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**Research Article**

Juan Campos, Andrea Corli and Luisa Malaguti*

# Saturated Fronts in Crowds Dynamics

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**Abstract:** We consider a degenerate scalar parabolic equation, in one spatial dimension, of flux-saturated type. The equation also contains a convective term. We study the existence and regularity of traveling-wave solutions; in particular we show that they can be discontinuous. Uniqueness is recovered by requiring an entropy condition, and entropic solutions turn out to be the vanishing-diffusion limits of traveling-wave solutions to the equation with an additional non-degenerate diffusion. Applications to crowds dynamics, which motivated the present research, are also provided.

**Keywords:** Traveling-Wave Solutions, Entropic Solutions, Nonlinear Convection-Diffusion Equations

**MSC 2010:** 35K65, 35C07, 35K57

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## 1 Introduction

We consider in this paper the scalar degenerate parabolic equation

$$u_t + f(u)_x = (g(u)\Phi(u_x))_x, \tag{1.1}$$

for $t \in \mathbb{R}$ and $x \in \mathbb{R}$. The unknown function $u = u(x, t)$ is valued in the interval $[0, 1]$. We assume that $f$, $g$ and $\Phi$ are $C^1$ functions and make the following hypotheses:
(H1) $f : [0, 1] \to \mathbb{R}$ with $f(0) = 0$.
(H2) $g : [0, 1] \to \mathbb{R}$ with $g(u) > 0$ for $u \in (0, 1)$ and $g(0) = g(1) = 0$.
(H3) $\Phi : \mathbb{R} \to (-1, 1)$ satisfies $\Phi' > 0$ in $\mathbb{R}$, $\Phi(0) = 0$ and $\Phi(w) \to \pm 1$ as $w \to \pm\infty$.
We refer to Figure 1 for possible plots of the functions $g$ and $\Phi$. For simplicity the values of $u$ and $\Phi$ have been normalized: the case when the intervals $[0, 1]$ and $(-1, 1)$ are replaced by other bounded intervals is dealt analogously. In particular, assumption (H3) makes (1.1) a *flux-saturated* porous media equation [8]. The second-order term in equation (1.1), which accounts for diffusion, is formally

$$g(u)\Phi'(u_x)u_{xx},$$

and so the product $g(u)\Phi'(u_x)$ represents the diffusion coefficient or *diffusivity*; because of assumptions (H2) and (H3) we have $g(u)\Phi'(u_x) \geq 0$. As a consequence, equation (1.1) is a nonlinear forward parabolic equation which degenerates both when $u = 0$ or $u = 1$ (because $g$ vanishes) and when $u_x = \pm\infty$ (because the diffusion term $\Phi$ is saturated), i.e., when the tangent to the graph of $u(\cdot, t)$ is vertical. An example of function $\Phi$

***Corresponding author: Luisa Malaguti,** Department of Sciences and Methods for Engineering, University of Modena and Reggio Emilia, Modena, Italy, e-mail: luisa.malaguti@unimore.it. https://orcid.org/0000-0002-8677-057X
**Juan Campos,** Department of Applied Mathematics, University of Granada, Granada, Spain, e-mail: campos@ugr.es. https://orcid.org/0000-0002-1228-830X
**Andrea Corli,** Department of Mathematics and Computer Science, University of Ferrara, Ferrara, Italy, e-mail: andrea.corli@unife.it. https://orcid.org/0000-0003-0600-4208

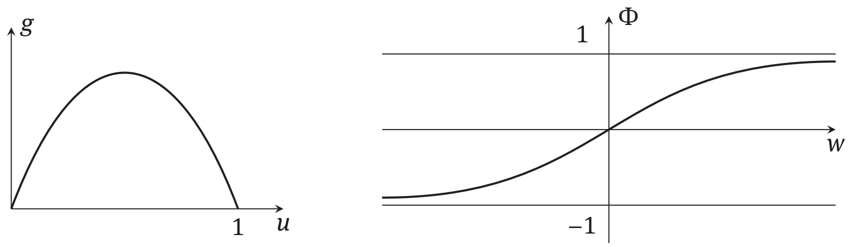


**Figure 1:** Two typical plots of the functions $g$ and $\Phi$.

satisfying assumption (H3) is

$$\widetilde{\Phi}(p) = \frac{p}{\sqrt{1+p^2}}. \tag{1.2}$$

In this case, the right-hand side of (1.1) becomes a degenerate version of the mean-curvature operator.

In this paper we are interested in the existence and uniqueness of a suitable class of *globally defined, monotone, non-constant traveling-wave solutions* to equation (1.1). With this name we mean solutions $u$ of (1.1) of the form $u(x,t) = \psi(x - \sigma t)$, with $\sigma \in \mathbb{R}$ and $\psi : \mathbb{R} \to [0,1]$; moreover, $\psi$ is not identically constant and it is monotone in the sense that $\xi_1 < \xi_2$ implies either $\psi(\xi_1) \le \psi(\xi_2)$ or $\psi(\xi_1) \ge \psi(\xi_2)$. Strictly monotone functions are characterized by strict inequalities. The function $\psi$ is the profile of $u$ and $\sigma$ is the speed of the traveling wave. Such a function $u$ is called a *wavefront solution*, WF for short, and we denote in the following with $\ell^\pm$ the limits of its profile at $\pm\infty$, i.e.,

$$\psi(-\infty) = \ell^-, \quad \psi(\infty) = \ell^+, \quad \ell^-, \ell^+ \in [0,1]. \tag{1.3}$$

For brevity we extend the name of WF also to $\psi$.

In the case $f = 0$, the existence and uniqueness of solutions to the initial-value problem for equation (1.1) with initial data $u_0$ is proved in [4, 23] in the non-degenerate case $g > 0$, for $u_0 \in L^\infty(\mathbb{R})$, $u_0$ strictly increasing; the case $g(0) = 0$ (indeed, in several space dimensions) is instead considered in [2, 11, 13], where analogous results are proved for $u_0 \in L^\infty \cap L^1(\mathbb{R})$ or $u_0 \in \mathrm{BV}(\mathbb{R})$, the space of functions with bounded variation. In both cases the solutions are obtained as vanishing viscosity limits. Recall that if $u \in \mathrm{BV}$, then $Du$ is a Radon measure that can be decomposed as $Du = D^a u + D^j u + D^c u$, where on the right-hand side we have the absolutely continuous part (with respect to the Lebesgue measure), the jump part, and the Cantor part of $Du$, respectively [1]. The solutions provided by the above authors can be discontinuous; since the term $\Phi(u_x)$ has no meaning in $\mathcal{D}'$ for such functions, it is understood [12] that equation (1.1) holds in $\mathcal{D}'$ with $\Phi(u_x)$ replaced by $\Phi(D^a u)$. Under this terminology, a WF $\psi$ in our class turns out to be a distributional solution to the equation

$$(g(\psi)\Phi(D^a\psi))' + \sigma\psi' - (f(\psi))' = 0, \tag{1.4}$$

where we omitted for simplicity the independent variable $\xi = x - \sigma t$ in the arguments of the functions. With $\psi' := D\psi$ we denoted here and in the following the (full) distributional derivative of $\psi$. The important issue concerning uniqueness is that it fails, in general, in presence of discontinuous solutions. However, uniqueness is recovered by requiring that solutions are *entropic*, a notion that is strictly related to that with the same name in the hyperbolic theory of conservation laws [6, 22, 28], see Section 4 for further details. We refer to [12, 13] for the general but rather technical definition of entropic solution for equation (1.1), while several additional informations and a bibliography on the subject can be found in the comprehensive survey [8]. In one space dimension, this definition reduces to the more explicit condition in [4].

As far as wavefronts are concerned, they can be discontinuous, too. Traveling waves for equation (1.1) have been studied in [15] (in the case $g(u) \sim u^n$, $g'(u) \ge 0$) and [27, 29] (for constant $g$), see also [32], with a particular emphasis to examples, physical motivations and numerics. An intuitive comparison between the case with convection $f$ and the case without convection ($f = 0$) is given in [14]. The references quoted in the papers mentioned just above also give some information for traveling waves in the latter case. We point out that equation (1.1) enters in the framework of none of those papers because of the assumption (H2).

Traveling waves have also been studied in [9] in the case $f = 0$ (see also [5, 24]), while [10] deals with the same case but adds a monostable source term to the equation; see also [30] for source terms with more than two zeros. In both cases, the results were provided for particular functions $\Phi$. Smooth fronts in presence of both convection and reaction, essentially in the case $g = 1$, were studied in [25]. To the best of our knowledge, the case when $f \neq 0$ has never been rigorously considered in this framework.

This paper, as the recent articles [17–20], aims at a better understanding of the wavefront solutions to degenerate parabolic equations modeling collective movements. The main motivation is to provide rigorous results to an issue motivated by [7], namely, roughly speaking, that profiles are smooth if $|\ell^+ - \ell^-|$ is small and possibly discontinuous otherwise. Such a result was justified in a special case in [29] and proved in [32] in the case $g = 1$.

**Content of the Paper.** Sections 2 to 4 provide a rigorous background to the class of solutions we are dealing with, without relying on the general setting of [2, 11, 13]. The main results are given in Section 5. First, we prove the existence and uniqueness of wavefronts in a simple setting; this allows us to completely characterize their singularities. The result is then extended to a general framework. Then, we show with an example that the entropy condition is necessary for the uniqueness. Indeed, this is known for general solutions since [23]. From a geometric point of view, the loss of uniqueness is due to multiple intersections of the functions $f - g$ and $s_\pm$, where $s_\pm$ is the line joining the points $(\ell^-, f(\ell^-))$ and $(\ell^+, f(\ell^+))$. This example motivates a further result: we prove that an entropic wavefront $\psi$ is the limit of smooth wavefronts $\psi_\varepsilon$, for $\varepsilon \to 0$. The latter profiles correspond to the equation

$$u_t + f(u)_x = (g(u)\Phi(u_x) + \varepsilon u_x)_x, \tag{1.5}$$

where the artificial viscosity $\varepsilon u_{xx}$ has been added to the original equation (1.1). This result also holds for general solutions [4], but we propose a different and simpler proof which is focussed on wavefronts and includes the presence of the term $f$. We recall that an analogous vanishing-viscosity criterion is used to uniquely select shock waves in a hyperbolic framework [6, 22, 28]. Since the profiles $\psi_\varepsilon$ are strictly monotone, this result justifies a posteriori our choice of focussing on monotone profiles since the beginning of the paper. Section 6 contains the proofs of the statements in Sections 3 and 5. At last, Section 7 shows an application to crowds dynamics suggested by a model in [7].

## 2 Classical Versus Singular Wavefront Solutions

In the following we always assume conditions (H1)–(H3) without any further mention. We begin with the definition of classical solution to equation (1.4), see [26], and briefly discuss its main properties; the parameter $\sigma$ in (1.4) is for the moment an arbitrary real value. Then we show how discontinuous solutions can arise and give some simple examples. For an open interval $I \subset \mathbb{R}$ we denote by $\mathrm{AC}_{\mathrm{loc}}(I)$ the set of locally absolutely-continuous functions in $I$, see [31].

**Definition 2.1.** A classical solution to equation (1.4) in an open interval $I \subseteq \mathbb{R}$ is a function $\psi \in C^1(I; [0, 1])$, with $\psi' \in \mathrm{AC}_{\mathrm{loc}}(I)$, such that equation (1.4) is satisfied a.e. in $I$.

If $\psi$ is a classical solution in $I$, then (1.4) can be integrated and reduced to the first-order equation

$$g(\psi)\Phi(\psi') + \sigma\psi - f(\psi) = c \quad \text{in } I, \tag{2.1}$$

for an arbitrary constant $c$. The same reduction holds if $g(\psi)\Phi(D^a\psi) + \sigma\psi - f(\psi) \in \mathcal{D}'(I)$ and (1.4) holds in the sense of distributions; in this case also equation (2.1) holds (with $D^a\psi$ replacing $\psi'$) in the same sense, see [33, Théorème I, Section 4, Chapitre II]. Any constant function $\psi$ solves (2.1) for some suitable constant $c$. By hypothesis (H2), equation (2.1) degenerates where either $\psi(\xi) = 0$ or $\psi(\xi) = 1$; for a wavefront $\psi$ we then define

$$J_\psi = J := \{\xi : 0 < \psi(\xi) < 1\}. \tag{2.2}$$

For simplicity, the subscript in the notation $J_\psi$ is often omitted in the following. Therefore, equation (2.1) can be written as

$$\Phi(\psi') = \frac{f(\psi) - (\sigma\psi - c)}{g(\psi)} \quad \text{in } J. \tag{2.3}$$

It is useful to introduce the function (as above, subscripts are dropped when unnecessary)

$$h_{\sigma,c}(u) = h(u) := \frac{f(u) - (\sigma u - c)}{g(u)}. \tag{2.4}$$

**Remark 2.1.** The existence and uniqueness of classical solutions $\psi$ to (2.1) is obvious in any interval $J$ as in (2.2). In this case, by (H3) equation (2.3) can be written as

$$\psi' = \Phi^{-1}(h(\psi)), \tag{2.5}$$

and then $\psi \in C^2(J)$. If $\sigma$ and $c$ are fixed, then either the solution $\psi$ is constant (and therefore $\psi \equiv \psi \in \mathbb{R}$ with $f(\psi) - \sigma\psi + c = 0$) or $\psi'(\xi) \neq 0$ for $\xi \in J$ (and so $\psi$ is either strictly increasing or strictly decreasing). If $(\alpha, \beta)$ is the maximal existence interval of (2.5) and $\alpha \in \mathbb{R}$ with $\psi(\alpha^+) \in (0, 1)$, then $\psi'(\xi) \to \pm\infty$ as $\xi \to \alpha^+$. The case $\beta \in \mathbb{R}$ is analogous.

Equation (2.5) hides a key feature of *formation of singularities* in the solutions, that we now show from a naive point of view. The domain of $\Phi^{-1}$ is the interval $(-1, 1)$ and then *classical* solutions $\psi$ to (2.5) can only take values in the region $\mathcal{A}_{\sigma,c} = \{u \in (0, 1) : |h(u)| < 1\}$. Consider for instance $\ell^\pm \in (0, 1)$ and $\ell^- < \ell^+$ (or $\ell^+ < \ell^-$); assume that $\ell^\pm$ are two consecutive zeros of $h$ belonging to a *same* interval contained in $\mathcal{A}_{\sigma,c}$; notice that this requires $f \neq 0$. Moreover, suppose $h > 0$ in $(\ell^-, \ell^+)$ ($h < 0$ in $(\ell^+, \ell^-)$, respectively). Then it is easy to show that there exists a unique (up to shifts) profile $\psi$ satisfying (1.3); moreover, $c = \sigma\ell^\pm - f(\ell^\pm)$, $h(\ell^\pm) = 0$ and

$$\sigma = \frac{f(\ell^+) - f(\ell^-)}{\ell^+ - \ell^-}. \tag{2.6}$$

On the contrary, assume $\ell^\pm \in (0, 1)$, let $\sigma$ and $c$ be as above and $h$ as in Figure 2 (i) for some $u_1 < u_2 \in (\ell^-, \ell^+)$. In this case $\mathcal{A}_{\sigma,c} \supset (\ell^-, u_1) \cup (u_2, \ell^+)$, and then no classical solution exists because $\ell^-$ and $\ell^+$ belong to two *different* intervals contained in $\mathcal{A}$.

The way to overcome this failure consists in finding maximal solutions $\psi_1$ and $\psi_2$ to (2.5), which are valued in $(\ell^-, u_1)$ and $(u_2, \ell^+)$, respectively, and to match them to obtain a single function $\psi$ defined in $\mathbb{R} \setminus \{\bar\xi\}$ as in Figure 2 (ii). Then $\psi$ is a classical solution to (2.5) in $\mathbb{R} \setminus \{\bar\xi\}$ with $\psi(\pm\infty) = \ell^\pm$; it has a discontinuity in $\bar\xi$ since

$$\lim_{\xi\to\bar\xi^-} \psi(\xi) = u_1 < \lim_{\xi\to\bar\xi^+} \psi(\xi) = u_2.$$

Moreover, we have $\lim_{\xi\to\bar\xi^\pm} \psi'(\xi) = \infty$ because $h(u_1) = h(u_2) = 1$, and then the graph of $\psi$ becomes vertical at $\bar\xi$ from both sides: at $\bar\xi$ the diffusion is saturated.

By subtracting the expressions $h(u_1) = 1$ and $h(u_2) = 1$, i.e., $f(u_i) - \sigma u_i + c = g(u_i)$, for $i = 1, 2$, and then computing $c$, we deduce

$$\sigma = \frac{(f - g)(u_2) - (f - g)(u_1)}{u_2 - u_1} \quad \text{and} \quad c = \sigma u_i - (f - g)(u_i). \tag{2.7}$$

Formula $(2.7)_1$ expresses the velocity of propagation of the discontinuity in the profile. However, formula (2.6) should also hold; this means that the propagation speed $(2.7)_1$ of the discontinuity in the profile must coincide with the speed (2.6) of the profile; on the other hand, $(2.7)_2$ implies that the points $(u_i, (f - g)(u_i))$, $i = 1, 2$, lie on the line joining $(\ell^-, f(\ell^-))$ and $(\ell^+, f(\ell^+))$. A geometrical interpretation is given in Figure 2 (iii).

If $h(u_1) = h(u_2) = -1$, with $u_1 \neq u_2$, then the corresponding profile $\psi$ is decreasing; in this case we have $\lim_{\xi\to\bar\xi} \psi'(\xi) = -\infty$ and the terms involving $g$ in the expression of $\sigma$ *change sign*; we now have

$$\sigma = \frac{f(u_2) - f(u_1) + (g(u_2) - g(u_1))}{u_2 - u_1}.$$

The case $u_1 = u_2$, i.e., when the function $h$ has a strict local maximum at $u_1$ with $h(u_1) = 1$ or strict local minimum with $h(u_1) = -1$, gives rise to a continuous profile whose graph has a vertical tangent with either $\psi'(\bar\xi) = \infty$ or $\psi'(\bar\xi) = -\infty$. We refer to Proposition 3.1 for a rigorous statement.

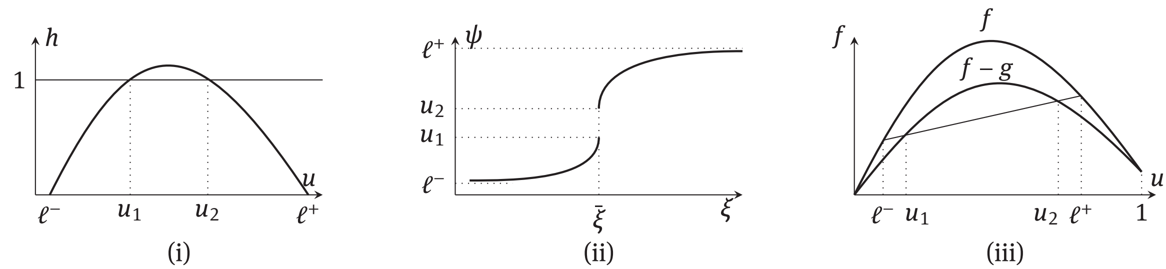


**Figure 2:** Formation of singularities in a profile. On the rightmost figure, a geometrical interpretation of (2.6) and (2.7).

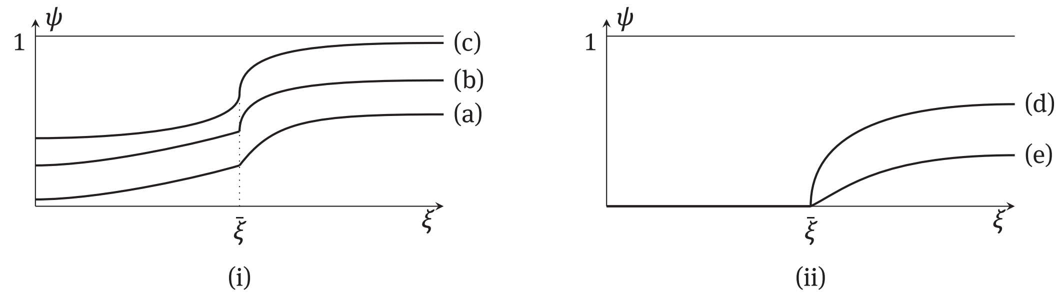


**Figure 3:** Candidates for continuous solutions with a singularity in the derivative at $\bar{\xi}$. Only cases (c) to (e) may occur.

The previous discussion is naive because it does not consider the case of several solutions to the equation $h(u) = 1$ and avoids the points where $g$ vanishes. Moreover, it bypasses the fact that equation (2.5) is not balanced where $\psi$ has a jump discontinuity: the right-hand side is a bounded function (at least if $u_i \notin \{0, 1\}$ for $i = 1, 2$) while the left-hand side $\psi'$ is a delta-like distribution. The same problem arises for the equation $(g(\psi)\Phi(\psi'))' + \sigma\psi' - f(\psi)' = 0$, because in this case the term $\Phi(\psi')$ has no meaning in the distribution sense.

**Example 2.1.** The weaker formulation (1.4) is already sufficient to include or to rule out some patterns of solutions, as we now show by some examples.

(i) Consider the case of an absolutely continuous function $\psi$ with a singularity in the derivative at $\bar{\xi}$ and $\psi(\bar{\xi}) \in (0, 1)$; assume moreover that $\psi$ is a classical solution in $(\bar{\xi} - \delta, \bar{\xi})$ and in $(\bar{\xi}, \bar{\xi} + \delta)$ for some $\delta > 0$, see Figure 3 (i). In this case $D^a\psi = \psi'$ and both $\psi'(\bar{\xi}^{\pm})$ exist by the equation.

- Case (a), where $\psi'(\bar{\xi}^-) \neq \psi'(\bar{\xi}^+)$ and both values are real, cannot occur: if it did, since $g(\psi(\bar{\xi}))$ does not vanish, then the term $g(\psi)\Phi(\psi')$ in (1.4) would give rise to a Dirac measure, which is not balanced by the other terms in the equation. Notice that this case can also be excluded by considering equation (2.1): if $\psi \in AC_{loc}$ is a solution, then $\sigma\psi - f(\psi)$ is continuous at $\bar{\xi}$ and then $\psi \in C^1$.
- Case (b), where $\psi'(\bar{\xi}^-) \in \mathbb{R}_+$ and $\psi'(\bar{\xi}^+) = \infty$, is ruled out analogously: indeed $\psi' \in L^1_{loc}$ because $\psi \in AC$, and terms do not match as above.
- Case (c), where $\psi'(\bar{\xi}^-) = \psi'(\bar{\xi}^+) = \infty$, is possible and $\psi$ becomes a solution in the sense of distributions to (1.4). In fact, for $\eta \in C_0^\infty(\bar{\xi} - \delta, \bar{\xi} + \delta)$, $\delta > 0$, we have

$$\int_{\bar{\xi}-\delta}^{\bar{\xi}+\delta} (g(\psi)\Phi(\psi') + \sigma\psi - f(\psi))\eta' \, d\xi = 0 \tag{2.8}$$

by integrating by parts and exploiting the fact that $\Phi(\infty) = 1$.

The same calculation shows that cuspon "solutions", where $\psi'(\bar{\xi}^{\pm}) = \mp\infty$, or $\psi'(\bar{\xi}^{\pm}) = \pm\infty$, cannot occur: in this case the left-hand side of (2.8) equals either $2g(\psi(\bar{\xi}))$ or $-2g(\psi(\bar{\xi}))$, respectively, and $g(\psi(\bar{\xi})) \neq 0$ because $\psi(\bar{\xi}) \in (0, 1)$. If $\psi(\bar{\xi}) \in \{0, 1\}$, we refer to Remark 3.2.

(ii) Let $\psi$ be a classical solution to (1.4) in its maximal existence interval $(\bar{\xi}, \beta)$; assume $\bar{\xi} \in \mathbb{R}$ and $\psi(\bar{\xi}^+) = 0$. We cannot have $\psi'(\bar{\xi}^+) = 0$, because then $\psi$ could be extended to the left of $\bar{\xi}$ as a classical solution. Then either $\psi'(\bar{\xi}^+)$ does not exist or it is positive (possibly $\infty$) and different from zero. In these cases, consider the extension $\bar{\psi}$ of $\psi$ defined by

$$\bar{\psi}(\xi) := \begin{cases} 0, & \xi \in (-\infty, \bar{\xi}], \\ \psi(\xi), & \xi \in (\bar{\xi}, \beta), \end{cases}$$

see Figure 3 (ii) for the cases $\psi'(\bar{\xi}^+) = \infty$ and $\psi'(\bar{\xi}^+) \in \mathbb{R}^+$. The function $\bar{\psi}$ is not a classical solution to (1.4) in $(-\infty, \beta)$; however, we claim that $\bar{\psi}$ is a solution in the sense of distributions. In fact, let $\eta \in C_0^\infty(\bar{\xi} - \delta, \bar{\xi} + \delta)$ with $\delta \in (0, \beta)$; we have

$$\begin{aligned} \int_{\bar{\xi}-\delta}^{\bar{\xi}+\delta} (g(\bar{\psi})\Phi(\bar{\psi}') + \sigma\bar{\psi} - f(\bar{\psi}))\eta' \, d\xi &= \lim_{\varepsilon\to 0^+} \int_{\varepsilon}^{\bar{\xi}+\delta} (g(\psi)\Phi(\psi') + \sigma\psi - f(\psi))\eta' \, d\xi \\ &= \lim_{\varepsilon\to 0^+} [-g(\psi(\varepsilon))\Phi(\psi'(\varepsilon)) - \sigma\psi(\varepsilon) + f(\psi(\varepsilon))]\eta(\varepsilon) \\ &\quad - \int_{\varepsilon}^{\delta} [(g(\psi)\Phi(\psi'))' + \sigma\psi' - f(\psi)']\eta \, d\xi. \end{aligned}$$

The term $\Phi(\psi'(\varepsilon))$ is bounded with respect to $\varepsilon$ and then $-g(\psi(\varepsilon))\Phi(\psi'(\varepsilon)) \to 0$ as $\varepsilon \to 0^+$ because $g(0) = 0$. Also the integral vanishes because $\psi$ is a classical solution. This proves the claim. We emphasize that these solutions are missing in the case $g$ does not vanish at 0.

(iii) Consider a continuous function $\psi$ defined on $(-\infty, \beta)$ and satisfying for some $\bar{\xi} < \beta$

$$\psi(\xi) \equiv \ell^- \in [0, 1] \quad \text{in } (-\infty, \bar{\xi}]. \tag{2.9}$$

How can $\psi$ be extended to the whole of $\mathbb{R}$ as a solution to (1.4)? By items (i) and (ii) above we deduce that $\psi$ can be extended either as a constant function or as a classical non-constant solution, but then $\psi(\bar{\xi}) \in \{0, 1\}$ and $\psi'(\bar{\xi}) = 0$ (bifurcation of a classical solution); we refer to the following Example 5.1 for the latter case. Moreover, if $\psi(\bar{\xi}) \in \{0, 1\}$, then $\psi$ can also be extended as a non-classical solution, see Figure 3 (ii) (bifurcation of a non-classical solution).

# 3 Admissible Wavefront Solutions

In this section we characterize, according to their singularities, the WFs we are going to deal with; we call them *admissible* WFs. The previous section provides us the motivations. The underlying idea is to deal with classical solutions as long as it is possible; otherwise, motivations have been provided in Section 2. More precisely, a typical profile $\psi$ under consideration is smooth except possibly at the finitely many points of its *singular set* $\mathcal{S}_\psi$; at these points, either $\psi$ is continuous but $\psi'$ is infinite, or $\psi$ has a jump discontinuity (the side limits of $\psi$ are finite but differ). For $\bar{\xi} \in \mathbb{R}$, we write for short $\psi'_\pm(\bar{\xi}) = \lim_{\xi\to\bar{\xi}^\pm} \psi'(\xi)$ and $\psi'(\bar{\xi}) = +\infty$ (resp., $-\infty$) if $\psi'_+(\bar{\xi}) = \psi'_-(\bar{\xi}) = +\infty$ (resp., $\psi'_+(\bar{\xi}) = \psi'_-(\bar{\xi}) = -\infty$).

Omitting for simplicity the subscript $\psi$, for $n \in \mathbb{N}$ we denote

$$\mathcal{S} := \{\xi_0 < \xi_1 < \cdots < \xi_n\} = \mathcal{C} \cup \mathcal{J}, \tag{3.1}$$

where the subsets $\mathcal{C}$ and $\mathcal{J}$ are defined below.

- We have $\xi_i \in \mathcal{C}$ if $\psi$ is *continuous* at $\xi_i$ and the following holds:
  ($\mathcal{C}_1$) if $\psi(\xi_i) \in (0, 1)$, then either $\psi'(\xi_i) = \infty$ or $\psi'(\xi_i) = -\infty$,
  ($\mathcal{C}_2$) if $\psi(\xi_i) \in \{0, 1\}$, then $i = 0$ or $i = n$ and $\psi'(\xi_i) \neq 0$.
- We have $\xi_i \in \mathcal{J}$ if $\psi$ has a *jump discontinuity* at $\xi_i$ and the following holds:
  ($\mathcal{J}_1$) if $\psi(\xi_i^\pm) \in (0, 1)$, then either $\psi'(\xi_i^\pm) = \infty$ or $\psi'(\xi_i^\pm) = -\infty$,
  ($\mathcal{J}_2$) if $\psi(\xi_0^-) \in \{0, 1\}$, then either $\psi'_+(\xi_0) = \infty$ or $\psi'_+(\xi_0) = -\infty$; if $\psi(\xi_n^+) \in \{0, 1\}$, then either $\psi'_-(\xi_n) = \infty$ or $\psi'_-(\xi_n) = -\infty$.

The two alternatives above (either ... or) depend on whether the WF is increasing or decreasing, respectively. The requirement $\psi'(\xi_0) \neq 0$ in $(\mathcal{C}_2)$ leaves open the possibilities that either $\psi'_+(\xi_0)$ does not exist or $\psi'_+(\xi_0) \in (0, \infty]$; analogous possibilities hold for $\psi'_-(\xi_n)$. These requirements exclude smooth WFs; in fact, in that situation, smooth profiles satisfy $\lim_{\xi\to\xi_0^+} \psi'(\xi) = 0$ or $\lim_{\xi\to\xi_n^-} \psi'(\xi) = 0$.

Notice the asymmetry of conditions $(\mathcal{C}_2)$ and $(\mathcal{J}_2)$: in the former case, only the value 0 for the derivative is excluded (otherwise $\psi$ is smooth); in the latter, the WF $\psi$ must have a vertical tangent at the discontinuity point. This depends on the fact that discontinuities in the profile can arise only if the diffusion is saturated to $\pm 1$, and in turn this only happens if $\psi' = \pm\infty$. Condition $(\mathcal{J}_1)$ implies, for WFs, the requirement [4, (2.1)], which is used there to give a meaning to discontinuous solutions to (1.1). Condition $(\mathcal{J}_2)$ (which is missing in the non-degenerate case considered in [4]) is introduced to cope with the vanishing of $g$ at 0 and 1. Both conditions together coincide with the definition given in [8, p. 187].

**Example 3.1.** We show that the case when $\psi'_+(\xi_0)$ does not exist can indeed occur in $(\mathcal{C}_2)$. Define

$$f(u) = u^3(1-u)\Big(\sin^2\frac{1}{u} + e^{-u}\Big) \quad \text{and} \quad g(u) = 4u^3(1-u).$$

Assume $\ell^- = 0$ and $\ell^+ = 1$, so that from (2.5) and (2.6) we have $\sigma = c = 0$, and $h(u) = \frac{1}{4}(\sin^2\frac{1}{u} + e^{-u}) \in (0, \frac{1}{2})$ if $u \in (0,1)$. A solution to the equation $\psi' = \Phi(h(\psi))$, with $\psi(-\infty) = 0$, $\psi(\infty) = 1$, is easily shown to exist. Moreover, we claim that the solution $\psi$ reaches the value 0 for some finite $\xi_0$; this claim can be proved as follows. Let $\psi$ be the solution of $\psi' = \Phi^{-1}(h(\psi))$ satisfying $\psi(0) = \psi$ for some $\psi \in (0,1)$. We notice that $\psi'(\xi) > 0$ for every $\xi$ such that $\psi(\xi) \in (0,1)$, and then we consider the inverse function $\xi = \xi(\psi)$. Then we define $\xi_0 = \inf\{\xi : \psi(\xi) > 0\} \in \mathbb{R} \cup \{-\infty\}$. We have

$$-\xi_0 = 0 - \xi_0 = \int_0^{\psi} \xi'(\zeta)\, d\zeta = \int_0^{\psi} \frac{1}{\Phi^{-1}(h(\zeta))}\, d\zeta. \tag{3.2}$$

Since $h(u) \le \frac{1}{2}$, for $u \in [0,1]$, by (H3) we deduce

$$\frac{1}{\Phi^{-1}(h(\psi))} \ge \frac{1}{\Phi^{-1}(\frac{1}{2})}, \quad \psi \in [0,1].$$

This implies that the integral in (3.2) is convergent; then $\xi_0$ is a real number and the claim is proved. In this case the limit $\lim_{u\to 0^+} \Phi^{-1}(h(u))$ does not exist. Since $0 < h(u) \le \frac{1}{2}$, we have $\mathcal{S} = \mathcal{C} = \{\xi_0\}$ for some $\xi_0 \in \mathbb{R}$ with $\psi(\xi_0^+) = 0$, but the limit $\lim_{\xi\to\xi_0^+} \psi'(\xi) = \lim_{\psi\to 0^+} \Phi^{-1}(h(\psi))$ does not exist.

**Definition 3.1.** Consider $\ell^-, \ell^+ \in [0,1]$, $\ell^- \neq \ell^+$, and a monotone function $\psi : \mathbb{R} \to [0,1]$. The function $\psi$ is an *admissible wavefront solution* to equation (1.1) with wave speed $\sigma \in \mathbb{R}$ and boundary conditions $\ell^-, \ell^+$ if
(i) $\psi$ satisfies (1.3),
(ii) there are points as in (3.1) such that $\psi$ is a classical solution to (1.4) in every interval $(-\infty, \xi_0)$, $(\xi_{i-1}, \xi_i)$ for $i = 1, \dots, n$, and $(\xi_n, \infty)$,
(iii) $\psi$ is a solution to equation (1.4) in $\mathcal{D}'(\mathbb{R})$.

**Remark 3.1.** The motivation for considering monotone profiles in Definition 3.1 relies on Remark 2.1: any profile $\psi$ in every interval as above and contained in $J$ has to be strictly monotone. Classical wavefront solutions are then monotone. On the other hand, at any point $\xi_i$ with $\psi(\xi_i) \in (0,1)$ the sign of the derivative does not change, and we shall see in Theorem 5.3 that solutions of the augmented problem (1.5) single out, in the vanishing-viscosity limit, only entropic (and then monotone) solutions. So, non-monotone profiles do not seem to provide interesting solutions.

In the following we always deal with admissible WFs. Remark that $\psi$ is strictly monotone in the interval $(\xi_0, \xi_n)$. Moreover, according to Definition 3.1 we have $D^c\psi = 0$, because $\psi$ is assumed to be smooth outside $\mathcal{S}$. The smoothness of the profiles in the above class is straightforward: if $\mathcal{S} = \emptyset$, then $\psi \in C(\mathbb{R}) \cap C^2(J)$; otherwise $\psi \in C^2(J \setminus \mathcal{S})$, being continuous at points in $\mathcal{C}$. The jumps of the WF at point in $\mathcal{J}$ are called *subshocks* [15].

**Remark 3.2.** Let $\psi$ be an admissible WF. We claim that if $\ell^- \in (0,1)$, then we have $\psi(\xi) \neq \ell^-$ for every $\xi \in \mathbb{R}$; an analogous statement holds for $\ell^+$. In fact, every admissible WF $\psi$ is classical in the half-line $(-\infty, \xi_0)$

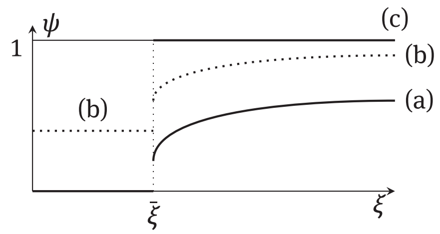


**Figure 4:** Profile (b) is not admissible, profiles (a) and (c) are.

and cannot be constant there by Examples 2.1 (iii) and 3.2. If $\ell^\pm \in (0,1)$, then $\psi(\xi) \in (\ell^-, \ell^+)$ for every $\xi \in \mathbb{R}$ and $\psi$ is strictly monotone in $\mathbb{R}$.

In particular, see Example 2.1 (iii), if $\psi$ satisfies (2.9) and has a jump discontinuity at $\bar\xi$, then necessarily $\ell^- \in \{0,1\}$, see Figure 4. This is also excluded by condition $(\mathcal{J}_1)$. The possibility that $\psi$ only assumes the values 0 and 1 is not excluded. We anticipate that these solutions are missing in the case $g$ does not vanish at 0, see Remark 3.4.

Also notice that cuspon "solutions" with a cusp either at 0 or 1, see Example 2.1 (i), are ruled out by Definition 3.1 because they are not monotone.

In the following proposition we characterize condition (iii) in Definition 3.1. Let $\psi$ be a WF of (1.1) with wave speed $\sigma$. In every interval where $\psi$ is classical, the equation can be integrated, see (2.1); hence, there exist finitely many $c_i \in \mathbb{R}$ such that

$$\begin{aligned} g(\psi(\xi))\Phi(\psi'(\xi)) + \sigma\psi(\xi) - f(\psi(\xi)) &= c_0, && \xi \in (-\infty, \xi_0),\\ g(\psi(\xi))\Phi(\psi'(\xi)) + \sigma\psi(\xi) - f(\psi(\xi)) &= c_i, && \xi \in (\xi_{i-1}, \xi_i),\ i = 1,\dots,n,\\ g(\psi(\xi))\Phi(\psi'(\xi)) + \sigma\psi(\xi) - f(\psi(\xi)) &= c_{n+1}, && \xi \in (\xi_n, \infty). \end{aligned}$$

**Proposition 3.1.** *For every monotone function $\psi$ satisfying conditions* (i) *and* (ii) *in Definition 3.1 the following three conditions are equivalent:*

(a) *$\psi$ satisfies condition* (iii) *in Definition 3.1.*

(b) *For every $\xi_i \in \mathcal{J}$ we have*

$$\sigma = \frac{(f(\psi(\xi_i^+)) - f(\psi(\xi_i^-))) \pm (g(\psi(\xi_i^+)) - g(\psi(\xi_i^-)))}{\psi(\xi_i^+) - \psi(\xi_i^-)}, \tag{3.3}$$

*where the sign + occurs for decreasing profiles and the sign – for increasing profiles.*

(c) *There exists a unique $\gamma \in \mathbb{R}$ such that $\psi$ satisfies*

$$g(\psi(\xi))\Phi(\psi'(\xi)) + \sigma\psi(\xi) - f(\psi(\xi)) = \gamma, \quad \xi \in \mathbb{R} \setminus \mathcal{S}. \tag{3.4}$$

**Example 3.2.** Assume that $\psi$ satisfies (2.9) and has a jump discontinuity at $\bar\xi$, see Remark 3.2 and Figure 4. By passing to the limit in (2.1) for $\xi \to \bar\xi^\pm$ we deduce $\sigma\psi(\bar\xi^+) = f(\psi(\bar\xi^+)) - g(\psi(\bar\xi^+)) + c$ and $\sigma\ell^- = f(\ell^-) + c$, whence

$$\sigma = \frac{f(\psi(\bar\xi^+)) - f(\ell^-) - g(\psi(\bar\xi^+))}{\psi(\bar\xi^+) - \ell^-}$$

by difference. This value coincides with (3.3) if and only if $\ell^- \in \{0,1\}$; if $\ell^- \in (0,1)$, then Proposition 3.1 confirms that $\psi$ is not a solution in the sense of distributions.

**Proposition 3.2.** *Every admissible wavefront $\psi$ satisfies*

$$\lim_{\xi\to\pm\infty} g(\psi(\xi))\Phi(\psi'(\xi)) = 0. \tag{3.5}$$

*Moreover, if* (1.3) *holds, then*

$$\sigma = \frac{f(\ell^+) - f(\ell^-)}{\ell^+ - \ell^-} \quad \textit{and} \quad \gamma = \sigma\ell^\pm - f(\ell^\pm) = \frac{\ell^- f(\ell^+) - \ell^+ f(\ell^-)}{\ell^+ - \ell^-}. \tag{3.6}$$

**Remark 3.3.** If $\ell^- \in (0,1)$, then $g(\ell^-) \neq 0$, and by formula (3.5) we deduce $\lim_{\xi\to-\infty} \psi'(\xi) = 0$; similarly, we have $\lim_{\xi\to\infty} \psi'(\xi) = 0$ when $g(\ell^+) \neq 0$. Formula $(3.6)_2$ means that the line $\sigma u - \gamma$ joins the points $(\ell^-, f(\ell^-))$ and $(\ell^+, f(\ell^+))$; we denote

$$s_\pm(u) := \sigma u - \gamma = \frac{f(\ell^+) - f(\ell^-)}{\ell^+ - \ell^-}(u - \ell^\pm) + f(\ell^\pm). \tag{3.7}$$

By (3.4), Proposition 3.2 and (3.7), we can write the function $h$ in (2.4) in a slightly different way and rewrite the equation for future reference: if $\psi$ is a classical solution to (1.4) satisfying (1.3), we have (by dropping the dependence of $h$ on $\ell^\pm$ as in (2.4))

$$\psi' = \Phi^{-1}(h(\psi)) \quad \text{for } h(u) := \frac{f(u) - s_\pm(u)}{g(u)}. \tag{3.8}$$

**Remark 3.4.** We have $h \in C^1(\ell^-, \ell^+)$ by hypotheses (H1) and (H2); moreover, if $\ell^- \in (0,1)$, then $h \in C^1[\ell^-, \ell^+)$ and $h(\ell^-) = 0$, while if $\ell^+ \in (0,1)$, then $h \in C^1(\ell^-, \ell^+]$ and $h(\ell^+) = 0$.

Under the notation in Definition 3.1, equation $(3.8)_1$ must be satisfied by an admissible profile $\psi$ in $\mathbb{R} \setminus \mathcal{S}$. Since we consider monotone profiles, it follows that $h(\psi)$ is either positive ($\geq 0$) or negative ($\leq 0$)in $\mathbb{R} \setminus \mathcal{S}$. Moreover, we have $\psi' > 0$ in $J$ by Remark 2.1 if $\ell^- \neq \ell^+$ and then either $h(\psi) > 0$ or $h(\psi) < 0$ in $(\mathbb{R} \setminus \mathcal{S}) \cap J$.

In Remark 3.2 we discussed the case of profiles vanishing on a half-line $(-\infty, \bar\xi)$ and then jumping to a positive value, say $u_1$, at $\bar\xi$, see Figure 4. These solutions are missing if $g(0) \neq 0$. Indeed, in that case we have $h = 1$ in $(0, u_1]$; since $h(\psi) = \frac{f(\psi) - \sigma\psi}{g(\psi)}$, if $g(0) \neq 0$, then $h(0) = 0$, in contradiction with the above equality.

## 4 Entropic Wavefront Solutions

As we shall see below, profiles are not unique in the class of admissible WFs. As for hyperbolic conservation laws, this depends on the presence of discontinuities and, more precisely, on the occurrence of more than two points where the function $h$ assumes the values 1 or $-1$. This was first noticed in [4, 23], where an entropy condition was introduced, in the case $f = 0$, to recover the uniqueness. For the case $f \neq 0$, we now provide an analogous condition.

**Definition 4.1.** Consider an admissible wavefront $\psi$ and let $u_1, u_2 \in [0,1]$, with $u_1 \neq u_2$, be two points such that $h(u_i) = 1$ (or $h(u_i) = -1$) for $i = 1, 2$ and $\psi$ has a jump discontinuity from $u_1$ to $u_2$. Then $\psi$ is *entropic* if

$$h(u) \geq 1 \quad \text{for } u \in (u_1, u_2) \quad (\text{resp., } h(u) \leq -1 \text{ for } u \in (u_2, u_1)). \tag{4.1}$$

(See Figure 5.)

We now comment on this definition by focusing on the case $h(u) > 1$ for $u \in (u_1, u_2)$; the case $h(u) < -1$ is analogous. In this case condition (4.1) is equivalent to

$$(f - g)(u) \geq s_\pm(u) \quad \text{for } u \in (u_1, u_2), \tag{4.2}$$

for $s_\pm$ as in (3.7). This means that the graph of the function $f - g$ must lie above the line $s_\pm$. Recall that if we have a discontinuity between $u_1$ and $u_2$, then necessarily the line $s_\pm$ passes through the points $(u_1, (f-g)(u_1))$, $(u_2, (f-g)(u_2))$ (see Section 2 and Figure 2 (iii)). Hence, condition (4.2) becomes

$$(f - g)(u) \geq \frac{(f-g)(u_2) - (f-g)(u_1)}{u_2 - u_1}(u - u_2) + (f-g)(u_2) \quad \text{for } u \in (u_1, u_2), \tag{4.3}$$

see (2.7). We refer to Figure 6 for a geometrical interpretation of conditions $h > 0$, $h \gtrless 1$ in terms of the functions $f$ and $g$; recall

$$h(u) > 0 \iff f(u) > s_\pm(u) \quad \text{and} \quad h(u) \gtrless 1 \iff f(u) - g(u) \gtrless s_\pm(u).$$

If $f = 0$, then (4.3) reduces to

$$g(u) \leq \frac{(g(u_2) - g(u_1))}{u_2 - u_1}(u - u_2) + g(u_2) \quad \text{for } u \in (u_1, u_2),$$

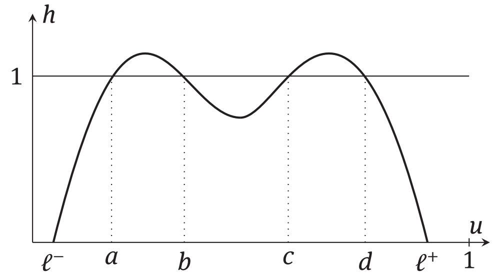


**Figure 5:** Jumps from $a$ to $b$, or from $c$ to $d$, are entropic; a jump from $a$ to $d$ is not.

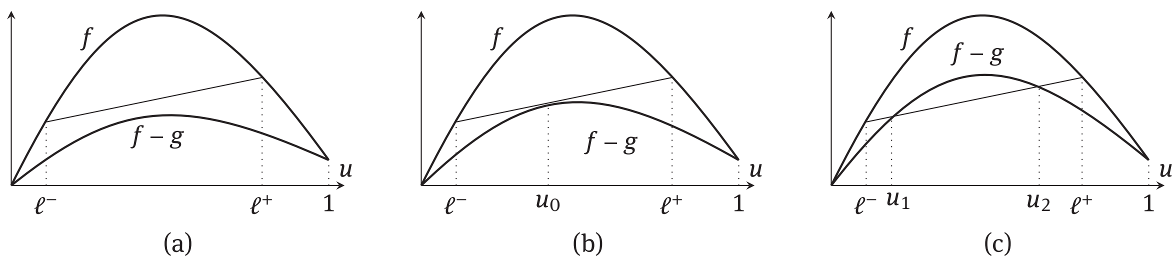


**Figure 6:** Geometrical interpretation of the conditions $h < 1$ (left), $h = 1$ (center) and $h > 1$ (right), in the case $h > 0$. The oblique line is the line $s_{\pm}(u)$.

which coincides with the definition of entropy solution in [4] (for wavefront solutions). On the other hand, if $g = 0$, then we find the usual entropy condition exploited in hyperbolic conservation laws [6, Theorem 4.4]. This shows that (4.1) fits to both parabolic and hyperbolic equations, and then the term "entropic" seems particularly suited to design this condition. Notice that condition (4.1) does not appear explicitly in some aforementioned papers, for instance in [8, Assumptions 13.2, p. 186], because $g$ is required to be *convex* there, and then (4.1) trivially holds.

# 5 Main Results

In this section we state and comment our main results. They concern the existence, uniqueness and the smooth approximation of entropic wavefront solutions to (1.1). We deal with the case of increasing profiles; analogous results for decreasing profiles can be obtained as well, see Remark 5.1. Since we focus on increasing profiles, we fix $\ell^{\pm} \in [0, 1]$ with $\ell^- < \ell^+$; as a consequence, this choice defines a function $h$ as in (3.8).

In the first result, Theorem 5.1, we essentially assume that the function $h$ either is valued in $[0, 1]$ or it is larger than 1 only in an interval. In this case the corresponding WFs are clearly entropic and their singular set $\mathcal{S}$ either is empty or contains only one point. This simple framework gives us the possibility of analyzing in detail all possible subcases.

**Theorem 5.1.** *We make assumptions* (H1)–(H3)*; fix $\ell^{\pm} \in [0, 1]$ with $\ell^- < \ell^+$ and assume*

$$h > 0 \quad \text{in } (\ell^-, \ell^+). \tag{5.1}$$

*Under the following conditions, equation* (1.1) *has a unique (up to shifts), increasing and entropic wavefront $\psi$, which satisfies* (1.3)*, with $\sigma$ given by* $(3.6)_1$ *and $\mathcal{S}$ specified below.*

(a) *Assume*

$$h < 1 \quad \text{in } (\ell^-, \ell^+), \tag{5.2}$$

$$\lim_{u \to (\ell^-)^+} h(u) = 0, \qquad \lim_{u \to (\ell^+)^-} h(u) = 0. \tag{5.3}$$

*In this case we have $\mathcal{S} = \emptyset$.*

(b) *Assume that there exists* $u_0 \in [\ell^-, \ell^+]$ *such that*

$$\begin{cases} h < 1 & \text{in } (\ell^-, \ell^+) \setminus \{u_0\}, \\ h(u_0) = 1, \end{cases} \tag{5.4}$$

*and one of the following conditions is satisfied:*

(1) $u_0 \in (\ell^-, \ell^+)$ *and* (5.3) *hold.*

(2) $u_0 = \ell^- = 0$ *and* $(5.3)_2$ *hold, but* $\lim_{u\to 0^+} h(u) \in (0, 1]$ *or* $h$ *has no limit if* $u \to 0^+$.

(3) $u_0 = \ell^+ = 1$ *and* $(5.3)_1$ *hold, but* $\lim_{u\to 1^-} h(u) \in (0, 1]$ *or* $h$ *has no limit if* $u \to 1^-$.

*In these cases we have* $\mathcal{S} = \mathcal{C} = \{\xi_0\}$ *for some* $\xi_0 \in \mathbb{R}$ *with* $\psi(\xi_0) = u_0$.

(c) *Assume that there exist* $u_1, u_2 \in [\ell^-, \ell^+]$, *with* $u_1 < u_2$, *such that*

$$\begin{cases} h < 1 & \text{in } (\ell^-, u_1) \cup (u_2, \ell^+), \\ h \geq 1 & \text{in } (u_1, u_2), \end{cases} \tag{5.5}$$

*and one of the following conditions is satisfied:*

(1) $u_1, u_2 \in (\ell^-, \ell^+)$ *and* (5.3).

(2) $u_1 = \ell^- = 0$, $u_2 < \ell^+$ *and* $(5.3)_2$.

(3) $u_2 = \ell^+ = 1$, $\ell^- < u_1$ *and* $(5.3)_1$.

(4) $u_1 = \ell^- = 0$ *and* $u_2 = \ell^+ = 1$.

*In these cases we have* $\mathcal{S} = \mathcal{J} = \{\xi_0\}$ *and* $\psi(\xi_0^-) = u_1$, $\psi(\xi_0^+) = u_2$.

*Conversely, assume again* $\ell^- < \ell^+$ *and* (5.1). *Moreover, assume that for some (every) increasing entropic WF* $\psi$ *that satisfies* (1.3) *the singular set* $\mathcal{S}$ *of* $\psi$ *is either empty or contains a single point* $\xi_0 \in \mathcal{C}$, *with* $u_0 := \psi(\xi_0)$ *(respectively,* $\xi_0 \in \mathcal{J}$ *and* $u_1 := \psi(\xi_0^-)$, $u_2 := \psi(\xi_0^+)$*). Then the conditions on* $h$ *given above in items* (a) *and* (b) *(respectively,* (c)*) hold.*

We collect here several comments on Theorem 5.1.

(i) Condition (5.1) only depends on $f$ and is the well-known necessary and sufficient condition for the existence of wavefronts in the case the diffusion term in (1.1) has the form $(g(u)u_x)_x$, with $g$ as in (H2) but with $g(0)$ and $g(1)$ not necessarily 0, see [26, Theorem 9.1]. It is always satisfied if $f$ is strictly concave in the interval $(\ell^-, \ell^+)$. Indeed, condition (5.1) also has a hyperbolic counterpart (i.e., when $g = 0$), which establishes that the piecewise constant discontinuous solution assuming the values $\ell^-$ for $x < 0$ and $\ell^+$ for $x > 0$ is entropic (in the Oleinik sense, see [6, Theorem 4.4]).

If (5.1) holds, then the function $\Phi^{-1}(h(u))$ can be undefined only when $h \geq 1$.

Assume that (5.1) fails at one point and, for instance, $h(u_0) = 0$ for some $u_0 \in (\ell^-, \ell^+)$; then the initial-value problem for equation (3.8) with datum $\psi(0) = u_0$ has the unique solution $\psi \equiv u_0$, and so no smooth profile joining $\ell^-$ with $\ell^+$ can exist. Indeed, neither discontinuous solutions can exists, because $\psi'(u_0) = 0$ contradicts $(\mathcal{J}_1)$. One can also argue as follows. Since $u_0 \in (0, 1)$, by Remark 3.2 any WF $\psi_-$ that connects $\ell^-$ with $u_0$ satisfies $\psi_- < u_0$ in $\mathbb{R}$. Analogously, the profile $\psi_+$ that connects $u_0$ to $\ell^+$ satisfies $\psi_- > u_0$ in $\mathbb{R}$. Then there is no possibility of connecting $\psi_-$ with $\psi_+$.

(ii) The conditions $h \gtreqless 1$ or $h = 1$ depend on both $f$ and $g$. They specify the *regularity* of the profile, which depends on whether the regime is "strongly parabolic" (large $g$, case (a)) or "weakly parabolic" (small $g$, case (c)).

(iii) Conditions (5.3) are always satisfied if $\ell^- > 0$ and $\ell^+ < 1$, because of Remark 3.4. Then they are really needed only in the cases $\ell^- = 0$ and $\ell^+ = 1$.

(iv) In the case $f = g$, see Section 7, the function $f$ must satisfy (H2); then $f \geq 0$ and $f(1) = 0$. Apart from the case $\ell^- = 0$ and $\ell^+ = 1$, we have $s_\pm > 0$ in $(\ell^-, \ell^+)$ and so (5.2) holds; only cases (a), (b2) or (b3) can occur, i.e., we are in the "strongly parabolic" regime. If $\ell^- = 0$ and $\ell^+ = 1$, then $s_\pm \equiv 0$ in $(0, 1)$ and $h \equiv 1$; the profile is a stationary step function.

(v) The shape of the profiles can be easily deduced from the conditions above.

- In case (a), if $\ell^- = 0$, the profile can be identically equal to 0 on a half-line $(-\infty, \xi_0]$; the case $\ell^+ = 1$ leads to an analogous situation.

- In case (b), first notice that $u_0 = \ell^-$ ($u_0 = \ell^+$) implies $\ell^- = 0$ (resp., $\ell^- = 1$) by Example 2.1 (iii). In case (b1) the profile has an inflection point with vertical tangent at $\xi_0$. In case (b2) the profile is identically equal to 0 in $(-\infty, \xi_0]$; we have $\psi'(\xi_0^+) > 0$ or $\psi'(\xi_0^+) = \infty$ according to $\lim_{u\to 0^+} h(u) \in (0,1)$ or $\lim_{u\to 0^+} h(u) = 1$.
- In case (c), observe that if $u_1 = \ell^-$, then $\psi = \ell^-$ in $(-\infty, \bar{\xi})$ for some $\bar{\xi} \in \mathbb{R}$, and then $\ell^- = 0$ by Remark 3.2; an analogous observation holds in the case $u_2 = \ell^+$. Case (c) contains some special subcases. For instance, if $\ell^- = u_1 = 0$ and $u_2 < \ell^+$, the profile equals 0 on $(-\infty, \xi_0)$ and has a jump discontinuity at $\xi_0$. If we have both $\ell^- = u_1 = 0$ and $\ell^+ = u_2 = 1$, the profile equals 0 on $(-\infty, \xi_0)$ and 1 on $(\xi_0, +\infty)$.

(vi) If $f \equiv 0$, then $\sigma = \gamma = 0$ and so $s_\pm \equiv 0$ and $h \equiv 0$. Equation (3.4) becomes $g(\psi)\Phi(D\psi) = 0$; then either $g(\psi) = 0$ or $\Phi(D\psi) = 0$, and therefore solutions are piecewise constant. Because of (3.3), in this case there is only one (up to shifts) increasing entropic WF $\psi$, which is 0 for $x < 0$ and 1 for $x > 0$.

(viii) The case when $g$ is strictly positive in $[0, 1]$ can be easily treated by dropping condition $(\mathcal{J}_2)$. We did not include this case in the paper to avoid long statements with enumeration of several cases. Then, our results extend and make precise those in [32], where $g$ is constant and $f$ strictly convex. The two latter conditions make (4.3) trivially satisfied. If moreover $f \equiv 0$, then the discontinuous profile of the previous item cannot occur, and so no entropic WFs exist. This result matches with what was pointed out in [4, below formula (1.8)].

**Example 5.1.** We show how to obtain classical WFs $\psi$ with $\ell^- = 0$ and $\psi(\underline{\xi}) = \psi'(\underline{\xi}) = 0$, for some $\underline{\xi} \in \mathbb{R}$, see Example 2.1 (iii). Notice that if such a $\psi$ exists, then $\gamma = 0$. Choose $\Phi$ as in (1.2), $\ell \in (0, 1)$ (for simplicity we write $\ell$ for $\ell^+$), a smooth function $f_r : [0, \ell] \to \mathbb{R}$, with $f_r > 0$ in $(0, \ell)$ and $f_r(0) = f_r(\ell) = 0$, and $\sigma > 0$. We define the flux $f$ as

$$f(u) = f_r(u) + \sigma u.$$

Then $\sigma$ is the candidate for the wave speed of a profile $\psi$ connecting 0 with $\ell$, and $f_r$ is the reduced flux of $f$. The inequality in (5.1) holds, while (5.2) is satisfied if

$$h(u) = \frac{f_r(u)}{g(u)} < 1 \quad \text{in } (0, \ell). \tag{5.6}$$

If (5.6) holds, then Theorem 5.1 (a) applies and there exists an increasing entropic WF $\psi$ with $\psi \in C(\mathbb{R}) \cap C^2(J)$: moreover, $\psi' > 0$ in $J$ by Remark 2.1. To have $\psi(\underline{\xi}) = \psi'(\underline{\xi}) = 0$ for some $\underline{\xi} \in \mathbb{R}$, we further assume

$$\frac{g(u)}{f_r(u)} \in L^1(0, \tfrac{\ell}{2}) \quad \text{and} \quad \lim_{u\to 0} \frac{f_r(u)}{g(u)} = 0. \tag{5.7}$$

We denote by $\bar{\xi} > 0$ the unique point satisfying $\psi(\bar{\xi}) = \frac{\ell}{2}$ and define $\underline{\xi} = \inf J$; then $\underline{\xi} < \bar{\xi}$. If $\xi(\psi)$ denotes the inverse function of the function $\psi$ in $J$, then by (5.7) we have

$$\begin{aligned} \bar{\xi} - \underline{\xi} &= \int_0^{\frac{\ell}{2}} \xi'(\psi)\, d\psi \\ &= \int_0^{\frac{\ell}{2}} \frac{1}{\Phi^{-1}\Big(\frac{f(\psi)-\sigma\psi}{g(\psi)}\Big)}\, d\psi \\ &= \int_0^{\frac{\ell}{2}} \frac{1}{\Phi^{-1}\Big(\frac{f_r(\psi)}{g(\psi)}\Big)}\, d\psi < \infty, \end{aligned}$$

because $\Phi^{-1}(u) \sim \alpha u$ for $u \to 0$, where $\alpha = (\Phi^{-1})'(0) = \frac{1}{\Phi'(0)}$. Then $\underline{\xi} \in \mathbb{R}$. At last, by (3.8) and $(5.7)_2$, we have

$$\lim_{\xi\to\underline{\xi}^+} \psi'(\xi) = \lim_{\xi\to\underline{\xi}^+} \Phi^{-1}\Big(\frac{f_r(\psi(\xi))}{g(\psi(\xi))}\Big) = 0.$$

Therefore $\psi \in C^1(\mathbb{R})$. As an example of functions satisfying (5.6) and (5.7), we can choose $f_r(u) = u^2(\ell - u)$ and $g(u) = u^{\frac{3}{2}}(1 - u)$.

The next result extends Theorem 5.1 to the case where $h$ crosses the horizontal line at height 1 (or −1) more than twice. This means that the set $\mathcal{J}$ contains more than one point.

**Theorem 5.2.** *We make assumptions* (H1)–(H3)*; fix* $\ell^\pm \in [0,1]$ *with* $\ell^- < \ell^+$ *and suppose* (5.1). *Moreover, assume that there exist distinct points* $v_l, u_k$ *in* $[\ell^-, \ell^+]$, *for* $l = 1, \dots, m$, $k = 1, \dots, 2n$, *and* $\varepsilon > 0$ *such that*
(i) $v_1 < v_2 < \dots < v_m$, $h(v_l) = 1$, $h < 1$ *in* $((v_l - \varepsilon, v_l + \varepsilon) \setminus \{v_l\}) \cap (\ell^-, \ell^+)$, *and conditions* (b1)–(b3) *in Theorem 5.1 hold with* $u_0$ *replaced by* $v_l$, $l = 1, \dots, m$ *in* (b1), *by* $v_1$ *in* (b2) *and by* $v_m$ *in* (b3),
(ii) $u_1 < u_2 < \dots < u_{2n-1} < u_{2n}$, $h \geq 1$ *in* $(u_{2k-1}, u_{2k})$ *for* $k = 1, \dots, n$, $h < 1$ *in* $(u_{2k}, u_{2k+1})$ *for* $k = 1, \dots, n-1$, *and conditions* (c1)–*(c4) in Theorem 5.1 hold with* $u_{2n}$ *replacing* $u_2$.

*Then equation* (1.1) *has a unique (up to shifts) increasing entropic wavefront* $\psi$, *which satisfies* (1.3), *with* $\sigma$ *given by* $(3.6)_1$. *Moreover, the profile* $\psi$ *is continuous but not differentiable at points* $v_l$ *and has precisely* $n$ *jumps from* $u_{2k-1}$ *to* $u_{2k}$.

Theorem 5.2 reduces to Theorem 5.1 either when $l = 1$ and there are no points satisfying conditions (ii) or when $n = 1$ and there are no points of type (i). Also in the framework of Theorem 5.2 a complete description of the singular set $\mathcal{S}$ can be done; we omit this detail for brevity. The proof of Theorem 5.2 exploits the techniques of the proof of Theorem 5.1 with minor modifications; as a consequence, it is omitted.

**Remark 5.1.** To obtain sufficient conditions for the existence of *decreasing* entropic WF to equation (1.1), it is sufficient to reverse the inequalities (5.2), (5.4) and (5.5); this amounts to require that now $h$ is valued in $(-1, 0)$, $[-1, 0)$ or in $(-\infty, 0)$. The proof is omitted.

Before introducing our last result, we provide a motivation. We assume $\ell^\pm \in [0,1]$, with $\ell^- < \ell^+$. Moreover, the functions $f$ and $g$ are chosen so that the corresponding function $h$ has the following property: there exist $u_i \in (\ell^-, \ell^+)$, $i = 1, \dots, 4$, with $\ell^- < u_1 < u_2 < u_3 < u_4 < \ell^+$ such that (see Figure 7)

$$\begin{cases} 0 < h(u) < 1 & \text{if } u \in (\ell^-, u_1) \cup (u_2, u_3) \cup (u_4, \ell^+), \\ \quad\; h(u) > 1 & \text{if } u \in (u_1, u_2) \cup (u_3, u_4). \end{cases} \tag{5.8}$$

Notice that we have

$$h(u) > 0 \quad \text{for } u \in (\ell^-, \ell^+). \tag{5.9}$$

This case enters into the framework of Theorem 5.2 and then there exists a unique (up to shifts) increasing entropic WF $\psi$ satisfying (1.3) with exactly two jump discontinuities, i.e., $\mathcal{J} = \{\xi_0, \xi_1\}$, such that, see Figure 8,

$$\psi(\xi_0^-) = u_1, \quad \psi(\xi_0^+) = u_2,$$
$$\psi(\xi_1^-) = u_3, \quad \psi(\xi_1^+) = u_4.$$

Notice, in particular, that $h(u_i) = 1$, $i = 1, \dots, 4$, is equivalent to

$$f(u_i) - g(u_i) = s_\pm(u_i), \quad i = 1, \dots, 4. \tag{5.10}$$

By subtracting (5.10) for $u_1$ and $u_2$ we obtain

$$\sigma = \frac{f(u_2) - f(u_1) - (g(u_2) - g(u_1))}{u_2 - u_1},$$

which is exactly the Rankine–Hugoniot condition for the jump of $\psi$ in $\xi_0$ (see (3.3)). Also subtracting (5.10) for $u_3$ and $u_4$, we obtain

$$\sigma = \frac{f(u_4) - f(u_3) - (g(u_4) - g(u_3))}{u_4 - u_3},$$

which is the Rankine–Hugoniot condition for the jump of $\psi$ in $\xi_1$. Consider now the function

$$\psi_1(\xi) = \begin{cases} \psi(\xi) & \text{if } \xi < \xi_0, \\ \psi(\xi - \xi_0 + \xi_1) & \text{if } \xi > \xi_0, \end{cases} \tag{5.11}$$

with only one jump discontinuity, i.e., $\mathcal{J} = \{\xi_0\}$, from $u_1$ to $u_4$, see Figure 8. The function $\psi_1$ is a $C^1$-solution

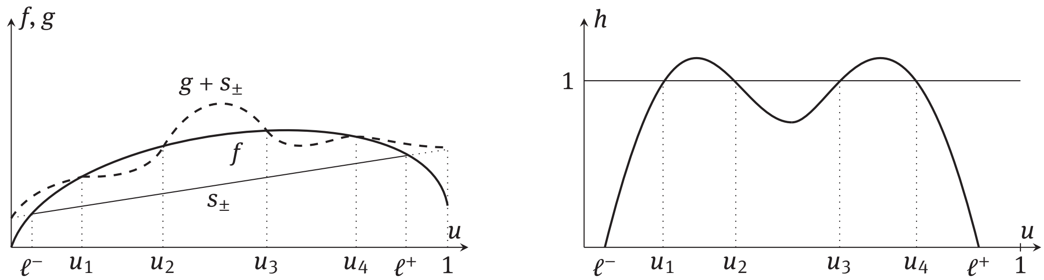


**Figure 7:** Left: the functions $f$, $s_\pm$ (thick lines) and $g + s_\pm$ (dashed). Right: the plot of the function $h$ in (5.8). Unit measures are different in the two figures.

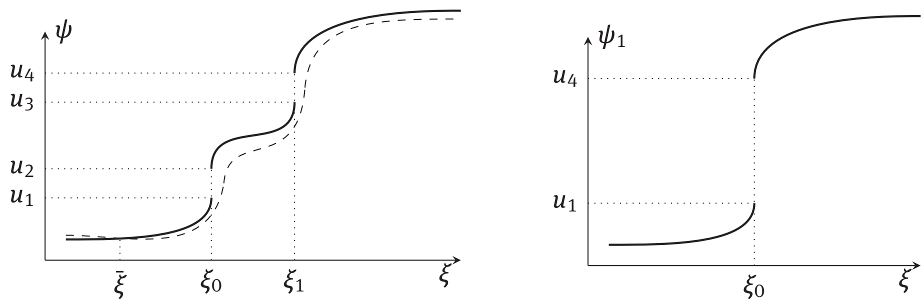


**Figure 8:** Left: the profiles $\psi$ (thick line) and $\psi_\varepsilon$ (dashed). Right: the profile $\psi_1$.

to (3.8) both in $(-\infty, \xi_0)$ and in $(\xi_0, \infty)$. Subtracting (5.10) for $u_1$ and $u_4$ we obtain

$$\sigma = \frac{f(u_4) - f(u_1) - (g(u_4) - g(u_1))}{u_4 - u_1},$$

that is, the Rankine–Hugoniot condition for the jump of $\psi_1$ in $\xi_0$. Clearly $\psi_1$ is admissible but not entropic, and this shows that uniqueness is lost if the entropy condition is dropped.

Our last result shows that entropic WFs are not only singled out uniquely but, moreover, are the limits of classical WFs that correspond to the equation

$$u_t + f(u)_x = (g(u)\Phi(u_x) + \varepsilon u_x)_x, \quad \varepsilon > 0, \tag{5.12}$$

where the non-degenerate diffusive term $\varepsilon u_{xx}$ has been added to the right-hand side of equation (1.1). The non-entropic profile $\psi_1$ in (5.11) has not this property. Notice that now the second-order term, which accounts for diffusion, is $(g(u)\Phi'(u_x) + \varepsilon u_x)_x$, which is no more degenerate. This result was first proved in [4] in the case $f = 0$; we provide here a different proof in the case of wavefront solutions. We first state a lemma about the existence of profiles to equation (5.12).

**Lemma 5.1.** *We make assumptions* (H1)–(H3) *and suppose* (5.1). *For every* $\varepsilon > 0$ *and* $\ell^\pm \in [0, 1]$ *with* $\ell^- < \ell^+$, *equation* (5.12) *has a unique (up to shifts) strictly increasing WF satisfying* (1.3)*; its profile* $\psi$ *is classical and solve*

$$g(\psi)\Phi(\psi') + \varepsilon\psi' = f(\psi) - s_\pm(\psi).$$

*where* $s_\pm$ *is defined as in* (3.7), *with* $\sigma$ *and* $\gamma$ *as in* (3.6).

Here follows our result. Notice that if $\psi$ is an increasing entropic WF as in Theorem 5.2 and $J = \emptyset$, then $\psi$ only assumes values 0 and 1, with a single jump at some $\bar{\xi} \in \mathbb{R}$.

**Theorem 5.3.** *We make the same assumptions of Theorem 5.2 and let* $\psi$ *be one of the corresponding increasing entropic wavefronts. If* $J \neq \emptyset$, *fix* $\bar{\xi} \in J \setminus \mathcal{S}$ *and denote* $\bar{\psi} = \psi(\bar{\xi})$. *Otherwise* $\mathcal{S} = \mathcal{J} = \{\bar{\xi}\}$ *for some* $\bar{\xi} \in \mathbb{R}$, *and we choose* $\bar{\psi} = \frac{1}{2}$. *In any case, let* $\psi_\varepsilon$ *be the WF to equation* (5.12) *provided by Lemma 5.1 and satisfying* $\psi_\varepsilon(\bar{\xi}) = \bar{\psi}$. *Then for any* $\xi \in \mathbb{R} \setminus \mathcal{J}$ *we have*

$$\lim_{\varepsilon \to 0} \psi_\varepsilon(\xi) = \psi(\xi). \tag{5.13}$$

**Remark 5.2.** Equation (5.12) admits a unique (up to shifts) WF satisfying (1.3) also when $h$ is valued in $(-\infty, 0)$ provided that $\ell^- > \ell^+$. Its profile $\psi_\varepsilon$ is strictly decreasing and, under suitable assumptions as in Remark 5.1, the same convergence as in (5.13) is true.

In conclusion, since the profiles $\psi_\varepsilon$ are always strictly monotone, it follows that also $\psi$ is monotone. Then, a posteriori, this result rigorously justifies the choice of considering monotone profiles of (1.1), see Remark 3.1.

# 6 Proofs

In this section we provide the proofs of the above statements, in particular of Theorems 5.1 and 5.3.

## 6.1 Proofs of Results in Section 3

*Proof of Proposition 3.1.* We split the proof into three parts. For simplicity we assume $\psi$ increasing.

(a) ⇒ (b) Let $\psi$ be an admissible WF with wave speed $\sigma$ and $\mathcal{J} \neq \emptyset$; consider $\xi_i \in \mathcal{J}$, for some $i = 1, \dots, n$. Moreover, fix $\delta > 0$ such that $\psi$ is a classical solution to (1.4) both in $[\xi_i - \delta, \xi_i)$ and in $(\xi_i, \xi_i + \delta]$, and consider $\eta \in C_0^\infty(\xi_i - \delta, \xi_i + \delta)$ with $\eta(\xi_i) \neq 0$. By (a) we have

$$
\begin{aligned}
0 &= \int_{\xi_i-\delta}^{\xi_i+\delta} (g(\psi)\Phi(D^a\psi) + \sigma\psi - f(\psi))\eta'\, d\xi \\
&= \int_{\xi_i-\delta}^{\xi_i} (g(\psi)\Phi(D^a\psi) + \sigma\psi - f(\psi))\eta'\, d\xi + \int_{\xi_i}^{\xi_i+\delta} (g(\psi)\Phi(D^a\psi) + \sigma\psi - f(\psi))\eta'\, d\xi.
\end{aligned}
\tag{6.1}
$$

Notice that $D^a\psi = \psi'$ in $(\xi_i - \delta, \xi_i) \cup (\xi_i, \xi_i + \delta)$ because $\psi$ is a classical solution there.

The function $\Phi(D^a\psi)$ is defined in $[\xi_i - \delta, \xi_i)$ but can be extended by continuity to $[\xi_i - \delta, \xi_i]$ with 0 if $\psi(\xi_i^-) = 0$ (and therefore $i = 0$), or with 1 if $\psi(\xi_i^-) > 0$. In any case we have

$$
\int_{\xi_i-\delta}^{\xi_i} = [(g(\psi)\Phi(D^a\psi) + \sigma\psi - f(\psi))\eta]_{\xi_i-\delta}^{\xi_i} - \int_{\xi_i-\delta}^{\xi_i} \{(g(\psi)\Phi(D^a\psi))' + \sigma\psi' - f(\psi)'\}\eta\, d\xi. \tag{6.2}
$$

The integral on the right-hand side of (6.2) vanishes because $\psi$ is a classical solution to (1.4) in $[\xi_i - \delta, \xi_i)$. Since $\eta(\xi_i - \delta) = 0$, if $\psi(\xi_i^-) \neq 0$, then $\psi'(\xi) \to \infty$ when $\xi \to \xi_i^-$, and we deduce

$$
\int_{\xi_i-\delta}^{\xi_i} = [g(\psi(\xi_i^-)) + \sigma\psi(\xi_i^-) - f(\psi(\xi_i^-))]\eta(\xi_i). \tag{6.3}
$$

If $\psi(\xi_i^-) = 0$, then (6.3) clearly still holds. In the same way we compute

$$
\int_{\xi_i}^{\xi_i+\delta} = -[g(\psi(\xi_i^+)) + \sigma\psi(\xi_i^+) - f(\psi(\xi_i^+))]\eta(\xi_i). \tag{6.4}
$$

By (6.1), (6.3) and (6.4) we obtain

$$
f(\psi(\xi_i^+)) - f(\psi(\xi_i^-)) - \big(g(\psi(\xi_i^+)) - g(\psi(\xi_i^-))\big) + \sigma\big(\psi(\xi_i^+) - \psi(\xi_i^-)\big) = 0.
$$

Hence, (b) is satisfied.

(b) ⇒ (c) If $\mathcal{S} = \emptyset$, then $\psi$ is a classical solution and (c) follows by integration. Otherwise, consider $\xi_i \in \mathcal{S}$ for some $i = 1, \dots, n$. Then there exists $\delta > 0$ such that $\psi$ is a classical solution to (1.4) in both $[\xi_i - \delta, \xi_i)$

and $(\xi_i, \xi_i + \delta]$, and hence there are $c_\pm \in \mathbb{R}$ such that

$$g(\psi(\xi))\Phi(\psi'(\xi)) + \sigma\psi(\xi) - f(\psi(\xi)) = c_-, \quad \xi \in [\xi_i - \delta, \xi_i), \tag{6.5}$$

and

$$g(\psi(\xi))\Phi(\psi'(\xi)) + \sigma\psi(\xi) - f(\psi(\xi)) = c_+, \quad \xi \in (\xi_i, \xi_i + \delta]. \tag{6.6}$$

If $\xi_i \in \mathcal{C}$, then we deduce $c_- = c_+$ by passing to the limit in (6.5) for $\xi \to \xi_i^-$ and in (6.6) for $\xi \to \xi_i^+$.

If $\xi_i \in \mathcal{J}$, by passing again to the limit in (6.5) and (6.6) as above, we obtain

$$c_+ - c_- = g(\psi(\xi_i^+)) - g(\psi(\xi_i^-)) + \sigma(\psi(\xi_i^+) - \psi(\xi_i^-)) - (f(\psi(\xi_i^+)) - f(\psi(\xi_i^-))),$$

and again $c_- = c_+$ by (3.3).

(c) $\Rightarrow$ (a) If $\mathcal{S} = \emptyset$, then (a) follows by differentiation. If $\mathcal{S} = \mathcal{C}$, then $D^a\psi = \psi'$ in $\mathbb{R} \setminus \mathcal{S}$ and (a) is satisfied analogously. If $\mathcal{J} \neq \emptyset$, let $\xi_i \in \mathcal{J}$ for some $i = 1, \dots, n$, and consider the intervals $[\xi_i - \delta, \xi_i)$, $(\xi_i, \xi_i + \delta]$ and the function $\eta$ as in the first step. We compute

$$\int_{\xi_i-\delta}^{\xi_i} = [g(\psi(\xi_i^-)) + \sigma\psi(\xi_i^-) - f(\psi(\xi_i^-))]\eta(\xi_i) = \gamma\eta(\xi_i),$$

and

$$\int_{\xi_i}^{\xi_i+\delta} = -[g(\psi(\xi_i^+)) + \sigma\psi(\xi_i^+) - f(\psi(\xi_i^+))]\eta(\xi_i) = -\gamma\eta(\xi_i).$$

By (6.1) we obtain (a). □

The proof of Proposition 3.1 shows that if $\psi : \mathbb{R} \to [0, 1]$ is a monotone function satisfying (i) and (ii) in Definition 3.1, then $\psi' \in L^1(\mathbb{R})$. Since

$$(g(\psi)\Phi(D^a\psi))' = (g(\psi)\Phi(\psi'))' = f'(\psi)\psi' - \sigma\psi' \quad \text{in } \mathbb{R} \setminus \mathcal{S},$$

we deduce that $(g(\psi)\Phi(D^a\psi))' \in L^1(\mathbb{R})$.

*Proof of Proposition 3.2.* We begin with (3.5); it is sufficient to consider the limit when $\xi \to -\infty$. If $\psi$ is constant in some interval $(-\infty, \hat{\xi})$, $\hat{\xi} \in \mathbb{R}$, then $\ell^- \in \{0, 1\}$ by Examples 2.1 (iii) and 3.2; hence (3.5) is trivially satisfied. Otherwise, assume that $\psi$ is constant in no left half-line; then $\psi(\xi) \in (0, 1)$ for all $\xi \in (-\infty, \hat{\xi})$, for some $\hat{\xi} \in \mathbb{R}$, and then $g(\psi(\xi)) > 0$ there. Recalling Definition 3.1, by possibly taking a smaller $\hat{\xi}$ we can assume that $\psi$ is a classical solution in $(-\infty, \hat{\xi})$; hence, $\psi$ satisfies (3.4) for some $\gamma \in \mathbb{R}$. Then we have

$$\lim_{\xi \to -\infty} g(\psi(\xi))\Phi(\psi'(\xi)) = \lambda := -\sigma\ell^- + f(\ell^-) + \gamma,$$

and (2.5) with $c = \gamma$ holds for $\xi < \hat{\xi}$. If, by contradiction, we have $\lambda \neq 0$, then $\psi'(\xi)$ has a nonzero limit for $\xi \to -\infty$ by (2.5) (again with $c = \gamma$). This contradicts the boundedness of $\psi$ and hence (3.5) is proved.

The proof of (3.6) follows by (3.5) by passing to the limit for $\xi \to \pm\infty$ in (3.4). □

## 6.2 Proofs of Results in Section 5

*Proof of Theorem 5.1.* First, we deal with the *sufficient* conditions for the existence of profiles. We consider separately each case in the statement.

Case (a). By (5.1) and (5.2), if $\psi \in (\ell^-, \ell^+)$, then the argument $h(\psi)$ of the function $\Phi^{-1}$ in (3.8) belongs to $(0, 1)$ and hence the right-hand side of (3.8) is well defined and positive by (H3). The Cauchy problem associated to (3.8) with initial condition

$$\psi(0) = \frac{\ell^- + \ell^+}{2}$$

has a unique classical solution $\psi$, see Remark 2.1, with $\psi \in C^2(J)$ and $\psi' > 0$ in $J$; let $(\alpha, \beta)$ be the maximal existence interval of $\psi$. Then $\psi(\xi) \to \ell^-$ as $\xi \to \alpha^+$ and $\psi(\xi) \to \ell^+$ as $\xi \to \beta^-$.

If $\ell^- \in (0,1)$, then we have $g(\ell^-) > 0$. Since the inverse function $\xi = \xi(\psi)$ is defined for $\psi \in (\ell^-, \ell^+)$ and $\xi(\psi) \in C^2(\ell^-, \ell^+)$, it follows that

$$-\alpha = \int_{\ell^-}^{\frac{\ell^-+\ell^+}{2}} \xi'(\psi)\,d\psi = \int_{\ell^-}^{\frac{\ell^-+\ell^+}{2}} \frac{1}{\Phi^{-1}(h(\psi))}\,d\psi. \tag{6.7}$$

By Remark 3.4 we deduce $h \in C^1[\ell^-, \ell^+)$ and $h(\ell^-) = 0$; hence, by (H3), $\Phi^{-1}(h(u)) = O(u - \ell^-)$ as $u \to (\ell^-)^+$. In particular, there exists $M > 0$ such that

$$0 < \Phi^{-1}(h(u)) \le M(u - \ell^-), \quad u \in \left(\ell^-, \frac{\ell^- + \ell^+}{2}\right).$$

By (6.7) we obtain $\alpha = -\infty$ and then $\psi$ is a classical solution to (1.4) on $(-\infty, \beta)$.

If $\ell^- = 0$, we may not exclude that $\alpha$ is a real value. However, if $\alpha$ is real, by $(5.3)_1$, we obtain that $\psi'(\alpha^+) = 0$ and then $\psi$ can be extended to $-\infty$ as a classical solution to (1.4) on $(-\infty, \beta)$. The reasoning near $\beta$ is similar and possibly involves condition $(5.3)_2$. In conclusion, we proved the existence and uniqueness of an increasing entropic WF $\psi$ of equation (1.1) satisfying (1.3).

Case (b). We further split the proof into three cases.

Assume first (b1). The Cauchy problem associated to (3.8) with initial condition

$$\psi(0) = \frac{\ell^- + u_0}{2}$$

is well defined and uniquely solvable, by (5.1), $(5.4)_1$, and Remark 2.1. Let $\psi_1$ be its classical solution and $(\alpha, \bar{\xi})$ the maximal existence interval of $\psi_1$, for some $\bar{\xi} > 0$. As in (a), possibly by means of $(5.3)_1$, we can show that $\psi_1$ is a classical solution to (1.4) on $(-\infty, \bar{\xi})$. Remark that $\psi(\xi) \to u_0$ as $\xi \to \bar{\xi}^-$; we claim that $\bar{\xi} \in \mathbb{R}$. In fact, we have

$$\lim_{\xi \to \bar{\xi}^-} \psi'(\xi) = \lim_{\xi \to \bar{\xi}^-} \Phi^{-1}(h(\psi(\xi))) = \lim_{\psi \to u_0^-} \Phi^{-1}(h(\psi)) = \infty.$$

If we had $\bar{\xi} = \infty$, then we would reach a contradiction with the boundedness of $\psi$; this proves the claim. As a consequence, we have $\psi_1(\bar{\xi}^-) = u_0$, $\psi_1'(\bar{\xi}^-) = \infty$, and $\psi_1$ is an increasing solution to (1.4) on $(-\infty, \bar{\xi})$. Moreover, by its construction $\psi_1$ is the unique solution of the above Cauchy problem satisfying the condition $\psi_1(-\infty) = \ell^-$. Similarly, the Cauchy problem associated to (3.8) with initial condition

$$\psi(0) = \frac{u_0 + \ell^+}{2}$$

is well defined and uniquely solvable, say by the function $\psi_2$, in $(\tilde{\xi}, \infty)$ with $\tilde{\xi} < 0$ and $\psi_2(\xi) \to \ell^+$ as $\xi \to \infty$. Moreover, $\tilde{\xi} \in \mathbb{R}$, $\psi_2(\tilde{\xi}^+) = u_0$, and $\psi_2'(\tilde{\xi}^+) = \infty$. If we define

$$\psi(\xi) := \begin{cases} \psi_1(\xi) & \text{if } \xi < \bar{\xi}, \\ u_0 & \text{if } \xi = \bar{\xi}, \\ \psi_2(\xi - \bar{\xi} + \tilde{\xi}) & \text{if } \xi > \bar{\xi}, \end{cases}$$

then $\psi$ is a continuous increasing entropic WF of (1.1) satisfying (1.3). This shows (b1) with $\xi_0 = \bar{\xi}$.

Assume (b2), in particular we have $u_0 = \ell^- = 0$. The function $\psi_2$ introduced in case (b1) is well defined also in this case and satisfies $\psi_2(\tilde{\xi}^+) = 0$, $\psi_2'(\tilde{\xi}^+) = \infty$. Therefore, the function

$$\psi(\xi) := \begin{cases} 0 & \text{if } \xi \le \tilde{\xi}, \\ \psi_2(\xi) & \text{if } \xi > \tilde{\xi}, \end{cases}$$

is a continuous increasing entropic WF of (1.1) satisfying (1.3). This proves (b2) with $\xi_0 = \tilde{\xi}$.

Assume (b3), and then $u_0 = \ell^+ = 1$. The function $\psi_1$ introduced in (b1) is well defined, $\psi_1(\bar{\xi}^-) = 1$, $\psi_1'(\bar{\xi}^-) = \infty$, and so

$$\psi(\xi) := \begin{cases} \psi_1(\xi) & \text{if } \xi < \bar{\xi}, \\ 1 & \text{if } \xi \ge \bar{\xi}, \end{cases}$$

is a continuous increasing entropic WF of (1.1) satisfying (1.3). This proves (b3) with $\xi_0 = \bar{\xi}$.

Case (c). Consider first subcase (c1). If $\ell^- < u_1$, then the Cauchy problem associated to (3.8) with initial condition

$$\psi(0) = \frac{\ell^- + u_1}{2}$$

is well defined and uniquely solvable. Let $\psi_1$ be its classical solution on the maximal existence interval $(\alpha, \bar{\xi})$. Since $h(u_1) = 1$, the function $\psi_1$ satisfies similar conditions as above in (b1). In particular, by making use of $(5.3)_1$ if $\ell^- = 0$, we deduce $\alpha = -\infty$ with $\psi_1(\xi) \to \ell^-$ as $\xi \to -\infty$, $\bar{\xi}$ is a real value with $\psi_1(\bar{\xi}^-) = u_1$ and $\psi_1'(\bar{\xi}^-) = \infty$.

Similarly, if $u_2 < \ell^+$, then the Cauchy problem associated to (3.8) with initial condition

$$\psi(0) = \frac{u_2 + \ell^+}{2}$$

is well defined and uniquely solvable, say by the function $\psi_2$, in $(\tilde{\xi}, \infty)$ with $\tilde{\xi} < 0$ and $\psi_2(\xi) \to \ell^+$ as $\xi \to \infty$ (by exploiting $(5.3)_2$ if $\ell^+ = 1$). Moreover, $\tilde{\xi}$ is a real value with $\psi_2(\tilde{\xi}^+) = u_2$. Since again $h(u_2) = 1$, we have that $\psi_2'(\tilde{\xi}^+) = \infty$ and $\psi_2$ is an increasing solution to (1.4) on $(\tilde{\xi}, \infty)$. The function $\psi$ defined by

$$\psi(\xi) := \begin{cases} \psi_1(\xi) & \text{if } \xi < \bar{\xi}, \\ \psi_2(\xi - \bar{\xi} + \tilde{\xi}) & \text{if } \xi > \bar{\xi}, \end{cases}$$

is an increasing entropic WF of equation (1.1) satisfying (1.3) and with $\xi_0 = \bar{\xi}$.

Consider now subcase (c2); let $\psi_2$ be the function defined in above in (c). The function

$$\psi(\xi) := \begin{cases} 0 & \text{if } \xi < \tilde{\xi}, \\ \psi_2(\xi) & \text{if } \xi > \tilde{\xi}, \end{cases}$$

is an increasing entropic WF of equation (1.1) satisfying (1.3) with $\xi_0 = \tilde{\xi}$. Similarly, when $u_2 = \ell^+ = 1$ and $\ell^- < u_1$, the function

$$\psi(\xi) := \begin{cases} \psi_1(\xi) & \text{if } \xi < \bar{\xi}, \\ 1 & \text{if } \xi > \tilde{\xi}, \end{cases}$$

with $\psi_1$ as above in (c) is an increasing entropic WF of (1.1) satisfying (1.3) with $\xi_0 = \bar{\xi}$. Subcase (c3) is dealt analogously.

At last, consider subcase (c4). It is easy to see that the function $\psi$ defined by $\psi(\xi) = 0$ if $\xi < \xi_0$ and $\psi(\xi) = 1$ if $\xi > \xi_0$, for some $\xi_0 \in \mathbb{R}$, is an increasing entropic WF of equation (1.1) satisfying (1.3).

Then we proved the existence of a profile; its uniqueness follows by the uniqueness of solutions to the above Cauchy problems.

Now, we deal with the *necessary* conditions for the existence of profiles. Assume $\ell^- < \ell^+$ and (5.1).

Case (a). Assume there exists an increasing entropic WF $\psi$ satisfying (1.3) with $\mathcal{S} = \emptyset$. Then $\psi$ is a classical solution to (1.4) and, hence, to (3.8); moreover, (5.2) is satisfied. Condition (5.3) is satisfied when $(\ell^-, \ell^+) \subset (0, 1)$ by Remark 3.4. We claim that (5.3) is true also in the remaining cases. In fact, let $\ell^- = 0$ and assume either $h(u) \to \lambda \in (0, 1]$ as $u \to 0^+$ or that such a limit does not exist. Then, by (3.8) there is $\xi_0 \in \mathbb{R}$ such that $\xi_0 \in \mathcal{C}$ and $\psi(\xi_0^+) = 0$, in contradiction with $\mathcal{S} = \emptyset$. The case $\ell^+ = 1$ is analogous. In conclusion, (5.3) is always satisfied.

Case (b). Assume $\psi$ is an increasing entropic WF satisfying (1.3) with $\mathcal{S} = \mathcal{C} = \{\xi_0\}$.

If $u_0 := \psi(\xi_0) \in (\ell^-, \ell^+)$, then $\psi$ is a classical solution to both (1.4) and (3.8) in $(-\infty, \xi_0) \cup (\xi_0, \infty)$ and $\psi'(\xi_0) = \infty$; hence, (5.4) is satisfied. Also (5.3) is proved to be satisfied as well, by arguing as in case (a) just above.

If $u_0 = \ell^- = 0$, then $\psi$ is a classical solution to (3.8) in $(-\infty, \xi_0) \cup (\xi_0, \infty)$ and $\psi(\xi_0) = 0$. Hence (b2) is satisfied. Similarly, if $u_0 = \ell^+ = 1$, then (b3) is satisfied.

Case (c). By assumption, *every* increasing entropic WF $\psi$ satisfying (1.3) has $\mathcal{S} = \mathcal{J} = \{\xi_0\}$. Let $\psi$ be one of these WFs and denote

$$u_1 := \psi(\xi_0^-) < \psi(\xi_0^+) =: u_2.$$

Then we have $h(u_1) = h(u_2) = 1$. If $\ell^- < u_1 < u_2 < \ell^+$, then there exists $\alpha \in [-\infty, \xi_0)$ such that $\psi > 0$ in $(\alpha, \xi_0)$ and $\psi$ is a classical solution to (3.8) in $(\alpha, \xi_0)$. If moreover $\alpha \in \mathbb{R}$, then $\psi'(\alpha^+) = 0$. In any case we deduce $h < 1$ in $(\ell^-, u_1)$. Since the discussion involving $u_2$ is analogous, we deduce $h < 1$ in $(\ell^-, u_1) \cup (u_2, \ell^+)$ and (5.3).

If $u_1 = \ell^-$ and $u_2 < \ell^+$, then $\ell^- = 0$ and $\psi = 0$ in $(-\infty, \xi_0)$; by arguing as just above we deduce $h < 1$ in $(u_2, \ell^+)$ and (5.3). The case $u_2 = \ell^+$ and $\ell^- < u_1$ is analogous.

If $u_1 = \ell^-$ and $u_2 = \ell^+$, then $(\ell^-, u_1) \cup (u_2, \ell^+) = \emptyset$.

In any case, condition $(5.5)_1$ is satisfied. It remains to prove $(5.5)_2$. Assume, by contradiction, that there exists $(u_3, u_4) \subset (u_1, u_2)$ with $u_3 < u_4$ such that $h < 1$ in $(u_3, u_4)$ and $h(u_3) = h(u_4) = 1$. We claim that there is a WF solution satisfying (1.3) whose singular set $S$ contains at least two points. This will be in contradiction with the assumption that every increasing entropic WF $\psi$ satisfying (1.3) has $\mathcal{S} = \mathcal{J} = \{\xi_0\}$, and then $(5.5)_2$ will be proved.

To prove the claim, consider the Cauchy problem associated to (3.8) with initial condition

$$\psi(0) = \frac{u_3 + u_4}{2},$$

which is well defined and uniquely solvable, see Remark 3.4. Denote by $\widehat{\psi}$ its classical solution on $(\alpha, \beta)$. Then, necessarily, $\widehat{\psi}(\xi) \to u_3$ if $\xi \to \alpha^+$ and $\widehat{\psi}(\xi) \to u_4$ if $\xi \to \beta^-$. By $h(u_3) = h(u_4) = 1$ we deduce

$$\widehat{\psi}'(\alpha^+) = \widehat{\psi}'(\beta^-) = \infty;$$

hence, $\alpha, \beta \in \mathbb{R}$ and

$$\widetilde{\psi}(\xi) := \begin{cases} \psi(\xi) & \text{if } \xi < \xi_0, \\ \widehat{\psi}(\xi - \xi_0 + \alpha) & \text{if } \xi_0 < \xi < \xi_0 + \beta - \alpha, \\ \psi(\xi - \beta + \alpha + \xi_0) & \text{if } \xi > \xi_0 + \beta - \alpha, \end{cases}$$

is an increasing entropic WF of equation (1.1) satisfying (1.3) and with $\mathcal{S}_{\widetilde{\psi}} = \mathcal{J} = \{\xi_0, \xi_0 + \beta - \alpha\}$. This shows the contradiction we sought for. □

We now prove Lemma 5.1.

*Proof of Lemma 5.1.* A classical WF $\psi$ of (5.12) is a solution of

$$(g(\psi)\Phi(\psi'))' + \varepsilon\psi'' + \sigma\psi' - (f(\psi))' = 0. \tag{6.8}$$

Assume for the moment that such a $\psi$ exists and it satisfies (1.3); then, as to obtain (2.1), we can integrate (6.8) and obtain the implicit ordinary differential equation

$$-\sigma\psi + f(\psi) = g(\psi)\Phi(\psi') + \varepsilon\psi' + c, \quad c \in \mathbb{R}. \tag{6.9}$$

We claim that the limits $\psi'(\pm\infty)$ exist for every solution $\psi$ to (6.9), and then $\psi'(\pm\infty) = 0$. To prove the claim, we argue by contradiction. Let $\psi$ be a solution to (6.9) satisfying (1.3) and assume that $\psi'(\infty)$ does not exist; then we find two sequences $\xi_n$ and $\eta_n$, $\xi_n, \eta_n \to \infty$ as $n \to \infty$ such that $\psi'(\xi_n) \to \lambda_1$ and $\psi'(\eta_n) \to \lambda_2$, with $\lambda_1, \lambda_2 \in [-\infty, \infty]$ and $\lambda_1 \neq \lambda_2$. By passing to the limit in (6.9) we find $g(\ell^+)(\Phi(\lambda_1) - \Phi(\lambda_2)) + \varepsilon(\lambda_1 - \lambda_2) = 0$. If $\ell^+ = 1$, then $g(\ell^+) = 0$ and so $\lambda_1 = \lambda_2$, a contradiction. If $\ell^+ \in (0, 1)$, then $g(\ell^+) \neq 0$ and we deduce again $\lambda_1 = \lambda_2$ because the function $p \mapsto g(\ell^+)\Phi(p) + \varepsilon p$ is strictly monotone for $p \in \mathbb{R}$. About $\psi'(-\infty)$ we argue analogously. This proves the claim.

Then by (6.9) we deduce

$$g(\psi)\Phi(\psi') + \varepsilon\psi' = f(\psi) - s_\pm(\psi), \tag{6.10}$$

instead of (3.8), with the same notation as in (3.7).

Now, we complete our reasoning by proving that (6.10) has strictly increasing solutions satisfying (1.3) for every $\ell^\pm \in [0, 1]$. Fix $\varepsilon > 0$, $\ell^\pm \in [0, 1]$, and define

$$G_\varepsilon : [\ell^-, \ell^+] \times \mathbb{R} \to \mathbb{R}, \quad G_\varepsilon(\psi, p) = g(\psi)\Phi(p) + \varepsilon p - f(\psi) + s_\pm(\psi).$$

By (5.1) we have $G_\varepsilon(\psi, 0) = -f(\psi) + s_\pm(\psi) < 0$ for $\psi \in (\ell^-, \ell^+)$ and $G(\ell^\pm, 0) = 0$ by the definition of $s_\pm$. Moreover, the map $p \mapsto G_\varepsilon(\psi, p)$ is strictly increasing, with $G_\varepsilon(\psi, \pm\infty) = \pm\infty$, and then equation $G_\varepsilon(\psi, p) = 0$ has a unique root $p = R_\varepsilon(\psi)$. Therefore equation (6.10) can be rewritten as

$$\psi' = R_\varepsilon(\psi),$$

for $R_\varepsilon : [\ell^-, \ell^+] \to \mathbb{R}$ and $R_\varepsilon(\ell^-) = R_\varepsilon(\ell^+) = 0$. We have $R_\varepsilon \in C^1(\ell^-, \ell^+)$ by the Implicit Function Theorem. Moreover, by differentiating the expression $G(\psi, R_\varepsilon(\psi)) \equiv 0$ with respect to $\psi \in (\ell^-, \ell^+)$ and then taking the limits for $\psi \to \ell^\pm$, we deduce that $R_\varepsilon \in C^1[\ell^-, \ell^+]$. At last, the inequality $R_\varepsilon(\psi) > 0$ holds for any $\psi \in (\ell^-, \ell^+)$ because it is equivalent to $G_\varepsilon(\psi, 0) < 0$, which follows from (5.9).

Now, fix $\bar\xi \in \mathbb{R}$, $\bar\psi \in (\ell^-, \ell^+)$ and let $\psi_\varepsilon$ be the solution of the boundary-value problem

$$\begin{cases} \psi' = R_\varepsilon(\psi), \\ \psi(\bar\xi) = \bar\psi. \end{cases} \tag{6.11}$$

It is easy to prove that $\psi_\varepsilon$ is defined in the whole of $\mathbb{R}$ and it satisfies (1.3); hence it is a classical WFs to (5.12), see Figure 8.

At last, it is clear that $\psi$ is strictly increasing because $R_\varepsilon > 0$ in $(\ell^-, \ell^+)$ and $(6.11)_1$ is a separable-variables equation. □

At last, we prove Theorem 5.3.

*Proof of Theorem 5.3.* Let $\psi_0$ be one of the increasing entropic wavefronts obtained in Theorem 5.2; to identify it, we fix $\bar\xi \in J \setminus \mathcal{S}$ and denote $\bar\psi = \psi_0(\bar\xi)$. We also denote

$$\mathcal{U} = \bigcup_{k=0}^{n} (u_{2k}, u_{2k+1}), \quad \mathcal{U}^{\complement} = [0,1] \setminus \mathcal{U} = \bigcup_{k=1}^{n} [u_{2k-1}, u_{2k}], \quad \mathcal{V} = \{v_l \in (0,1) : l = 1, \dots, m\},$$

where we set $u_0 := \ell^- \le u_1$ and $u_{2n+1} := \ell^+ \ge u_{2n}$. We have $h > 1$ in the interior $(\mathcal{U}^{\complement})^\circ$ of $\mathcal{U}^{\complement}$, that is, in $\bigcup_{k=1}^{n} (u_{2k-1}, u_{2k})$, $h = 1$ at points $u_1, \dots, u_{2n}$ and in $\mathcal{V}$, and $0 < h < 1$ in $\mathcal{U} \setminus \mathcal{V}$.

First, we define the map $\theta_0 = \psi_0^{-1} : (\ell^-, \ell^+) \to \mathbb{R}$, where we understand $\psi_0$ as a (possibly) multivalued map. We claim that $\theta_0 \in C^1(\mathbb{R})$ and

$$\theta_0'(\psi) = \begin{cases} \frac{1}{\Phi^{-1}(h(\psi))} & \text{if } \psi \in \mathcal{U} \setminus \mathcal{V}, \\ 0 & \text{if } \psi \in \mathcal{U}^{\complement} \cup \mathcal{V}. \end{cases} \tag{6.12}$$

Notice that the case $v_1 = 0$ or $v_m = 1$ is excluded since $\theta_0$ is only defined in the open interval $(\ell^-, \ell^+)$. To prove the claim, first notice that clearly $\theta_0$ is a continuous function. Moreover, if $\psi \in (\mathcal{U}^{\complement})^\circ$, then $\theta_0$ is constant and so (6.12) is evident. If $\psi \in \{u_1, \dots, u_{2n}\} \cup \mathcal{V}$, then $\lim_{s\to\psi} \theta_0'(\psi) = 0$ by Definition 3.1. When $\psi \in \mathcal{U} \setminus \mathcal{V}$, then (6.12) is obtained from (3.8). This proves the claim. Notice that we have $\theta_0(\bar\psi) = \bar\xi$.

Second, since the function $\psi_\varepsilon$ is strictly monotone, we can also define its inverse function

$$\theta_\varepsilon = \psi_\varepsilon^{-1} : (\ell^-, \ell^+) \to \mathbb{R}.$$

By the Inverse Function Theorem we deduce

$$\theta_\varepsilon'(\psi) = \frac{1}{R_\varepsilon(\psi)}, \quad \psi \in (\ell^-, \ell^+), \tag{6.13}$$

with $R_\varepsilon$ introduced in the proof of Lemma 5.1. Moreover, we have $\theta_\varepsilon(\bar\psi) = \bar\xi$. The core of the proof consists in showing that

$$\lim_{\varepsilon\to0^+} R_\varepsilon(\psi) = \begin{cases} \Phi^{-1}(h(\psi)) & \text{if } \psi \in (\mathcal{U} \setminus \mathcal{V}) \cap (\ell^-, \ell^+), \\ \infty & \text{if } \psi \in (\mathcal{U}^{\complement} \cup \mathcal{V}) \cap (\ell^-, \ell^+). \end{cases} \tag{6.14}$$

We observe that $R_\varepsilon(\psi)$ is strictly decreasing in $\varepsilon > 0$, for $\psi \in (\ell^-, \ell^+)$, because the map

$$(p > 0, \varepsilon > 0) \mapsto g(\psi)\Phi(p) + \varepsilon p$$

is strictly increasing in both variables; so, the limit $\lim_{\varepsilon\to0^+} R_\varepsilon(\psi)$ exists. To prove (6.14), denote

$$R_0(\psi) := \lim_{\varepsilon\to0^+} R_\varepsilon(\psi), \quad \psi \in (\ell^-, \ell^+).$$

From the proof of Lemma 5.1 we deduce $R_0(\psi) \ge 0$ for $\psi \in (\ell^-, \ell^+)$. By the definition of $R_\varepsilon$ we have

$$G_\varepsilon(\psi, R_\varepsilon(\psi)) = g(\psi)\Phi(R_\varepsilon(\psi)) + \varepsilon R_\varepsilon(\psi) - f(\psi) + s_\pm(\psi) = 0.$$

By passing to the limit for $\varepsilon \to 0^+$, we see that if $p = R_0(\psi) < \infty$, then $p$ must solve the equation

$$g(\psi)\Phi(p) - f(\psi) + s_\pm(\psi) = 0. \tag{6.15}$$

If $\psi \in (\mathcal{U} \setminus \mathcal{V}) \cap (\ell^-, \ell^+)$, then $\bar{p} = R_0(\psi)$ is finite and $\bar{p} = \Phi^{-1}(h(\psi))$ satisfies $(6.14)_1$. If $\psi \in (\mathcal{U}^\complement \cup \mathcal{V}) \cap (\ell^-, \ell^+)$, then equation (6.15) has no roots because $g(\psi) \le f(\psi) - s_\pm(\psi)$; then $R_0(\psi) = \infty$ and so $(6.14)_2$ follows. This completely proves (6.14).

From (6.13) and (6.14) we obtain

$$\lim_{\varepsilon \to 0^+} \theta'_\varepsilon(\psi) = \begin{cases} \frac{1}{\Phi^{-1}(h(\psi))} & \text{if } \psi \in \mathcal{U} \setminus \mathcal{V}, \\ 0 & \text{if } \psi \in \mathcal{U}^\complement \cup \mathcal{V}, \end{cases}$$

and then

$$\lim_{\varepsilon \to 0} \theta'_\varepsilon(\psi) = \theta'_0(\psi), \quad \psi \in (\ell^-, \ell^+).$$

By the Monotone Convergence Theorem and $\theta_\varepsilon(\bar{\psi}) = \theta_0(\bar{\psi})$ we have $\theta_\varepsilon \to \theta_0$ as $\varepsilon \to 0$ in every compact subset of $(\ell^-, \ell^+)$ and then (5.13) for any $\xi \in J \setminus \mathcal{J}$. This concludes the proof in the case $\xi \in J \setminus \mathcal{J}$.

We now deal with the case when $\psi_0 = 1$ in an interval $(\widehat{\xi}, \infty)$, with $\psi_0(\xi) < 1$ if $\xi < \widehat{\xi}$. We show that $\psi_\varepsilon(\xi) \to 1$, as $\varepsilon \to 0$, for any $\xi > \widehat{\xi}$.

First, if $\psi_0$ is continuous at $\widehat{\xi}$, then we consider a sequence $\xi_k \to \widehat{\xi}$ with $\xi_k < \widehat{\xi}$. Then

$$\varliminf_{\varepsilon \to 0} \psi_\varepsilon(\xi) \ge \lim_{\varepsilon \to 0} \psi_\varepsilon(\xi_k) = \psi_0(\xi_k), \quad \xi > \widehat{\xi}.$$

By the continuity of $\psi_0$ at $\widehat{\xi}$ we deduce $\psi_0(\xi_k) \to 1$, which finishes the proof in this case.

Second, if $\widehat{\xi} \in \mathcal{J}$, then we denote $\widehat{\psi} = \psi_0(\widehat{\xi}^-) < 1$; therefore $\theta_0(\widehat{\psi}) = \widehat{\xi}$. Moreover, we have $h \ge 1$ in $(\widehat{\psi}, 1)$; then we deduce $\theta_0 = \widehat{\xi}$, and so $\theta'_0 = 0$, in $(\widehat{\psi}, 1)$. Fix any $\xi > \widehat{\xi}$ and assume by contradiction that $\psi_\varepsilon(\xi) \not\to 1$; then there is a sequence $\varepsilon_p \to 0^+$ such that $\psi_{\varepsilon_p}(\xi) \to \psi_1 < 1$. Now we consider again a sequence $\xi_k \to \widehat{\xi}$ with $\xi_k < \widehat{\xi}$; then $\psi_0(\xi_k) \to \widehat{\psi}$. We have $\psi_{\varepsilon_p}(\xi) > \psi_{\varepsilon_p}(\xi_k)$ for every $\xi > \widehat{\xi}$ and $k \in \mathbb{N}$; by taking the limit $p \to \infty$, we deduce $\psi_1 \ge \psi_0(\xi_k)$, whence $\psi_1 \ge \widehat{\psi}$. But this gives a contradiction since we would have, for every $\xi > \widehat{\xi}$,

$$\xi = \theta_{\varepsilon_p}(\psi_{\varepsilon_p}(\xi)) \to \theta_0(\psi_1) = \widehat{\xi}.$$

The remaining case when $\psi_0 = 0$ in an interval $(-\infty, \widetilde{\xi})$, with $\psi_0 > 0$ if $\xi > \widetilde{\xi}$, can be treated similarly. The theorem is now completely proved. □

## 7 An Application to Crowds Dynamics

In this section we show an application of equation (1.1) to the modeling of crowds dynamics; we refer to [3] for an introductory survey to this subject. Since we mainly refer to the unpublished paper [7], we first provide some introductory details and then state our results.

A model for crowds dynamics in $\Omega \subset \mathbb{R}^2$ is proposed in [16, 21] and can be written as

$$\rho_t + \operatorname{div}(\rho v(\rho)(\nu + \mathcal{I}(\rho))) = 0.$$

Here $\rho(x_1, x_2, t)$ is the crowd density at point $(x_1, x_2) \in \mathbb{R}^2$ and time $t$, with $0 \le \rho(x_1, x_2, t) \le \overline{\rho}$, where $\overline{\rho}$ is the maximum density. Moreover, $v = v(\rho)$ is the scalar pedestrians' velocity in absence of environmental constraints; usually $v$ is assumed to be a decreasing function and $v(\overline{\rho}) = 0$. The unit vector $\nu = \nu(x_1, x_2) \in \mathbb{R}^2$ is the preferred direction of the pedestrian at $(x_1, x_2)$, while $\mathcal{I}(\rho)$ describes how a pedestrian deviates from the direction $\nu$ by trying to avoid high crowd densities $\rho$. Then, the pedestrians' speed is

$$V(\rho; x_1, x_2) = v(\rho)(\nu + \mathcal{I}(\rho)).$$

The vector operator $\mathcal{I}$ can be nonlocal and involve terms of the form $\nabla\rho * \eta$, where $\eta$ is a suitable mollifier. If $\eta$ is chosen to be the Dirac measure, one recovers the model proposed in [7] for $\widetilde{\mathcal{I}}(\rho) = -\varepsilon\nabla\rho/(\sqrt{1 + \|\nabla\rho\|^2})$, namely

$$\rho_t + \operatorname{div}(\nu\rho v(\rho)) = \varepsilon \operatorname{div}\left(\rho v(\rho) \frac{\nabla A(\rho)}{\sqrt{1 + \|\nabla A(\rho)\|^2}}\right) \tag{7.1}$$

for $\varepsilon > 0$ and $A(\rho) = \rho$; more generally, $A : [0, \overline{\rho}] \to \mathbb{R}$ can be a suitable weight function. In general, one may assume

$$a(\rho) := A'(\rho) > 0, \quad \rho \in (0, \overline{\rho}),$$

so that the deviation term is still directed against $\nabla\rho$. We have $V(\overline{\rho}; \cdot) = 0$. If $\varepsilon < 1$, this means that in (7.1) the diffusion cannot counterbalance the movement toward the preferred direction: pedestrians stop rather than reversing their direction. This a consequence of the saturation of the diffusion and of the smallness of $\varepsilon$.

In the case $v$ is a constant vector and $\Omega = \mathbb{R}^2$ we can look for *plane-wave* solutions, which are solutions of the form $\rho(x_1, x_2, t) = u(\mu \cdot \vec{x}, t)$, where $u = u(x, t)$ is some function in $\mathbb{R}^2$ and $\mu \in \mathbb{R}^2$ is a unit vector. In this case $u$ must satisfy the equation

$$u_t + \mu \cdot v(uv(u))_x = \varepsilon \left( uv(u) \frac{A(u)_x}{\sqrt{1 + |A(u)_x|^2}} \right)_x, \tag{7.2}$$

which is of the form (1.1) for $f(u) = \mu \cdot vuv(u)$, $g(u) = \varepsilon uv(u)$, $\Phi(w) = \widetilde{\Phi}(A(w))$ and $\overline{\rho} = 1$. Notice that the convection term disappears if $\mu$ is orthogonal to $v$. It is immediate to see that $\rho(x_1, x_2, t) = \psi(\mu \cdot \vec{x} - \sigma t)$ is a WF solution of (7.1) if and only if $u(x, t) = \psi(x - \sigma t)$ is a WF solution of (7.2).

We now focus on an issue that was motivated by [7], namely, that a WFs $u$ to (1.1) with profile $\psi$ is

(a) continuous when $|\ell^+ - \ell^-|$ is sufficiently small,
(b) possibly discontinuous when $|\ell^+ - \ell^-|$ is large.

Partial answers to this issue have been reported in the Introduction. From an hyperbolic point of view this means, roughly speaking, that small shock waves have *smooth* viscous profiles, while large shock waves have possibly *discontinuous* profiles. To this aim, motivated by [7], in addition to (H1)–(H3) we further assume

(H4) $f$ is strictly concave in $[0, 1]$.

Assume $\ell^\pm \in [0, 1]$; we claim that, under (H4), every entropic WF $\psi$ connecting $\ell^-$ with $\ell^+$ is increasing. In fact, if $\psi$ is an entropic WF then

$$f(u) - s_\pm(u) > 0, \quad u \in (\ell^-, \ell^+),$$

by (3.6) and (H4). Since $\psi$ satisfies (3.8) in every interval where it is classical, by Definition 3.1, (H2) and (H3) we deduce that it is increasing.

**Theorem 7.1.** *Assume* (H1)–(H4).

(i) *If* $\ell^- \in (0, 1)$, *then there exists* $\eta = \eta(\ell^-) \in (0, 1 - \ell^-]$ *such that for every* $\ell^+ \in (\ell^-, \ell^- + \eta)$ *the corresponding entropic WF* $\psi$ *satisfying* (1.3) *is of class* $C^2(\mathbb{R})$.
(ii) *If* $\ell^- = 0$, *the same result holds under the further assumption* $f'(u) - f'(0) = o(g'(u))$ *for* $u \to 0^+$.

The further condition in the case $\ell^- = 0$ requires, roughly speaking, that the diffusive flux is larger than the convective flux; in other words, is "parabolicity" prevails on "hyperbolicity".

*Proof of Theorem 7.1.* First, we extend the function $h$ in (3.8) to the case $\ell^- = \ell^+$ and define the function $H : [0, 1] \times [0, 1] \times (0, 1) \to \mathbb{R}$ by

$$H(\ell^-, \ell^+, u) = \begin{cases} \dfrac{f(u) - \frac{f(\ell^+) - f(\ell^-)}{\ell^+ - \ell^-}(u - \ell^-) - f(\ell^-)}{g(u)} = h(u) & \text{if } \ell^- \neq \ell^+, \\[2ex] \dfrac{f(u) - f'(\ell^-)(u - \ell^-) - f(\ell^-)}{g(u)} & \text{if } \ell^- = \ell^+. \end{cases}$$

The function $H$ is well defined and continuous by (H1) and (H2).

(i) Assume $\ell^- \in (0, 1)$; then $H(\ell^-, \ell^-, \ell^-) = 0$. By the continuity of $H$ we can find $\eta = \eta(\ell^-) > 0$ such that $H(\ell^-, \ell^+, u) \in (-1, 1)$ when $(\ell^+, u) \in (\ell^-, \ell^- + \eta) \times (\ell^-, \ell^- + \eta)$; see Figure 9. Fix $\ell^+ \in (\ell^-, \ell^- + \eta)$. Since $(\ell^-, \ell^+) \subset (\ell^-, \ell^- + \eta)$, we have $H(\ell^-, \ell^+, u) \in (-1, 1)$ for $u \in (\ell^-, \ell^+)$. Indeed, by conditions (H2) and (H4), we deduce $H(\ell^-, \ell^+, u) \in (0, 1)$ for $u \in (\ell^-, \ell^+)$. Then condition (5.2) is satisfied and $\psi \in C^2(J)$. Moreover, we have $J = \mathbb{R}$ because $\ell^\pm \in (0, 1)$, see Remark 3.2. The smoothness of $\psi$ follows by Remark 2.1.

(ii) Assume $\ell^- = 0$. In this case we have

$$H(0, 0, u) = \frac{f(u) - f'(0)u}{g(u)}, \quad u \in (0, 1),$$

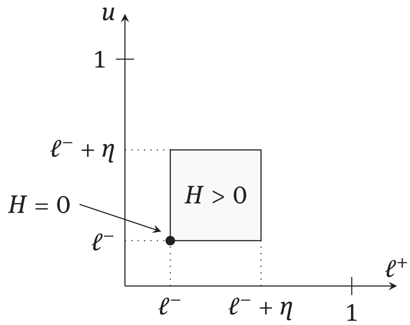


**Figure 9:** For the proof of case (i) of Theorem 7.1.

and then by the further assumption in (ii) we deduce

$$\lim_{u\to 0^+} H(0,0,u) = \lim_{u\to 0^+} \frac{f'(u)-f'(0)}{g'(u)} = 0,$$

so that the proof of item (i) works again. The proof of the theorem is complete. □

Notice that the condition $f'(u) - f'(0) = o(g'(u))$ for $u \to 0^+$ is certainly satisfied if $g'(0) > 0$. This is the case if $g(u) = uv(u)$, with $v(0) > 0$, as it is the case in (7.2).

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