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On the evolution of a social network

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In this paper we show that the small world and weak ties phenomena can spontaneously emerge in a social network of interacting agents. This dynamics is simulated in the framework of a simplified model of opinion diffusion in an evolving social network where agents are made to interact, possibly update their beliefs and modify the social relationships according to the opinion exchange.

Keywords: Opinion dynamics; social network; small world; weak ties.

1. Introduction

Modeling social phenomena represents a major challenge that has in recent years attracted a growing interest. Insight into the problem can be gained by resorting, among others, to the so called *Agent Based Models*, an approach that is well suited to bridge the gap between hypotheses concerning the microscopic behavior of individual agents and the emergence of collective phenomena in a population composed of many interacting heterogeneous entities.

Constructing sound models deputed to return a reasonable approximation of the scrutinized dynamics is a delicate operation, given the degree of arbitrariness in assigning the rules that govern mutual interactions. In the vast majority of cases, data are scarce and do not sufficiently constrain the model, hence the provided answers can be questionable. Despite this intrinsic limitation, it is however important to inspect the emerging dynamical properties of abstract models, formulated so to incorporate the main distinctive traits of a social interaction scheme. In this paper we aim at discussing one of such models, by combining analytical and numerical techniques. In particular, we will focus on characterizing the evolution of the underlying social network in terms of dynamical indicators.

It is nowadays well accepted that several social groups display two main features: the *small world property* [18] and the presence of *weak ties* [12]. The first property implies that the network exhibits clear tendency to organize in large, densely connected, clusters. As an example, the probability that two friends of mine are also, and independently, friends to each other is large. Moreover, the shortest path between two generic individuals is small as compared to the analogous distance computed for a random network made of the same number of individuals and inter-links connections. This observation signals the existence of short cuts in the social tissue. The second property is related to the cohesion of the group which is mediated by small groups of well tied elements, that are conversely weakly connected to other groups. The skeleton of a social community is hence a hierarchy of subgroups.

A natural question arise on the ubiquity of the aforementioned pecularior aspects, distinctive traits of a real social networks: how can they eventually emerge, starting from an finite group of randomly connected actors? We here provide an answer to this question in the framework of a minimalistic opinion dynamics model, which exploit an underlying substrate where opinions can flow. More specifically, the network that defines the topological structure is imagined to evolve, coupled to the opinions and following a specific set of rules: once two agents reach a compromise and share a common opinion, they also increase their mutual degree of acquaintance, so strengthing the reciprocal link. In this respect, the model that we are shortly going to introduce hypothesize a co-evolution of opinions and social structure, in the spirit of a genuine adaptive network [13, 19].

Working within this framework, we will show that an initially generated random group, with respect to both opinion and social ties, can evolve towards a final state where small worlds and weak ties effects are indeed present. The results of this paper constitute the natural follow up of a series of papers [3, 9, 8], where the time evolution of the opinions and affinity, together with the fragmentation vs. polarization phenomena, have been discussed.

Different continuous opinion dynamics models have been presented in literature, see for instance [10, 11], dealing with the general consensus problem. The aim is to shed light onto the assumptions that can eventually yield to fixation, a final mono-clustered configuration where all agents share the same belief, starting from an initial condition where the inspected population is instead fragmented into several groups. In doing so, and in most cases, a fixed network of interactions is a priori imposed [2], and the polarization dynamics studied under the constraint of the imposed topology. At variance, and as previously remarked, we will instead allow the underlying network to dynamically adjust in time, so modifying its initially imposed characteristics. Let us start by revisiting the main ingredients of the model. A more detailed account can be found in [3].

Consider a closed group of N agents, each one possessing its own opinion on a given subject. We here represent the opinion of element i as a continuous real variable $O_i \in [0,1]$. Each agent is also characterized by its affinity score with respect to the remaining $N-1$ agents, namely a vector α_{ij}, whose entries are real number defined in the interval $[0,1]$: the larger the value of the affinity α_{ij}, the more reliable the relation of i with the end node j.

Both opinion and affinity evolve in time because of binary encounters between agents. It is likely that more interactions can potentially occur among individuals that are more affine, as defined by the preceding indicator, or that share a close opinion on a debated subject. Mathematically, these requirements can be accommodated for by favoring the encounters between agents that minimizes the *social metric* $D^t_{ij} = |\Delta O^t_{ij}|(1-\alpha^t_{ij}) + \mathcal{N}_j(0,\sigma)$, where $\Delta O^t_{ij} = O^t_i - O^t_j$ is the opinions' difference of agents i and j at time t, and the last term is a stochastic contribution, normally distributed with zero mean and variance σ. For a more detailed analysis on the interpretation of σ as a *social temperature* responsible of a increased mixing ability of the population, we refer to [3, 9, 8].

Once two agents are selected for interaction they possibly update their opinions (if they are affine enough) and/or change their affinities (if they have close enough opinions), following:

$$\begin{cases} O_i^{t+1} &= O_i^t - \frac{1}{2}\,\Delta O^t_{ij}\,\Gamma_1\left(\alpha^t_{ij}\right) \\ \alpha_{ij}^{t+1} &= \alpha^t_{ij} + \alpha^t_{ij}(1-\alpha^t_{ij})\,\Gamma_2\left(\Delta O^t_{ij}\right)\ , \end{cases} \tag{1}$$

being:

$$\Gamma_1(x) = \frac{\tanh(\beta_1(x-\alpha_c))+1}{2} \quad \text{and} \quad \Gamma_2(x) = -\tanh(\beta_2(|x|-\Delta O_c))\,, \tag{2}$$

two *activating functions* which formally reduce to step functions for large enough the values of the parameters β_1 and β_2, as it is the case in the numerical simulations reported below.

Despite its simplicity the model exhibits an highly non linear dependence on the involved parameters, α_c, ΔO_c and σ, with a phase transition between a polarized and fragmented dynamics [3].

A typical run for $N = 100$ agents is reported in the main panel of Fig. 1, for a choice of the parameters which yields to a consensus state. The insets represent three successive time snapshots of the underlying social network: The N nodes are the individuals, while the links are assigned based on the associated values of the affinity. The figures respectively refer to a relatively early stage of the evolution $t = 1000$, to an intermediate time $t = 5000$ and to the convergence time $T_c = 10763$. Time is here calculated as the number of iterations (not normalized with respect to N). The corresponding three networks can be characterized using standard topological indicators [1, 5] (see Table 1), e.g. the mean degree $< k >$, the network clustering coefficient C and the average shortest path $< \ell >$. An explicit definition of those quantities will be given below.

In the forthcoming discussion we will focus on the evolution of the network topology, limited to a choice of the parameters that yield to a final mono cluster.

Table 1. Topological indicators of the social networks presented in Fig. 1. The mean degree $< k >$, the network clustering C and the average shortest path $< \ell >$ are reported for the three time configurations depicted in the figure.

	$< k > (t)$	$C(t)$	$< \ell > (t)$
$t = 1000$	0.073	0.120	3.292
$t = 5000$	0.244	0.337	2.013
$t = T_c$	0.772	0.594	1.228

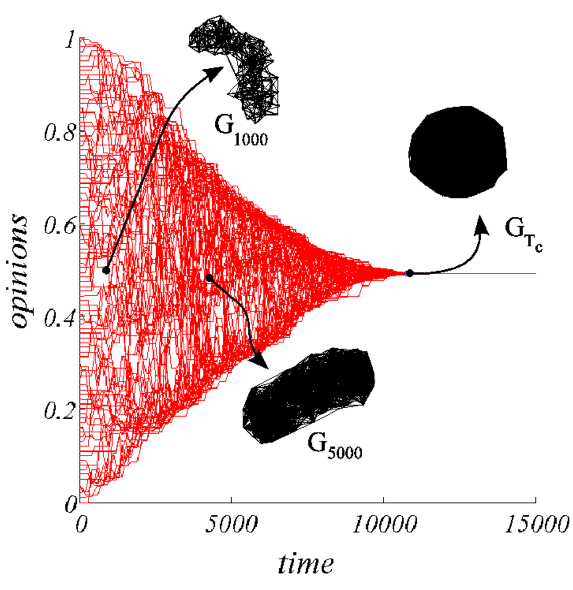

Fig. 1. Opinions as function of time. The run refers to $\alpha_c = 0.5$, $\Delta O_c = 0.5$ and $\sigma = 0.01$. The underlying network is displayed at different times, testifying on its natural tendency to evolve towards a single cluster of affine individuals. Initial opinions are uniformly distributed in the interval $[0, 1]$, while α_{ij}^0 are randomly assigned in $[0, 1/2]$ with uniform distribution.

2. The social network

The affinity matrix drives the interaction via the selection mechanism. It hence can be interpreted as the *adjacency* matrix of the underlying *social network*, i.e. the network of social ties that influences the exchange of opinions between acquaintances, as mediated by the encounters. Because the affinity is a dynamical variable of the model, we are actually focusing on an *adaptive* social network [13, 19] : The network topology influences in turn the dynamics of opinions, this latter providing a feedback on the network itself and so modifying its topology. In other words, the evolution of the topology is inherent to the dynamics of the model because of the proposed self-consistent formulation and not imposed a priori as an additional, external ingredient, (as e.g. rewire and/or add/remove links according to a given probability [14, 15] once the state variables have been updated).

Remark 1. (Weighted network) Let us observe that the affinity assumes positive real values, hence we can consider a weighted social networks, where agents weigh the relationships. Alternatively, one can introduce a cut-off parameter, α_f: agents i and j are socially linked if and only if the recorded relative affinity is large enough, meaning $\alpha_{ij} > \alpha_f$. Roughly, the agent chooses its closest friends among all his neighbors.

The first approach avoids the introduction of non–smooth functions and it is suitable to carry on the analytical calculations. The latter results more straightforward for numerical oriented applications.

As anticipated, we are thus interested in analyzing the model, for a specific choice of the parameters, α_c, σ and ΔO_c, yielding to consensus, and studying the evolution of the network topology, here analyzed via standard network indicators: the average value of *weighted degree*, the *cluster coefficient* and the *averaged shortest path*. These quantities will be quantified for (i) a fixed population, monitoring their time dependence; (ii) as a function of the population size, photographing the dynamics at convergence, namely when consensus has been reached.

2.1. *Time evolution of weighted degree*

The simplest and the most intensively studied one–vertex (i.e. local) characteristic is the node *degree* [a]: the *total number of its connections* or its nearest neighbors. Because we are dealing with a weighted network we can also introduce the *weighted node degree*, also called *node strength* [4], namely $s_i(t) = \sum_j \alpha_{ij}^t/(N-1)$. Its mean

[a] Let us observe that the affinity may not be symmetric and thus the inspected social network will be directed. One has thus to distinguish between *In–degree*, k_{in}, being the number of *incoming edges* of a vertex and *Out–degree*, k_{out}, being the number of its *outgoing edges*. In the following we will be interested only in the outgoing degree, from here on simply referred to as to degree.

value averaged over the whole network reads:

$$< s > (t) = \frac{1}{N}\sum_{i=1}^{N} s_i(t)\,. \tag{3}$$

Let us observe that the normalization factor $N-1$ holds for a population of N agents, self-interaction being disregarded, $< s >$ belongs hence to the interval $[0,1]$ and having eliminated the relic of the population size, one can properly compare quantities calculated for networks made of different number of agents.

All these quantities evolve in time because of the dynamics of the opinions and/or affinities. Passing to continuous time and using the second relation of (1), we obtain:

$$\frac{d}{dt} < s >= \frac{1}{N(N-1)}\sum_{i,j=1}^{N}\frac{d}{dt}\alpha_{ij}^{t}\,. \tag{4}$$

Let us observe that the evolution of affinity and opinion can be decoupled when $\Delta O_c = 1$. For $\Delta O_c < 1$, this is not formally true. However on can argue for an approximated strategy [9], by replacing the step function Γ_2 by its time average counterpart γ_2, where the dependence in ΔO_{ij}^{t} has been silenced. In this way, we obtain form (4)

$$\frac{d}{dt} < s >= \frac{\gamma_2}{N(N-1)}\sum_{i,j=1}^{N}\alpha_{ij}^{t}(1-\alpha_{ij}^{t}) = \gamma_2(< s > - < s^2 >)\,, \tag{5}$$

where $< s^2 >= \sum \alpha_{ij}^2/(N(N-1))$. Let us observe that γ_2 is of the order of $1/N^2$ times, a factor taking care of the asynchronous dynamics [9].

In [6] authors proved that (5) can be analytically solved once we provide the initial distribution of node strengths (see Appendix A for a short discussion of the involved methods). Assuming $s_i(0)$ to be uniformly distributed in $[0,1/2]$, we get the following solution (see Fig. 2):

$$< s > (t) = \frac{e^{\gamma_2 t}}{e^{\gamma_2 t}-1} - \frac{2e^{\gamma_2 t}}{(e^{\gamma_2 t}-1)^2}\log\left(\frac{e^{\gamma_2 t}+1}{2}\right)\,, \tag{6}$$

Using similar ideas we can prove [6] that the variance $\sigma_s^2(t) =< s^2 > - < s >^2$ is analytically given by

$$\sigma^2(t) = \frac{2e^{2\gamma_2 t}}{(e^{\gamma_2 t}-1)^2(e^{\gamma_2 t}+1)} - \frac{4e^{2\gamma_2 t}}{(e^{\gamma_2 t}-1)^4}\left[\log\left(\frac{e^{\gamma_2 t}+1}{2}\right)\right]^2\,. \tag{7}$$

The comparison between analytical and numerical profiles is enclosed in Fig. 2, where the evolution of $< s > (t)$ is traced. Let us observe that here γ_2 will serve as a fitting parameter, when testing the adequacy of the proposed analytical curves versus direct simulations, instead of using its computed numerical value [9]. The qualitative correspondence is rather satisfying, so confirming the correctness of the analytical results reported above.

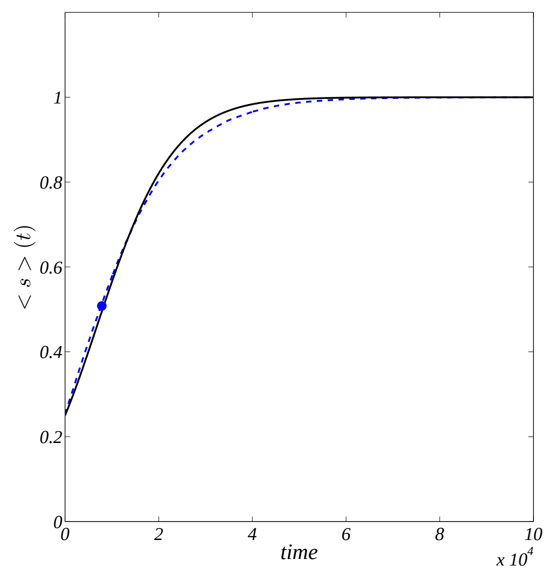

Fig. 2. Evolution of $< s > (t)$. Dashed line (blue on-line) refers to numerical simulations with parameters $\alpha_c = 0.5$, $\Delta O_c = 0.5$ and $\sigma = 0.3$. The full line (black on-line) is the analytical solution (6) with a best fitted parameter $\gamma_2 = 1.6\,10^{-4}$. The dot denotes the convergence time in the opinion space to the consensus state, for the used parameters affinities did not yet converge. Let us observe in fact that affinities and opinions do converge on different time scale [9].

Assume T_c to label the time needed for the consensus to be reached. Clearly, T_c depends on the size of the simulated system [b]. From the above relation (6), the average node strength at convergence as an implicit function of the population size N reads:

$$< s > (T_c(N)) = \frac{e^{\gamma_2(N)T_c(N)}}{e^{\gamma_2(N)T_c(N)} - 1} - \frac{2e^{\gamma_2(N)T_c(N)}}{(e^{\gamma_2(N)T_c(N)} - 1)^2} \log\left(\frac{e^{\gamma_2(N)T_c(N)} + 1}{2}\right) , \tag{8}$$

where we emphasized the dependence of γ_2 and T_c on N. However, as already observed $\gamma_2(N) = \mathcal{O}\left(N^{-2}\right)$ and $T_c(N) = \mathcal{O}\left(N^a\right)$, with $a \in (1,2)$. Hence $\gamma_2(N)T_c(N) \to 0$ when $N \to \infty$ and thus $< s > (T_c(N))$ is predicted to be a decreasing function of the population size N, which converges to the asymptotic value $1/4$, a value identical to the initial average node strength (see Fig. 3), given the selected initial condition. In sociological terms this means that even when consensus is achieved the larger the group the smaller, on average, the number of local

[b] In [3, 7] it was shown that T_c scales faster than linearly but slower than quadratically with respect to the population size N.

acquaintances. This is a second conclusion that one can reach on the basis of the above analytical developments.

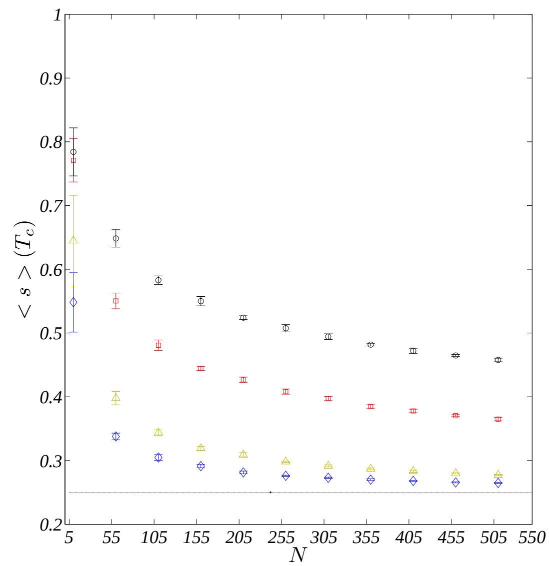

Fig. 3. Average node strength at convergence as a function of the population size. Parameters are $\Delta O_c = 0.5$, $\sigma = 0.5$ and four values of α_c have been used : (◊) $\alpha_c = 0$, (△) $\alpha_c = 0.25$, (□) $\alpha_c = 0.5$ and (◯) $\alpha_c = 0.75$. Vertical bars are standard deviations computed over 10 replicas of the numerical simulation using the same initial conditions.

2.2. *Small world*

Several social networks exhibit the remarkable property that one can reach an arbitrary far member of the community, via a relatively small number of intermediate acquaintances. This holds true irrespectively of the size of the underlying network. Experiments [16] have been devised to quantify the "degree of separation"in real system, and such phenomenon is nowadays termed the "small world"effect, also referred to as the "six degree of separation".

On the other hand several, models have been proposed [18, 17] to construct complex networks with the small world property. Mathematically, one requires that the average shortest path grows at most logarithmic with respect to the network size, while the network still displays a large clustering coefficient. Namely, the network has an average shortest path comparable to that of a random network, with the same number of nodes and links, while the clustering coefficient is instead significantly larger.

In this section we present numerical results aimed at describing the time evolution of both the average shortest path and the clustering coefficient of the social network emerging from the model. As before, the parameters are set so to induce the convergence to a consensus state in the opinion space.

We will be particularly interested in their asymptotic solutions, terming the associated values respectively $< \ell > (T_c)$ and $C(T_c)$ once the consensus state has been achieved.

In Fig. 4 we report these quantities (normalized to the homologous values estimated for a random network with identical number of nodes and links) versus the system size. The (normalized) clustering coefficient is sensibly larger than one, this effect being more pronounced the smaller the value of α_c. On the other hand the (normalized) average shortest path is always very close to 1.

Based on the above we are hence brought to conclude that the social network emerging from the opinion exchanges, has the small world property. This is a remarkable feature because the social network evolves guided by the opinions, as it does in reality, and not result from an artificially imposed recipe.

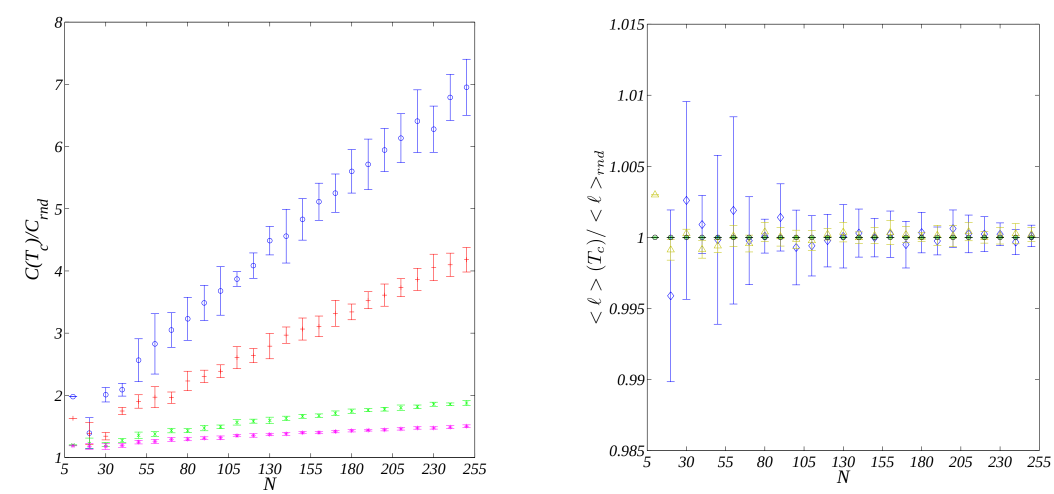

Fig. 4. Normalized clustering coefficient (left panel) and normalized average mean path (right panel) as functions of the network size at the convergence time. Parameters are $\Delta O_c = 0.5$, $\sigma = 0.5$ and four values of α_c have been used : ($\Diamond$) $\alpha_c = 0$, ($\triangle$) $\alpha_c = 0.25$, ($\square$) $\alpha_c = 0.5$ and ($\bigcirc$) $\alpha_c = 0.75$. Vertical bars are standard deviations computed over 10 repetitions.

2.3. *Weak ties*

Social networks are characterized by the presence of hierarchies of well tied small groups of acquaintances, that are possibly linked to other such groups via "weak ties". According to Granovetter [12] these weak links are fundamental for the cohesion of the society, being at the basis of the social tissue, so motivating the statement "the strength of weak ties".

The smallest group in a social network is composed by three individuals sharing high mutual affinities, in term of network theory they form a *clique* [1], i.e. a maximal complete graph composed by three nodes. This can of course be generalized to larger maximal complete graphs, defining thus m-cliques.

The degree of cliqueness of a social network is hence a measure of its cohesion/fragmentation: the presence of a large number of m-cliques together with very few, m'-cliques, for $m' > m$, means that the population is actually fragmented into small pieces, of size m not strongly interacting each other.

We are interested in studying such phenomenon within the social network emerging from the opinion dynamics model here considered, still operating in consensus regime. To this end we proceed as follows. We introduce a cut–off parameter α_f used to binarize the affinity matrix, which hence transforms into a an adjacency matrix a. More precisely, agents i and j will be connected, i.e. $a_{ij} = 1$, if and only if $\alpha_{ij} \geq \alpha_f$. Once the adjacency matrix is being constructed, we compute the number of m–cliques in the network. Let us observe that this last step is highly time consuming, being the clique problem NP-complete. We thus restrict our analysis to the cases $m \in \{3, 4, 5\}$.

For small values of α_f the network is almost complete, while for large ones it can in principle fragment into a vast number of finite small groups of agents. As reported in the inset of right panel of Fig. 5, for $\alpha_f \sim 1$ only 3–cliques are present. Their number rapidly increases as α_f is lowered. On the other hand for $\alpha_f \sim 0.98$ few 4–cliques emerge while 5–cliques appear around $\alpha_f \sim 0.73$. This means that the social networks is mainly composed by 3–cliques, i.e. agents sharing high mutual affinities, that are connected together to form larger cliques, for instance 4 and 5–cliques by weaker links, i.e. whose mutual affinities are lower than the above ones.

Results reported in left panel of Fig. 5 show that for specific parameter values, still falling into the class deputed to the consensus dynamics, the model does not present the weak ties phenomenon: 3, 4 and 5-cliques are all present at the same time for large values of α_f. This is an important point that will deserve future investigations. Let us observe here that the observed differences stem from the social temperature.

3. Conclusion

Social system and opinion dynamics models are intensively investigated within simplified mathematical schemes. One of such model is here revisited and analyzed. The evolution of the underlying network of connections, here emblematized by the

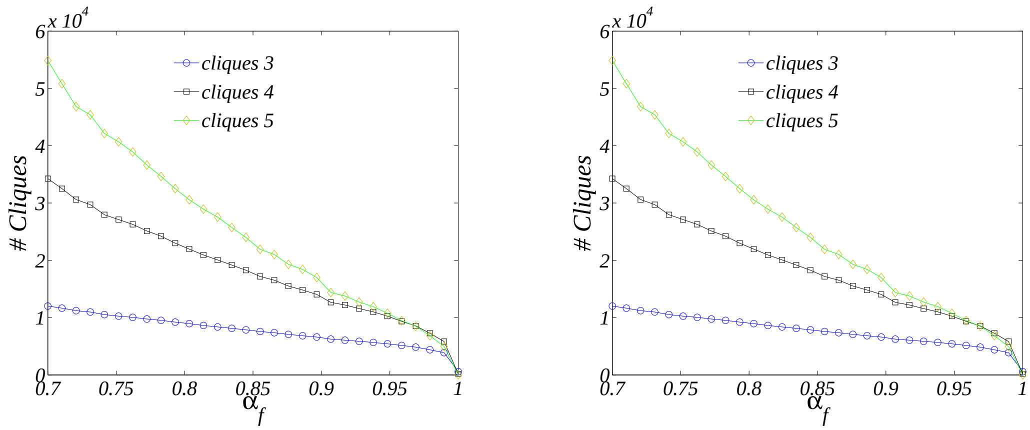

Fig. 5. Number of 3, 4 and 5–cliques in the social network once consensus has been achieved. Parameters are $N = 100$, $\Delta O_c = 0.5$, $\alpha_c = 0.5$. Right panel, $\sigma = 0.5$, the network exhibits the weak ties property. Left panel, $\sigma = 0.1$, the network does not display the weak ties phenomenon.

mutual affinity score, is in particular studied. This is a dynamical quantity which adjusts all along the system evolution, as follows a complex coupling with the opinion variables. In other words, the embedding social structure is adaptively created and not a priori assigned, as it is customarily done. Starting from this setting, the model is solved analytically, under specific approximations. The functional dependence on time of the networks mean characteristics are consequently elucidated. The obtained solutions correlate with direct simulations, returning a satisfying agreement. Moreover, the structure of the social network is numerically monitored, via a set of classical indicators. Small world effects, as well weak ties connections, are found as an emerging property of the model. It is remarkable that such properties, ubiquitous in nature, are spontaneously generated within a simple scenario which accounts for a minimal number of ingredients, in the context of a genuine self-consistent formulation.

Appendix A. On the momenta evolution

The aim of this section if to present and sketch the proof of the result used to study the evolution of the momenta of the affinity distribution. We refer the interested reader to [6] where a more detailed analysis is presented in a general setting.

For the sake of simplicity, let us label the $N(N-1)$ affinities values α_{ij} by x_l, upon assigning a specific re-ordering of the entries. Hence $\vec{x}$ is a vector with $M = N(N-1)$ elements. As previously recalled (5), we assume each x_l to obey a first order differential equation of the logistic type, once time has been rescaled, namely:

$$\frac{dx_l}{dt} = x_l(1 - x_l)\,. \tag{A.1}$$

The initial conditions will be denoted as x_l^0.

Let us observe that each component x_l evolves independently from the other. We can hence imagine to deal with M replicas of a process ruled by by (A.1) whose initial conditions are distributed according to some given function. We are interested in computing the momenta of the x variable as functions of the initial distribution. The m-th momentum is given by:

$$< x^m > (t) = \frac{(x_1(t))^m + \dots + (x_M(t))^m}{M}\, , \tag{A.2}$$

and its time evolution is straightforwardly obtained deriving (A.2) and making use of Eq. (A.1):

$$\begin{aligned} \frac{d}{dt} < x^m > (t) &= \frac{1}{M}\sum_{i=1}^{M} \frac{dx_l^m}{dt} = \frac{m}{M}\sum_{l=1}^{N} x_l^{m-1}\frac{dx_l}{dt} \\ &= \frac{m}{M}\sum_{l=1}^{N} x_l^{m-1} x_l(1-x_l) = m\left(< x^m > - < x^{m+1} >\right)\, . \end{aligned} \tag{A.3}$$

To solve this equation we introduce the *time dependent moment generating function*, $G(\xi, t)$,

$$G(\xi, t) := \sum_{m=1}^{\infty} \xi^m < x^m > (t)\, . \tag{A.4}$$

This is a formal power series whose Taylor coefficients are the momenta of the distribution that we are willing to reconstruct, task that can be accomplished using the following relation:

$$< x^m > (t) := \frac{1}{m!}\frac{\partial^m G}{\partial \xi^m}\Big|_{\xi=0}\, . \tag{A.5}$$

By exploiting the evolution's law for each x_l, we shall here obtain a partial differential equation governing the behavior of G. Knowing G will eventually enables us to calculate any sought momentum via multiple differentiation with respect to ξ as stated in (A.5).

On the other hand, by differentiating (A.4) with respect to time, one obtains :

$$\frac{\partial G}{\partial t} = \sum_{m\geq 1} \xi^m \frac{d < x^m >}{dt} = \sum_{m\geq 1} m\xi^m\left(< x^m > - < x^{m+1} >\right)\, , \tag{A.6}$$

where used has been made of Eq. (A.3). We can now re-order the terms so to express the right hand side as a function of G [c] and finally obtain the following non–homogeneous linear partial differential equation:

$$\partial_t G - (\xi - 1)\partial_\xi G = \frac{G}{\xi}\, . \tag{A.7}$$

[c]Here the following algebraic relations are being used:

$$\xi\partial_\xi G(\xi, t) = \xi \sum_{m\geq 1} m\xi^{m-1} < x^m > = \sum_{m\geq 1} m\xi^m < x^m >\, ,$$

Such an equation can be solved for ξ close to zero (as in the end of the procedure we shall be interested in evaluating the derivatives at $\xi = 0$, see Eq. (A.5)) and for all positive t. To this end we shall specify the initial datum:

$$G(\xi, 0) = \sum_{m\geq 1} \xi^m < x^m > (0) = \Phi(\xi)\,, \tag{A.8}$$

i.e. the initial momenta or their distribution.

Before turning to solve (A.7), we first simplify it by introducing

$$G = e^g \quad \text{namely} \quad g = \log G\,, \tag{A.9}$$

then for any derivative we have $\partial_* G = G\partial_* g$, where $* = \xi$ or $* = t$, thus (A.7) is equivalent to

$$\partial_t g - (\xi - 1)\partial_\xi g = \frac{1}{\xi}\,, \tag{A.10}$$

with the initial datum

$$g(\xi, 0) = \phi(\xi) \equiv \log \Phi(\xi)\,. \tag{A.11}$$

This latter equation can be solved using the *method of the characteristics*, here represented by:

$$\frac{d\xi}{dt} = -(\xi - 1)\,, \tag{A.12}$$

which are explicitly integrated to give:

$$\xi(t) = 1 + (\xi(0) - 1)e^{-t}\,, \tag{A.13}$$

where $\xi(0)$ denotes $\xi(t)$ at $t = 0$. Then the function $u(\xi(t), t)$ defined by:

$$u(\xi(t), t) := \phi(\xi(0)) + \int_0^t \frac{1}{1 + (\xi(0) - 1)e^{-s}}\, ds\,, \tag{A.14}$$

is the solution of (A.10), restricted to the characteristics. Observe that $u(\xi(0), 0) = \phi(\xi(0))$, so (A.14) solves also the initial value problem.

Finally the solution of (A.11) is obtained from u by reversing the relation between $\xi(t)$ and $\xi(0)$, i.e. $\xi(0) = (\xi(t) - 1)e^t + 1$:

$$g(\xi, t) = \phi\left((\xi - 1)e^t + 1\right) + \lambda(\xi, t)\,, \tag{A.15}$$

and

$$\begin{aligned}\xi\partial_\xi \frac{G(\xi,t)}{\xi} &= \xi\partial_\xi \sum_{m\geq 1} \xi^{m-1} < x^m > = \xi \sum_{m\geq 1} (m-1)\xi^{m-2} < x^m > \\ &= \sum_{m\geq 1} (m-1)\xi^{m-1} < x^m >\end{aligned}$$

Renaming the summation index, $m - 1 \to m$, one finally gets (note the sum still begins with $m = 1$):

$$\xi\partial_\xi \frac{G(\xi,t)}{\xi} = \sum_{m\geq 1} m\xi^m < x^{m+1} > \,.$$

where $\lambda(\xi,t)$ is the value of the integral in the right hand side of (A.14).

This integral can be straightforwardly computed as follows (use the change of variable $z=e^{-s}$):

$$\lambda = \int_0^t \frac{1}{1+(\xi(0)-1)e^{-s}}\,ds = \int_1^{e^{-t}} \frac{-dz}{z}\frac{1}{1+(\xi(0)-1)z}\,, \tag{A.16}$$

which implies

$$\begin{aligned}\lambda &= -\int_1^{e^{-t}} dz\left(\frac{1}{z}-\frac{\xi(0)-1}{1+(\xi(0)-1)z}\right) = -\log z + \log(1+(\xi(0)-1)z)\Big|_1^{e^{-t}} \\ &= t+\log(1+(\xi(0)-1)e^{-t})-\log\xi(0)\,. \end{aligned}\tag{A.17}$$

According to (A.15) the solution g is then

$$g(\xi,t)=\phi\left((\xi-1)e^t+1\right)+t+\log\xi-\log((\xi-1)e^t+1)\,, \tag{A.18}$$

from which G straightforwardly follows:

$$G(\xi,t)=\Phi\left((\xi-1)e^t+1\right)\frac{\xi e^t}{(\xi-1)e^t+1}\,. \tag{A.19}$$

As anticipated, the function G makes it possible to estimate any momentum (A.5). As an example, the mean value correspond to setting $m=1$, reads:

$$\begin{aligned}< x > (t) = \partial_\xi G\Big|_{\xi=0} &= \Big[\Phi'\left(1+(\xi-1)e^t\right)e^t\frac{\xi e^t}{(\xi-1)e^t+1} \\ &+ \Phi\left(1+(\xi-1)e^t\right)e^t\frac{(\xi-1)e^t+1-\xi e^t}{\left(1+(\xi-1)e^t\right)^2}\Big]\Big|_{\xi=0} \\ &= \frac{e^t}{1-e^t}\Phi(1-e^t)\,. \end{aligned}\tag{A.20}$$

In the following section we shall turn to considering a specific application in the case of uniformly distributed values of affinities.

A.1. *Uniform distributed initial conditions*

The initial data x_l^0 are assumed to span uniformly the bound interval $[0,0.5]$, thus the probability distribution $\psi(x)$ clearly reads [d]:

$$\psi(x)=\begin{cases}2 & \text{if } x\in[0,1/2]\\ 0 & \text{otherwise}\end{cases}\,, \tag{A.21}$$

and consequently the initial momenta are:

$$< x^m > (0) = \int_0^1 \xi^m\psi(\xi)d\xi = \int_0^{1/2} 2\xi^m\,d\xi = \frac{1}{m+1}\frac{1}{2^m}\,. \tag{A.22}$$

[d] We hereby assume to sample over a large collection of independent replica of the system under scrutiny (M is large). Under this hypothesis one can safely adopt a continuous approximation for the distribution of allowed initial data. Conversely, if the number of realizations is small, finite size corrections need to be included [6].

Hence the function Φ as defined in (A.8) takes the form:

$$\Phi(\xi) = \sum_{m\geq 1} \frac{1}{m+1}\frac{\xi^m}{2^m}\,. \tag{A.23}$$

A straightforward algebraic manipulation allows us to re-write (A.23) as follows:

$$\sum_{m\geq 1} \frac{y^m}{m+1} = \frac{1}{y}\int_0^y \sum_{m\geq 1} z^m\, dz = \frac{1}{y}\int_0^y \frac{z}{1-z}\, dz = -1 - \frac{1}{y}\log(1-y)\,, \tag{A.24}$$

thus

$$\Phi(\xi) = -1 - \frac{2}{\xi}\log\left(1-\frac{\xi}{2}\right)\,. \tag{A.25}$$

We can now compute the time dependent moment generating function, $G(\xi,t)$, given by (A.19) as:

$$G(\xi,t) = \frac{\xi e^t}{(\xi-1)e^t+1}\left[-1 - \frac{2}{(\xi-1)e^t+1}\log\left(1-\frac{(\xi-1)e^t+1}{2}\right)\right]\,, \tag{A.26}$$

and thus recalling (A.5) we get

$$\begin{aligned} <x>(t) &= \frac{e^t}{e^t-1} - \frac{2e^t}{(e^t-1)^2}\log\left(\frac{e^t+1}{2}\right) \\ <x^2>(t) &= \frac{e^{2t}}{(e^t-1)^2} + \frac{4e^{2t}}{(e^t-1)^3}\log\left(\frac{e^t+1}{2}\right) + \frac{2e^{2t}}{(e^t-1)^2(e^t+1)}\,. \end{aligned} \tag{A.27}$$

Let us observe that $<x>(t)$ deviates from the logistic growth to which all the single variable $x_i(t)$ does obey.

For large enough times, the distribution of the variable outputs is in fact concentrated around the asymptotic value 1 with an associated variance (calculated from the above momenta) which decreases monotonously with time.

Let us observe that a naive approach would suggest interpolating the averaged numerical profile with a solution of the logistic model whose initial datum $\hat{x}^0$ acts as a free parameter to be adjusted to its best fitted value: as it is proven in [6] this procedure yields a significant discrepancy, which could be possibly misinterpreted as a failure of the underlying logistic evolution law. For this reason, and to avoid drawing erroneous conclusions when ensemble averages are computed, attention has to be payed on the role of initial conditions.

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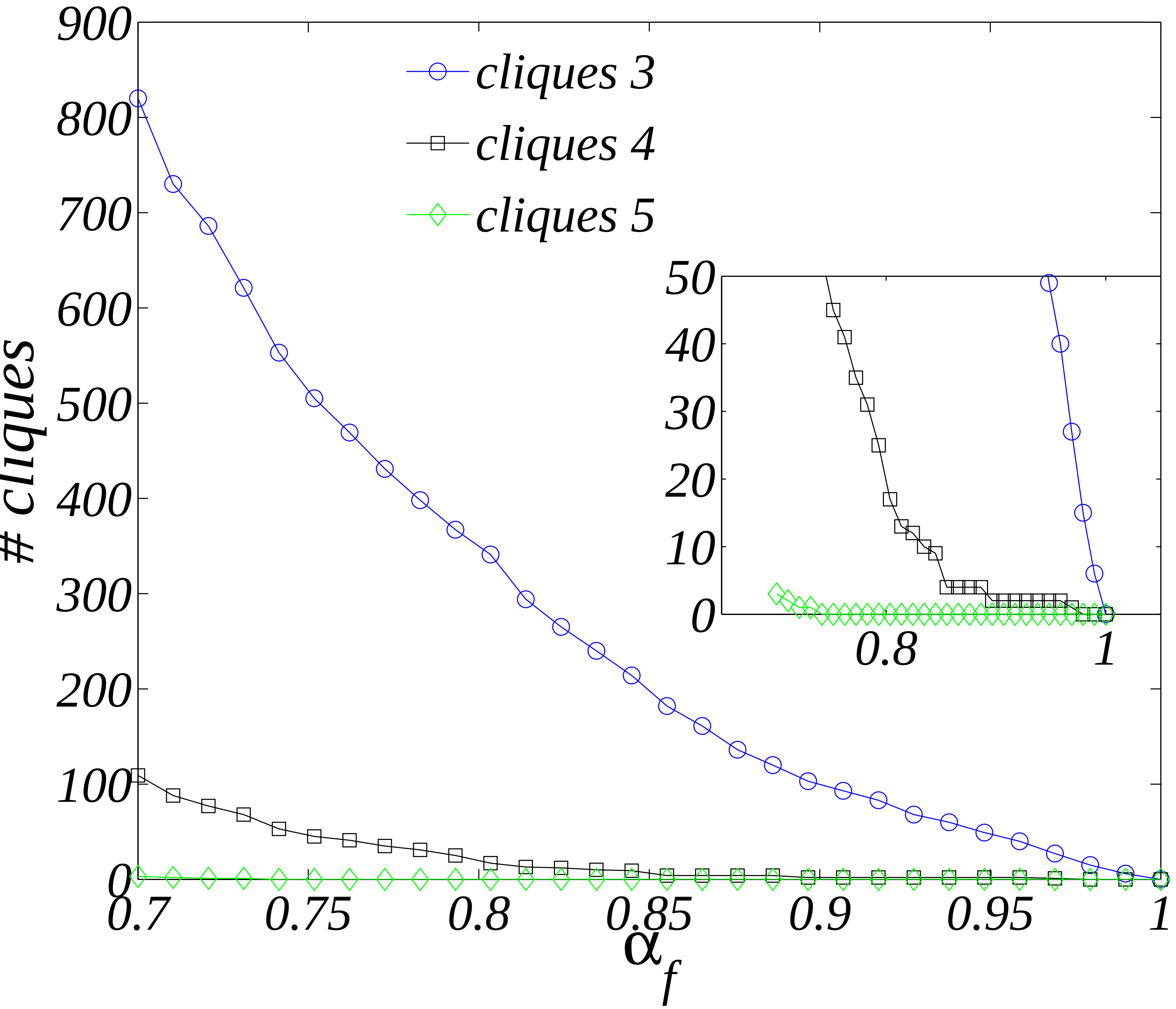

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