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AN EXTENSION OF THE BOLSINOV-FOMENKO THEOREM ON ORBITAL CLASSIFICATION OF INTEGRABLE HAMILTONIAN SYSTEMS

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ABSTRACT. The main result of the paper is an extension of the Bolsinov-Fomenko theorem on topological orbital classification of nondegenerate integrable Hamiltonian systems with two degrees of freedom on three-dimensional constant energy manifolds (1994). Namely, it is shown that their restriction that the integral has no critical circles with nonorientable separatrix diagrams can be omitted. Our proof is based on an analogue of obstruction theory for certain types of Seifert fibrations.

1. Introduction. In 1994 Bolsinov and Fomenko [**3**] proved a theorem on topological orbital classification of nondegenerate integrable Hamiltonian systems with two degrees of freedom on three-dimensional constant energy manifolds. For motivation and a short survey see [**5**, Section 1], [**3**, Section 1]. They showed that two such systems are equivalent if certain invariants are. The invariant is a graph with a number of additional labels associated to its vertices and edges, Section 2. One of the restrictions they had to impose was that the Hamiltonian systems do not have unstable isolated periodic orbits with a nonorientable separatrix. Since the existence of such orbits is, in some sense, a generic property which appears in many classical integrable cases, e.g., in the Kovalevskaya top, it is desirable to remove this restriction. In this paper we show that the Bolsinov-Fomenko theorem is also true without this restriction.

Theorem 1.1 (cf. [**3**, Theorem 4.1]). *Let X be the set of nondegenerate integrable Hamiltonian systems with two degrees of freedom*

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on constant energy oriented three-manifolds, up to topological orbital equivalence preserving orientation. Then there is an injection from X to the set of t-labeled graphs W up to t-equivalence, see definitions in Section 3.

The definition of t-equivalence of invariants is dispersed all over the long paper [**3**]. So for the reader's convenience we give in Section 2 a precise and self-contained formulation of the (extended) theorem of Bolsinov-Fomenko. In fact, the more general situation does not require any additions and changes with respect to [**3**] except that P-labels can be atoms with stars, see below (the change in the definition of the Ξ-invariant is only a technical improvement and has nothing to do with the new situation, and the definition of coloring was implicitly contained in [**3**]).

Observe that in [**3**, Sections 12.3 and 13.5] there is a description of the image of the injection from Theorem 1.1 and the dependence of t-labels upon the orientation of the constant energy 3-manifold. Also constructed in [**3**, Section 13] was in a certain sense simpler frame on W, called a t-molecule. Again, all these are the same in our more general situation, except that P-labels can be atoms with stars. But we will not recite the definitions of the image of the injection and of the t-molecule from [**3**] in order to avoid very extensive and technical notation.

To extend the Bolsinov-Fomenko theorem, we prove, in fact, a purely topological result on Seifert fibrations, classification theorem 1.5 below, which is also interesting in itself and could perhaps be applied to some other problems. To state this result and explain its relation to Hamiltonian systems, recall some general observations. A bifurcation of Liouville tori in an integrable, with a Bott integral, Hamiltonian system can be described by a neighborhood of $F^{-1}(c)$, where F is an additional integral and c is its critical value. If the critical submanifold of F, corresponding to c, is a circle, then this neighborhood is a Seifert fibration Q over a 2-surface P with boundary [**6**]. More precisely, we have the following:

Definition 1.2. Of a double. A *double* P^* is an orientable 2-surface with boundary with an involution χ on P^* having only finitely many, possibly zero, fixed points called *stars*. Let $P = P^*/\chi$, P^* is called the double of P. Denote by $p : P^* \to P$ the projection. Denote by N the p-image of the set of fixed points of χ, i.e., stars. Denote by $\tilde{P}$ a closed surface obtained by attaching disks to the boundary circles of P.

While reading this paper it is helpful to keep in mind the following examples:

1) P has no stars, i.e., χ is fixed points free. For this case the results of Section 3, for a double $P \sqcup P$ of P, are contained in [**3**], in a slightly simpler formulation.

2) $P = A^*$, i.e., P is the regular neighborhood of a circle K in the plane with the only star lying on it. In this case there are *two* doubles P^*, both homeomorphic to the regular neighborhood of the figure eight K^* in the plane. The doubles are not homeomorphic *over* p, which is especially evident if one colors annuli of $P \backslash K$ and $P^* \backslash K^*$ black and white, respectively.

Definition 1.3. Of a 3-atom (cf. [**5**, Definition 2.2]). A *3-atom* is a fiber bundle over S^1 with the fiber P^* and the sewing map χ, i.e.,

$$Q(P^*) \cong P^* \times I / \{(a, 0) \sim (\chi a, 1)\}$$

[**9**, p. 33]. By [**8**], [**5**, Definition 2.2], $Q(P^*)$ depends only on P, hence not also on P^*. So in the sequel we shall denote $Q(P^*)$ by $Q(P)$ or even by Q. Define a map $\pi : Q \to P$ by $\pi[(a, t)] = p(a)$ (a Seifert fibration having singular fibers only over the stars and only of type (2,1)).

To study the bifurcation of Liouville tori, we construct a Poincaré section of the flow on Q [**3**]. If the critical circle has an *orientable* separatrix diagram, or equivalently if P has no stars, then $Q \cong P \times S^1$ and the Poincaré section can be chosen to be a cross-section. Therefore Poicaré sections are classified by methods of the classical obstruction theory. If the critical circle has a *nonorientable* separatrix diagram, or equivalently P has stars, then the Seifert fibration is not a locally trivial fibration. Nevertheless, the Poincaré section is a Seifert analogue of a cross-section.

Definition 1.4. Of a Seifert section. An embedding $f : P^* \to Q$ is called a *Seifert section* if $\pi \circ f = p$.

In a smooth category we shall assume additionally that f is transversal to the fibers of π. In [**3**] Seifert sections were called transversal 2-surface elements. They were assumed to be transversal to the Hamiltonian flow. But every section, transversal to the fibers of π, is isotopic to the one transversal to the Hamiltonian flow. So the difference between the two transversality conditions does not affect our results.

Every singular fiber of π intersects $f(P^*)$ in one point, and every nonsingular fiber of π intersects $f(P^*)$ in two points. If $P^* = P \sqcup P$ is a double (with the involution exchanging corresponding points from the two copies of P), then P has no stars, $Q \cong P \times S^1$ and π is trivial, but a Seifert section is not a cross-section. But, since a Seifert section $f : P \sqcup P \to Q$ is an embedding, it follows that the restrictions of f to the copies of P are *equivalent* cross-sections. Thus there is a one-to-one correspondence between Seifert sections and cross-sections of the trivial bundle $P \times S^1 \to P$. In the case when P has stars, Seifert section is a Seifert analogue of a cross-section. Just as cross-sections define representations of a bundle space as a direct product of a base and a fiber, Seifert sections define representations of Q as a skew product of P^* and S^1.

The heart (and the only significantly new point) of our proof is a classification of Seifert sections of the Seifert fibration, Theorem 1.5 below. In this chapter we omit $\mathbf{Z}$-coefficients from the notation of cohomology groups. For a space with involution the symmetric (co)homology groups are denoted by adding index S to the usual notation.

Classification theorem 1.5. *For a fixed double P^*, the set X of Seifert sections up to isotopy over π is in one-to-one correspondence with $H^1(P)$. In fact, there is a difference map $d : X \times X \to H^1(P)$, i.e., a map such that $d(f, \cdot)$ is a one-to-one correspondence and $d(f,g) + d(g,h) = d(f,h)$.*

Proof. (See Figure 1.) The idea is to reduce classification of Seifert sections to that of classical sections. Define a map

$$q : P^* \times S^1 \cong P^* \times I/\{(a,0) \sim (a,1)\} \longrightarrow P^* \times I/\{(a,0) \sim (\chi a,1)\} \cong Q$$

as

$$q[(a,t)] = \begin{cases} [(a,2t)] & 0 \le t \le 1/2 \\ [(\chi a, 2t-1)] & (1/2) \le t \le 1, \end{cases}$$

cf. [**9**, p. 33]. Since χ is an involution, it follows that q is well defined and continuous.

Let $f : P^* \to Q$ be a Seifert section. For each $x \in P^* \backslash N$ there is a unique point $f'(x) \in P^* \times S^1$ such that $qf'(x) = f(x)$ and $p_1 f'(x) = x$. For each $x \in N$ there are two points $s, t \in S^1$ such that $q(x, s) = q(x, t) = f(x)$. Since the small deleted disk neighborhood of x in P^* is connected, we can choose $f'(x)$ to be either (x, s) or (x, t) so that the map $f' : P^* \to P^* \times S^1$ will be continuous. This map f' is a classical section of the trivial bundle $P^* \times S^1 \to P^*$. Since f is an embedding, it follows that $p_2 f'(x)$ and $p_2 f'(\chi x)$ are not antipodes for each point $x \in P^*$. Here $p_2 : P^* \times S^1 \to S^1$ is the projection. Therefore f' can be canonically homotoped to a *symmetric* section f'', i.e., a section f'' such that $p_2 f''(x) = p_2 f''(\chi x)$ for each $x \in P^*$.

Also, for each symmetric section $F : P^* \to P^* \times S^1$, the map $q \circ F$ is a Seifert section and $(q \circ F)'' = F$. Evidently, Seifert sections f and g are isotopic over π if and only if the corresponding symmetric sections f'' and g'' are symmetrically homotopic, or equivalently, isotopic. Then X is in one-to-one correspondence with the set X'' of symmetric sections of the trivial bundle $P^* \times S^1 \to P^*$ up to symmetric homotopy.

The latter in turn is in one-to-one correspondence with $H^1_S(P^*)$; moreover, there exists a difference map $d : X'' \times X'' \to H^1_S(P^*)$. By Lemma 3.9, $H^1_S(P^*) \cong H^1(P)$. Or, alternatively, we can construct from f'' a section f''' of the trivial bundle $P \times S^1 \to P$ and check that the correspondence $f'' \mapsto f'''$ induces a one-to-one correspondence on the sets of sections up to equivalence. □

Corollary 1.6. *For the fixed double P^* and $f|_{\partial P^*}$ the set X_∂ of Seifert sections up to isotopy over π is in one-to-one correspondence with $H^1(\tilde{P}) \cong H^1(P, \partial P)$. In fact, there exists a difference map $d : X_\partial \times X_\partial \to H^1(\tilde{P})$.*

The proof of Theorem 1.1 modulo Theorem 1.5 is essentially the same as in [**3**]. Nevertheless, in Section 3 we sketch the proof in some detail in order to make clear all the points where the proof of our extension is not the same but only analogous to [**3**] and to provide a well-structured guideline through the very long proof in [**3**]. Of course the reader who wants to go into details of the proof should consult [**3**]. But since that

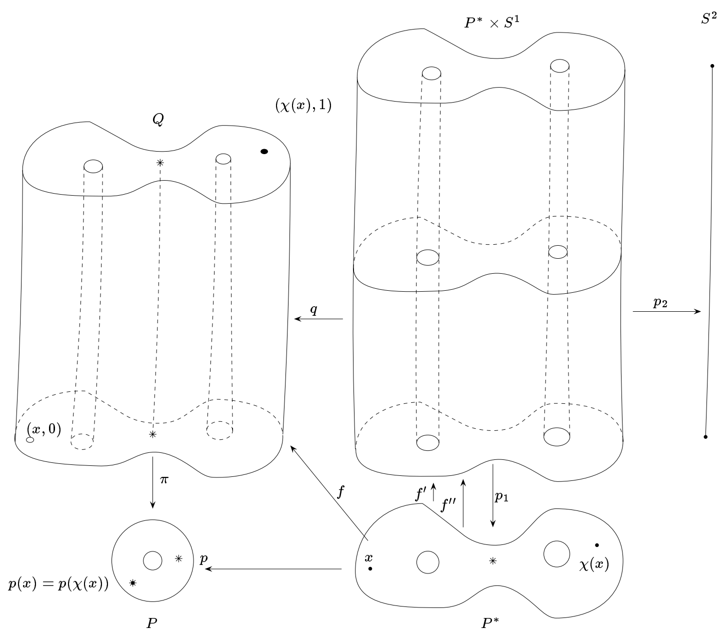

FIGURE 1.

paper does not contain the idea and the plan of the entire proof, it would be easier first to learn them from Section 3 of the present paper.

In Section 4, which is elementary and independent of Sections 2 and 3, we present several results and conjectures related to Theorem 1.5 and Construction 3.2. In particular, we show that although the proof of Theorem 1.5 is very short, it is nontrivial: we use essentially all the hypotheses, and neither the proof nor the formulation can be generalized to similar problems.

Conjecture 1.7. *The main result of* [**2**] *is valid without the restriction that the integral has no critical circles with nonorientable separatrix diagrams.*

2. Formulation of Theorem 1.1. Let (M^4, ω) be a symplectic 4-manifold. Let $H : M \to \mathbf{R}$ be Hamiltonian and $v = \operatorname{sgrad} H$ a

Hamiltonian vector field on M^4. Let $M_h = H^{-1}(h)$ be a regular constant energy 3-manifold. Suppose that the Hamiltonian system on M_h is *Liouville integrable*, in the sequel we shall call it shortly *integrable*, i.e., there is a smooth additional integral $F : O_M M_h \to \mathbf{R}$, independent almost everywhere on H. By the classical Liouville theorem, every connected component of a compact regular surface $(H \times F)^{-1}(h, a)$ is a two-dimensional, Liouville, torus.

Definition 2.1. Of a Liouville foliation [**3**, Section 3]. A partition of a compact constant energy 3-manifold M_h into Liouville tori and connected components of singular fibers $(H \times F)^{-1}(h, a)$ is called a *Liouville foliation*.

Definition 2.2. Of a Liouville equivalence [**3**, Section 3]. Two integrable Hamiltonian vector fields v and v' on compact constant energy manifolds M_h and M'_h are said to be *Liouville equivalent* if there exists a diffeomorphism $g : M_h \to M'_h$ carrying the leaves of the Liouville foliation of v to those of the Liouville foliation of v'.

Definition 2.3. Of a topological orbital equivalence [**3**, Section 3]. Two vector fields v and v' on manifolds M and M' are said to be *topologically orbitally* (*trajectory*) *equivalent* if there exists a homeomorphism $g : M \to M'$ carrying the oriented trajectories of v to those of v'.

Definition 2.4. Of a nondegenerate system (cf. [**3**, Section 1]. An integrable Hamiltonian system on M_H is said to be *nondegenerate* if:

(1) M_h is a smooth closed connected orientable 3-manifold;

(2) M_h is *Liouville stable*, i.e., the system remains Liouville equivalent to that on M_h under small variations of h;

(3) The additional integral is a *Bott* function F. Moreover, every one of its critical manifolds in M_h is nondegenerate and is a circle;

(4) Every saddle critical circle of F is a *hyperbolic* orbit of v, i.e., for each periodic orbit of v, which is a critical circle of F, the differential of the Poincaré map is neither the identity nor ‘minus the identity’;

(5) The system v is *nonresonant*, i.e., the irrational Liouville tori are everywhere dense in M_h;

(6) For every regular family $T(t)$ of Liouville tori, the rotation function

$$\rho : (0,1) \longrightarrow \mathbf{R}P^1,$$

see below, has only a finite number of return points (cusps). Although $\rho(t)$ depends on the choice of $\lambda(t)$ and $\mu(t)$, this property does not [**3**, important remark in Section 6].

Definition 2.5. Of a rotation function (cf. [**3**, Section 5.2]. Suppose that $T(t)$, $t \in (0,1)$ is an arbitrary regular family of Liouville tori in M_h. Suppose that $(\lambda(t), \mu(t))$ is a smooth family of bases in $H_1(T(t), \mathbf{Z})$. On each rational torus $T(t)$ the orbits are closed and isotopic to each other. Therefore any orbit γ on $T(t)$ uniquely defines the integers $p(t)$ and $q(t)$ such that $\gamma = p(t)\lambda(t) + q(t)\mu(t)$. Let $\rho(t) = (p(t) : q(t)) \in \mathbf{R}P^1$. Since our system is nonresonant, it follows that almost all Liouville tori are irrational, but rational tori are everywhere dense among $T(t)$. Hence our mapping ρ can be continuously extended to a mapping $\rho : (0,1) \to \mathbf{R}P^1$ [**3**, Lemma 5.1], which is called the *rotation function*.

Definition 2.6. Of an atom [**5**, Definition 2.1]. Let K be a connected graph whose vertices have degrees 2 or 4. We call the vertices of degree 2 *stars*. Let P be an oriented 2-surface with boundary containing K; note that in general P is not uniquely defined by K. We call the pair (P, K) an *atom*, and denote it by P, if $P \backslash K$ is a union of annuli $S^1 \times [0,1)$ and these annuli can be colored in black and white so that each edge of K is in the boundary of one black and one white annulus. We call $(P, K) = (D^2$, its center$)$ an *atom* A.

Definition 2.7. Of t-labels (cf. [**3**, Definition 11.1]. *t-labels* P, Λ, Δ, Ξ, s, C and R on a graph W are defined as follows. Each vertex c is labelled with an atom P_c; we shall say shortly that a vertex is an atom P_c. Moreover, c is labelled with a one-to-one correspondence α_c between the boundary circles of P_c and the edges of W incident to c. Following the tradition, and to shorten the notation, we shall omit α_c from the notation of the t-frame. Let $S(P)$ be the set of vectors with real positive components corresponding to vertices of P up to proportionality, i.e., up to a common positive scalar factor. Each vertex c is also labelled with elements $\Lambda_c \in S(P_c)$, $\Delta_c \in B_0(P_c, \mathbf{R})$, $\Xi_c \in H_1(\tilde{P}_c, S^1)$. Each edge i is labelled with an orientation s_i, a (2×2)-matrix $C_i = \left(\begin{smallmatrix}\alpha_i & \beta_i \\ \gamma_i & \delta_i\end{smallmatrix}\right)$ of integers such that $\det C_i = -1$ and a vector R_i of an arbitrary length with components from $\mathbf{R}\cup\{\pm\infty\}$. The vector R_i is such that:

1) All but the boundary infinities can be split into successive pairs;

2) No three successive components of R_i are monotonous;

3) If the end, respectively the beginning, of the edge i is not A, then the last, respectively the first, component of R is $\pm\infty$, respectively $-(\delta_i/\gamma_i)$.

We should remark that $|S(A)| = |B_0(A, \mathbf{R})| = |H_1(\tilde{A}, S^1)| = 1$, and hence there are actually no Λ, Δ, Ξ-labels on vertices A.

Definition 2.8. Of a t-equivalence. Two graphs W and W' with assigned t-labels, $P, \Lambda, \Delta, \Xi, s, C, R$ and $P', \Lambda', \Delta', \Xi', s', C', R'$, respectively, are said to be *t-equivalent* if:

1) There is an isomorphism $W' \cong W$ (in the sequel, W and W' are identified via this isomorphism) such that for each vertex c of W, $P'_c \cong P_c$, $\Lambda'_c = \Lambda_c$ and $\alpha'_c = \alpha_c$. The last equality means that the homeomorphism $P'_c \cong P_c$ carries every circle of $\partial P'_c$ corresponding to an edge i of W, i is adjacent to c, to the circle of ∂P_c corresponding to the same edge i of W;

2) $\Delta', \Xi', s', C', R'$-labels can be obtained from Δ, Ξ, s, C, R-labels by applying (several times) operations l, (k^+, k^-), see below.

Definition 2.9. Of the operation l (cf. [**3**, Section 13.5]. Let W be a t-labelled graph, and let $l \in C^1(W, \mathbf{Z}_2)$ be a cochain. For each edge j of W, make the following modification of the s, C, R-labels:

$$(s'_j, C'_j, R'_j) = \begin{cases} (s_j, C_j, R_j) & l_j = 0 \\ \text{(opposite to } s_j,\ C_j^{-1}\text{, the vector} & \\ \text{obtained from } C_jR_j \text{ (see below) by} & \\ \text{rewriting its components in the reverse order)} & l_j = 1. \end{cases}$$

Definition 2.10. Of CR. Suppose that R is a vector of an arbitrary length with components from $\mathbf{R}\cup\{\pm\infty\}$ satisfying properties 1, 2 and 3 from the definition of t-labels. To each such vector R there corresponds a function $\rho : (0,1) \to \mathbf{R}\cup\{\pm\infty\}$ having only a finite number of local minima, maxima and poles [**3**, Section 6]. Set $\rho' = (\alpha\rho+\beta)/(\gamma\rho+\delta)$. Let $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} R$ be the vector obtained by varying t from 0 to 1 and writing out successively the limit of ρ' at 0, the values of ρ' at all poles, local maxima and minima, and the limit of ρ' at 1. Since R is actually a class of ρ up to conjugation [**3**, Section 6], it follows that CR is well defined.

Definition 2.11. Of an admissible pair. Suppose that W is a t-framed graph, and let c be a vertex of it. Consider the cell decomposition of $\tilde{P}$ generated by the graph K. The 2-cells of $\tilde{P}_c$ can be identified with the circles of ∂P_c and hence with edges of W adjacent to c. For a pair $k^+, k^- \in C_1(W,\mathbf{Z})$ define a class $k_c \in C^2(\tilde{P}_c,\mathbf{Z})$ by

$$k_c(i) = \begin{cases} k_i^+ & \text{if the edge } i \text{ goes in } c \\ k_i^- & \text{if the edge } i \text{ goes out of } c. \end{cases}$$

Note that the pair (k^+,k^-) is uniquely defined by the collection $\{k_c\}$. A pair (k^+,k^-) is called *admissible* if, for each vertex c of W such that $P_c \neq A$, the sum of k_i^+ over edges going *out* of c equals the sum of k_i^- over edges going in c, or equivalently, if $k_c \in B^2(\tilde{P}_c,\mathbf{Z})$ for each c such that $P_c \neq A$.

Definition 2.12. Of a coloring corresponding to P,α,C,R,s. Let i be an oriented edge of W, and let b,e be its beginning and end, respectively. If the last component of R_i is $+\infty$ (respectively $-\infty$), then color the annulus of $P_e\backslash K_e$ corresponding to the edge i in white (respectively black). If the first component of C_iR_i is $+\infty$ (respectively $-\infty$), then color the annulus of $P_b\backslash K_b$ corresponding to the edge i in white (respectively black).

It follows from property 3 of the vector R_i that the coloring is well defined. We assume that P,α,C,R,s are such that for each vertex c, each edge of K is in the boundary of one black and one white annulus. This will be used in the definition of operations (k^+,k^-), and it is easy to see that the labels constructed from Hamiltonian systems, Section 3, satisfy this condition (so our assumption does not restrict the class of nondegenerate integrable Hamiltonian systems). Note that this restriction on the t-labels is not present in the definition of admissible labels in [**3**].

Definition 2.13. Of the operation (k^+, k^-) (cf. [**5**, Section 3.2]. Let W be a t-framed graph and

$$k^+, k^- \in C_1(W, \mathbf{Z})$$

an admissible pair. Let $A_j^\pm = \left(\begin{smallmatrix} 1 & 0 \\ k_j^\pm & 1 \end{smallmatrix}\right)$. Take a coloring corresponding to P, α, C, R, s. For each vertex c such that $P_c \neq A$ and for each edge j of W, make the following modification of the Δ, Ξ, C, R-labels:

$$\begin{aligned} &\Delta_c' = \Delta_c + \varphi(k_c), &&\Xi_c' = \Xi_c + \psi(k_c), \\ &C_j' = (A_j^+)^{-1} C_j A_j^-, &&R_j' = R_j + k_j^+ = A_j^T R_j. \end{aligned}$$

Here $\varphi : B^2(\tilde{P}, \mathbf{Z}) \to B_0(P, \mathbf{R})$ and $\psi : B^2(\tilde{P}, \mathbf{Z}) \to H^1(\tilde{P}, S^1)$ are the linear operators defined below. They depend on P_c, Λ_c and the coloring of P_c, and hence on α, C, R, s. But we omit this from the notation of φ and ψ.

Definition 2.14. Of the linear maps φ and ψ. Suppose that P is a colored atom, $P \neq A$ and $\Lambda \in S(P)$. Let a be a vertex of P. Let 1, 2, 3, 4 be the four faces of $\tilde{P}$ containing a in their boundary; they are not necessarily distinct. If a is a star, then there are two faces of $\tilde{P}$, containing a, and we assume $1 = 3$, $2 = 4$. Suppose that 1 and 3 are white, while 2 and 4 are black. Let Λ_1 be the sum of Λ_b over vertices b of P, contained in the face 1. Analogously, $\Lambda_2, \Lambda_3, \Lambda_4$ are defined, up to proportionality. Define a linear operator $\varphi : B^2(\tilde{P}, \mathbf{Z}) \to B_0(P, \mathbf{R})$ by

$$\varphi(k)(a) = \Lambda_a \left(\frac{k(1)}{\Lambda_1} - \frac{k(2)}{\Lambda_2} + \frac{k(3)}{\Lambda_3} - \frac{k(4)}{\Lambda_4} \right).$$

To define a linear operator $\psi : B^2(\tilde{P}, \mathbf{Z}) \to H_1(\tilde{P}, S^1)$, we begin with the definition of a linear operator $L : C^1(\tilde{P}, \mathbf{R}) \to C_1(\tilde{P}, \mathbf{R})$. Take any edge of K. It suffices to define the L-image of the characteristic cochain

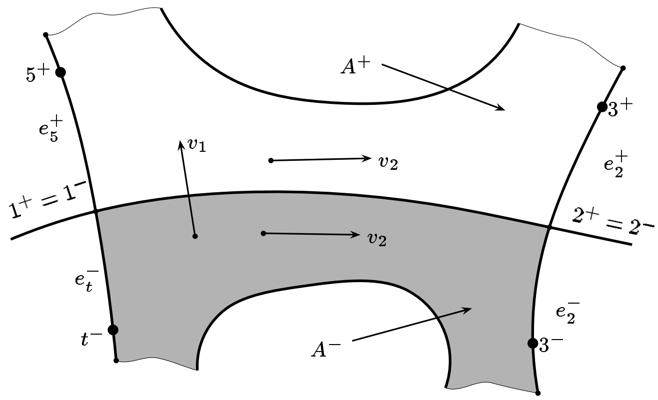

FIGURE 2.

that is 1 on this edge and 0 elsewhere. Denote the white and the black annulus adjacent to this edge by G^+ and G^-, respectively (Figure 2). The orientation on P and the coloring define the direction on the annuli G^+ and G^-, so that a vector v_1 looking from black to white and a vector v_2 defining the direction on either G^+ or G^- constitute a basis defining the orientation on P. Denote the vertices of G^+ (respectively G^-) in their order along the direction by $1^+, \dots, s^+$ (respectively $1^-, \dots, t^-$), so that $1^+ = 1^-$ and $2^+ = 2^-$ (Figure 2). Note that $1^+, \dots, s^+$ and $1^-, \dots, t^-$ are not necessarily distinct. Denote by $e_i^\pm$ the edge of K joining $i^\pm$ and $(i+1)^\pm$ and going along the direction of $G^\pm((s+1)^+ = 1^+ = (t+1)^- = 1^-)$. Let the L-image of the above characteristic cochain be the chain l defined by

$$l_e = \sum_{e_i^+=e} \frac{\Lambda_{2^+} + \cdots + \Lambda_{i^+}}{\Lambda_{1^+} + \cdots + \Lambda_{s^+}} - \sum_{e_i^-=e} \frac{\Lambda_{2^-} + \cdots + \Lambda_{i^-}}{\Lambda_{1^-} + \cdots + \Lambda_{t^-}}.$$

Take a nondegenerate scalar product on $C_1(\tilde{P}, \mathbf{R})$ with an orthogonal basis formed by chains $1 \times e$ where e is an edge of K. Denote by $\mathrm{pr} : C_1(\tilde{P}, \mathbf{R}) \to Z_1(\tilde{P}, \mathbf{R})$ the orthogonal projection. Let $\xi : H_1(\tilde{P}, \mathbf{R}) \to H_1(\tilde{P}, S^1)$ be the projection. In [**3**, Sections 9, 10] it was proved that the linear operator ψ is well defined under the condition that the two paths on the following diagram from $C^1(\tilde{P}, \mathbf{Z})$

to $H_1(\tilde{P}, S^1)$ yield the same composite homomorphism:

$$\begin{tikzcd}
C^1(\tilde{P},\mathbf{R}) \arrow[d, equal] & C^1(\tilde{P},\mathbf{Z}) \arrow[l, "\supset"'] \arrow[r, equal] & C^1(\tilde{P},\mathbf{Z}) \arrow[d, "\delta"] & Z^1(\tilde{P},\mathbf{Z}) \arrow[l, "\supset"'] \arrow[r, equal] & Z^1(\tilde{P},\mathbf{Z}) \arrow[d] \\
C^1(\tilde{P},\mathbf{R}) \arrow[r, "\partial"'] \arrow[d, "L"'] & B_0(\tilde{P},\mathbf{R}) & B^2(\tilde{P},\mathbf{Z}) \arrow[l, "\varphi"] \arrow[r, "\psi"'] & H_1(\tilde{P},S^1) & H^1(\tilde{P},\mathbf{Z}) \arrow[d, "D"] \\
C_1(\tilde{P},\mathbf{R}) \arrow[r, "\mathrm{pr}"'] & Z_1(\tilde{P},\mathbf{R}) \arrow[r] & H_1(\tilde{P},\mathbf{R}) \arrow[r, equal] & H_1(\tilde{P},\mathbf{R}) \arrow[u, "\xi"'] & H_1(\tilde{P},\mathbf{Z}). \arrow[l, "\supset"]
\end{tikzcd}$$

Note that it was proved in [**3**, Sections 9, 10] that the linear operators φ (respectively D) are well defined under the conditions that the two paths on the diagram from $C^1(\tilde{P}, \mathbf{Z})$ to $C_0(\tilde{P}, S^1)$ (respectively from $Z^1(\tilde{P}, \mathbf{Z})$ to $H_1(\tilde{P}, S^1)$), yield the same composite homomorphism. Moreover D is the Poincaré duality and φ coincides with the operator defined directly in the beginning of this definition.

Problem 2.15. *Find a direct definition of* ψ (*begin with the case when* $\Lambda_i = 1$).

For fixed W the above operations l and (k^+, k^-) define an action on the set of t-frames by the product of $C^1(W, \mathbf{Z}_2)$ and the subgroup of admissible pairs in $C^1(W, \mathbf{Z}) \times C^1(W, \mathbf{Z})$. Thus the Hamiltonian systems are classified by the orbits of this action.

3. Proof of Theorem 1.1. The structure of the proof of Theorem 1.1 is as follows. The heart of the proof is the construction of a t-framed graph corresponding to a given nondegenerate integrable Hamiltonian system with two degrees of freedom on three-dimensional constant energy manifolds (in the sequel we shall refer to them simply as *Hamiltonian system*). First we define the graph W and the P, α-labels on its vertices. For each vertex c we construct the double P_c^* of P_c as in Construction 3.2, taking each arc on P_c in the *white* annulus (any other double also will do, but the image of X in the set of t-labelled graphs W depends on this choice of double). Note that we define them directly from the Hamiltonian system. However, to define $\Lambda, \Delta, \Xi, C, R$-labels we need an additional structure on that system. Namely, for each vertex c we fix an *admissible collection of bases* on every boundary torus of every 3-atom Q_c, see the definition below. Then we can define Λ, Δ and Ξ-labels. If we additionally fix orientations on edges of W, then we can define C- and R-labels. Let us sketch the definition of Λ, Δ and Ξ-labels in some details. We additionally fix a *Seifert section* for every vertex c agreeing with the given admissible collection of bases, or roughly, we fix an element of $H^1(\tilde{P}_c, \mathbf{Z})$, cf. Corollary 1.6.

Using the classical Poincaré section idea we define a Hamiltonian system on a double P_c^* from that on Q_c and a Seifert section. Then we define Λ, Δ, Z-invariant which classify Hamiltonian systems on the neighborhood of K_c^* in P_c^* up to conjugation near K^*. We observe how this triple of invariants depends on the choice of a Seifert section for fixed admissible collection of bases, in fact, Λ- and Δ-invariants do not depend. Based on these observations we define Λ, Δ, Ξ-labels which depend only on the choice of admissible collection of bases, not of Seifert sections. Thus a t-labelled graph corresponding to a Hamiltonian system, orientations on edges of W and a family of admissible collections of bases are constructed.

Now Hamiltonian systems on M_h are orbitally topologically equivalent regarding the orientation of M_h if and only if the corresponding t-labeled graphs W and W' are the same for some choice of orientations and collections of admissible cycles. In conclusion we verify that: 1) t-labeled graphs corresponding to the same Hamiltonian system and constructed from different orientations and the same collections of admissible cycles, are t-equivalent by operation l; 2) t-labelled graphs constructed from different collections of admissible cycles and the same orientations are t-equivalent by operation (k^+, k^-).

Note that the choices of the additional structures, admissible collections of bases and orientations, appear in the definitions of t-labels and t-equivalence quite differently. The transformation of t-labels corresponding to a change of admissible cycles depends on the choice of the orientations. That is why orientations appear as labels and why we cannot introduce t-labels independent of the choice of orientations. In contrast, the transformation of t-labels corresponding to a change of the orientation does not depend on the choice of the admissible cycles. That is why admissible cycles do not appear as labels. In fact, we can introduce a t-frame independent of the choice of admissible cycles (it was constructed and called a t-molecule in [**3**]). We will not do this to avoid technicalities and because a t-molecule is not so natural an object as t-labels.

Now we are going to realize the above plan in detail. We use definitions and notation from Sections 1 and 2. Before Definition 3.7 we denote by F the restriction of the second integral to M_h. In this section we omit $\mathbf{R}$-coefficients from the notation of chain and homology groups.

Definition 3.1. Of W, P, α (cf. [**5**]). We may assume that for each critical value c of F the set of critical points of F corresponding to c is connected. Then vertices of W are critical values c of F. For each c and sufficiently small $\varepsilon > 0$ there is a unique atom P_c such that $F^{-1}[c-\varepsilon, c+\varepsilon] \cong Q(P_c) = Q_c$ and $F^{-1}(c) = \pi_c^{-1} K_c$ [**6**], [**5**, Lemma 2.1]. For $P_c \neq A$ the map π is uniquely defined, while for $P_c = A$ it is not. Orientation on the fibers of π is given by the flow. The annuli of $P_c \backslash K_c$ for π-preimages for which $F > c$ (respectively $F < c$) are colored in white (respectively black). Since P_c is oriented and at each point of P_c there is a vector from black to white, there exists a χ-symmetric orientation on edges of K^* which defines directions on annuli of $P^* \backslash K^*$.

Then ∂Q_c is a union of tori. Two vertices b and e of W are joined by an edge if there is a family of Liouville tori $T \times I \subset Q$ such that $T \times 0$ and $T \times 1$ are boundary tori of Q_b and Q_e, respectively. A bijection α_c is given by the one-to-one correspondence between edges of W adjacent to c and boundary tori of Q_c, or equivalently, boundary circles of P_c.

In the sequel if index c is fixed, we shall omit it.

Construction 3.2. Of a double. For each star on an atom P, take an arc joining the star to ∂P, transverse to K and going through the white annulus (Figure 3). Cut P along all such arcs. Take another copy of such a cut-off atom. For each star glue together different edges of the cut from different copies of P to get the surface P^*. There is an involution χ on P^*, exchanging corresponding points from different copies of P. Evidently, the resulting surface P^* is a double of P.

The construct double contains a χ-invariant *spine* K^*, i.e., a graph such that $P^* \backslash K^*$ is a union of annuli, with vertices of degree 4 and such that each star is a vertex of K^* and the annuli of $P^* \backslash K^*$ can be colored in black and white so that each edge of K^* is in the boundary of one black and one white annulus. This K^* is the graph obtained

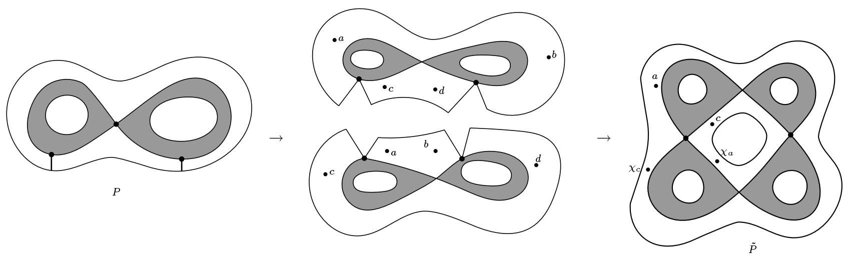

FIGURE 3.

from two copies of K by the gluings described above. Analogously to Definition 3.1, there exists a χ-symmetric orientation on edges of K^* that define directions on annuli of $P^* \backslash K^*$.

Definition 3.3. Of an admissible collection of bases (cf. [**5**, Section 3]. Denote by T_j tori of ∂Q. Recall that we have an orientation on the fibers of π and a direction on the annuli of $P^* \backslash K^*$. For $P = A$ let $\lambda_j \in H_1(T_j, \mathbf{Z})$ be the cycle that is null-homotopic in $Q \cong D^2 \times S^1$ and oriented along the annulus $P \backslash A$. Let $\mu_j \in H_1(T_j, \mathbf{Z})$ be any cycle such that (λ_j, μ) is a basis. For $P \neq A$ let $\lambda_j \in H_1(T_j, \mathbf{Z})$ be the cycle defined by the oriented fiber of π. Suppose that we are given a Seifert section $f : P^* \to Q$. For a torus T_j of ∂Q, if $f(P^*) \cap T_j$ is the union of two circles, then let $\mu_j = \partial f_j \in H_1(T_j, \mathbf{Z})$ be the oriented cycle defined by any of them (since the circles are disjoint, it follows that they are homologous). If $f(P^*) \cap T_j$ is one circle, then denote by S_j the oriented cycle defined by that circle. Set $\mu_j = \partial f_j = (S_j - \lambda_j)/2 \in H_1(T_j, \mathbf{Z})$ (since S_j is connected and intersects λ_j at two points, it follows that $S_j - \lambda_j$ is divisible by 2). We call the collection obtained $\{(\lambda_j, \mu_j)\}_j$ an *admissible collection of bases* (corresponding to the given Seifert section f, when $P \neq A$).

Note that the first element λ_j of a basis from an admissible collection is the same for distinct collections, but the second element μ_j is not. Also ∂f_j is not the cycle in $H_1(T_j, \mathbf{Z})$ represented by $f^{-1} T_j \subset \partial P^*$.

Lemma 3.4. (cf. [**5**, Section 3], [**3**, Section 11.1]). *Suppose that* $\{(\lambda_j, \mu_j)\}_j$ *is an admissible collection of bases. Then we have*:

1) (λ_j, μ_j) *is a basis in* $H_1(T_j, \mathbf{Z})$;

2) *For $P \neq A$ the formula $\partial f_j = \partial f'_j + k_j \lambda_j$ holds. Here k_j is the value of $\delta d(f, f') \in B^2(\tilde{P}, \mathbf{Z})$ on the face of $\tilde{P}$ corresponding to T_j. Since $\delta(B^1(\tilde{P}, \mathbf{Z})) = 0$, it follows that $\delta : H^1(P, \mathbf{Z}) = C^1(\tilde{P}, \mathbf{Z})/B^1(\tilde{P}, \mathbf{Z}) \to B^2(\tilde{P}, \mathbf{Z})$ is well defined.*

3) *For each admissible collection of bases $\{(\lambda'_j, \mu'_j)\}_j$, there exists a homeomorphism $h : Q \to Q$ over π such that $h_*(\lambda_j, \mu_j) = (\lambda'_j, \mu'_j)$, for each j, if and only if $\lambda'_j = \lambda_j$ and there exists an element $k \in \mathbf{Z}$ for $P = A$ and $k \in B^2(\tilde{P}, \mathbf{Z})$ for $P \neq A$ such that $\mu'_j = \mu_j + k_j \lambda_j$, for each j. Here for $P = A$, $k_j = k$ and for $P \neq A$, k_j is the value of $k \in B^2(\tilde{P}, \mathbf{Z})$ on the face of $\tilde{P}$ corresponding to T_j.*

The proof is analogous to [**5**, Section 3.1]. Note that our construction is different from [**5**, Section 3.1]. Nevertheless, in the sequel we shall use only properties 1, 2, 3 of admissible collection of bases, instead of its definition.

Definition 3.5. Of gluing matrices C and rotation vectors R (cf. [**5**], [**3**, Sections 5, 6]. Fix orientations s on edges of W and a family of admissible collection of bases $\{(\lambda_j, \mu_j)\}_j$. Take any edge j of W with beginning b and end e, possibly $b = e$. There is a family $T_j \times I$ of Liouville tori such that $T_j \times 0$ and $T_j \times 1$ are boundary tori of the 3-atoms Q_b and Q_e, respectively. Let $(\lambda_j^-, \mu_j^-) \in H_1(T_j \times 0, \mathbf{Z})$ and $(\lambda_j^+, \mu_j^+) \in H_1(T_j \times 1, \mathbf{Z})$ be the bases from the above family of admissible collections. We may assume that the basis (λ_j^-, μ_j^-) is extended smoothly to all tori $T_j \times t$, $t \in I$. Let C_j be the matrix consisting of the coordinates of (λ_j^+, μ_j^+) in the basis $(\lambda_j^-(1), \mu_j^-(1))$. Take a rotation function $\rho_j : (0, 1) \to \mathbf{R} \cup \{\pm\infty\}$ corresponding to the basis $(\lambda_j^-(t), \mu_j^-(t))$, defined similarly to the definition in Section 1. Let R_j be a vector obtained by varying t from 0 to 1 and writing out successively the limit of ρ_j at 0, the values of ρ_j at all poles, local maxima and minima, and the limit of ρ_j at 1 (the limits at 0 and 1 exist by [**3**, Sections 5, 6]). Note that our R_j is R_j^+ of [**3**].

Reduction theorem 3.6. a) (cf. [**3**, Proposition 7.3]). *If $P \neq A$, then to each flow on Q and Seifert section $f : P^* \to Q$ there corresponds an integrable Hamiltonian system, 'Poincaré flow,' on P^*. Moreover, flows on Q and Q' are topologically orbitally equivalent over π if and* *only if for each Seifert section $f : P^* \to Q$ there exists a Seifert section $f' : P^* \to Q'$ such that the Poincaré flows on P^* constructed from f and f' are conjugate equivalent regarding the involution χ, i.e., for the conjugation g we have $\chi \circ g = g \circ \chi$. Or, equivalently, there exist Seifert sections $f : P^* \to Q$ and $f' : P^* \to Q'$ such that the Poincaré flows constructed from f and f' are conjugate equivalent regarding the involution χ.*

b) (cf. [**3**, Proposition 7.2]). *If $P = A$, then to each flow on $Q \cong D^2 \times S^1$ and admissible basis (λ, μ) in $H_1(\partial D^2 \times S^1, \mathbf{Z})$ there corresponds an integrable Hamiltonian system, 'Poincaré flow,' on P. Moreover, flows on Q and Q' are topologically orbitally equivalent over π if and only if for each admissible basis (λ, μ) there exists an admissible basis (λ', μ') such that the Poincaré flows on P constructed from (λ, μ) and (λ', μ') are conjugate equivalent regarding the involution χ. Or, equivalently, there exist admissible bases (λ, μ) and (λ', μ') such that the Poincaré flows constructed from (λ, μ) and (λ', μ') are conjugate equivalent regarding the involution χ.*

Note that the hypothesis $P \neq A$ is used and the conclusion 'over π' is actually proved and used in the sequel in [**3**, Proposition 7.3]. The proof of Theorem 3.6 for the case $P \neq A$ differs from [**3**, Proof of Proposition 7.3] only formally, since when P has no stars, Seifert sections are not exactly cross-sections. Case $P = A$ was actually proved in [**3**, Proposition 7.2].

Definition 3.7. Of Λ, Δ, Z-invariants (cf. [**3**, Sections 8.1–8.2]). Suppose that $P \neq A$ and there is a symplectic 2-form ω on P^*. Let F be a smooth Morse function on P^* such that $F^{-1}(0) = K^*$ and the Hamiltonian vector field $v = \operatorname{sgrad} F$ on P^* is χ-invariant.

For a vertex i of K^* take a linear operator that is the linearization of v in the neighborhood of i. The eigenvalues of this linearization have the same module and the opposite signs. Let Λ_i be the inverse of the positive eigenvalue. Define Λ-*invariant* $\Lambda \in S(P^*)$ to be the collection $\{\Lambda_i\}$ up to proportionality. Since v is χ-invariant, it follows that in fact $\Lambda \in S^{\mathbf{Z}_2}(P^*) \cong S(P)$.

Let G be a white annulus that is the closure of a connected component of $P^* \backslash K^*$. Let $K_1, \dots, K_s$ be the oriented edges of K^*, contained in G.

Let N_1 be an arc, transverse to v, equivalently to the fibers of F, and joining a point $x_1^+ \in \mathring{K}_1$ to ∂P^*. Then N_1 defines uniquely an 'angle' function $\vartheta : G \backslash K^* \to S^1 = \mathbf{R} \bmod 2\pi$ such that $N_1 = \vartheta^{-1}(0)$. For each $i = 2, \dots, s$, let $N_i = \mathrm{Cl}\,(\vartheta^{-1}(2\pi(\Lambda_1 + \cdots + \Lambda_i)/(\Lambda_1 + \cdots + \Lambda_p)))$ and $x_i^+ = N_i \cap K_i$; N_i is an arc joining x_i^+ to ∂P^* [**3**, Lemma 8.13]. Analogously, construct points x_i^- on the boundary of every black annulus. If the annulus G is χ-invariant, then s is even and $N_i = N_{(s/2)+i}$. If the annulus G is χ-symmetric to another annulus G', then we take the 'initial' arcs $N_1' = \chi N_1$. Thus the points x_i^+ and $x_{\chi(i)}^+$, x_i^- and $x_{\chi(i)}^-$ are χ-symmetric.

Let l_i be the time in which the point x_i^- goes to x_i^+ under the flow σ^t of v. These l_i define a chain $l \in C_1(K^*)$. Actually, $l \in C_1^{\mathbf{Z}_2}(K^*) \cong C_1(K)$. Note that the chain l is constructed uniquely up to the choice of the arc N_1 on each annulus G, i.e., up to elements from $B_1(K)$ [**3**, Section 8]. Let $\Delta = \partial l \in B_0(K)$ be the Δ-*invariant*. Take a nondegenerate scalar product on $C_1(K)$ with an orthogonal basis formed by chains $1 \times K_i$. Define the Z-*invariant* to be the class of $\mathrm{pr}\, l$ in $H_1(\tilde{P})$, where $\mathrm{pr} : C_1(K) = Z_1(K) \to Z_1(\tilde{P})$ is the orthogonal projection.

Theorem 3.8 (cf. [**3**, Theorem 8.1]). *Suppose that $P \neq A$ and P^* and $(P^*)'$ are smooth symplectic doubles. Suppose that v and v' are symmetric Hamiltonian vector fields, with Morse Hamiltonians F and F' such that $K^* = F^{-1}(0)$ and $(K^*)' = (F')^{-1}(0)$, on P^* and $(P^*)'$, respectively. Let (Λ, Δ, Z) and (Λ', Δ', Z') be the corresponding triples of invariants. Let $g : P^* \to (P^*)'$ be a homeomorphism preserving the orientation, colorings, involutions and graphs. Then there exist χ-invariant and invariant under the flow regular neighborhoods U and U' of K and K' and an isotopic to g conjugation of Hamiltonian systems on U and U', preserving χ, K and colorings, if and only if g carries the first triple to the second one.*

Note that the conditions that the conjugation be between U and U', not between P^* and $(P^*)'$, and that it is isotopic to g are assumed, actually proved and used in the sequel in [**3**, Theorem 8.1]. Proof of the sufficient condition (respectively, the necessary condition) part of Theorem 3.6 is the same as (respectively, analogous to) [**3**, Proof of Theorem 8.1]. In the necessary condition we apply the first isomorphism from Lemma 3.9 below. For an involution χ we denote the set of its fixed points by $\operatorname{fix}\chi$. Denote by $\tilde{P}^*$ a closed surface obtained by attaching disks to the boundary circles of P^*. The involution χ on P^* obviously extends to an involution on $\tilde{P}^*$ which we will denote by $\tilde{\chi}$. Then there is a unique 'new fixed point,' i.e., a point of $\operatorname{fix}\tilde{\chi}\backslash\operatorname{fix}\chi$ in the interior of each disk from $\tilde{P}^*\backslash P^*$ having χ-invariant boundary circle. Moreover, these are the only 'new fixed points.' Also $\tilde{P}^*/\tilde{\chi} \cong \tilde{P}$.

Lemma 3.9. *$H_1^S(\tilde{P}^*) \cong H_1(\tilde{P})$, $H_S^1(P^*) \cong H^1(P)$ for each coefficient group.*

Proof. The second isomorphism is analogous to the first one. Let $E = \operatorname{fix}\tilde{\chi}$. The first isomorphism follows since $\dim E = 0$ and therefore E does not affect one-dimensional homologies, so we have $H_1^S(\tilde{P}^*) \cong H_1(\tilde{P})$ as if $E = \varnothing$. More precisely, since $\operatorname{fix}\tilde{\chi}|_{\tilde{P}^*\backslash E} = \varnothing$, it follows that $p_* : H_1^S(\tilde{P}^*\backslash E) \to H_1(\tilde{P}\backslash pE)$ is an isomorphism. Denote by $a_1, \dots , a_k$ the elements of $H_1^S(\tilde{P}^*\backslash E)$ represented by, arbitrarily oriented, small χ-invariant circles going around points of E. Since $H_1^S(\tilde{P}^*) = H_1^S(\tilde{P}^*\backslash E)/\langle a_1, \dots , a_k\rangle$ and $H_1^S(\tilde{P}) \cong H_1(P\backslash pE)/\langle pa_1, \dots , pa_k\rangle$, the map $p : H_1^S(\tilde{P}^*) \to H_1(P)$ is an isomorphism; here $p : \tilde{P}^* \to \tilde{P} \cong \tilde{P}^*/\tilde{\chi}$ is the projection. $\square$

Lemma 3.10 (cf. [**3**, Propositions 11.2 and 9.1 and corollary of the proof of Proposition 10.4]). *For $P \neq A$ and symmetric Hamiltonian systems on P^*, induced from that on Q by the Seifert sections $f, f' : P^* \to Q$ with homologous boundary, we have $\Lambda' = \Lambda$, $\Delta' = \Delta$, $Z' = Z + D(d(f, f'))$. Here $D : H^1(\tilde{P}, \mathbf{Z}) \to H_1(\tilde{P}, \mathbf{Z})$ is the Poincaré duality.*

The proof of Lemma 3.10 is analogous to [**3**, Section 9]. It is based on the notion of *pasting-gluing* operation Φ_d, $d \in H^1(P)$ [**3**, Section 9]. The Hamiltonian system induced by f' is obtained from that induced by f by means of applying the operation $\Phi_{d(f',f)}$. In the proof of $\Delta' = \Delta$ an equivalent definition of Δ [**3**, Lemma 8.4] is used.

Definition 3.11. Of Λ, Δ, Ξ-labels. ($P \neq A$). Take any collection $\{f_c\}$ of Seifert sections. By reduction theorem 3.6 there corresponds to f_c a Hamiltonian system on P_c^*. Let $\Lambda_c = \Lambda(f_c)$, $\Delta_c = \Delta(f_c)$, $\Xi_c = \xi(Z(f_c))$. Here the homomorphism

$$\xi : H_1(\tilde{P}, \mathbf{R}) \longrightarrow H_1(\tilde{P}, S^1)$$

is the projection.

Proposition 3.12 (cf. [**3**, Proposition 11.1]). *Two nondegenerate integrable Hamiltonian systems v and v' on oriented constant energy 3-manifolds M_h and M_h' are orbitally topologically equivalent regarding the orientations of M_h and M_h' if and only if the corresponding graphs W and W' are isomorphic (we fix an isomorphism and identify W and W'), and there exist orientations on edges of W and two families of admissible collections of bases on boundary tori of 3-atoms of W such that the corresponding t-labels coincide.*

Proof of Proposition 3.12 for the case $P \neq A$ is analogous to [**3**, Proof of Proposition 11.1]. Case $P = A$ was not actually proved in [**3**], but it follows easily from Theorem 3.6b and, since the system on $Q \cong D^2 \times S^1$ is uniquely defined by the limit of R-vector on the 'hanging' edge of W, corresponding to this atom A. In the proof we also use the fact that, from the long coefficient exact sequence, corresponding to the short exact sequence $0 \to \mathbf{Z} \to \mathbf{R} \to S^1$, one can obtain that $H^1(P, \mathbf{R})/H^1(P, \mathbf{Z}) \cong H^1(P, S^1)$.

Lemma 3.13. a) (cf. [**3**, Theorem 11.1]). *Let $\{(\lambda_{cj}, \mu_{cj})\}, \{(\lambda_{cj}, \mu'_{cj}\}$ be two families of admissible collections of bases. Let s be orientations of edges of the graph W. Let $k_c = \delta d(f_c', f_c)$ for $P \neq A$ and $k_c \in \mathbf{Z}$ be such that $\mu_c' = \mu_c + k_c \lambda_c$ for $P = A$. Construct $k^+, k^- \in C_1(W, \mathbf{Z})$ from $\{k_c\}$ as in the definition of admissible pairs. Then the t-labels corresponding to s, $\{(\lambda_{cj}, \mu_{cj})\}$ are obtained from those corresponding to s, $\{(\lambda_{cj}, \mu'_{cj})\}$ by the operation (k^+, k^-).*

b) (cf. [**3**, Section 13.5]). *Let μ be a family of admissible collection of bases. Let $l \in C^1(W, \mathbf{Z}_2)$ and s, s' be orientations of edges of W such that $s_j' = s_j$, respectively opposite to s_j if $l_j = 0$, respectively $l_j = 1$. Then the t-labels corresponding to s' and μ are obtained from those corresponding to s and μ by operation l.*

Proof of Lemma 3.13 is analogous to [**3**, Proof of Theorem 11.1, Section 13.5].

The proof that every *admissible* t-labeled graph [**3**, Definition 12.1] can be represented by some Hamiltonian system is analogous to [**3**, Proofs of Propositions 12.1 and 15.1]. The only alterations are that we take a symmetric integrable Hamiltonian system realizing the given triple of invariants not on P but on (no matter which) double P^* of P. We define $\tilde{M} = P^* \times [0, 2\pi] \times (-1, 1)$, $\Omega = \omega + dH\Lambda\, d\varphi$ and the point (p, H, φ) is identified to the point $(\chi p, H, \varphi + 2\pi)$.

4. Epilogue: Obstruction theory for certain Seifert fibrations.

Problem 4.1. *Find analogues of Theorem* 1.5 *for more general Seifert fibrations* [**8**]; *in particular, one-dimensional analogues.*

Note that the Euler class of a Seifert fibration [**5**] is an obstruction to existence of a *classical* cross-section outside neighborhoods of singular fibers.

In most parts of this section we sketch new proofs of Theorem 1.5. Although the proof from Section 1 is shorter, the new ones, they are in fact original, better clarify the matter. Conjecture 4.3 and Theorem 4.8 are of independent interest, and so are steps towards Problem 4.1 and the third proof of Theorem 1.5. We also consider a generalization of Construction 3.2, when the arcs along which to cut go in arbitrary (not necessarily white) annuli. Note that not every double can be obtained by this construction, for example, P has no stars. In Lemma 4.10 we show how P^* depends on the choices in this construction (the answer is in terms of regluing operation, cf. Definition 4.4).

We use definitions and notation of Section 1 and Construction 3.2. We assume that a double P^* contains a χ-invariant *spine* K^*, i.e., a graph such that $P^*\backslash K^*$ is a union of annuli, with vertices of degree 4 and such that each star is a vertex of K^* and the annuli of $P^*\backslash K^*$ can be colored in black an white so that each edge of K^* is in the boundary of one black and one white annulus (but P^* is not necessarily obtained by Construction 3.2). By $\mathring{A}$ we denote the interior of A.

Problem 4.2. a) *Prove that each double P^* contains a χ-invariant spine K^* with vertices of degree* 4 *and such that each star is a vertex of K^* and the annuli of $P^*\backslash K^*$ can be colored in black and white so that each edge of K^* is in the boundary of one black and one white annulus.*

b) *Find conditions on a triple (P^*, K^*, N) of a surface P^*, its spine K^* and a finite subset N of K under which there exists an involution $\chi : P^* \to P^*$, preserving K^* and whose set of fixed points is N.*

Observe that triples (P^*, K^*, χ) and $((P^*)', (K^*)', \chi')$ are homeomorphic if and only if triples $(P^*, K^*, \mathrm{fix}\,\chi)$ and $((P^*)', (K^*)', \mathrm{fix}\,\chi')$ are (the homeomorphism between the first two triples can be deformed to that between the second two first on χ-invariant regular neighborhood of stars in P^* then on the union of this neighborhood with K^*, and at last on P^*). This is interesting with respect to Theorem 3.8; it shows that a homeomorphism g in Theorem 3.8 may be assumed to involve not involutions but merely fixed points.

Sketch of the second proof of Theorem 1.5. The idea is to make an isotopy of Seifert sections f, g so that f and g will coincide on neighborhoods of all vertices of K, including stars, and then to define $d(f, g)$ directly.

Fix an orientation on the fibers of π. Note that the correspondence d in Theorem 1.5 and in Corollary 1.6 depends on this choice.

Suppose that $f, g : P^* \to Q$ are Seifert sections. By isotoping them over π we can obtain new Seifert sections, which we still denote by f, g, coinciding on the union $p^{-1}U$ of small disk neighborhoods in P^* of vertices of K^*. In fact, for neighborhoods of vertices of K^* which are not stars, this is obvious. For neighborhoods of stars, this is not so obvious and even cannot generally be done for graphs (Figure 4.).

But for each star in P^*, π is a trivial bundle over $p\partial D^2$ for some small disk neighborhood D^2 of this star. Therefore we can make an isotopy so that $f = g$ on ∂D^2. Since the set of homotopy classes $(D^2, \partial D^2) \to (S^1, \text{ a point})$ is trivial, it follows that the Seifert section $f|_{\partial D^2}$ can be uniquely up to isotopy extended to a Seifert section on D^2. Accurate proof of this fact would of course involve ideas from the first proof, Section 1, for the partial case, $(P^*, \chi) \cong$ (the unit disk in $\mathbf{R}^2$, central symmetry). Since the set of homotopy classes

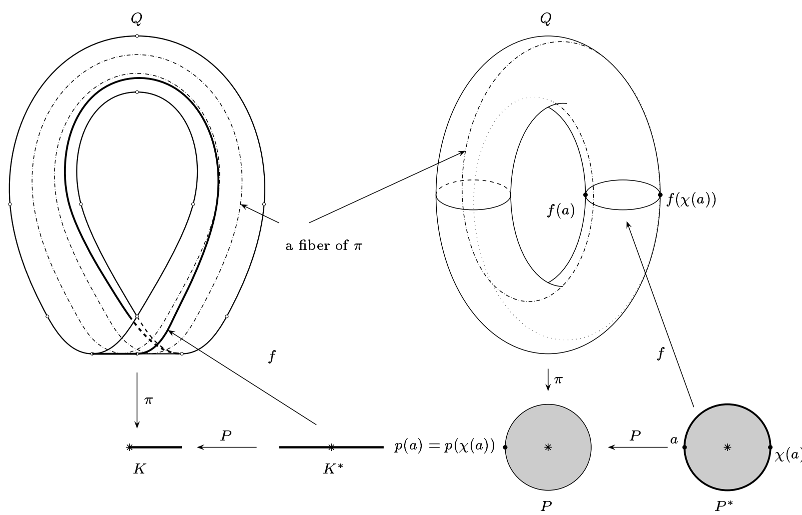

FIGURE 4.

$(D^1, \partial D^1) \to (S^1$, a point) is nontrivial, it follows that the Seifert section $f|_{\partial D^1}$ can be nonuniquely up to isotopy extended to a Seifert section on D^1 (for example, Figure 5a). Note that there are Seifert sections $f : \partial D^2 \to Q$ nonextendable to Seifert sections $f : D^2 \to Q$ (Figure 5b).

For each edge k of K, $\pi^1(k \setminus \mathring{U})$ is an annulus (Figure 5). Note that $\pi^{-1}(k)$ is not always an annulus, as claimed in [**3**, Section 11.2]. Let λ be the unit cycle on this annulus, whose orientation is defined by

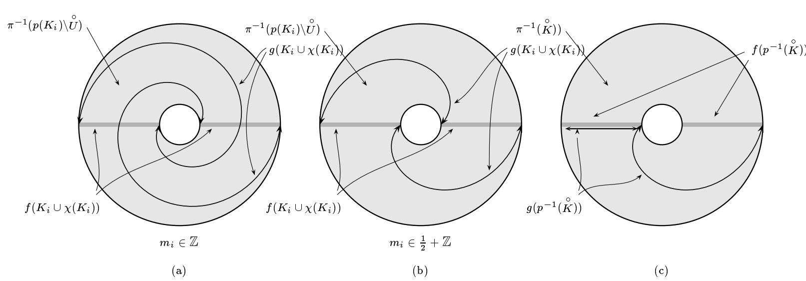

FIGURE 5.

the fibers of π. Since $f = g$ on $p^{-1}U$ and f, g are embeddings, it follows that the cases of Figure 5b,c are impossible. Therefore the oriented cycle '$fp^{1}(k\setminus\mathring{U}) - gp^{-1}(k\setminus\mathring{U})$' equals $2m\lambda$ for some $m \in \mathbf{Z}$. Let $m(f,g) \in C^1(K)$ be the cochain constituted by the above m's. The cochain $m(f,g)$ depends on the choice of the isotopy from the previous paragraph up to coboundaries from $B^1(K)$. Let $d(f,g) \in H^1(K)$ be the class of $m(f,g)$. It is now easy to verify that d is a difference map. Hence X is in one-to-one correspondence with $H^1(K) \cong H^1(P)$. $\square$

Let

$$H^1_N\left(K, \frac{1}{2}\mathbf{Z}\right) = Z^1_N\left(K, \frac{1}{2}\mathbf{Z}\right) / \delta C^0_N\left(K, \frac{1}{2}\mathbf{Z}\right)$$

and

$$H^1_N(K, \mathbf{Z}_2) = Z^1_N(K, \mathbf{Z}_2)/\delta C^0_N(K, \mathbf{Z}_2),$$

where index N means that $1/2$ and 1 appear only on stars or edges, adjacent to stars. Actually, $H^1_N(K, (\mathbf{Z}/2)) = H^1(K, \{\pi_1(\pi^{-1}x)\})$, and the same for $\mathbf{Z}_2$-coefficients. From the long coefficient exact sequence, corresponding to the short exact sequence $0 \to \mathbf{Z} \to (\mathbf{Z}/2) \to \mathbf{Z}_2 \to 0$, we can obtain the short exact sequence

$$0 \longrightarrow H^1(K, \mathbf{Z}) = H^1_N(K, \mathbf{Z}) \xrightarrow{\varphi} H^1_N\left(K, \frac{1}{2}\mathbf{Z}\right) \xrightarrow{\psi} H^1_N(K, \mathbf{Z}_2) \longrightarrow 0.$$

Sketch of the third proof of Theorem 1.5. Fix an orientation on the fibers of π. For Seifert sections $f, g : P^* \to Q$ make an isotopy so that they will coincide on vertices of K^*. Define $d(f|_{K^*}, g|_{K^*})$ as in the sketch of the second proof of Theorem 1.5 (Figure 5). Since $g|_{K^*}$ extends to P^*, by Conjecture 4.3a it follows that the case from Figure 5c is still impossible. But the case from Figure 5b is now possible for edges k adjacent to stars. Therefore, $d(f|_{K^*}, g|_{K^*}) \in H^1_N(K, (\mathbf{Z}/2))$. It actually follows from Conjecture 4.3b that $\psi d(f|_{K^*}, g|_{K^*}) = 0$. Hence, by exactness, $d(f,g) \in \operatorname{Im}\varphi \cong H^1(K)$. It is now easy to verify that d is a difference map. Hence X is in one-to-one correspondence with $H^1(K) \cong H^1(P)$. $\square$

Hereafter, until the end of Section 4, we omit the graph K and $\mathbf{Z}_2$-coefficients from the notation of (co)chain and (co)homology groups, and we indicate $\mathbf{Z}$-coefficients. Note that

$$H^1_N = C^1_N/B^1_N = C^1_N/(C^1_N \cap B^1) \cong (C^1_N + B^1)/B^1 \subset H^1.$$

Extension conjecture 4.3. *Let $f : P^* \to Q$ be a Seifert section.*

a) *The Seifert section $g' : K^* \to \pi^{-1}K$ extends to a Seifert section $g : V^* \to Q$ for some double V^* of P^*/χ if and only if $d(f|_{K^*}, g') \in H^1_N$, that is, the case of Figure* 5c *is impossible for each edge $k \subset K$.*

b) *The Seifert section $g' : K^* \to \pi^{-1}K$, $K^* \subset P^*$, extends to a Seifert section $g : P^* \to Q$ if and only if $\psi d(f|_{K^*}, g') = 0 \in H^1_N$.*

On Figure 5a there is an example of a Seifert section $f : K^* \to \pi^{-1}(K) \subset Q(P)$ nonextendable to a Seifert section $f : P^* \to Q(P)$ for any double P^* of P. The informal formulation of Conjecture 4.3a is that all examples are of such type.

Conjecture 4.3b follows from Conjecture 4.3a, Lemma 4.5 and Theorem 4.6 below. Lemma 4.5 is proved analogously to [**3**, Proposition 11.2].

Definition 4.4. Of the regluing operation Ψ_d (cf. [**3**, Section 9]). Let P^* be a double of P. Suppose that we are given a cochain $c \in C^1$, i.e., edges of K are framed with zeros and units so that an edge k is framed with zero whenever no vertex of k is a star. For each pair of χ-symmetric edges k and χk of a given graph K^* such that $c(pk) = 1$ make the following operation, Figure 6. Cut the surface P^* along χ-symmetric arcs, transverse to k and χk and going to ∂P^*. Glue edges of different cuts that are not χ-symmetric so that the resulting surface is orientable. The operation Ψ_c obviously change involution χ, graph K and projection p. Evidently, $\Psi_c(K^*)$ is a double of K and $\Psi_b \circ \Psi_c = \Psi_{b+c}$. Also $\Psi_c = \mathrm{id}$ for $c \in B^1$; it suffices to check this for characteristic coboundaries for which this is obvious. So for $[c] \in H^1$, let $\Psi_{[c]}$ be the operation Ψ_c.

Note that not all doubles of a given P can be obtained from one double by regluings Ψ_d, $d \in H^1_N$, for example, P has no stars. All

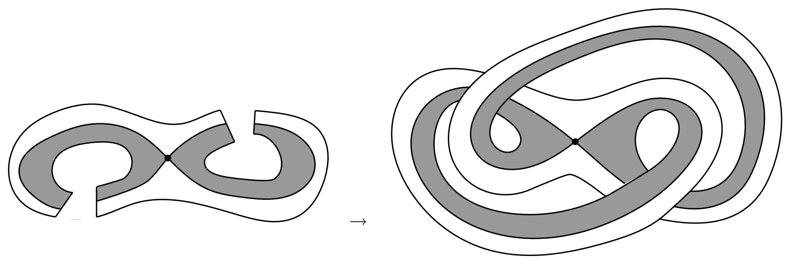

FIGURE 6.

doubles can be obtained by regluings Ψ_d, $d \in H^1$. This fact, Lemma 4.5 and Conjecture 4.8 imply that $Q(P^*)$ depends only on P not on P^* [**9**], [**5**, Definition 2.2].

Lemma 4.5. *Let P^*, V^* be doubles of P and $f : P^* \to Q$, $g : V^* \to Q$ Seifert sections. Then $V^* = \Psi_{\psi d(g,f)}(P^*)$.*

Theorem 4.6. *Let P^* be a double of P. Then $d \mapsto \Psi_d(P^*)$ is a one-to-one correspondence between H^1 and the set of doubles of P up to homeomorphism over p. In particular, $\Psi_d(P^*) \cong P^*$ over p if and only if $d = 0 \in H^1$.*

Proof. Let us prove the injectivity of the above map. For each vertex of P^* take the four edges of K^*, adjacent to this vertex. Cut the surface P^* along two pairs of arcs, transverse to these edges and going to ∂P^*; if the edges are χ-symmetric, then the arcs are χ-symmetric. Make the same operation for $\Psi_d(P)$. It is easy to define a homeomorphism h, over p, between the corresponding connected components of such cut doubles. Once h is fixed, there is an element $c \in C^1_N$ that is zero if and only if h can be extended to a homeomorphism, over p, of doubles. This c depends on the choice of h up to coboundaries from B^1_N. So the class of c in H^1_N is the complete obstruction to $\Psi_d(P^*) \cong P^*$ over p. The surjectivity is proved by the same ideas. $\square$

Observe that $\Psi_d(K^*) \cong K^*$ over p if and only if $d \in H^1_N$.

Definition 4.7. Suppose that we have two doubles P^*, V^* of P. Two Seifert sections $f : P^* \to Q$ and $g : V^* \to Q$ are *equivalent* if there exists a homeomorphism $h : P^* \to V^*$ over the projection to P such that the Seifert section $g \circ h$ is isotopic to f over π.

Second classification theorem 4.8. *The set X_P of Seifert sections $f : P^* \to Q$ with fixed P, up to equivalence, is in one-to-one correspondence with $H^1(P, (\mathbf{Z}/2))$. In fact, there is a difference map $d : X_P \times X_P \to H^1(P, (\mathbf{Z}/2))$.*

Theorem 4.8 follows from Theorems 1.5 and 4.6.

Sketch of another proof of Theorem 4.8. Analogously to the third proof of Theorem 1.5, the only alteration is that we can make an isotopy of Seifert sections $f : P^* \to Q$ and $g : V^* \to Q$ so that for the new maps, denote them also by f, g, we obtain $f(p_1^{-1}a) = g(p_2^{-1}a)$ for each vertex $a \in K$. So the case from Figure 5b is now possible for each edge $k \subset K$. Since $g|_{K^*}$ extends to P^*, by Conjecture 4.3a it follows that the case from Figure 5c is still impossible. Also we need to remark that every double P^* has the only nontrivial autohomeomorphism over p, that is, χ, and $f \circ \chi$ is isotopic to f over π for each Seifert section f. $\square$

Definition 4.9. Of the regluing operation Ψ_α. Let P^* be a double of P constructed as in Construction 3.2, only the arcs along which to cut are arbitrary, not necessarily white, annuli. For a chain $\alpha \in C_0(N)$, let $\Psi_\alpha(P^*)$ be a new double of P^* which is constructed by making cuts from any star a in the same direction (as in the construction of P^*) if $\alpha(a) = 0$ and in the opposite direction if $\alpha(a) = 1$. Operation Ψ_α obviously changes involution χ, graph K and projection p. Evidently, $\Psi_\alpha \circ \Psi_\beta = \Psi_{\alpha+\beta}$.

Lemma 4.10 a) *Suppose that $\alpha \in C_0(N)$. For each $A_j \in N$ take any edge $k(A_j)$ with vertex A_j. For each edge k of K, let $d(k) = \sum_{k(A_j)=k} \alpha(A_j)$. Then $\Psi_\alpha = \Psi_{[d]}$.*

b) *Suppose that $d \in C^1_N$. For each edge k of K adjacent to a star take*

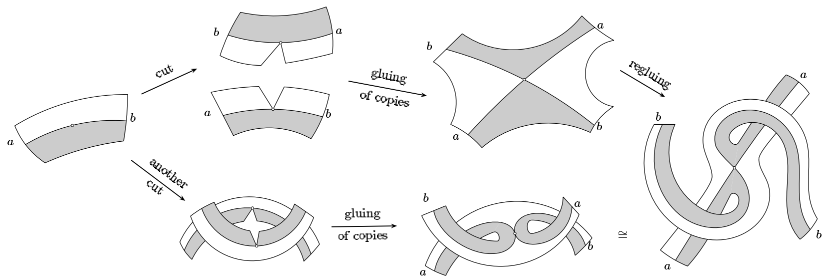

FIGURE 7.

any star $a(k) \in k \cap N$. For each $A_j \in N$ let $\alpha(A_j) = \sum_{a(k)=A_j} d(k)$. Then $\Psi_{[d]} = \Psi_\alpha$.

Sketch of the proof. It suffices to prove only a) and only for the case of a characteristic chain α (see Figure 7). $\square$

Lemma 4.11. *The subgroup of chains $\alpha \in C_0(N)$ for which $\Psi_\alpha(P^*) \cong P^*$ over p is generated by*

$$\{\partial_B \chi_B \mid B \text{ is the closure of a connected component of } K \backslash N\}.$$

Here $\chi_B \in C_1(B)$ is the characteristic chain of B and ∂_B is the composition $C_1(B) \xrightarrow{\partial} C_0(B) \supset C_0(N \cap B) \xrightarrow{i} C_0(N)$, $\operatorname{Im} \partial \in C_0(N \cap B)$.

Sketch of the proof. Follows by Theorem 4.6 and Lemma 4.10. $\square$

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REFERENCES

1. H.T. Baues, *Obstruction theory: On homotopy classification of maps*, Springer Verlag, Berlin, 1977.

2. A.V. Bolsinov, *Smooth trajectorial classification of integrable Hamiltonian systems with two degrees of freedom*, Mat. Sb. **186** (1995), 3–28, in Russian.

3. A.V. Bolsinov and A.T. Fomenko, *Orbital equivalence of integrable Hamiltonian systems with two degrees of freedom. A classification theorem*, I, II, Mat. Sb. **185** (1994), 27–80, in Russian; Russian Acad. Sci. Sbornik Math. **81 82** (1995), in English.

4. ———, *Introduction into topology of integrable Hamiltonian systems*, Nauka, Moscow, 1997, in Russian.

5. A.V. Bolsinov, A.T. Fomenko and S.V. Matveev, *Topological classification of integrable Hamiltonian systems with two degrees of freedom*, Uspekhi Mat. Nauk **45** (1990), 49–77, in Russian; Russian Math. Surveys **45** (1990), 59–94, in English.

6. A.T. Fomenko, *Topology of constant energy surfaces of integrable Hamiltonian systems and obstructions to integrability*, Izv. Akad. Nauk SSSR Ser. Mat. **50** (1986), 1276–1307, in Russian; Math. USSR-Izv. **29** (1987), in English.

7. A.T. Fomenko and H. Zieschang, *A topological invariant and a criterion for the equivalence of integrable Hamiltonian systems with two degrees of freedom*, Izv. Akad. Nauk SSSR Ser. Mat. **54** (1990), 546–575, in Russian; Math. USSR-Izv. **36** (1991), in English.

8. W.H. Jaco and P.B. Shalen, *Seifert fibered spaces in* 3*-manifolds*, Mem. Amer. Math. Soc., **21**, Providence, RI, 1979.

9. P. Orlik, *Seifert manifolds*, Springer Verlag, Berlin, 1972.

10. C.P. Rourke and B.J. Sanderson, *Introduction to piecewise-linear topology*, Ergeb. Math. Grenzgeb. (3) **69**, Springer-Verlag, Berlin, 1972.

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