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On the non-existence of a projective plane of order 15 with an A_4-invariant oval[☆]

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Abstract

Let π be a projective plane of order 15 with an oval Ω. Assume π admits a collineation group G fixing Ω such that G is isomorphic to A_4 and the action of G on Ω yields precisely two orbits Ω_1 and Ω_2 with $|\Omega_2| = 4$. We prove that the Buekenhout oval arising from Ω cannot exist.

MSC: 51E15; 05B25

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1. Introduction

The exhaustive search for the existence of a finite projective plane of a given order n can be very time-consuming even for small values of n. The troubled story of the case $n = 10$ (see [8]) does not seem to have discouraged attempts for the next values of n for which non-existence is not covered by the Bruck–Ryser Theorem. Apart from the legitimate curiosity related to the prime power conjecture, there might be other indirect reasons for wanting to investigate a specific value of n. Classification theorems in finite geometries often have the shape of a general statement handling all but finitely many values of the involved parameter. Typically, when dealing with planes, this parameter is the order of the plane.

The "exceptional" values usually require special treatment, which sometimes can only be purely combinatorial, in absence of suitable theoretical tools: unless the value in question is indeed very small, it is likely that the only approach left is the computational one. Shortcuts in the combinatorial search for a plane of a given order may be possible when the plane has additional structure or when symmetries come into play: the former situation imposes more combinatorial constraints than the bare structure of a projective plane, the latter one allows substantial branchings of the search-tree through the principle of "isomorph rejection," see [8].

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The problem we are addressing in this paper is the existence of a projective plane π of order $n = 15$. The additional structure that we require is the existence of an oval Ω; the symmetries involved are those of a collineation group G of π fixing Ω such that G is isomorphic to the alternating group A_4 and the action of G on Ω yields precisely two orbits Ω_1 and Ω_2 with $|\Omega_2| = 4$. The two aspects that we mentioned in the previous paragraph are thus strictly linked in our case. It is the purpose of this paper to prove that a plane with these properties cannot exist.

The motivation for the study of this specific case comes from an attempt of classifying projective planes of odd order admitting an oval which is left invariant by a collineation group having two orbits on the oval, one of which is assumed to be faithful and primitive. The case where the collineation group fixes a triangle off the oval is treated in [1]: under the additional assumption that the group fixes no points nor lines, it is proved that the order of the plane cannot exceed 27; furthermore, the groups and the planes that do occur are determined in some detail. The case we are considering in the present paper is "exceptional" in that classification in the sense that we mentioned above.

The approach that we follow is based on the concept of a Buekenhout oval (*B-oval*), that is a family of involutory permutations of degree 16 with certain properties, see the original paper of F. Buekenhout [3] or section 3.4 of the survey article [7] for the definitions and details. An oval in a finite projective plane naturally defines a *B*-oval and the *B*-ovals arising in this manner are usually called *projective B*-ovals in this context [4]. Our proof consists in showing that the projective *B*-oval arising from the oval Ω with the described properties cannot exist.

In the end, we tackle the problem from a strictly combinatorial point of view, in the sense that we generate all possible candidates for a suitable subfamily of our projective *B*-oval and show by an exhaustive computer search that none of them can be completed to a full *B*-oval. We have performed all our computer calculations using the computer algebra system GAP [6], which allowed us efficient handling of involutory permutations.

Isomorph rejection occurs at all levels by exploiting some useful geometric and algebraic properties of the group G established in the next section. We would like to stress the circumstance that, upon replacement of A_4 by the symmetric group S_4 with the corresponding assumptions, a non-existence proof can be given in a purely theoretical manner. The reason for that probably lies in the fact that the group S_4 would have two distinct conjugacy classes of involutory homologies, a situation which has made life easier in a number of similar situations, see for instance chapter 4 in [7].

2. The basics

Let π be a finite projective plane of odd order n with an oval Ω. For each point X of Ω denote by t_X the tangent to Ω at X. Let P be a point of $\pi\backslash\Omega$. We denote by j_P the involutory permutation on Ω mapping each point $Q \in \Omega$ to the other point of intersection of the line PQ with Ω, if this line is a secant, or to itself, if the line PQ is a tangent, respectively. The involution j_P will therefore have 0 or 2 fixed points on Ω according as P is an internal or an external point with respect to Ω. Let P and L be two distinct points of $\pi\backslash\Omega$. If, for a permutation g on Ω, we denote by $\mathrm{Fix}(g)$ the set of all fixed points of g on Ω, then the relation $|\mathrm{Fix}(j_P j_L)| \leqslant 2$ will hold, as the line PL has at most two points in common with Ω. The set $\mathscr{F}$ of all involutory permutations j_P, as P runs over $\pi\backslash\Omega$, is a projective *B*-oval in the sense of [4], see also [3]. For convenience we shall use the term partial *B*-oval to denote any set of involutory permutations on Ω such that the product of any two involutions in the set has at most two fixed points. We shall begin with a very elementary but useful property.

Proposition 1. *Assume g is a collineation of π fixing Ω and let P be a point in $\pi\backslash\Omega$. We have the relation $g j_P g^{-1} = j_{g(P)}$.*

As a consequence we have that if G is a collineation group of π fixing Ω then the *B*-oval $\mathscr{F}$ must contain all G-conjugates of any one of its involutions.

From now on assume $n = 15$, unless otherwise stated. The projective *B*-oval $\mathscr{F}$ arising from Ω consists of 225 involutory permutations altogether—120 of these will have two fixed points on Ω and 105 of these will be fixed-point-free. The set $\mathscr{I}_2$ of involutions with two fixed points on the 16 elements of Ω has cardinality $\binom{16}{2} \cdot 1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11 \cdot 13 = 16,216,200$; the set $\mathscr{I}_0$ of fixed-point-free involutions on the 16 elements of Ω has cardinality $1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11 \cdot 13 \cdot 15 = 2,027,025$. We have to choose 225 involutions in the set $\mathscr{I} = \mathscr{I}_0 \cup \mathscr{I}_2$ in a suitable manner: the number of possible choices for our *B*-oval $\mathscr{F}$ is thus excessive for anyone wishing to generate them all and that is why we need to exploit the symmetries of the problem.

Therefore let $G \cong A_4$ be a collineation group of π with the properties stated in the Introduction. The three involutions in the Klein subgroup K of G are involutory homologies. An involutory homology fixing a given oval in a projective plane of odd order has its center off the oval; furthermore, any two such homologies have distinct centers and distinct axes, see [2, Proposition 2.1]. It follows then from [5, Section 3.1.7] that the centers and axes of the involutory homologies in K form the vertices and sides of a triangle. We denote the vertices by A, B, C and the involutory homologies by h_A, h_B, h_C, where h_A has center A and axis

BC, h_B has center B and axis AC and h_C has center C and axis AB, respectively. The vertices A, B, C are the unique points of the plane which are fixed by K, consequently $\{A, B, C\}$ is setwise fixed by G.

Proposition 2. *Each collineation of order* 3 *in G has precisely one fixed point on* π *and this point lies on* Ω_2.

Proof. It follows from $|\Omega_2| = 4$ that each point in Ω_2 is fixed by some collineation of order 3 in G and by no involution. If a point of Ω_2 is fixed by collineations of order 3 in distinct Sylow 3-subgroups of G, then this point is fixed by G, a contradiction. Since each collineation of order 3 in G has at least one fixed point on Ω_2, we conclude that each collineation of order 3 in G has exactly one fixed point on Ω_2.

Let g be a collineation of order 3 in G and let R be its fixed point on Ω_2. It follows from $|G| = |\Omega_1| = 12$ that the action of G on Ω_1 is regular, consequently g has no fixed points on Ω_1. The tangent t_R to Ω through R is also fixed by g. If there were a fixed point Q of g off Ω and not on t_R then the line QR should be a secant and consequently meet Ω at a further fixed point of g on Ω, a contradiction. Assume g fixes a point S on t_R other than R. The tangent to Ω through S other than t_R must be fixed by g and consequently g also fixes the point W at which this further tangent touches Ω. As $R \neq W$, we have a contradiction again. □

As a consequence G acts on Ω_2 as A_4 in its natural permutation representation on four objects; furthermore, G is transitive on $\{A, B, C\}$, in other words $\{A, B, C\}$ is a point-orbit and $\{AB, AC, BC\}$ is a line-orbit under the action of G.

Proposition 3. *The G-orbit of a point* $W \neq A, B, C$ *on the sides of the triangle has length six with two points on each side. Every other point-orbit under G has length twelve. Dually, the G-orbit of a line* $q \neq AB$, *AC, BC through one of the vertices of the triangle has length six with two lines through each vertex. Every other line-orbit under G has length twelve.*

Proof. The stabilizer of a point on the side BC other than B, C consists of the identity and of the involutory homology h_A; transitivity of G on $\{AB, AC, BC\}$ shows that the given G-orbit has points on each side of the triangle. The stabilizer of a point off the sides of the triangle reduces to the identity. The argument for lines is quite similar. □

Proposition 4. *The points A, B, C are internal, the lines AB, AC, BC, are external with respect to* Ω.

Proof. Suppose that the axes of the homologies h_A, h_B, h_C are secant lines with respect to Ω. The points of intersection of each axis with Ω lie in the same orbit (for instance h_A exchanges the points of intersection of AB with Ω). These six points are pairwise distinct and they all lie in the same G-orbit, as G is transitive on $\{A, B, C\}$. Since $|\Omega_2| = 4$ we conclude that the orbit containing these six points is Ω_1. On the other hand, K is normal in G and the K-orbits on Ω_1 form blocks of imprimitivity for G on Ω_1. In particular these orbits must have the same length. The six points mentioned above form three K-orbits on Ω_1 of length two each: if there were a further point in Ω_1, its K-orbit would have length four, as such a point is not fixed by any one of the involutory homologies in K. We conclude that Ω_1 consists precisely of these six points and the plane π should have order 9, a contradiction.

If the axis of a homology fixing Ω is an external line, then its center must be an internal point, as it is immediately checked, see again [2, Proposition 2.1]. □

Proposition 5. *Through each one of A, B, C there exist precisely six secants meeting* Ω *at two points of* Ω_1 *and two secants meeting* Ω *at two points of* Ω_2.

Proof. Assume ℓ is a secant through A with $\ell \cap \Omega = \{A_1, A_2\}$ and $A_1 \in \Omega_1$, $A_2 \in \Omega_2$. The homology h_A fixes Ω and ℓ and consequently fixes each one of A_1, A_2, a contradiction. Since A cannot lie on a tangent, the assertion follows. □

Set $\Omega_2 = \{Y_1, Y_2, Y_3, Y_4\}$. The points Y_1, Y_2, Y_3, Y_4 form a quadrangle of which A, B, C are the diagonal points by Proposition 5, say $\{A\} = Y_1Y_2 \cap Y_3Y_4$, $\{B\} = Y_1Y_3 \cap Y_2Y_4$, $\{C\} = Y_1Y_4 \cap Y_2Y_3$.

Define $\{P\} = t_{Y_3} \cap t_{Y_4}$, $\{Q\} = t_{Y_1} \cap t_{Y_2}$, $\{R\} = t_{Y_2} \cap t_{Y_4}$, $\{S\} = t_{Y_1} \cap t_{Y_3}$, $\{T\} = t_{Y_2} \cap t_{Y_3}$, $\{U\} = t_{Y_1} \cap t_{Y_4}$.

Proposition 6. *The points P, Q are on the line BC, the points R, S are on the line AC and the points T, U are on the line AB.*

Proof. The involutory homology h_A exchanges Y_1 with Y_2 and Y_3 with Y_4, consequently h_A fixes both P and Q, which lie thus on the axis BC. Similar arguments with the involutory homologies h_B and h_C yield the rest of the assertion. □

The external points P, Q, R, S, T, U form a single G-orbit of length six; consequently, once we have a candidate for j_P in $\mathscr{F}$, we know that j_P must have precisely six G-conjugates, all of which lie in $\mathscr{F}$.

The next lemma is formulated in a slightly more general setting than we require.

Proposition 7. *Let π be a finite projective plane of order $n \equiv -1 \bmod 4$ with an oval Ω. Let h be an involutory homology fixing Ω with center W and axis ℓ, where W is an internal point. If Z is an internal point on ℓ then the line WZ is external.*

Proof. Since h fixes ℓ pointwise we know from Proposition 1 that the relation $hj_Zh^{-1} = j_Z$ holds, whence also $j_W j_Z j_W^{-1} = j_Z$. We conclude that j_Z commutes with j_W and consequently $j_W j_Z$ is also an involution. Since both j_Z and j_W are fixed-point-free on Ω, they induce even permutations on Ω as the number of 2-cycles is $(n+1)/2$; consequently $j_Z j_W$ also induces an even permutation showing that $j_Z j_W$ cannot have two fixed points on Ω. In other words the line ZW cannot be a secant. □

The following lemmas will be useful in the reconstruction process described in the next section.

Proposition 8. *Let Z be an internal point on BC other than B, C. After a suitable relabelling of points in Ω we have*

$$j_A = (W_1, X_1)(W_2, X_2)(W_3, X_3)(W_4, X_4)(W_5, X_5)(W_6, X_6)(Y_1, Y_2)(Y_3, Y_4),$$
$$j_Z = (Y_1, W_1)(Y_2, X_1)(Y_3, W_2)(Y_4, X_2)(W_3, W_4)(X_3, X_4)(W_5, W_6)(X_5, X_6).$$

Proof. Join Z to an arbitrary point Y of Ω_2; the line YZ is a secant, meeting Ω at a further point W_1. Let Y' be the other point of Ω_2 in the 2-cycle of j_A in which Y appears, that is $j_A = (Y, Y')\ldots$. We cannot have $W_1 \in \Omega_2$ since otherwise the line $YZ = W_1Z$ should be a secant through two points of Ω_2, hence should be incident with one of the vertices A, B or C, but that is impossible, as none of the lines ZA, ZB, ZC is a secant by Proposition 7. Set $X_1 = h_A(W_1)$; we have $h_A(Z) = Z$, $h_A(Y) = Y'$, $h_A(W_1) = X_1$, hence X_1 is the further point at which the line ZY' meets Ω. We conclude that X_1 lies in Ω_1 and that (W_1, X_1) is also a 2-cycle of j_A. The involution j_Z has thus the following form:

$$j_Z = (Y_1, W_1)(Y_2, X_1)(Y_3, W_2)(Y_4, X_2)\ldots,$$

where (W_1, X_1) and (W_2, X_2) are two distinct 2-cycles of j_A on Ω_1. A quite similar argument shows that, up to relabelling, the remaining four 2-cycles of j_Z on Ω_1 are (W_3, W_4), (X_3, X_4), (W_5, W_6), (X_5, X_6) where (W_3, X_3), (W_4, X_4), (W_5, X_5), (W_6, X_6) are the remaining four 2-cycles of j_A on Ω_1. □

Proposition 9. *Let Z be an external point on BC. After a suitable relabelling of points in Ω we have*

$$j_A = (P_1, Q_1)(P_2, Q_2)(P_3, Q_3)(P_4, Q_4)(P_5, Q_5)(P_6, Q_6)(P_7, Q_7)(P_8, Q_8),$$
$$j_Z = (P_2, Q_2)(P_3, P_4)(Q_3, Q_4)(P_5, P_6)(Q_5, Q_6)(P_7, P_8)(Q_7, Q_8).$$

Proof. Since Z is fixed by the involutory homology h_A we see that the two tangents t_{P_1}, t_{Q_1} to Ω through Z are exchanged by h_A, which means the fixed points P_1 and Q_1 of j_Z are exchanged by h_A. In other words, (P_1, Q_1) is a 2-cycle of j_A.

By Proposition 7 the point Z is on a secant through A, meeting Ω at, say, P_2 and Q_2. That means j_A and j_Z share the 2-cycle (P_2, Q_2). If P_3P_4 is a further secant through Z, then h_A will exchange it with another secant Q_3Q_4 through A, hence, up to relabelling, P_3, Q_3, A and P_4, Q_4, A are two triples of collinear points, showing that (P_3, Q_3) and (P_4, Q_4) are 2-cycles of j_A. The assertion follows. □

3. Reconstructing the plane

In this section, we want to prove that the projective B-oval $\mathscr{F}$ described in the previous section does not exist. We do so by reconstructing parts of the putative plane in the sense that we determine the candidates for the involutions j_Z for suitable points Z in $\pi\backslash\Omega$. We continue with this reconstruction process, until we reach a stage in which the adjunction of any possible candidate to the current partial B-oval produces a set of involutions which no longer is a partial B-oval. We begin with the points lying on the sides of the fixed triangle ABC.

We have already remarked in the proof of Proposition 2 that G acts regularly on Ω_1 and acts on Ω_2 as the alternating group A_4 in its natural action on four objects. Up to relabelling we may assume $\Omega_1 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ and $\Omega_2 = \{13, 14, 15, 16\}$. Up to conjugation in $\mathrm{Sym}(\Omega) = S_{16}$ we may assume that the permutation representation of G on Ω is generated by the permutations

$$(1, 4)(2, 6)(3, 5)(7, 11)(8, 10)(9, 12)(13, 14)(15, 16),$$
$$(1, 2, 3)(4, 7, 10)(5, 9, 11)(6, 8, 12)(13, 14, 15).$$

The involutions j_A, j_B and j_C are obtained by restricting the action of the homologies h_A, h_B and h_C to Ω, respectively. Consequently j_A, j_B, j_C are precisely the three involutions of A_4 in its permutation representation on Ω just given. We may assume the labelling of points to be such that the relations

$$j_A = (1,4)(2,6)(3,5)(7,11)(8,10)(9,12)(13,14)(15,16),$$
$$j_B = (1,12)(2,11)(3,10)(4,9)(5,8)(6,7)(13,15)(14,16),$$
$$j_C = (1,9)(2,7)(3,8)(4,12)(5,10)(6,11)(13,16)(14,15)$$

hold. With the notation of the previous section we may also assume $Y_1 = 13$, $Y_2 = 14$, $Y_3 = 15$, $Y_4 = 16$.

We shall say that an involution j in $\mathscr{I}$ is *adequate* if its G-orbit is a partial B-oval; two involutions j, j' in $\mathscr{I}$ are said to be *compatible* if the union of the G-orbit of j with the G-orbit of j' is a partial B-oval. In other words, adequacy is a necessary condition for an involution to lie in a G-invariant B-oval and compatibility is a necessary condition for two involutions to lie in a G-invariant B-oval together.

The following property is very elementary, but it will be used over and over again to test adequacy and compatibility, respectively.

Proposition 10. *An involution j in $\mathscr{I}$ with G-conjugacy class Δ is adequate if and only if the relation $|\mathrm{Fix}(jk)| \leqslant 2$ holds for all $k \in \Delta \backslash \{j\}$. Two adequate involutions j, j' in distinct G-conjugacy classes Δ, Δ' are compatible if and only if the relation $|\mathrm{Fix}(jk')| \leqslant 2$ holds for all $k' \in \Delta'$.*

A counting argument based on Proposition 9 shows that there are 840 candidates for the involution j_P: the two fixed points of j_P are 15, 16; since P is on a secant through A, one of the 2-cycles of j_A other than (15, 16), say (W, X), must also be a 2-cycle of j_P; the remaining six 2-cycles of j_P are obtained by pairing the six 2-cycles of j_A different from (15, 16) and (W, X) in the manner described in Proposition 9. The conjugacy class under G of each such candidate has size 6, but it must be further tested whether such a conjugacy class is a partial B-oval. The GAP program that we have written for this purpose reveals that only 352 candidates are adequate. Each one of these 352 adequate candidates is compatible with j_A and we tested that the 352 partial B-ovals of size 9 arising in this manner fall into 17 classes of conjugate sets under S_{16}. The reconstruction process can therefore begin from a partial B-oval of size 9 chosen from a set of representatives for these 17 classes. For $i = 1, 2, \ldots, 17$ the ith choice for j_P is shown in the following table.

i	involution	i	involution
1.	(1, 2)(3, 5)(4, 6)(7, 10)(8, 11)(9, 13)(12, 14)	10.	(1, 2)(3, 9)(4, 6)(5, 12)(7, 14)(8, 10)(11, 13)
2.	(1, 2)(3, 5)(4, 6)(7, 10)(8, 11)(9, 14)(12, 13)	11.	(1, 2)(3, 13)(4, 6)(5, 14)(7, 12)(8, 10)(9, 11)
3.	(1, 2)(3, 5)(4, 6)(7, 12)(8, 13)(9, 11)(10, 14)	12.	(1, 4)(2, 5)(3, 6)(7, 8)(9, 13)(10, 11)(12, 14)
4.	(1, 2)(3, 7)(4, 6)(5, 11)(8, 10)(9, 13)(12, 14)	13.	(1, 4)(2, 5)(3, 6)(7, 9)(8, 13)(10, 14)(11, 12)
5.	(1, 2)(3, 7)(4, 6)(5, 11)(8, 10)(9, 14)(12, 13)	14.	(1, 4)(2, 5)(3, 6)(7, 9)(8, 14)(10, 13)(11, 12)
6.	(1, 2)(3, 7)(4, 6)(5, 11)(8, 12)(9, 10)(13, 14)	15.	(1, 4)(2, 8)(3, 9)(5, 12)(6, 10)(7, 13)(11, 14)
7.	(1, 2)(3, 9)(4, 6)(5, 12)(7, 11)(8, 13)(10, 14)	16.	(1, 4)(2, 8)(3, 9)(5, 12)(6, 10)(7, 14)(11, 13)
8.	(1, 2)(3, 9)(4, 6)(5, 12)(7, 11)(8, 14)(10, 13)	17.	(1, 4)(2, 8)(3, 11)(5, 7)(6, 10)(9, 13)(12, 14)
9.	(1, 2)(3, 9)(4, 6)(5, 12)(7, 13)(8, 10)(11, 14)		

Once j_P has been chosen we may assume that $\mathscr{F}$ contains the partial B-oval

$$\overline{\mathscr{F}} = \{j_A, j_B, j_C\} \cup \{g j_P g^{-1} : g \in G\} = \{j_A, j_B, j_C, j_P, j_Q, j_R, j_S, j_T, j_U\}.$$

We shall say for short that $\overline{\mathscr{F}}$ is our *prescribed* partial B-oval.

Our next step is the reconstruction of the remaining points on the sides of the fixed triangle ABC. We begin with internal points.

In order to reconstruct the involutions in the B-oval arising from the internal points on the line BC other than B and C, we observe by a direct counting argument that there are 1440 fixed-point-free involutions on Ω which have the form described by Proposition 8. We have written a GAP program that generates all of these candidates and tests each one of them to see if it is adequate and if its G-orbit has length six. Among the surviving candidates, those which are compatible with all the involutions in a given prescribed partial B-oval are singled out and representatives from distinct G-orbits are chosen. The next table gives the size of the resulting set of involutions for each prescribed partial B-oval (numbering as before):

i	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.	12.	13.	14.	15.	16.	17.
#	181	176	180	179	178	168	177	180	175	178	178	180	180	181	175	178	178

The six internal points on BC other than B, C can be divided into three pairs from distinct G-orbits. For $i = 1, 2, \dots, 17$ we denote by $\mathscr{T}^{(i)}$ the set of all triples of pairwise compatible involutions from the ith list described above. The GAP program that we have written for generating all such triples has returned the following cardinalities (numbering as before):

i	1.	2.	3.	4.	5.	6.	7.	8.	9.
#	17049	14053	16760	14030	13004	13269	14119	17317	13207
i		10.	11.	12.	13.	14.	15.	16.	17.
#		13007	12756	16760	14782	16480	12021	13707	12756

We now look at the external points on BC other than P and Q. The same arguments of Proposition 6 show that there exists an external point E_1 on BC such that the tangents through E_1 touch Ω at 1 and 4, respectively; similarly, there exists an external point E_2 on BC such that the tangents through E_2 touch Ω at 2 and 6, respectively; finally, there exists an external point E_3 on BC such that the tangents through E_3 touch Ω at 3 and 5, respectively.

For $r = 1, 2, 3$ we have $h_B(E_r) = h_C(E_r)$ and if we set $E'_r = h_B(E_r)$, then the points of contact with Ω of the two tangents through E'_r are the h_B-images of the corresponding points of contact of the two tangents through E_r. The previous observation, together with the relations $(j_B(1), j_B(4)) = (9, 12)$, $(j_B(2), j_B(6)) = (7, 11)$, $(j_B(3), j_B(5)) = (8, 10)$, shows that the points in the G-orbits of E_1, E_2 and E_3 account for all external points on the sides of the triangle ABC other than P, Q, R, S, T, U.

Another piece of the putative plane is reconstructed by considering for $i = 1, 2, \dots, 17$ the set $\mathscr{E}_r^{(i)}$ of all candidates for the involution j_{E_r}, $r = 1, 2, 3$, which are compatible with the ith prescribed partial B-oval. The GAP program that we have written for this purpose generates all the involutory permutations on Ω fixing the two points of contact and acting on the remaining elements in the manner described by Proposition 9. Compatibility with a given prescribed partial B-oval is also checked. The cardinalities are summarized in the following table (numbering as before):

i	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.	12.	13.	14.	15.	16.	17.
$r = 1$	224	224	228	224	224	264	232	234	232	234	228	246	251	251	249	249	246
$r = 2$	232	234	228	232	234	264	249	249	224	224	228	228	234	232	224	224	228
$r = 3$	249	249	246	251	251	264	224	224	249	249	246	228	224	224	232	234	228

In order to fully reconstruct the three lines AB, AC, BC we have to extend each triple from $\mathscr{T}^{(i)}$ by adding one involution from $\mathscr{E}_r^{(i)}$, $r = 1, 2, 3$, in such a way that the resulting sextuple consists of pairwise compatible involutions: once such a sextuple is found we obtain a partial B-oval of size 45 by taking the ith prescribed partial B-oval together with the G-conjugacy class of each involution in the sextuple. We denote by $\mathscr{V}^{(i)}$ the set of all such sextuples extending the ith prescribed partial B-oval, $i = 1, \dots, 17$.

Before we reach the final contradiction, we need to push our reconstruction process just one step further.

The G-orbits of the secant lines through A are easily found by looking at the G-orbits of unordered pairs of points in Ω occurring in one and the same 2-cycle of j_A. In particular, if we denote by s_1 the secant through A containing 1 and 4 and by s_2 the secant through A containing 2, 6, respectively, we see that s_1 and s_2 lie in two distinct G-orbits of length 6 each.

In much the same way as we have done before, we want to generate the candidates for the involutions arising from the internal points on the lines s_1 and s_2. One such candidate for s_1 must be the product of the 2-cycle $(1, 4)$ times a fixed-point-free involution on the remaining 14 elements of Ω: we have written a GAP program testing each one of the 135, 135 such involutions to see if it is adequate, if its G-orbit has length 12 and if it is compatible with a given prescribed partial B-oval. From the set of involutions passing all these tests we select representatives for the distinct G-conjugacy classes and end up with sets of candidates, the cardinalities of which are summarized in the following table (numbering as before):

i	1.	2.	3.	4.	5.	6.	7.	8.	9.
#	7776	7757	7640	7749	7741	7661	7812	7820	7828
i		10.	11.	12.	13.	14.	15.	16.	17.
#		7822	7648	4580	4624	4642	4655	4704	4828

A similar process for the secant s_2 yields the following cardinalities:

i	1.	2.	3.	4.	5.	6.	7.	8.	9.
#	7775	7758	7640	7755	7760	7661	4658	4852	7737
i		10.	11.	12.	13.	14.	15.	16.	17.
#		7743	7648	7645	7775	7751	7732	7758	7649

Since the six internal points on s_1 other than A lie in three distinct G-orbits, two points in each orbit, we have to look for triples of pairwise compatible involutions from s_1 which are compatible with the given prescribed partial B-oval. Similarly, we need the triples of pairwise compatible involutions from s_2 which are compatible with the given prescribed partial B-oval. In order to continue the reconstruction of the plane, we need to extend a given sextuple from $\mathscr{V}^{(i)}$ by adding a triple from s_1 and a triple from s_2 in such a way that the 12 involutions thus generated are pairwise compatible.

The final GAP program that we have written takes a triple in $\mathscr{T}^{(i)}$ and constructs all sextuples in $\mathscr{V}^{(i)}$ extending it. Then it generates the subset consisting of the involutions representing internal points on s_1 which are compatible with each involution in a given sextuple. The corresponding operation is done for s_2. Triples of pairwise compatible involutions are generated within these smaller subsets. In order to obtain the required list of 12 involutions extending the original sextuple we have to see if it is possible to match one of the generated triples from s_1 with one of the generated triples from s_2, in such a way that each involution in the first triple is compatible with each involution in second triple. The program revealed that no such matching is possible for any one of the sextuples in $\mathscr{V}^{(i)}$, $i = 1, 2, \ldots, 17$.

Our reconstruction process has thus come to a dead end: the projective B-oval with the required properties does not exist.

4. Final remark

The GAP programs described in this paper ran on different PC's under GAP 4.2 for Windows. For each one of the 17 prescribed B-ovals several days of CPU time were required to carry out all the calculations. GAP source code can be obtained from either author by E-mail.

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