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Periodic Orbits for Vector Fields with Nondegenerate First Integrals

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We give a geometric proof, in the particular case of nondegenerate first integrals, of a theorem by Moser about the existence of periodic orbits on each level set of the integral, in a neighbourhood of a singular point of a vector field satisfying a nonresonance hypothesis. We use the same geometric approach to deal with the resonance case, obtaining a generalization of previous results by Sweet. © 2001 Academic Press

1. INTRODUCTION

Let O be a singular point of a vector field X. Let us suppose that X admits a first integral I in a neighbourhood of O. A classical problem is the existence of periodic trajectories on each level manifold of the integral. In the case of a linear Lagrangian system this gives rise to the study of normal modes in small oscillations theory, while in the more general setting of nonlinear and not necessarily Hamiltonian systems, it is the subject of the Lyapunov centre theorem.

Weinstein [12] removed the nonresonance condition from the hypotheses of the Lyapunov centre theorem in the case of a positive definite Hamilton function. Moser [6] extended Weinstein's existence result to non-Hamiltonian systems admitting a first integral and gave [7] an example of a Hamiltonian system with a nondegenerate Hamilton function without nontrivial periodic solutions of fixed energy. Actually, the theorem in [6] referred to below as Theorem 1.1 can be considered as a refinement of previous results by Seifert [8] and Fuller [3, 4]: the close relationship among these papers is a major motivation for this article.

Moser's result in [6] can be formulated as follows:

THEOREM 1.1 (Moser [6], Seifert–Fuller). *Let X be a C^1 vector field in a neighbourhood of $O \in \mathbf{R}^m$. Let O be a singular point of X such that*

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det $DX(O) \neq 0$, and $\mathbf{R}^m = E \oplus F$ where E, F are $DX(O)$-invariant subspaces of $\mathbf{R}^m$. Let us suppose that X admits a C^2 first integral I such that $DI(O) = 0$. Let $DX(O)_{|E}$ be the infinitesimal generator of a linear flow which is a multirotation of minimal period 2π, while none of the trajectories of the linear flow generated by $DX(O)_{|F}$ is a (nontrivial) periodic trajectory of (not necessarily minimal) period 2π. Assume, moreover, that the restricted Hessian of the first integral $D^2I(O)_{|E}$ is positive definite. Then for a sufficiently small $\varepsilon > 0$ every level set $I = I(O) + \varepsilon$ contains at least one periodic solution having minimal period close to 2π.

We will refer to the hypothesis in the above theorem concerning the periods of the periodic solutions of the linear flows generated by $DX(O)_{|E} = A$ and $DX(O)_{|F} = B$ (or equivalently on the spectra of A and B) as the *nonresonance blocks condition.*

In this paper we consider a (more) geometric approach to Moser's theorem in order to obtain a simpler proof of it in the particular but significant case of a nondegenerate first integral. As a consequence of that we will be able to deal with the case of vector fields violating the nonresonance blocks condition, and we obtain the following theorem which can be considered a generalization of results obtained by Sweet in [9]. Let $\mathscr{A} \subset J_0^3 X$ be a family of the 3-jets at O of C^4 vector fields satisfying the hypotheses in Moser's theorem except the nonresonance blocks condition, *i.e.*, for instance each element of $\mathscr{A}$ satisfies $DX(O) = A \oplus B$ where both A and B generate a 1-parameter group which is a multirotation with the same (minimal) period, say 2π. Then if I is a smooth—say C^3—regular function, such that $I(O) = 0$, $DI(O) = 0$, and $D^2I(O) \neq 0$, we have

THEOREM 1.2. *There exists a family $\mathscr{A}$ of 3-jets of the above specified type such that all the extensions of an element of $\mathscr{A}$, which still admit I as a nondegenerate first integral, have a periodic orbit of (minimal) period close to 2π for each level set of the first integral $\{I = c\}$, c sufficiently small; the family $\mathscr{A}$ is a semialgebraic subset of the 3-jet space of vector fields at O.*

Remark. The proof of the above theorem gives a geometric insight of the quoted result by Sweet [9]: moreover, that result is improved—even if an extra hypothesis of non-degeneration of the first integral has been introduced—by removing any condition on the resonance type (Sweet's result exclude the case of resonance blocks with the same minimal periods, *i.e.*, the case of *weakly coupled linear oscillators* with the same frequencies) and on the dimension of the resonance blocks (in [9] only 2-dimensional resonance blocks are considered).

We end this section by giving a sketch of the proof of Theorem 2.3, *i.e.*, Moser's theorem in the case of nondegenerate first integrals, and of the corresponding idea leading to the proof of Theorem 3.1. We begin by considering a particular situation, when $m = 2n$ and

$$I(x) = x_1^2 + \cdots + x_{2n}^2,$$

i.e., $D^2 I(O)$ is positive definite. We choose such a situation because it makes evident and easy to expose the two arguments on which the proof is based, still being a case of interest; see, for instance, Weinstein's theorem in [12]. The two mentioned arguments of the proof are a *blow up adapted to the first integral* and the use of a topological argument, namely the *Fuller index of periodic orbits*. In the case we are now considering, the adapted blow up is just the spherical one, of course. After blowing up we obtain a phase space fibered by S^{2n-1}-spheres, each invariant by the flow generated by the blown up vector field $\tilde{X}$. In particular, $\tilde{X}$ has an extension up to the divisor D of the blow up—again diffeomorphic to a S^{2n-1} sphere. The resulting dynamics of $\tilde{X}$ on D, in the hypothesis that $DX(O)$ generates a multirotation and if $n > 1$, defines the Hopf fibration of $D \simeq S^{2n-1}$. Moreover, if $M_h \simeq S^{2n-1}$ is the pull-back through the blow up of the level-spheres $\{I = h\}$, the vector field $\tilde{X}$ defines a smooth 1-parameter family of vector fields

$$\tilde{X}_h = \tilde{X}_{|M_h}: S^{2n-1} \simeq M_h \mapsto TS^{2n-1}$$

$h \geqslant 0$. Of course, $\tilde{X}_0$ is the restriction of $\tilde{X}$ along the divisor.

It is well known [3, 4] and easily understood by homotopically perturbing the Hopf fibration that the Fuller index of $\tilde{X}_0$ with respect to the set $\Omega = S^{2n-1} \times]\pi, 3\pi[$ is $i(\tilde{X}_0, \Omega) = \chi(\mathbf{CP}^{n-1}) = n$: here $]\pi, 3\pi[$ concerns the periods of the periodic orbits we are detecting by Fuller index, while $\chi(\mathbf{CP}^{n-1})$ is the Euler characteristic of the $n-1$-dimensional complex projective space, which is the orbit space of the Hopf fibration. Therefore for sufficiently small h we still have that $n = i(\tilde{X}_0, \Omega) = i(\tilde{X}_h, \Omega)$, hence implying that there exists at least one periodic trajectory on each level set of the first integral. Actually we will show that "generically" we get n periodic trajectories of minimal period approximatly 2π on each level set. The proof in the general nondegenerate case follows the same lines, with some technical parts due to the generalization of the standard blowing up to non-spherical cases, and with the use of the nonresonance blocks condition, which is here immaterial. The resonance case, *i.e.*, Theorem 3.1, then follows regarding an appropriate choice of a family of 3-jets as the nonlinear analogous of the nonresonance linear part of the vector field in the proof of Theorem 2.3.

The paper is organized as follows: in the next section we introduce the adapted blow up and recall the definition and the used properties of the Fuller index. We use these tools to give the proof of Theorem 2.3 (the non-resonance case) and state some consequences concerning the generic case.

In the third section we state and prove Theorem 3.1 (the resonance case), while in the fourth section we collect some technical parts on regularity properties of the blowing up of vector fields.

2. PERIODIC ORBITS WITH FIXED ENERGY

Let $X: \mathbf{R}^m \mapsto T\mathbf{R}^m$ be a C^k vector field, $m \geqslant 3$[1], $k \geqslant 2$, and let O be a singular point of X. We denote by $DX(O)$ the linear part of X at O: we will always suppose that $\det DX(O) \neq 0$. Let $I: \mathbf{R}^m \mapsto \mathbf{R}$ be a C^k first integral of X, *i.e.*, let

$$L_X I \equiv 0.$$

We will always suppose that $I(O) = 0$—this is not restrictive, of course—and that $DI(O) = 0$—a justification for that being the relevant Hamiltonian case. We will suppose that I is *nondegenerate* at O, *i.e.*, that the Hessian $D^2 I(O)$ is a nondegenerate bilinear form. We denote by $M_h = \{I = h\}$ the level sets of the integral: if $h \neq 0$ is sufficiently small and in a sufficiently small neighbourhood of O they are actually codimension one manifolds. Let $\mathbf{R}^m = E \oplus F$ be the splitting induced by $DX(O) = A \oplus B$, where E and F are respectively A and B invariant subspaces. We will suppose that $D^2 I(O)_{|E}$ is positive definite—therefore $\dim E = 2k$—and that e^{tA} is a (proper) multirotation. As a consequence of the last assumption, all the eigenvalues of A must be purely imaginary and rationally dependent numbers, and A must be diagonalizable. *We will always suppose that A has eigenvalues $\pm i$*: the general case could be treated by the same arguments and is only notationally more involved.

As a consequence of nondegeneracy of $D^2 I(O)$ and of the Morse lemma, in a sufficiently small neighbourhood U of O we can choose coordinates in such a way that $x(O) = 0$ and (here one should read $p = 2k$ in the situations relevant for this article)

$$I(x) = x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_m^2.$$

Throughout all the rest of the paper we will consider our differential system inside the neighbourhood U and refer to the above specified coordinate system. In particular this choice of coordinates makes clear the topological features of the level set of the integral: let us call $M_h = \{I = h\}$, then[2]

[1] In the two-dimensional case the questions considered are trivial as a consequence of the classical Poincaré centre theorem and some generalizations of it [11].

[2] It is probably worthwhile specifying that in the following claim, in the case $p = 1$, S^0 stays for the 0-dimensional manifold consisting of two points.

(i) if $h>0$ then $M_h \simeq S^{p-1} \times \mathbf{R}^{m-p}$

(ii) if $h<0$ then $M_h \simeq \mathbf{R}^p \times S^{m-p-1}$

(iii) if $h=0$ then M_0 is singular at 0, and $M_0 - \{0\} \simeq S^{p-1} \times (\mathbf{R}^{m-p} - \{0\})$.

Let $\Gamma = \{I>0\}$ and let $\bar{\Gamma} = \{I \geqslant 0\}$ be its closure: $\Gamma = \{I>0\}$ is a manifold foliated by the M_h's, while $\bar{\Gamma} = \{I \geqslant 0\}$ is a piecewise-regular manifold, carrying a singular foliation by the M_h's, $h \geqslant 0$. A fundamental tool in the local study of vector fields is the blow up technique: in view of our applications we will use a *pseudospherical blow up* which will turn out to be well suited for the geometry of the level sets of the first integral. Let us define

$$\pi\colon \mathbf{R}_0^+ \times M_1 \simeq \mathbf{R}_0^+ \times S^{p-1} \times \mathbf{R}^{m-p} \mapsto \bar{\Gamma} = \{I \geqslant 0\}$$

where

$$\begin{aligned}
x_1 &= h\sqrt{1 + w_{2k+2}^2 + \cdots + w_m^2}\, u_1 \cosh\chi \\
&\vdots \\
x_p &= h\sqrt{1 + w_{2k+2}^2 + \cdots + w_m^2}\, u_p \cosh\chi \\
x_{p+1} &= h\sqrt{1 + w_{2k+2}^2 + \cdots + w_m^2}\, \sinh\chi \\
x_{p+2} &= hw_{p+2} \\
&\vdots \\
x_m &= hw_m,
\end{aligned}$$

where $p = 2k$ and $S^{p-1} = \{u_1^2 + \cdots + u_p^2 = 1\}$, $h \in \mathbf{R}_0^+$ and $(\chi, w_{p+2}, ..., w_m) \in \mathbf{R}^{m-p}$. We leave the details of this transformation to Section 4: to help the geometric intuition of it, we remark that M_1 is the manifold defined by

$$w_1^2 + \cdots + w_p^2 - w_{p+1}^2 - \cdots - w_m^2 = 1.$$

Namely it is the standard hyperquadric diffeomorphic to $S^{p-1} \times \mathbf{R}^{m-p}$, and the region Γ is foliated by the homogeneous deformations of M_1. The blow up π simply realizes such a deformation: it is an analytic diffeomorphism when restricted to positive values of the h-variable, having in this case Γ as image. Mathematical objects originally defined in Γ and pulled back by π will be denoted by an added tilde: for instance, the domain where π is invertible is foliated by manifolds $\tilde{M}_h = \pi^{-1}(M_h)$. The *divisor* of this blow up is $\Lambda = \pi^{-1}(M_0)$: it is a regular manifold, diffeomorphic to $S^{p-1} \times \mathbf{R}^{m-p}$. Finally we define the submanifold of the divisor Λ

$$S^{p-1} = \{0\} \times S^{p-1} \times \{(0, ..., 0)\}.$$

In the fourth section we will prove:

LEMMA 2.1. *Let $\tilde{X}$ be the pull-back of the original C^r vector field X, having a singular point at the origin. The following propositions hold*:

(i) *the vector field $\tilde{X}$ has a C^{r-1} extension up to the divisor $\{h=0\} = \Lambda$: we still name such an extension $\tilde{X}$.*

(ii) *For a vector field X admitting a first integral of the type specified above the divisor Λ is an invariant set of $\tilde{X}$.*

(iii) *If the hypotheses in Theorem 2.3 on $DX(0)$ are assumed, with* $\dim E = 2k = p$, *the vector field $\tilde{X}$ is invariant on the $(2k-1)$-dimensional sphere S^{2k-1}, where the flow $\tilde{X}$ defines the Hopf fibration of the sphere.*

In order to give the proof of the main result of this section, we need now to recall the definition and basic properties of an important homotopy invariant of periodic trajectories, the Fuller index [2, 3]. Let M be a manifold and Y be a vector field on it, with everything at least C^1-regular. Let Ω be a bounded set in $M \times \mathbf{R}^+$ such that its projection on M is bounded away from the set of singular points of Y: Ω is named an *admissible neighbourhood* if it satisfies the following property. Let

$$\Pi(Y) = \{(p, t) \in M \times \mathbf{R}^+ : \varphi(p, t) = p\},$$

where φ denotes the flow of Y. We ask for Ω to satisfy

$$\partial\Omega \cap \Pi(Y) = \varnothing.$$

Let γ be the image in the phase space M of a (nontrivial) hyperbolic periodic trajectory of Y, with minimal period T_γ, and let V be a neighbourhood of γ in M, not containing any other closed trajectory having minimal period close to 2π. Then for any positive integer k we define for $\Omega = V \times \{kT_\gamma\}$ the *Fuller index*

$$i(Y, \Omega) = \frac{(-1)^\sigma}{k},$$

where σ is the number of the eigenvalues of the differential of the Poincaré map of γ which lie in the open half line $(1, +\infty)$. If $\varepsilon \mapsto Y_\varepsilon$ is a smooth homotopy we ask that

$$\partial\Omega \cap \Pi(Y_\varepsilon) = \varnothing$$

for every $0 \leqslant \varepsilon \leqslant 1$: in this case we say that Ω is admissible for that homotopy. It is well known that:

THEOREM 2.2 [3]. *$i(Y, \Omega)$ is a rational-valued function of Y and Ω, which is invariant for C^1-homotopies $\varepsilon \mapsto Y_\varepsilon$, until $\partial\Omega \cap \Pi(Y_\varepsilon) = \emptyset$. Moreover $i(Y, \Omega)$ is additive, meaning that if Ω_1, Ω_2 are two compactly disjoint subsets of $M \times \mathbf{R}^+$, then*

$$i(Y, \Omega_1) + i(Y, \Omega_2) = i(Y, (\Omega_1 \cup \Omega_2)).$$

We are now in position to state and prove, in a geometrical way different from the one used in the original Moser proof, the following result that is a particular case of Theorem 1.1.

THEOREM 2.3 [6]. *Let X and I be as in Theorem* (1.1), *except for the hypothesis that I is nondegenerate and X is a C^2 vector field. Then for h sufficiently small and for each level manifold $\{I = h\}$ of the first integral there exists at least one closed orbit $\gamma_h \subset \{I = h\}$ of X, with minimal period of γ_h tending to 2π and with γ_h tending to the singular point* (*Hausdorff topology*) *when h tends to zero.*

Proof. During this proof we will consider the region Γ only as our original phase space: moreover, to keep the same notations as in the previous section, here we will always consider $p = 2k$. We perform the adapted blow up as specified in Lemma 2.1, hence obtaining a pulled-back vector field $\tilde{X}$ of class C^{k-1} defined in $\mathbf{R}_0^+ \times M_1 \simeq \mathbf{R}_0^+ \times S^{p-1} \times \mathbf{R}^{m-p}$, for sufficiently small values of the parameters h, χ and $w_{p+2}, ..., w_m$; see the fourth section for the definitions. The phase space of $\tilde{X}$ is foliated by $\tilde{X}$-invariant manifolds

$$\tilde{M}_h = \{h\} \times M_1 \simeq \{h\} \times S^{p-1} \times \mathbf{R}^{m-p}.$$

Therefore we can define a smooth 1-parameter family of vector fields on M_1

$$\tilde{X}_h = \tilde{X}_{|\{h\} \times M_1}.$$

The vector field $\tilde{X}_0$ is the pull-back of the linear vector field defined by $DX(O)$, therefore $C = S^{p-1} \times \{\underline{0}\}$ is a compact of periodic orbits for $\tilde{X}_0$. We are going to show that the nonresonance blocks condition permit us to find an admissible neighbourhood Ω of C which turns out to be stable with respect to the smooth homotopy

$$h \mapsto X_h.$$

Then a computation will show that the Fuller index $i(\tilde{X}_h, \Omega)$ is nonzero for $h \geqslant 0$, h sufficiently small, and this implies that on each level set M_h of the first integral there exists a periodic orbit of period approximately 2π, thus concluding the proof.

The manifold M_1 is a trivial fiber bundle diffeomorphic to an open subset of $S^{p-1} \times \mathbf{R}^{m-p}$. Let

$$p\colon M_1 \mapsto \mathbf{R}^{m-p}$$

be the canonical projection of this bundle: then in suitable local coordinates $p(C) = \underline{0}$. Let B_r be an open ball of radius[3] r in $\mathbf{R}^{m-p}$ centered at $\underline{0}$. Let $\tilde{\Omega} = p^{-1}(B_r)$ and let us define $\Omega = \tilde{\Omega} \times]\pi, 3\pi[$. Hence $\tilde{\Omega} \simeq S^{p-1} \times B_r$; therefore

$$\partial\Omega \simeq (S^{p-1} \times \partial B_r \times]\pi, 3\pi[\,) \cup (\tilde{\Omega} \times \{\pi\}) \cup (\tilde{\Omega} \times \{3\pi\}).$$

By elementary considerations [3] on the vector fields of the smooth homotopy $h \mapsto X_h$ it is easy to realize that to prove that Ω is an admissible neighbourhood for the homotopy it is enough to show that for a sufficiently small h we have (up to a diffeomorphism of the first set on the left member of the following equality)

$$S^{p-1} \times \partial B_r \times]\pi, 3\pi[\, \cap \Pi(X_h) = \varnothing.$$

More explicitly, we must prove that for sufficiently small h there are no closed orbits with periods between π and 3π passing through points $q \in M_1$ such that $p(q) \in \partial B_r$. Let φ_t^h be the flow of X_h. The nonresonance blocks condition implies that

$$\min\{\, \|\varphi_t^0(q) - q\| \colon q \in \partial\tilde{\Omega},\ \pi \leqslant t \leqslant 3\pi \} = 2c > 0.$$

Moreover

$$\tilde{X}_h(q) = \tilde{X}_0(q) + F(q, h),$$

where $F(q, 0) \equiv 0$ and $F(q, h) = O(h)$ uniformly with respect to $q \in \bar{\tilde{\Omega}}$. A straightforward application of Gronwall's lemma, gives that, for sufficiently small h,

$$\|\varphi_t^h(q) - \varphi_t^0(q)\| < c$$

[3] The choice of r is of no importance as long as a nonresonance hypothesis is assumed.

for each $0 < t < 4\pi$ and each $q \in \bar{\tilde{\Omega}}$. Therefore for h sufficently small and for $q \in \partial\tilde{\Omega}$ and $\pi \leqslant t \leqslant 3\pi$, we get

$$\|\varphi_t^h(q) - q\| \geqslant \|\varphi_t^0(q) - q\| - \|\varphi_t^h(q) - \varphi_t^0(q)\| \geqslant c$$

showing that $\tilde{\Omega}$ is an admissible neighbourhood for the homotopy $h \mapsto X_h$.

To end the proof we must compute $i(\tilde{X}_0, \Omega)$ and prove that it is not zero. This can be accomplished generalizing a geometrical argument given in [3, 4]. We consider first the case $k > 1$: the case $k = 1$, namely the case when Lyapunov's theorem holds true, will be considered later and in a similar way. Let $L = DX(O)$ and let X_0 be its restriction to M_1. Let us recall that (here and in the following we use that $p = 2k$) if

$$S^{2k-1} = S^{2k-1} \times \{(0, ..., 0)\}$$

then S^{2k-1} is X_0-invariant and its flow on S^{2k-1} generates the Hopf fibration σ: $S^{2k-1} \mapsto \mathbf{CP}^{k-1}$. Let us consider the smooth homotopy

$$\varepsilon \mapsto X_{0,\varepsilon}$$

defined as follows. The trivial fiber bundle structure p: $M_1 \simeq S^{2k-1} \times \mathbf{R}^{m-2k} \mapsto \mathbf{R}^{m-2k}$ induces the decomposition of the tangent spaces in *vertical* and *horizontal* components

$$T_q M_1 = T_q S^{2k-1} \times T_q \mathbf{R}^{m-2k}$$

and

$$X_{0,\varepsilon} = X_0 + X_{0,\varepsilon}^{hor} + X_{0,\varepsilon}^{ver},$$

where X_0 is the linear vector field restricted to M_1 and

$$X_{0,\varepsilon}^{ver} = \varepsilon\sigma^*(\delta)$$

and

$$X_{0,\varepsilon}^{hor} = -\varepsilon p^*(E_{m-2k}),$$

where δ is a gradient field[4] on $\mathbf{CP}^{k-1}$ and E_{m-2k} is the identity on $\mathbf{R}^{m-2k}$. Moreover, $\sigma^*(\delta)$ and $p^*(E_{m-2k})$ are two vector fields, having for instance zero components along the S^1-fiber or, respectively, the S^{2k-1}-fiber, and

[4] δ can be defined as follows: we embed $\mathbf{CP}^{k-1}$ as a $2(k-1)$-dimensional real manifold in a Euclidean space and choose in this space a coordinate system such that the height of the points of $\mathbf{CP}^{k-1}$ with respect to one coordinate of the ambient pace is a Morse function f and define $\delta = \nabla f$. It turns out that this vector field has exactly k hyperbolic singular points each of index 1.

projecting to the base spaces $\mathbf{CP}^{k-1}$ and $\mathbf{R}^{m-2k}$ onto δ and E_{m-2k}: their existence easily follows from standard differential geometric arguments (see [3, 4]). It is easy to check that for initial data in Ω and for sufficiently small $\varepsilon > 0$ the vector field $X_{0,\varepsilon}$ has only k hyperbolic periodic orbits $\gamma_1, ..., \gamma_k$, each of minimal period 2π: these trajectories correspond to the pull-back through σ of the k singular points of δ. If U_j is a sufficiently small tubular neighbourhood of γ_j, a simple computation in local trivializing coordinates permits us to conclude that

$$i(X_{0,\varepsilon}, U_j \times]\pi, 3\pi[\,) = i(\delta, \sigma(\gamma_j)) = 1,$$

where $i(\delta, \sigma(\gamma_j))$ is the Poincaré index of δ at $\sigma(\gamma_j)$. Moreover, the same argument based on the nonresonance blocks condition, used to claim that $h \mapsto X_h$ has Ω as an admissible neighbourhood, permits us to consider Ω as an admissible neighbourhood of $X_{0,\varepsilon}$, too, when ε is sufficiently small. Hence

$$i(X_{0,\varepsilon}, \Omega) = k$$

and this ends the proof when $k > 1$. In the case $k = 1$ the same argument applies with the obvious substitutions $M_1 = S^1 \times \mathbf{R}^{m-2}$, $X^{vert}_{0,\varepsilon} = 0$ and $X^{hor}_{0,\varepsilon} = -\varepsilon p^*(E_{m-2})$. ∎

Remark. The nondegeneracy hypothesis of the first integral has been used in the above proof in the description of the topology of the level manifolds of the first integral. If we assume only the hypotheses of Theorem 1.1, a similar description of the codimension one foliation defined by the first integral in full generality is a hopeless problem of singularity theory. Nevertheless the approach we just exposed to the problem of existence of closed orbit on level sets of the integral can be useful to deal with the resonance case, as we will see in the next section, as well as in particular examples, of specific interest, when more degenerate singularities are considered: this could be an object of further research.

We end this section considering the "generic" situation in the case of positive definite first integral, namely in the case when the first integral in Morse normal form is

$$I(x) = x_1^2 + \cdots + x_{2n}.$$

The analogous general case should be dealt in the same way, when one has obtained a pseudospherical version of the following theorem by Takens

THEOREM 2.4 [10]. *Let $\tilde{X}$ be a vector field on $S^{2k-1} \times \mathbf{R}$ of the form*

$$\tilde{X} = \sum_{l \leqslant N} h^l g_l(u)\, \tilde{R} + \sum_{k \leqslant N} \sum_{i,j=1}^{2n} h^k f_{i,j,k}(u)\, \tilde{V}_{i,j},$$

where $S^{2k-1}=\{u_1^2+\cdots+u_{2k}^2=1\}$, *h is a coordinate on* $\mathbf{R}$, $R=\sum_{j=1}^{2n} x_j(\partial/\partial x_j)$, $V_{i,j}=\frac{1}{2}(x_i(\partial/\partial x_j)-x_j(\partial/\partial x_i))$ *and* $\tilde{R}=\pi^*(R)$, $\tilde{V}_{i,j}=\pi^*(V_{i,j})$, *where* π *is the spherical blow up map. Let* $T\colon S^{2k-1}\times\mathbf{R}\mapsto TS^{2k-1}\times\mathbf{R}$ *be the involution*

$$T(u_1, ..., u_{2n}, h)=(-u_1, ..., -u_{2n}, -h).$$

If $T_*(\tilde{X})=\tilde{X}$ *then there is a vector field X on* $\mathbf{R}^{2n}$ *and an integer m such that*

$$\tilde{X}=h^m\pi^*(X).$$

We will say that a vector field $X=Lx+R_2(x)$, with L generating the 2π-periodic rotation given in complex coordinates by

$$t\mapsto(e^{it}, e^{-it}, ..., e^{it}, e^{-it}),$$

is *generic* if $\tilde{X}=\pi^*(X)$ can be written in local trivializing coordinates of the Hopf bundle as (here $\varepsilon=h$)

$$\begin{aligned}\dot{\varphi}&=1+\varepsilon f(\varphi, P, \varepsilon)\\ \dot{P}&=\varepsilon g(\varphi, P, \varepsilon)\\ \dot{\varepsilon}&=0,\end{aligned}$$

where φ is a coordinate on $S^1=\mathbf{R}/2\pi\mathbf{Z}$, P is a coordinate on $\mathbf{CP}^{n-1}$, and, as follows from the argument of Theorem 2.4, f and g are π-periodic with respect to φ; moreover, if

$$g(\varphi, P, \varepsilon)=g^{(0)}(\varphi, P)+\varepsilon\{\cdots\}$$

we ask that the averaged vector field

$$G\colon \mathbf{CP}^{n-1}\mapsto T\mathbf{CP}^{n-1}$$

defined in local coordinates by

$$G(P)=\frac{1}{2\pi}\int_0^{2\pi} g^{(0)}(\varphi, P)\,d\varphi$$

has only a finite number (hence at least n) of nondegenerate zeroes: $G(P_0)=0$ and

$$\det\frac{\partial}{\partial P}_{|P=P_0} G\neq 0.$$

From Theorem 2.4 it easily follows that, for instance, the subset of infinitely smooth vector fields Y such that their blow up $\tilde{Y}$ satisfies the above condition is an open dense subset, for instance in the C^2-topology, of the set of vector fields of the type $X = Lx + R_2(x)$, with L generating a 2π-periodic rotation: this remark justifies the used terminology. Then a straightforward application of basic results of averaging theory (see Theorem 2 in [5] or paragraph 2.3 in [1]) permit us to state:

THEOREM 2.5. *Let $X(x) = Lx + \{\cdots\}$ be a generic vector field satisfying the hypotheses of Theorem* 2.3. *Then there exist at least n hyperbolic closed orbits of X for each level manifold $\{I = h\}$, $h > 0$.*

3. THE RESONANCE CASE

The existence of periodic orbits on the level manifolds of the first integral in the nonresonance case (*i.e.*, when the nonresonance blocks condition is satisfied) is based on the interpretation of the vector field X near the singular point as a perturbation of the linear vector field Lx, $L = DX(O)$: the nonresonance hypothesis implies that Lx has nonzero Fuller index with respect to an admissible neighbourhood of a compact set of closed orbits, as a vector field on a manifold diffeomorphic to the typical leaf of the foliation induced by the first integral. In the resonance case such a property of the linear vector field is no longer true: to obtain an existence result for periodic orbits on the level sets of the first integral we must substitute the linear part of X with a suitable jet of it, and we must prove that any extension of that jet has a nonzero Fuller index, with respect to some admissible neighbourhood of a compact of periodic orbits. This is the argument on which we will base the proof of Theorem 3.1.

Throughout this section we will always suppose that the vector fields are C^4. Let G^3 be the space of 3-*jets* at $O \in \mathbf{R}^m$. The 3-jet of a given vector field X will be indicated as $\hat{X}$, while given a 3-jet $\hat{X}$ an *extension* X of $\hat{X}$ verifies $j^3 X = \hat{X}$. Let

$$I(x) = x_1^2 + \cdots + x_{2k}^2 - (x_{2k+1}^2 + \cdots + x_m^2)$$

be a fixed first integral in Morse normal form: if X is a smooth vector field having I as a first integral and

$$X(x) = Lx + P_2(x) + P_3(x) + \cdots,$$

where the $P_j(x)$'s are homogeneous vector polynomials, then

$$Lx \cdot \nabla I \equiv P_2(x) \cdot \nabla I \equiv P_3(x) \cdot \nabla I \equiv \cdots \equiv 0.$$

Let $\mathscr{A}$ be the subset of G^3 defined, in x-coordinates, by the following three conditions:

(i) $P_2(x) \equiv 0$[5]

(ii) $Lx \cdot \nabla I \equiv P_3(x) \cdot \nabla I \equiv 0$

(iii) this condition has an elementary geometrical meaning: it is a condition of *dissipativity*, *e.g.*, in the sense of Levinson, on the base of M_1 viewed as a trivial fiber bundle $M_1 \simeq S^{2k-1} \times \mathbf{R}^{m-2k}$. Its precise analytic expression is the following. If $P_3(x) = \sum_{i=1}^{m} P_{3,1}(x)\, \partial/\partial x_1 + \cdots + P_{3,m}(x)\, \partial/\partial x_m$ then

$$P_{3,2k+1}(x)\, x_{2k+1} + \cdots + P_{3,m}(x)\, x_m < 0$$

if $x_{2k+1} \cdots x_m \neq 0$.

Remark. It is easy to see that the above conditions do not depend on the choice of the coordinates of the Morse normal form of I, in the sense that they define a semialgebraic set in the 3-jet space of the vector fields at O, independent of the choice of coordinates. We will also give concrete examples of systems having 3-jet satisfying the above conditions.

THEOREM 3.1. *Let X be any smooth (or only C^4) vector field which is an extension of a 3-jet in $\mathscr{A}$. Moreover, let X admit I as a first integral and let $DX(O) = L$ admit the decomposition $L = A \oplus B$ where A generates a multirotation and defines a maximal invariant subspace E, $D^2 I_{|E}(O)$ is definite, and A, B do not necessarily satisfy the nonresonance blocks condition. Then X admits a family of periodic orbits γ_h satisfying the same properties in the statement of Theorem 2.3.*

Proof. As in the proof of Theorem 2.3 we consider Γ as the phase space. We prove the theorem in the worst case, namely the case of complete resonance, when B generates the same multirotation as A: all the other cases reduce to this one by simple considerations. Let us remark that in the considered situation $m = 2n$. Let us consider M_1 as the trivial fiber bundle

$$p\colon M_1 \simeq S^{2k-1} \times \mathbf{R}^{2(n-k)} \mapsto \mathbf{R}^{2(n-k)}$$

with fiber diffeomorphic to S^{2k-1}. Condition (ii) in the definition of $\mathscr{A}$ implies that the vector field

$$Z(x) = Lx + P_3(x)$$

[5] By the Poincaré–Dulac normal form theorem it is always possible, in the completely resonant case, to choose normalizing coordinates satisfying this property. If normalizing coordinates and Morse coordinates agree at the 3-jet level this hypothesis is ineffective: we do not know if this is generally true.

admits I as a first integral. With a slight abuse of notation we will denote by $Z(x)$ the 3-jet class at O of vector fields to which $Z(x)$ belongs. We will show that for any extension X of Z

$$j^3 X = Z$$

such that X still admits I as a first integral and for every $0 < h < \bar{h}$ the vector field X_h has a periodic orbit of period approximately 2π, hence concluding the proof. We begin defining the smooth homotopy $\varepsilon \mapsto Z_\varepsilon$, where $\varepsilon \geqslant 0$, satisfying $Z_0 = Z$, and such that any Z_ε still admits I as a first integral. It is easier to carry on such construction for the blown up vector fields $\tilde{Z}_\varepsilon$. Let

$$\tilde{Z}_h = \tilde{Z}_{|\tilde{M}_h} : M_1 \mapsto TM_1$$

and

$$\tilde{Z}_{h,\varepsilon} = \tilde{Z}_h + Z_\varepsilon^{vert} + Z_\varepsilon^{hor},$$

where the vertical and horizontal vector fields

$$Z_\varepsilon^{vert} = \varepsilon \sigma^*(\delta)$$
$$Z_\varepsilon^{hor} = -\varepsilon p^*(E_{2(n-k)})$$

are defined as the analogous vector fields in the proof of Theorem 2.3. Writing $X \in \mathscr{A}$ as $X = Z + N$, we define

$$\varepsilon \mapsto Z_\varepsilon + (1-\varepsilon) N = X_\varepsilon.$$

The particular homogeneous form of I implies that it is a first integral for X_ε, too. For every $h > 0$ let $\tilde{X}_{h,\varepsilon} : M_1 \mapsto TM_1$ be defined as

$$\tilde{X}_{h,\varepsilon} = \tilde{X}_{\varepsilon|\tilde{M}_h} = \tilde{Z}_{h,\varepsilon} + (1-\varepsilon)\, \pi_*^{-1} N.$$

If $p: M_1 \simeq S^{2k-1} \times \mathbf{R}^{2(n-k)} \mapsto \mathbf{R}^{2(n-k)}$ is the projection of the bundle we consider[6]

$$D_{r,h} = \{ v \in \mathbf{R}^{2(n-k)} : v_1^2 + \cdots + v_{2(n-k)}^2 < r^2 \}$$

and define

$$\tilde{\Omega} = p^{-1}(D_{r,h}).$$

[6] The choice of r is of no importance for us: it only influences the size of $\bar{h}$.

Let us consider as usual the splitting

$$T_q M_1 = T_q S^{2k-1} \times T_q \mathbf{R}^{m-2k}$$

induced by the trivial bundle structure of M_1. Hypothesis (iii), together with the fact that $\pi_*^{-1}(N) = o(h^2)$ uniformly in the closure of $\tilde{\Omega}$, imply that for sufficiently small h the vector field $\tilde{X}_h = \tilde{X}_{|\tilde{M}_h}$ has component along $T_q \mathbf{R}^{m-2k}$ forming an angle with $\partial D_{r,h}$ which is always greater of a given positive number (say one half of the minimum angle formed by the component of $\tilde{Z}_{0,0}$ along $T_q \mathbf{R}^{m-2k}$ with $\partial D_{r,h}$). Therefore if $q \in \partial\tilde{\Omega}$, or equivalently if $p(q) \in \partial D_{r,h}$ and if φ_t^h is the flow of $\tilde{X}_h$ then $p(\varphi_t^h(q)) \in D_{r,h}$ for every positive t. The definition of Z_ε^{hor} implies that this conclusion still holds true for the flow of $\tilde{X}_{h,\varepsilon}$ for $0 \leqslant \varepsilon \leqslant 1$ and for every sufficiently small h. Therefore no closed orbit of period close to 2π of $\tilde{X}_{h,\varepsilon}$ can pass through $\partial\tilde{\Omega}$, and it follows that there exists $\bar{h} > 0$ such that for $0 < h < \bar{h}$ the set $\Omega = \tilde{\Omega} \times]\pi, 3\pi[$ is admissible for the smooth homotopy $\varepsilon \mapsto \tilde{X}_{h,\varepsilon}$. If we prove that for a fixed $0 < \varepsilon < 1$ there exists $h(\varepsilon) < \bar{h}$ such that if $0 < h < h(\varepsilon)$, then $i(\tilde{X}_{h,\varepsilon}, \Omega) = k$, we can conclude that $i(X_{|M_h}, \pi(\Omega)) = k$, and this ends the proof. Let $\tilde{L} = \pi_*^{-1} L$ be the lifted linear vector field. The perturbation

$$\tilde{L}_\varepsilon = \tilde{L} + Z_\varepsilon^{vert} + Z_\varepsilon^{hor}$$

has exactly k periodic hyperbolic orbits with minimal period 2π (see [3, 4]), $\gamma_{1h}, ..., \gamma_{kh}$, contained in $\tilde{M}_h \simeq S^{2k-1} \times \mathbf{R}^{2(n-k)}$. Let us remark that $\tilde{X}_{h,\varepsilon} = \tilde{L}_\varepsilon + o(1)$, $o(1) \mapsto 0$ as $h \mapsto 0$, and that this vector field defines, in local coordinates near one of the γ_{jh}'s, the system

$$\dot{u} = Au + \varepsilon\tilde{\delta}(u) + hF(\varepsilon, u, v, h)$$

$$\dot{v} = (B - \varepsilon E_{2(n-k)})\, v + hG(\varepsilon, u, v, h)$$

where $u = (u_1, ..., u_{2k})$, $u_1^2 + \cdots u_{2k}^2 = 1$, $v = (v_1, ..., v_{2(n-k)})$, $\varepsilon\tilde{\delta} = Z_\varepsilon^{vert}$ and $F, G = o(1)$ as $h \mapsto 0$. Let us stress the fact that for a fixed positive ε and for $h = 0$, such a system has k hyperbolic 2π-periodic orbits. Therefore for a fixed positive ε, there exists $h(\varepsilon)$, $h(\varepsilon) < \bar{h}$, such that for $0 < h < h(\varepsilon)$, $\tilde{X}_{h,\varepsilon}$ has k hyperbolic periodic orbits $\gamma_{jh}(\varepsilon)$ of approximate period 2π. Moreover, it is easy to check that, at least if $h(\varepsilon)$ is sufficiently small, $\tilde{X}_{h,\varepsilon}$ has no other closed orbit having period in $]\pi, 3\pi[$. Therefore $i(\tilde{X}_{h,\varepsilon}, \Omega) = i(\tilde{X}_h, \Omega) = k$. The case of general B exhibiting a resonance relation with A needs only some obvious modifications of the above arguments. Finally, the case when $k = 1$ is covered by the same argument, where the perturbed vector field Z_ε is obtained through a horizontal perturbation Z_ε^{hor} only, as in the proof of Theorem 2.3. ∎

Remark. It is probably worthwhile to observe that the conditions (i)–(iii) are rather crude and can be substituted, as the reader can easily understand, by less restrictive, but more cumbersome, ones: for instance the conditions of types (ii) and (iii) could be expressed through the averaged vector field along a time interval of length 2π, etc. We content ourselves with the above considerations, having in mind mostly the geometric interpretation of the results in [9] and the already given improvements of it with respect to the generality of the permitted resonances. For instance the geometric interpretation given by (iii) explains why the condition for the existence of period orbits is given on the 3-jets and not on the 2-jet: in fact the analogous of (iii) for the 2-jets leads to a condition of definitiveness for a 3-form, which is impossible. On the other hand the hypotheses (i)–(iii) are semialgebraic and they are consistent, as the following example shows. We consider the class of 3-jets at O of vector fields in $\mathbf{R}^4$, with the first integral in Morse normal form $I(x)=x_1^2+\cdots+x_4^2$ and with the linear part given by the two standard uncoupled oscillators

$$L=I_1 \oplus I_2,$$

where

$$I_1=I_2=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

satisfying

$$\begin{aligned} P_2 &\equiv 0 \\ P_{3,1}(x) &= x_1(x_3^2+x_4^2) \\ P_{3,2}(x) &= x_2(x_3^2+x_4^2) \\ P_{3,3}(x) &= -x_3(x_1^2+x_2^2) \\ P_{3,4}(x) &= -x_4(x_1^2+x_2^2). \end{aligned}$$

This 3-jet satifies the conditions (i)–(iii); hence every extension of it, still admitting I as a first integral, has a periodic orbit on each level manifold of the integral.

4. PSEUDOSPHERICAL BLOW UP

The aim of this section is to give a proof of Lemma 2.1. Of course in this section we refer to the analogous tractation given by Takens in [10] for the spherical blow up, even if our arguments will be slightly different —obviously only from the computational point of view—from those of Takens. We recall the definition of the pseudospherical blow up as it has been given in the first section: the pseudospherical blow up map is (here as usual $p=2k$)

$$\pi\colon \mathbf{R}_0^+ \times M_1 \simeq \mathbf{R}_0^+ \times S^{p-1} \times \mathbf{R}^{m-p} \mapsto \bar{\Gamma} = \{I \geqslant 0\},$$

where

$$\begin{aligned}
x_1 &= h\sqrt{1+w_{p+2}^2+\cdots+w_m^2}\, u_1 \cosh\chi \\
&\vdots \\
x_p &= h\sqrt{1+w_{p+2}^2+\cdots+w_m^2}\, u_p \cosh\chi \\
x_{p+1} &= h\sqrt{1+w_{p+2}^2+\cdots+w_m^2}\, \sinh\chi \\
x_{p+2} &= hw_{p+2} \\
&\vdots \\
x_m &= hw_m,
\end{aligned}$$

where $S^{p-1}=\{u_1^2+\cdots+u_p^2=1\}$, $h\in\mathbf{R}^+$, and $(\chi, w_{p+2}, ..., w_m)\in\mathbf{R}^{m-p}$. We study now the vector field

$$\tilde{X}=\pi^* X$$

which is defined *a priori* only for $h>0$. We will prove that $\tilde{X}$ admits a C^{r-1} extension up to the divisor $h=0$. Let ϑ be the complex of the variables after the blow up: $\vartheta=\pi^{-1}(x)$. The computation of π_*^{-1} leads to the following expression, valid for $h>0$

$$\tilde{X}(\vartheta)=\pi_*^{-1}Lx(\vartheta)+\pi_*^{-1}P_2(x(\vartheta))+\cdots+\pi_*^{-1}P_k(x(\vartheta))+\pi_*^{-1}R_k(x(\vartheta)),$$

where $R_k(x)=o(\|x\|^k)$. From direct computation we get

$$\pi_*^{-1}(x(\vartheta))=h^{-1}\Theta,$$

where Θ is the matrix

$$\begin{pmatrix}
ha(w)\,u_1\cosh(\chi) & \cdots & ha(w)\,u_{2k}\cosh(\chi) & ha(w)\sinh(\chi) & -hw_{2k+2} & \cdots & -hw_m \\
(1-u_1^2)F & \cdots & -u_1u_{2k}F & 0 & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
-u_{2k}u_1F & \cdots & (1-u_{2k}^2)F & 0 & 0 & \cdots & 0 \\
f_1 & \cdots & f_{2k} & f_{2k+1} & f_{2k+2} & \cdots & f_m \\
M_{1,l} & \cdots & M_{lp} & N_1 & 1-w_{2k+2}^2 & \cdots & -w_{2k+2}w_m \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
M_{m-2,l} & \cdots & M_{m-2,p} & -w_mN_p & -w_mw_{2k+2} & \cdots & (1-w_m^2)
\end{pmatrix}$$

where a, F, f_j, M_{ke}, N are real analytic functions of ϑ, not depending on h; their explicit expressions of are no importance for us. We observe that the ϑ-coordinates are homogeneous in h; hence

$$\tilde{X}(\vartheta)=L^*(\vartheta)+hP_2^*(\vartheta)\cdots+h^{k-1}P_k^*(\vartheta)+h^{-1}\Theta R_k,$$

where L^* and each P_j^* is regular up to $h=0$, and the regularity problem reduces to prove that

LEMMA 4.1. *$h^{-1}\Theta R_k$ is C^{k-1}-regular and verifies*

$$h^{-1}\Theta R_k=o(h^{k-1}).$$

Proof. Let $G=h^{-1}\Theta R_k$: the statement of the lemma is proved if we prove that for each differential operator $D_{\underline{S}}{}^{(l)}$ in the variables of ϑ, of order l, $l\leqslant k-1$ we get

$$\lim_{h\mapsto 0} D_{\underline{s}}{}^{(l)}G(\vartheta)=0$$

when $\sqrt{1+w_{2k+2}^2+\cdots+w_m^2}<constant$, $|\chi-\chi_0|<constant$ or, in other words, that the $(k-1)$th order jet of G is zero along the divisor

$$J^{k-1}_{\{h=0\}}G=0.$$

This is a classical argument of division lemma type (see [11] for details): the key step is the proof (by an easy induction) of the equality

$$\frac{\partial^l}{\partial h^l}\frac{R_k}{h}=\frac{R_k^{(l)}}{h^{2^l}},$$

where

$$R_k^{(l)}=o(h^{2^l+k-1-l})$$

from which the statement easily follows. ∎

Remark. From the above lemma it follows that the pseudospherical blow up leads to a lifted vector field $\tilde{X}$ defined up to the divisor. It is worthwhile observing explicitly the following two facts. First, as we already said the above proof is only slightly different from the analogous one given by Takens in [10] in the case of the spherical blow up, nevertheless it shows a difference between spherical and pseudospherical blow up, namely the non-uniformity of the limit

$$\lim_{h \mapsto 0} D^{(l)}_{\underline{S}} G(\vartheta) = 0$$

in the pseudospherical case: the condition that in the above limit

$$|\chi - \chi_0| < \text{constant}$$

simply reflects the fact that the pseudospherical blow up corresponds to a pseudo-Riemannian metric and points at zero distance—with respect to this metric—from the origin O need not be topologically close to O. Second, and not unrelated to the previous remark, in the case of pseudospherical blow up the dynamics of $\tilde{X}$ along the divisor $\{h = 0\}$ do *not* reflect the dynamics of the original vector field X along the 0-level set of the first integral (they are the dynamics of the linear part).

Finally let us observe that, from the expression of the Jacobian matrix $h^{-1}\Theta$ of the pseudospherical change of coordinates, we easily get that the submanifold of M_1 given by $\{x_{2k+1} = \cdots = x_m = 0\} \simeq S^{2k-1}$ is invariant by the flow of $\tilde{X}_0$, and this flow generates on $\{x_{2k+1} = \cdots = x_m = 0\}$ the Hopf fibration on the sphere S^{2k-1}, with periodic orbits of period 2π. This concludes the proof of Lemma 2.1.

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The approach to the problem by blowing up is only a refinement of classical arguments, while the relation of the considered problem with the Fuller index theory is suggested by Moser in [6]. The link between the averaging method, entering in the generic case treatment, and Fuller index theory has been suggested by Anosov's comments in [1].

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