \chi(\{h_n(s)\}_{n=1}^{\infty})\, ds \leq \int_0^t \beta(s)\chi(\{v_n(s)\}_{n=1}^{\infty})\, ds = 0.\end{aligned}$$

Thus, we have that $\gamma(\{u_n\}_{n=1}^{\infty}) = \sup_{t\in[0,T]} e^{-Lt}\chi(\{u_n(t)\}_{n=1}^{\infty}) = 0$. Furthermore, by Lemma 3.13 we have that $\{u_n\}_{n=1}^{\infty}$ is a sequence of equicontinuous functions, thus $\mathrm{mod}_C(\{u_n\}_{n=1}^{\infty}) = 0$. Getting $\nu(\{u_n\}_{n=1}^{\infty}) = (0, 0)$. Obtaining the relative compactness of the sequence $\{u_n\}_{n=1}^{\infty}$ and thus that G is a quasicompact operator.

Analogously, it is possible to prove that for every $\lambda \in [0, 1]$ and $v \in C([0, T]; \overline{D(A)})$, the set $G(\lambda, v)$ is relatively compact. Moreover, by Proposition 5.2 it follows that for every $\lambda \in [0, 1]$ and $v \in C([0, T]; \overline{D(A)})$, the set $G(\lambda, v)$ is closed and hence compact.

In conclusion, by Proposition 2.4 G is an upper semicontinuous multimap. □

Now we prove that G has R_δ values following the same lines of the proof of [34, Theorem 3.1].

Proposition 5.4. *The operator* $G : [0, 1] \times C([0, T]; \overline{D(A)}) \multimap C([0, T]; \overline{D(A)})$ *has* R_δ *values.*

Proof. In Proposition 5.3 we have proven that G has compact values.

Next we will show that G has contractible values, thus obtaining that G has R_δ values. Let $\lambda \in [0,1]$, $v \in C([0,T];\overline{D(A)})$ and let $C = G(\lambda, v)$, fix $\overline{f} \in S_v^B$ and define $h : [0,1] \times C \to C$ by

$$h(\eta, u)(t) = \begin{cases} u(t), & \text{if } t \in [0, \eta T), \\ \overline{u}(t; \eta T, u(\eta T)), & \text{if } t \in [\eta T, T], \end{cases}$$

where $\overline{u}(\cdot; t_0, x_0)$ is the solution of

$$\begin{cases} w'(t) \in Aw(t) + \overline{f}(t) & t \in [t_0, T] \\ w(t_0) = x_0. \end{cases}$$

Since $u = K_{\overline{g}(\lambda,v)}(\lambda f)$ for some $f \in S_v^B$, we have $h(\eta, u) = K_{\overline{g}(\lambda,v)}(\lambda \widetilde{f})$ with $\widetilde{f} := f\chi_{[0,\eta T]} + \overline{f}\chi_{[\eta T,T]} \in S_v^B$, hence the range of h is contained in C. Indeed, let $\eta \in [0,1]$ and $u \in C$, the map $h(\eta, u)$ is a continuous map and $h(\eta, u)(t) \in \overline{D(A)}$ for every $t \in [0,T]$. Now we prove that $h(\eta, u)$ is a solution of (3.1).

For the cases $0 \le s \le t \le \eta T$ and $\eta T \le s \le t \le T$ is trivial by the definition of u and $\overline{u}(\cdot; \eta T, u(\eta T))$.

So, let $0 \le s \le \eta T \le t \le T$, $x \in D(A)$ and $y \in Ax$, we have

$$\begin{aligned} \|h(\eta,u)(t) - x\|^2 &= \|\overline{u}(t;\eta T, u(\eta T)) - x\|^2 \le \|\overline{u}(\eta T; \eta T, u(\eta T)) - x\|^2 \\ &\quad + 2\int_{\eta T}^{t} \langle J(\overline{u}(\tau;\eta T, u(\eta T)) - x), \lambda \overline{f}(\tau) + y\rangle \, d\tau \\ &= \|u(\eta T) - x\|^2 + 2\int_{\eta T}^{t} \langle J(\overline{u}(\tau;\eta T, u(\eta T)) - x), \lambda \overline{f}(\tau) + y\rangle \, d\tau \\ &\le \|u(s) - x\|^2 + 2\int_{s}^{\eta T} \langle J(u(\tau) - x), \lambda f(\tau) + y\rangle \, d\tau \\ &\quad + 2\int_{\eta T}^{t} \langle J(\overline{u}(\tau;\eta T, u(\eta T)) - x), \lambda \overline{f}(\tau) + y\rangle \, d\tau \\ &= \|h(\eta,u)(s) - x\|^2 + 2\int_{s}^{t} \langle J(h(\eta,u)(\tau) - x), \lambda \widetilde{f}(\tau) + y\rangle \, d\tau. \end{aligned}$$

We prove now that h is a continuous map. Let $\eta_1, \eta_2 \in [0,1]$, $\eta_2 > \eta_1$ and $u_1, u_2 \in C$.

For $t \in [0, \eta_1 T]$

$$\|h(\eta_1, u_1)(t) - h(\eta_2, u_2)(t)\| = \|u_1(t) - u_2(t)\|;$$

for $t \in [\eta_1 T, \eta_2 T]$, by Lemma 3.6, we have

$$\begin{aligned} \|h(\eta_1, u_1)(t) - h(\eta_2, u_2)(t)\| &= \|\overline{u}(t; \eta_1 T, u_1(\eta_1 T)) - u_2(t)\| \\ &\le \|u_1(\eta_1(T)) - u_2(\eta_1 T)\| + \int_{\eta_1 T}^{t} \|\overline{f}(s) - f_2(s)\| \, ds \end{aligned}$$

By (H_F^3) with $\ell = \|v\|$, we have that

$$\|h(\eta_1, u_1)(t) - h(\eta_2, u_2)(t)\| \le \|u_1(\eta_1 T) - u_2(\eta_1 T)\| + 2\|\nu_\ell\|_{L^1[\eta_1 T, \eta_2 T]};$$

for $t \in [\eta_2 T, T]$, again by Lemma 3.6, we have

$$\begin{aligned}
\|h(\eta_1, u_1)(t) - h(\eta_2, u_2)(t)\| &= \|\overline{u}(t; \eta_1 T, u_1(\eta_1 T)) - \overline{u}(t; \eta_2 T, u_2(\eta_2 T))\| \\
&\leq \|\overline{u}(\eta_2 T; \eta_1 T, u_1(\eta_1 T)) - u_2(\eta_2 T)\| \\
&\leq \|\overline{u}(\eta_1 T; \eta_1 T, u_1(\eta_1 T)) - u_2(\eta_1 T)\| + \int_{\eta_1 T}^{\eta_2 T} \|\overline{f}(s) - f_2(s)\| \, ds \\
&\leq \|u_1(\eta_1 T) - u_2(\eta_1 T)\| + \int_{\eta_1 T}^{\eta_2 T} \|\overline{f}(s) - f_2(s)\| \, ds \\
&\leq \|u_1(\eta_1 T) - u_2(\eta_1 T)\| + 2\|\nu_\ell\|_{L^1[\eta_1 T, \eta_2 T]}.
\end{aligned}$$

Now, by the absolute continuity of the integral function of ν_ℓ, for every $\varepsilon > 0$ there exists $\gamma(\varepsilon)$ such that for every $E \subset [0, T]$ with Lebesgue measure $\mu(E) < \gamma(\varepsilon)$ it holds

$$2 \int_E \nu_\ell(s) \, ds \leq \varepsilon.$$

Moreover, C being compact is an equicontinuous set of functions, thus for every $\varepsilon > 0$ there exists $\beta(\varepsilon) > 0$ such that for every $|t - s| \leq \beta(\varepsilon)$

$$\|u(t) - u(s)\| \leq \varepsilon, \quad \forall \quad u \in C.$$

Thus, for every $\varepsilon > 0$, choosing $\eta_1, \eta_2 \in [0, 1]$, $\eta_2 > \eta_1$ such that $(\eta_2 - \eta_1)T < \min\{\beta(\varepsilon), \gamma(\varepsilon)\}$ and $u_1, u_2 \in C$ such that $\|u_1 - u_2\|_0 < \varepsilon$ we obtain for every $t \in [0, T]$

$$\|h(\eta_1, u_1)(t) - h(\eta_2, u_2)(t)\| \leq 2\varepsilon,$$

hence the continuity of the map h.

Moreover, $h(0, u) = K_{\overline{g}(\lambda, v)}(\lambda \overline{f})$, $h(1, u) = K_{\overline{g}(\lambda, v)}(\lambda f) = u$. Therefore, C is contractible. □

We will prove that the operator G is condensing with respect to the measure of non compactness defined in (2.2).

Proposition 5.5. *The operator* $G : [0, 1] \times C([0, T]; \overline{D(A)}) \to C([0, T]; \overline{D(A)})$ *is* ν*-condensing.*

Proof. Let $L > 0$ such that

$$q := \sup_{t \in [0, T]} \int_0^t e^{-L(t-s)} \beta(s) \, ds < 1.$$

We notice that $q < 1$ for every $L > 0$ sufficiently big.

The set $\Omega \subset C([0, T]; \overline{D(A)})$ is bounded and such that

$$\nu(G([0, 1] \times \Omega)) \geq \nu(\Omega),$$

we will prove that Ω is a relatively compact set.

Let $\{w_n\}_{n=1}^{\infty} \subset G([0,1]\times\Omega)$ be a sequence which realizes the maximum in the definition of $\nu(G([0,1]\times\Omega))$, i.e.

$$\nu(G([0,1] \times \Omega)) = (\gamma(w_n), \text{mod}_C(w_n)).$$

Since $\{w_n\}_{n=1}^{\infty} \subset G([0,1] \times \Omega)$, there exist $\{v_n\}_{n=1}^{\infty} \subset \Omega$ and $\{\lambda_n\}_{n=1}^{\infty} \subset [0,1]$ such that $\|v_n(t)\| \leq M$ for some $M > 0$ and for every $t \in [0,T]$, $w_n \in G(\lambda_n, v_n)$. Given the assumption that

$$\nu(G([0,1] \times \Omega)) \geq \nu(\Omega)$$

and that $v_n \subset \Omega$, we then deduce that

$$(\gamma(w_n), \text{mod}_C(w_n)) \geq (\gamma(v_n), \text{mod}_C(v_n))$$

and, in particular

$$\gamma(w_n) \geq \gamma(v_n).$$

Denote by $\{h_n\}_{n=1}^{\infty}$ the sequence of functions $h_n : [0,T] \to X$ such that $h_n \in S_{v_n}^B$. By assumption (H_F^4)

$$\begin{aligned}\chi(\{h_n(s)\}_{n=1}^{\infty})) \leq \beta(s)\chi(\{v_n(s)\}_{n=1}^{\infty}) &= e^{Ls}\beta(s)e^{-Ls}\chi(\{v_n(s)\}_{n=1}^{\infty}) \\ &\leq e^{Ls}\beta(s) \sup_{\xi\in[0,T]} e^{-L\xi}\chi(\{v_n(\xi)\}_{n=1}^{\infty}) \\ &= e^{Ls}\beta(s)\gamma(\{v_n\}_{n=1}^{\infty}).\end{aligned}$$

By assumption (H_F^3) we have that the sequence $\{\lambda_n h_n\}_{n=1}^{\infty}$ is uniformly integrable. Moreover, by the compactness of g, it follows that the sequence $\{g_n\}_{n=1}^{\infty}$, defined as $g_n = \overline{g}(\lambda_n, v_n)$, is relatively compact in $\overline{D(A)}$. Hence, applying Lemma 3.12 we have

$$\begin{aligned}\chi(\{w_n(t)\}_{n=1}^{\infty}) &= \chi(\{K_{g_n}(\lambda_n h_n)(t)\}_{n=1}^{\infty}) \leq \int_0^t \chi(\{\lambda_n h_n(s)\}_{n=1}^{\infty})\, ds \\ &\leq \int_0^t \chi(\{h_n(s)\}_{n=1}^{\infty})\, ds \\ &\leq \gamma(\{v_n\}_{n=1}^{\infty}) \int_0^t e^{Ls}\beta(s)\, ds.\end{aligned}$$

Now it follows that

$$\begin{aligned}\gamma(\{v_n\}_{n=1}^{\infty}) \leq \gamma(\{w_n\}_{n=1}^{\infty}) &= \sup_{t\in[0,T]} e^{-Lt}\chi(\{w_n(t)\}_{n=1}^{\infty}) \\ &\leq \left(\sup_{t\in[0,T]} \int_0^t e^{-L(t-s)}\beta(s)\, ds\right) \gamma(\{v_n\}_{n=1}^{\infty} = q\, \gamma(\{v_n\}_{n=1}^{\infty}.\end{aligned}$$

Since $q < 1$ we obtain $\gamma(\{v_n\}_{n=1}^{\infty}) = 0$ and, as a consequence, $\gamma(\{w_n\}_{n=1}^{\infty}) = 0$. Furthermore, by Lemma 3.13 we have that $\{w_n\}_{n=1}^{\infty}$ is a sequence of equicontinuous functions, thus $\text{mod}_C(\{w_n\}_{n=1}^{\infty}) = 0$. Getting $\nu(G([0,1] \times \Omega)) = (0,0)$ and in conclusion $\nu(\Omega) = (0,0)$. $\square$

Notice that, by assumption $\overline{D(A)}$ is convex and, as a consequence, $C([0,T],\overline{D(A)})$ is convex. Following the notation of Section 2, let

$$D = X = C([0,T],\overline{D(A)}) \quad U = \{u \in C([0,T],\overline{D(A)}),: \|u(t)-\overline{x}\| < r \; \forall\, t \in [0,T]\}$$

with $r > 0$ and $\overline{x}$ from (H_F^5). By Propositions 5.3, 5.4 and 5.5 we have that the map $G(1,\cdot)$ is upper semicontinuous and condensing with R_δ values. If there exists $u \in \partial U$ such that $u \in G(1,u)$ we get a fixed point of the multioperator G at level 1 and thus a solution of Problem (1.1), i.e. the claimed result. So, we can assume $u \notin G(1,u)$ for every $u \in \partial U$. Thus, the index $\mathrm{ind}(G(1,\cdot),X,U)$ is well defined.

Moreover, the unique integral solution of the problem

$$\begin{cases} u'(t) \in Au(t) & t \in [0,T] \\ u(0) = \overline{x} \end{cases}$$

is given by $u(t) = S(t)\overline{x}$. Being $r > \|S(t)\overline{x}-\overline{x}\|$ we have that $S(t)\overline{x} \in U$. Thus, $\mathrm{ind}(G(0,\cdot),X,U) = 1$.

5.1. Proof of Theorem 4.10

We are able now to prove the existence of at least one mild solution of problem (1.1).

By Propositions 5.3, 5.4 and 5.5 we have that the map G is upper semicontinuous and condensing with R_δ values. In order to prove that the map G is an homotopy between the map $G(0,\cdot) \equiv 0$ and $G(1,\cdot)$ we have to show that $\mathrm{Fix}G(\lambda,\cdot)\cap\partial U = \emptyset$ for every $\lambda \in [0,1)$. Assume by contradiction that there is $u \in C([0,T],\overline{D(A)})$ with $\|u-\overline{x}\|_0 = r$ and $\overline{\lambda} \in [0,1)$ such that $u \in G(\overline{\lambda},u)$. Notice that $\overline{\lambda} \neq 0$, indeed $G(0,u) \equiv S(t)\overline{x}$ and by assumption $\|S(t)\overline{x}-\overline{x}\| < r$. Thus, we can assume $\overline{\lambda} \in (0,1)$. Let $t_0 \in [0,T]$ be such that $\|u(t_0)-\overline{x}\| = r$. Notice that $t_0 \neq 0$. Indeed, since $\|\overline{x}\| < r$, it follows

$$\begin{aligned} \|u(0)-\overline{x}\| &= \|\overline{\lambda}g(u) + (1-\overline{\lambda})\overline{x} - \overline{x}\| \\ &= \overline{\lambda}\|g(u)-\overline{x}\| \le \overline{\lambda}r < r. \end{aligned}$$

So, let $t_0 \in (0,T]$. By the continuity of u we can find $t_1 \in (0,t_0]$ satisfying $\|u(t_1)-\overline{x}\| = r$ and $\|u(t)-\overline{x}\| < r$ for $t \in [0,t_1)$. In particular, corresponding to $\varepsilon > 0$ from (H_F^5) there exists $\delta > 0$ such that

$$r - \varepsilon < \|u(t)-\overline{x}\| < r$$

for every $t \in [t_1-\delta, t_1)$.

Let $h \in S_u^B$ such that $u = K_{\overline{g}(\overline{\lambda},u)}(\overline{\lambda}h)$ and apply the definition of integral solution with $x = \overline{x}$ and $y = \overline{y}$. We have

$$\begin{aligned} r^2 &= \|u(t_1)-\overline{x}\|^2 \le \|u(t_1-\delta)-\overline{x}\|^2 + 2\int_{t_1-\delta}^{t_1} \langle J(u(\tau)-\overline{x}), \overline{\lambda}h(\tau)+\overline{y}\rangle\, d\tau \\ &< r^2 - \eta, \end{aligned}$$

with $-\eta = 2\int_{t_1-\delta}^{t_1} \langle J(u(\tau)-\overline{x}), \overline{\lambda}h(\tau)+\overline{y}\rangle\, d\tau \le 0$ by (H_F^5), hence a contradiction. By the normalization and homotopy invariance property of the index we get

$$\mathrm{ind}(G(1,\cdot),X,U) = \mathrm{ind}(G(0,\cdot),X,U) = 1.$$

Thus we obtain a fixed point of $G(1,\cdot)$, i.e. a solution of (5.1) and, as a consequence, a solution of (1.1).

6. Applications

We will apply the existence result in Theorem 4.10 to the following class of partial differential inclusions in a domain $\Omega \subset \mathbb{R}^n$ with smooth boundary:

$$\begin{cases} u_t \in a\,\Delta u(t,x) - \partial\varphi(u(t,x)) - bu(t,x) + [f_1(t,u(t,x)), f_2(t,u(t,x))] & \text{in } \Omega_T \\ -\dfrac{\partial u}{\partial \mu}(t,x) \in \partial j(u(t,x)) & \text{in } \partial\Omega_T \\ u(0,x) = \displaystyle\int_0^T \int_\Omega h(s,x,\xi,u(s,\xi))\,d\xi\,ds & \text{in } \Omega \end{cases} \tag{6.1}$$

where $\Omega_T = \Omega\times]0,T[$, $\partial\Omega_T = \partial\Omega \times [0,T]$, $\partial\varphi$ is the subdifferential of φ, $\Delta u(t,x)$ is the Laplacian of $u(t,\cdot)$, $a \geq 0$ and $\dfrac{\partial u}{\partial \mu}(t,x)$ is the normal derivative of $u(t,\cdot)$ at $x \in \partial\Omega$.

We consider problem (6.1) under the following assumptions:

(1) $\varphi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is a proper, convex, lower semicontinuous function such that
  - (1_i) $\varphi(0) = 0 = \min_{s\in\mathbb{R}} \varphi(s)$;
  - (1_{ii}) for every $v \in L^2(\Omega)$ $\varphi \circ v \in L^1(\Omega)$;

(2) $j : \mathbb{R} \to \mathbb{R}$ is a nonnegative, convex, continuous function such that
  - (2_i) $j(0) = \min_{s\in\mathbb{R}} j(s)$;
  - (2_{ii}) there exists a constant $C > 0$ such that

$$0 \leq j(s) \leq C(1 + s^2), \quad s \in \mathbb{R};$$

(3) $f_1, f_2 : [0,T] \times \mathbb{R} \to \mathbb{R}$ satisfy
  - (3_i) $f_1(t,u) \leq f_2(t,u)$ for every $(t,u) \in [0,T] \times \mathbb{R}$;
  - (3_{ii}) for every $u \in \mathbb{R}$, $f_1(\cdot,u) : \mathbb{R} \to \mathbb{R}$ is measurable and $f_1(\cdot,0) : \mathbb{R} \to \mathbb{R}$ is continuous with $L = \max_{t\in[0,T]} |f_1(t,0)|$;
  - (3_{iii}) for every $u \in \mathbb{R}$, $f_2(\cdot,u) : \mathbb{R} \to \mathbb{R}$ is measurable and $f_2(\cdot,0) \in L^1([0,T])$;
  - (3_{iv}) there exist $\alpha_1 > 0$ and $\alpha_2 \in L^1([0,T],\mathbb{R}_+)$ such that

$$|f_1(t,u_1) - f_1(t,u_2)| \leq \alpha_1|u_1 - u_2|$$

and

$$|f_2(t,u_1) - f_2(t,u_2)| \leq \alpha_2(t)|u_1 - u_2|,$$

for a.e. $t \in [0,T]$ and for every $u_1, u_2 \in \mathbb{R}$;

(4) $h : [0,T] \times \Omega \times \Omega \times \mathbb{R} \to \mathbb{R}$ is such that
  - (4_i) for a.e $(t,\xi) \in [0,T] \times \Omega$ the map $h(t,\cdot,\xi,\cdot) : \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous;
  - (4_{ii}) for every $(x,u) \in \Omega \times \mathbb{R}$ the map $h(\cdot,x,\cdot,u) : [0,T] \times \Omega \to \mathbb{R}$ is measurable;
  - (4_{iii}) there exists a function $\eta \in L^1([0,T],\mathbb{R}_+)$, with $\|\eta\|_1 \leq \frac{1}{|\Omega|}$, such that

$$|h(t,x,\xi,u)| \leq \eta(t)(1 + |u|) \quad \text{for every } (t,x,\xi,u) \in [0,T] \times \Omega \times \Omega \times \mathbb{R}.$$

Theorem 6.1. *If $b > \alpha_1$, for every $r > \max\left\{ \frac{\|\eta\|_1|\Omega|\sqrt{|\Omega|}}{1-\|\eta\|_1|\Omega|}, \frac{L\sqrt{|\Omega|}}{b-\alpha_1} \right\}$, the nonlocal problem (6.1) has a solution $u \in C([0,T], L^2(\Omega))$, with $\|u(t)\|_2 \le r$ for every $t \in [0,T]$.*

Proof. Let $X = L^2(\Omega)$. Consider the operators $\Phi : X \to \mathbb{R}$ and $\Psi : X \to \mathbb{R} \cup \{+\infty\}$ defined respectively as

$$\Phi(v) = \int_\Omega \varphi(v(x))\, dx$$

and

$$\Psi(v) = \begin{cases} \frac{1}{2} a \int_\Omega |\nabla v(x)|^2\, dx + a \int_{\partial\Omega} j(v(x))\, dx, & v \in H^1(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

It is possible to prove that they are proper, convex, lower semicontinuous functionals with

$$D(\Phi) = \{v \in L^2(\Omega), \varphi \circ v \in L^1(\Omega)\} = L^2(\Omega) \quad D(\Psi) = H^1(\Omega).$$

Notice that $D(\Phi) = L^2(\Omega)$, comes directly from assumption (1_{ii}). Moreover, $f \in \partial\Phi(v)$ if and only if

$$v, f \in L^2(\Omega),\ f(x) \in \partial\varphi(v(x)),\ \text{a.e. } x \in \Omega$$

and $\psi \in \partial\Psi(v)$ if and only if

$$\psi = -a\Delta v \quad \text{in } L^2(\Omega) \quad \text{and} \quad 0 \in \frac{\partial v}{\partial \mu}(t,x) + \partial j(v) \quad \text{in } L^2(\partial\Omega),$$

see [31, Example 2B, 2E, pp. 163-165] and [32, Example 1.6.2]. Furthermore, $\partial\Phi + \partial\Psi$ is m-accretive and equal to $\partial(\Phi + \Psi)$, see [31, Example 2F, p. 167].

We write problem (6.1) as the following abstract problem

$$\begin{cases} u'(t) \in A(u(t)) + F(t, u(t)) & t \in]0,T[\\ u(0) = g(u) \end{cases} \tag{6.2}$$

where $A = -\partial(\Phi + \Psi)$, the multimap $F : [0,T] \times L^2(\Omega) \to L^2(\Omega)$ is defined as $F(t,u) = -bu + G(t,u)$ with

$$G(t,u) = \left\{ v \in L^2(\Omega) : \exists\lambda \in [0,1] \text{ such that } v(x) = \lambda f_1(t, u(x)) + (1-\lambda) f_2(t, u(x)) \right\}$$

and $g : L^2(\Omega) \to L^2(\Omega)$

$$g(u)(x) = \int_0^T \int_\Omega h(s, x, \xi, u(\xi))\, d\xi\, ds, \quad x \in \Omega.$$

By Theorem 1.6.1 in [32], we have that $D(A)$ is a dense subset of $D(\Psi + \Phi)$, thus

$$\overline{D(A)} = \overline{D(\Psi + \Phi)} = \overline{D(\Psi) \cap D(\Phi)} = \overline{H^1(\Omega)} = L^2(\Omega).$$

Clearly, $\overline{D(A)}$ is a convex set and, since A is in the form of a subdifferential, by Theorem 1.6.2, Theorem 1.8.1 and Corollary 1.9.1 in [32], it is a m-dissipative operator generating an equicontinuous semigroup $\{S(t)\}_{t\geq 0}$. So, the operator $A : D(A) \subset L^2(\Omega) \to L^2(\Omega)$ satisfies the assumption (H_A).

Now, we prove that the multimap F satisfies the properties $(H_F^0) - (H_F^4)$ of Theorem 4.10. Notice that, according to its definition, it is enough to show that G satisfies such properties.

Let $u \in L^2(\Omega)$ and $t \in [0,T]$. We have that

$$\begin{aligned} \int_\Omega \|f_1(t,u(x))\|^2\,dx &\leq 2\int_\Omega \|f_1(t,u(x)) - f_1(t,0)\|^2\,dx + 2\int_\Omega \|f_1(t,0)\|^2\,dx \\ &\leq 2\alpha_1^2 \int_\Omega \|u(x)\|^2\,dx + 2L^2|\Omega| \\ &\leq 2\alpha_1^2\|u\|_2^2 + 2L^2|\Omega|, \end{aligned} \tag{6.3}$$

thus $f_1(t,u(\cdot)) \in L^2(\Omega)$ and so the set $G(t,u)$ is nonempty.

The set $G(t,u)$ is compact. Indeed, since given $\{v_n\} \in G(t,u)$, we have that there exists a sequence $\{\lambda_n\} \subset [0,1]$ such that

$$v_n(x) = \lambda_n f_1(t,u(x)) + (1-\lambda_n) f_2(t,u(x)) \quad \text{for a.e. } x \in \Omega.$$

The sequence $\{\lambda_n\}$ has a convergent subsequence, $\lambda_{n_k} \to \overline{\lambda}$. Thus, the corresponding subsequence, $\{v_{n_k}\}$, defined as

$$v_{n_k}(x) = \lambda_{n_k} f_1(t,u(x)) + (1-\lambda_{n_k}) f_2(t,u(x)) \quad \text{for a.e. } x \in \Omega,$$

converges in $L^2(\Omega)$ to $\overline{v} : \Omega \to \mathbb{R}$ defined as

$$\overline{v}(x) = \overline{\lambda} f_1(t,u(x)) + (1-\overline{\lambda}) f_2(t,u(x)) \quad \text{for a.e. } x \in \Omega.$$

Reasoning as for $f_1(t,u(\cdot))$ we can prove that $f_2(t,u(\cdot)) \in L^2(\Omega)$, getting $\overline{v} \in G(t,u)$. Now, let $\eta \in (0,1)$ and consider $v_1, v_2 \in G(t,u)$, we have that $\eta v_1 + (1-\eta)v_2 \in L^2(\Omega)$ as a convex combination of L^2- functions. Moreover it is easy to show that $\eta v_1 + (1-\eta)v_2 \in G(t,u)$, implying that $G(t,u)$ is a convex set and so condition (H_F^0) is satisfied.

For every $u \in L^2(\Omega)$, the function $f_1^u : [0,T] \to L^2(\Omega)$ defined as

$$f_1^u(t)(x) = f_1(t,u(x)) \quad \text{for a.e. } x \in \Omega$$

is a measurable selection of $G(t,u)$. Indeed, by (6.3) we have for a.e. $t \in [0,T]$ that $f_1^u(t)(\cdot) \in L^2(\Omega)$ and, by assumptions (3_{ii}) and (3_{iv}), for every $g \in L^2(\Omega)$ the map $gf_1^u \in L^1([0,T]\times\Omega)$. Hence, by Fubini's Theorem, the map

$$t \to \langle g, f_1^u(t)\rangle = \int_\Omega g(x) f_1(t,u(x))\,dx$$

is measurable. In conclusion, since $L^2(\Omega)$ is separable, by the Pettis measurability Theorem (see [31, Theorem 1.1, p. 103]), the map $f_1^u : [0,T] \to L^2(\Omega)$ is measurable. Moreover, by previous reasonings we have that $f_1^u(t) \in G(t,u)$ for a.e. $t \in [0,T]$. Hence condition (H_F^1) holds.

To prove assumption (H_F^2), let $t \in [0,T]$. Consider $\{u_n\} \subset L^2(\Omega)$, $u_n \to u$ and $\{v_n\} \subset L^2(\Omega)$, $v_n \in G(t,u_n)$. The fact that $v_n \in G(t,u_n)$ implies that there exists a sequence $\{\lambda_n\} \subset [0,1]$ such that

$$v_n(x) = \lambda_n f_1(t, u_n(x)) + (1 - \lambda_n) f_2(t, u_n(x)) \quad \text{for a.e. } x \in \Omega.$$

By the compactness of the interval $[0, 1]$, we have that there exists a convergent subsequence $\{\lambda_{n_k}\}$, $\lambda_{n_k} \to \overline{\lambda}$, moreover there exist a subsequence $\{u_{n_k}\}$ a.e. convergent to u and a function $\ell \in L^2(\Omega)$ such that

$$|u_{n_k}(x)| < \ell(x) \quad \text{for every } k \text{ and for a.e. } x \in \Omega.$$

So, by the continuity of the maps $f_1(t, \cdot)$, $f_2(t, \cdot)$, it follows that

$$v_{n_k}(x) = \lambda_{n_k} f_1(t, u_{n_k}(x)) + (1 - \lambda_{n_k}) f_2(t, u_{n_k}(x)) \to \overline{\lambda} f_1(t, u(x)) + (1 - \overline{\lambda}) f_2(t, u(x)), \quad \text{a.e. in } \Omega.$$

Thus, we have

$$\begin{aligned} |\lambda_{n_k} f_1(t, u_{n_k}(x)) + (1 - \lambda_{n_k}) f_2(t, u_{n_k}(x))| &\leq \alpha_1 |u_{n_k}(x)| + |f_1(t, 0)| + \alpha_2(t) |u_{n_k}(x)| + |f_2(t, 0)| \\ &\leq (\alpha_1 + \alpha_2(t)) \ell(x) + |f_1(t, 0)| + |f_2(t, 0)|. \end{aligned}$$

Hence, by the Lebesgue Convergence Theorem, $\{v_{n_k}\}$ converges in $L^2(\Omega)$, obtaining that $G(t, \cdot)$ is a quasi-compact map. Moreover, notice that $\{v_{n_k}\}$ converges to v, defined as

$$v(x) = \overline{\lambda} f_1(t, u(x)) + (1 - \overline{\lambda}) f_2(t, u(x)) \quad \text{for a.e. } x \in \Omega,$$

hence $v \in G(t, u)$. Thus the graph of $G(t, \cdot)$ is closed in $L^2(\Omega) \times L^2(\Omega)$. Finally, by Proposition 2.4, $G(t, \cdot)$ is u.s.c., then it is u.s.c. from $L^2(\Omega)$ to $(L^2(\Omega))^w$.

Now, let $\ell > 0$, $u \in B_\ell(0)$ and $t \in [0, T]$. For every $v \in G(t, u)$, by (3_{iv}) we have

$$\begin{aligned} \|v\|_2^2 &= \int_\Omega |v(x)|^2 \, dx = \int_\Omega |\lambda f_1(t, u(x)) + (1 - \lambda) |f_2(t, u(x))|^2 \, dx \\ &\leq 2 \int_\Omega \left(|f_1(t, u(x))|^2 + |f_2(t, u(x))|^2 \right) \, dx \\ &\leq 4 \int_\Omega (|f_1(t, u(x)) - f_1(t, 0)|^2 + |f_1(t, 0)|^2) \, dx \\ &\quad +4 \int_\Omega (|f_2(t, u(x)) - f_2(t, 0)|^2 + |f_2(t, 0)|^2) \, dx \\ &\leq 4(\alpha_1^2 + \alpha_2^2(t)) \|u\|_2^2 + 4|f_1(t, 0)|^2 |\Omega| + 4|f_2(t, 0)|^2 |\Omega| \\ &\leq 4(\alpha_1^2 + \alpha_2^2(t)) \ell^2 + 4(|f_1(t, 0)|^2 + 4|f_2(t, 0)|^2) |\Omega|, \end{aligned}$$

thus, by conditions (3_{ii}) and (3_{iii}), assumption (H_F^3) is satisfied as well, with $\alpha_\ell \in L^1([0, T])$, $\alpha_\ell(t) = 4(\alpha_1^2 + \alpha^2(t))\ell^2 + 4L + 4|f_2(t, 0)|^2 |\Omega|, t \in [0, T]$.

Now we prove assumption (H_F^4). Let $t \in [0, T]$, $D \subset L^2(\Omega)$ be a bounded set, by Remark 4.5 we have that

$$\chi(f_1(t, D)) \leq \alpha_1 \chi(D) \quad \text{and} \quad \chi(f_2(t, D)) \leq \alpha_2(t) \chi(D).$$

Hence, by (2.1), we have

$$\begin{aligned} \chi(G(t, D)) &\leq \chi \left(\cup_{\lambda \in [0,1]} (\lambda f_1(t, D) + (1 - \lambda) f_2(t, D)) \right) \\ &\leq \chi(f_1(t, D)) + \chi(f_2(t, D)) \leq (\alpha_1 + \alpha_2(t)) \chi(D), \end{aligned}$$

and then (H_F^4) is satisfied, with $\beta(\cdot) = \alpha_1 + \alpha(\cdot)$. In order to prove condition (H_F^5), we recall that for $w \in L^2(\Omega)$ with $\|w\|_2 > 0$, we have

$$\langle J(w), v\rangle = \int_\Omega w(\xi)\, v(\xi)\, d\xi.$$

Moreover, $0 \in D(A)$ and $0 \in A(0)$. Indeed, since

$$\partial\varphi(x) = \{z \in \mathbb{R}\ ;\ \varphi(x) \le \varphi(y) + (x-y)z, \text{ for every } y \in \mathbb{R}\}$$

and by (1_i), $0 = \varphi(0) = \min \varphi$, we have that $0 \in \partial\varphi(0)$ and so $0 \in \partial\Phi(0)$. Similarly by (2_i), we can prove that $0 \in \partial j(0)$. So, since for $v \equiv 0$, $-a\Delta v = 0$, we have that $0 \in \partial\Psi(0)$ as well, obtaining the claimed result. Moreover, we have that $\|S(t)0 - 0\| = 0$. Indeed, by Theorem 1.8.1 [32], $\|S(t)0 - 0\| \le t|A0|$ and $|A0| = \inf\{\|y\|; y \in A0\} = 0$.

Therefore, given $r > \frac{L\sqrt{|\Omega|}}{b-\alpha_1}$, for a.e. $t \in [0,1]$ and $w \in L^2(\Omega)$, $0 < \|w\|_2 < r$:

$$\begin{aligned}\langle J(w), f_1(t,w)\rangle &= \int_\Omega w(\xi) f_1(t, w(\xi))\, d\xi \\ &\le \int_\Omega |w(\xi)|(\alpha_1|w(\xi)| + |f_1(t,0)|)\, d\xi \le \alpha_1\|w\|_2^2 + |f_1(t,0)|\|w\|_2\sqrt{|\Omega|} \\ &\le \alpha_1\|w\|_2^2 + L\|w\|_2\sqrt{|\Omega|}.\end{aligned}$$

Consequently,

$$\begin{aligned}\langle J(w), -bw + f_1(t,w)\rangle &= \langle J(w), -bw\rangle + \langle J(w), f_1(t,w)\rangle \\ &\le -\|w\|_2\left((b-\alpha_1)\|w\|_2 - L\sqrt{|\Omega|}\right) \le 0,\end{aligned}$$

provided

$$\frac{L\sqrt{|\Omega|}}{b-\alpha_1} < \|w\|_2 < r.$$

So assumption (H_F^5) is satisfied with $\overline{x} = \overline{y} = 0$ and $\lambda = 1$ (see Remark 4.1).

By the above reasonings the assumptions (H_A) and $(H_F^0)-(H_F^5)$ of Theorem 4.10 are satisfied. Moreover, applying Proposition 4.1, Chapter 5 of [28], we have that g is a compact and continuous map from $L^2(\Omega)$ to $L^2(\Omega) = \overline{D(A)}$. Finally, for every $u \in L^2(\Omega)$ such that $\|u\|_2 \le r$

$$\begin{aligned}\|g(u)\|_2 &= \left(\int_\Omega \left|\int_0^T \int_\Omega h(t,x,\xi,u(\xi))\, d\xi\, dt\right|^2 dx\right)^{1/2} \\ &\le \left(\int_\Omega \left|\int_0^T \int_\Omega \eta(t)(1+|u(\xi)|)\, d\xi\, dt\right|^2 dx\right)^{1/2} \\ &= \sqrt{|\Omega|}\left(\int_0^T \int_\Omega \eta(t)\, d\xi\, dt + \int_0^T \int_\Omega \eta(t)|u(\xi)|\, d\xi\, dt\right) \\ &\le \sqrt{|\Omega|}\left(\|\eta\|_1|\Omega| + \|\eta\|_1\|u\|_2\sqrt{|\Omega|}\right) \\ &\le \sqrt{|\Omega|}\left(\|\eta\|_1|\Omega| + \|\eta\|_1 r\sqrt{|\Omega|}\right) \le r.\end{aligned}$$

Exploiting the fact that $\|\eta\|_1 < \frac{1}{|\Omega|}$ (see condition (4_{iii})), the last inequality holds for r sufficiently big. Thus, also the assumption (H_g) holds and we get the claimed result. $\square$

Remark 6.2. Notice that if $\varphi : \mathbb{R} \to \mathbb{R}$ is such that

$$0 \le \varphi(s) \le D(1+s^2), \quad s \in \mathbb{R}$$

for some constant $D > 0$, the assumption (1_{ii}) is satisfied.

We will see now an example of an operator $\Phi : X \to \mathbb{R}$ such that $\partial\Phi$ does not generate a compact semigroup. Hence, assuming $a = 0$ in problem (6.1), we obtain that the operator $A = -\partial\Phi$ generates an equicontinuous non compact semigroup.

Example 6.3. Let $K = \{v \in L^2(\Omega) : v(x) \ge 0 \text{ a.e. } x \in \Omega\}$ and

$$\Phi(v) = \begin{cases} 0 & v \in K \\ +\infty & v \notin K \end{cases}$$

In this case $D(\Phi) = D(\partial\Phi) = K$ and

$$\partial\Phi(u) = \{w \in L^2(\Omega) : \langle u - v, w\rangle \ge 0, \ \forall v \in K\}.$$

Thus, let $k > 0$ we have that the level set

$$\{u \in K : \|u\|_2 + \Phi(u) \le k\} = B_{L^2(\Omega)}(k)$$

is not relatively compact in X. Hence, by Proposition 2.2.2 in [32], the semigroup generated by $\partial\Phi$ is non compact. In this case the map $\varphi : \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ is defined as

$$\varphi(v) = \begin{cases} 0 & v \ge 0 \\ +\infty & v < 0 \end{cases}$$

Remark 6.4. The function φ in Example 6.3 does not satisfy the property (1_{ii}). However, such a condition is only needed to guarantee that the nonlocal initial condition $g : L^2(\Omega) \to L^2(\Omega)$ takes values in $\overline{D(A)}$. In Example 6.3 this property can be alternatively obtained when assuming that $h : [0,T] \times \Omega \times \Omega \times \mathbb{R} \to \mathbb{R}$ has non negative values. Indeed, in this case, $g(u) \in K = D(\Phi) = D(A) = \overline{D(A)}$ for every $u \in L^2(\Omega)$.

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