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Local Lipschitz continuity for energy integrals with slow growth and lower order terms

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ABSTRACT

We consider integral functionals with slow growth and explicit dependence on u of the Lagrangian; this includes many relevant examples as, for instance, in elastoplastic torsion problems or in image restoration problems. Our aim is to prove that the local minimizers are locally Lipschitz continuous. The proof makes use of recent results concerning the Bounded Slope Conditions.

1. Introduction and statement of the main result

Nowadays there is renewed interest regarding Lipschitz regularity results for local minimizers of integral functionals or weak solutions to a class of nonlinear elliptic partial differential equations in divergence form with non-standard growth conditions, see for example the recent contributions [1–10]. Our paper fits into this research line, i.e. with the present paper our aim is to prove local Lipschitz regularity results for integral functionals of the type

$$\mathcal{F}(u) = \int_\Omega f(Du) + g(x,u)\,dx. \tag{1.1}$$

We emphasize our interest in dealing with the explicit dependence on u of the Lagrangian. This includes many significant functionals involved, for instance, in elastoplastic torsion problems or in image restoration problems (see [11] for explicit examples). Moreover this class of functionals has been already considered in literature, see for instance [6,12] concerning regularity of local minimizers of a class of integrals of the Calculus of Variations, see also [13] where the functionals considered do not necessarily satisfy the Euler–Lagrange equation. On the other hand, in [14] the motivation to introduce an explicit u-dependence on the coefficients in the differential equation comes from several recent studies on nonlinear elliptic and parabolic equations with general growth conditions.

In this work we have been inspired by the papers [8,15] dealing with functionals depending only on Du with general growth assumptions, respectively fast and slow. In both papers the authors prove suitable a priori estimates and then apply classical results on the Bounded Slope Condition to get the local Lipschitz continuity. We generalize these techniques in order to include also the lower order terms. To this aim we need to exploit some recent results that extend the classical ones on the Bounded Slope Conditions (BSC) to our type of functionals [11,16]. We also recall some related results in [17,18].

Our aim is to study the regularity of local minimizers of the functional (1.1) and, as it is commonly used in literature, we say that $u \in W^{1,1}_{loc}(\Omega)$ is a *local minimizer* of the integral functional $\mathcal{F}$ in (1.1) if $f(Du) + g(\cdot,u) \in L^1_{loc}(\Omega)$ and

$$\int_{\Omega'} f(Du) + g(x,u)\,dx \le \int_{\Omega'} f(Du + D\varphi) + g(x, u+\varphi)\,dx$$

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for every open set Ω', $\overline{\Omega'} \subset \Omega$ and for every $\varphi \in W_0^{1,1}(\Omega')$.

In order to state the main result of the paper, we need to introduce the assumptions on the Lagrangian. Let $f : \mathbb{R}^n \to [0, +\infty)$ be a convex function in $C(\mathbb{R}^n) \cap C^2(\mathbb{R}^n \setminus B_{t_0}(0))$ for some $t_0 \geq 0$, satisfying the following growth conditions: there exist two continuous functions $h_1, h_2 : [t_0, +\infty) \to (0, +\infty)$ and positive constants C_1, C_2, α, β and $\mu \in [0, 1]$ such that

- (F1) $h_1(|\xi|)\,|\lambda|^2 \leq \sum_{i,j=1}^n f_{\xi_i\xi_j}(\xi)\,\lambda_i\lambda_j \leq h_2(|\xi|)\,|\lambda|^2, \ \forall \lambda, \xi \in \mathbb{R}^n, \ |\xi| \geq t_0$;
- (F2) $t \mapsto t^\mu h_2(t)$ is decreasing and $t \mapsto t h_1(t)$ is increasing;
- (F3) $\left(h_2(t)\right)^{2/2^*} \leq C_1 t^{2\beta} h_1(t), \quad \beta < \frac{2}{n}, \ \forall\, t \geq t_0$;
- (F4) $h_2(|\xi|)|\xi|^2 \leq C_2\,[1 + f(\xi)]^\alpha, \ \ \alpha > 1, \ \forall \xi \in \mathbb{R}^n, \ |\xi| \geq t_0$;

where, in (F3), $2^* = \frac{2n}{n-2}$ if $n \geq 3$ while in the case $n = 2$, it must be replaced with any fixed positive number greater than $\frac{2}{1-\beta}$.

Moreover we assume that $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying

- (G1) there exists L such that $|g(x, \eta_1) - g(x, \eta_2)| \leq L|\eta_1 - \eta_2|$ for a.e. x in Ω and every $\eta_1, \eta_2 \in \mathbb{R}$;
- (G2) $g(\cdot, 0) \in L^1_{loc}(\Omega)$;
- (G3) $u \mapsto g(x, u)$ is convex for a.e. $x \in \Omega$;
- (G4) there exists a positive constant K such that for a.e. $x, y \in \mathbb{R}^n$, $\forall u, v \in \mathbb{R}$

$$v \geq u + K|y - x| \Rightarrow g_v^+(y, v) \geq g_v^+(x, u)$$

where g_v^+ denotes the right derivative of g with respect to the second variable.

Condition (F2) means that we are considering *slow growth conditions.* Indeed in the model case of p, q-growth, the map $t \mapsto t^\mu h_2(t)$ is decreasing if and only if $q \leq 2$. We notice that (F2) is inspired by similar assumptions that appear in [8]; however here the monotonicity of $t \mapsto t h_1(t)$ implies that $f(\xi)$ grows at least as $|\xi| \log |\xi|$ while in [8] there was no growth requirement from below, therefore the additional assumption of superlinearity was needed in order to perform the approximation argument.

It is moreover worth to highlight that we require uniform convexity and growth assumptions on $f = f(\xi)$ only for large values of $|\xi|$ [7,19–22].

We finally point out that the assumptions on g are needed as far as we have to use the results on the Bounded Slope Condition for functionals depending also on (x, u) and we refer to [11,16,23] for more details. We will discuss the meaning of this set of assumptions in Section 5.

The main result of the paper can be stated as follows. Notice that, here and in the sequel, we denote by B_R a generic ball of radius R compactly contained in Ω and by B_ρ a ball of radius $\rho < R$ concentric with B_R.

Theorem 1.1. *Let $u \in W^{1,1}_{loc}(\Omega) \cap L^\infty_{loc}(\Omega)$ be a local minimizer of the functional (1.1). Suppose that f satisfies the growth assumptions* (F1)–(F4)*, with the parameters α, β, μ related by the condition*

$$2 - \mu - \alpha(n\beta - \mu) > 0\,. \tag{1.2}$$

Assume moreover that g fulfills assumptions (G1)–(G4).

Then u is locally Lipschitz continuous in Ω and there exists $\bar{R} > 0$ such that for every $0 < \rho < R < \bar{R}$, there exist two positive constants C and κ depending on the data of the problem, with C depending also on κ, such that

$$\|Du\|_{L^\infty(B_\rho;\mathbb{R}^n)} \leq C \left\{ \frac{1}{(R-\rho)^n} \left(\int_{B_R} f(Du) + g(x, u)\, dx + \kappa \right) \right\}^\theta \tag{1.3}$$

where $\theta = \frac{(2-\mu)\alpha}{2-\mu-\alpha(n\beta-\mu)}$.

The paper is organized as follows. In Section 2 are listed some notations and preliminary results. In particular we notice the use of the polar function of f to construct suitable barriers for the minimizers. These barriers have been introduced by Cellina in [24] and satisfy a Comparison Principle [16]. Section 3 contains the proof of the a priori estimate, namely Theorem 3.1; this proof is new since here we have to deal also with lower order terms. We also notice that we proved it under assumptions on f that are slightly different from those in [8]. This is due to the fact that, in subsequent Section 4, we have to construct barriers for the local minimizer. We underline that, as far as the a priori estimate is concerned, the proof still holds also under the same assumptions of f of [8]; in particular condition (1.2) is the same of [8, Theorem 2.1] and it is not affected by the presence of the lower order term g.

Section 4 is devoted to the proof of Theorem 1.1, which is divided in three steps: approximation; estimates for the boundedness of the gradient; passage to the limit. In this last part the crucial tools are the results concerning the (BSC) that still hold for a more general functional explicitly depending on x and u. Here assumption (F2) plays a crucial role. Finally Section 5 aims to present some additional results related to Theorem 1.1. In Theorem 5.1 we extend the main result to the case in which assumption (G1) is replaced by a local Lipschitz condition in the second variable. In Theorem 5.2 we present a regularity result for a Dirichlet problem in which it is not necessary to assume the a priori boundedness of the local minimizer. Finally, Theorem 5.3 shows how, in the case the Lagrangian depends on the modulus of the gradient, we allow slower growth conditions, more precisely condition (1.2) is always satisfied.

We conclude this introduction with some remarks showing a comparison between our results and the classical non-standard growth conditions.

We begin with the widely studied case of (p,q)–growth, namely

$$h_1(t) \sim t^{p-2} \qquad h_2(t) \sim t^{q-2};$$

in this case we obtain the usual gap condition

$$\frac{q}{p} < 1 + \frac{2}{n}.$$

On the other hand, if f is strongly anisotropic as, for example,

$$f(\xi) = \sum_{i=1}^{n} \left(1+\xi_i^2\right)^{\frac{p_i}{2}}, \quad f(\xi) = \sqrt{\sum_{i=1}^{n} \left(1+|\xi_i|^2\right)^{p_i}}, \quad f(\xi) = |\xi|^p + \sqrt{\sum_{i=1}^{n} |\xi_i|^{2p_i}},$$

then we obtain, even in the presence of lower order terms, the regularity with the same conditions on the exponents p_i, obtained in [8, Examples 3.3, 4.1 and 4.4].

As we previously mentioned, assumption (F2) is crucial to construct barriers for a minimizer of an approximated problem. This, in particular, requires to evaluate the principal curvatures of the boundary of suitable sets, see (4.6). In the case where f is radially symmetric we obtain better estimates on the curvatures and this allows us to deal also with very slow growth. For example, in this case we cover also the case $f(\xi) = (|\xi|+1)L_k(|\xi|)$, $k \in \mathbb{N}$, L_k defined inductively as

$$L_1(t) = \log(1+t), \qquad L_{k+1}(t) = \log\left(1 + L_k(t)\right),$$

(see also [25]). As a final remark we notice that, still in the radially symmetric case, for the (p,q)-growth, we obtain, as in [26], the local Lipschitz regularity of the minimizers if

$$\frac{q}{p} < \frac{n}{n-2}.$$

2. Notations and preliminary results

Let $f : \mathbb{R}^n \to [0,+\infty)$ be a convex function. we denote $\partial f(x)$ the subdifferential of f at the point x. We indicate by f^* the polar, or Fenchel transform of f, see [27], defined by

$$f^*(x) := \sup_{\xi\in\mathbb{R}^n} \{x\cdot\xi - f(\xi)\}, \quad \forall x \in \mathbb{R}^n.$$

We list here some properties that will be useful in the rest of the paper.

Proposition 2.1. *Let $f : \mathbb{R}^n \to [0,+\infty)$ be a convex function. Therefore:*

(i) if f is superlinear, then $f^*(x) \in \mathbb{R}$ for every $x \in \mathbb{R}^n$;
(ii) if f is such that $f(\xi) \in \mathbb{R}$ for every $\xi \in \mathbb{R}^n$, then $f^*(x)$ is superlinear;
(iii) $\partial f(\xi) = (\partial f^*)^{-1}(\xi)$ for every $\xi \in \mathbb{R}^n$;
(iv) if f is superlinear $\partial f^*(x) = (\partial f)^{-1}(x)$ for every $x \in \mathbb{R}^n$;
(v) if f is in $C(\mathbb{R}^n) \cap C^2(\mathbb{R}^n \setminus B_{t_0}(0))$ for some $t_0 \geq 0$, superlinear and satisfies assumption (F1), then there exists $s_0 \in \mathbb{R}$, such that $f^* \in C^2(\mathbb{R}^n \setminus B_{s_0}(0))$. Moreover, for every $x \in \mathbb{R}^n \setminus B_{s_0}(0)$, $D^2 f^*(x) = \left(D^2 f\right)^{-1}(Df^*(x))$ and

$$\frac{1}{h_2(|Df^*(x)|)}|\lambda|^2 \leq \sum_{i,j=1}^{n} f^*_{x_i x_j}(x)\lambda_i\lambda_j \leq \frac{1}{h_1(|Df^*(x)|)}|\lambda|^2, \ \forall \lambda, x \in \mathbb{R}^n, \ |x| \geq s_0. \tag{2.1}$$

Proof. Statements (i) and (ii) follow from Lemma 3.1 in [11], observing that it is not restrictive to assume that $f(0) = 0$. Properties (iii) and (iv) are proved in [27, Theorem 11.3] and (v) is a consequence of [27, Theorem 13.21] and subsequent observation. □

The next two lemmas will play an important role in the third step of the proof of the main theorem.

Lemma 2.2. *Let A be an open bounded subset of $\mathbb{R}^n$ with regular boundary. Fix $\varphi \in L^\infty(A)$ and assume that $f : \mathbb{R}^n \to \mathbb{R}$ is convex and superlinear. Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying (G1). Let u be a minimizer of*

$$\int_A f(Du) + g(x,u)\,dx$$

in the class $\varphi + W_0^{1,1}(A)$. Then u is essentially bounded on A.

Proof. We start proving that there exists a constant K^- such that $u(x) \geq K^-$ a.e. on A. We fix $x_0 \in \mathbb{R}^n$ and we consider the function

$$\omega_L(x) = \frac{n}{L} f^*\left(\frac{L}{n}(x-x_0)\right) + \inf_{\partial A} \varphi(x) - \sup_{\partial A} \frac{n}{L} f^*\left(\frac{L}{n}(x-x_0)\right) \tag{2.2}$$

and we observe that, thanks to Proposition 2.1(i), it is well defined for every $x \in \mathbb{R}^n$. Moreover the definition of ω_L implies that

$$\omega_L(x) \le \varphi(x) \text{ on } \partial A$$

so that Theorem 2.4 in [16], see also Theorem 2.4 in [11], implies that then

$$u(x) \ge K^- = \inf_A \omega_L(x) = \inf_A \frac{n}{L} f^*\left(\frac{L}{n}(x-x_0)\right) + \inf_{\partial A} \varphi(x) - \sup_{\partial A} \frac{n}{L} f^*\left(\frac{L}{n}(x-x_0)\right)$$

a.e. in A. The proof of the fact that there exists K^+ such that $u(x) \le K^+$ a.e. in A follows in an analogous way using the function

$$\omega_{-L}(x) = -\frac{n}{L} f^*\left(-\frac{L}{n}(x-x_0)\right) + \sup_{\partial A} \varphi(x) - \inf_{\partial A}\left(-\frac{n}{L} f^*\left(-\frac{L}{n}(x-x_0)\right)\right) \tag{2.3}$$

and choosing

$$K^+ = \sup_A \omega_{-L}(x) = \sup_A \left(-\frac{n}{L} f^*\left(-\frac{L}{n}(x-x_0)\right)\right) + \sup_{\partial A} \varphi(x) - \inf_{\partial A}\left(-\frac{n}{L} f^*\left(-\frac{L}{n}(x-x_0)\right)\right). \quad \square$$

Before stating the next lemma, we need to recall the following classical definition.

Definition 2.3 (*BSC*)**.** The function ϕ satisfies the *Bounded Slope Condition* of rank $m \ge 0$ if for every $\gamma \in \partial\Omega$ there exist z_γ^-, $z_\gamma^+ \in \mathbb{R}^n$ and $m \in \mathbb{R}$ such that

$$\forall \gamma' \in \partial\Omega \quad \phi(\gamma) + z_\gamma^- \cdot (\gamma' - \gamma) \le \phi(\gamma') \le \phi(\gamma) + z_\gamma^+ \cdot (\gamma' - \gamma) \tag{2.4}$$

and $|z_\gamma^\pm| \le m$ for every $\gamma \in \partial\Omega$.

Lemma 2.4. *Assume that $f_-, \tilde{f}, f_+ : \mathbb{R}^n \to \mathbb{R}$ are convex and superlinear. Moreover assume that*

$$f_-(\xi) \le \tilde{f}(\xi) \le f_+(\xi). \tag{2.5}$$

Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ a Carathéodory function satisfying (G1). *Let φ satisfy the* (BSC) *and $\tilde{u}$ be a minimizer of*

$$\int_{B_R} \tilde{f}(Dv) + g(x,v)\,dx \qquad v \in \varphi + W_0^{1,1}(B_R). \tag{2.6}$$

Then there exists $\bar{K} = \bar{K}(L, f_-, f_+, \|\varphi\|_{L^\infty(B_R)})$ such that $\|\tilde{u}\|_{L^\infty(B_R)} \le \bar{K}$.

Proof. Assumption (2.5) implies that

$$f_+^*(\xi) \le \tilde{f}^*(\xi) \le f_-^*(\xi). \tag{2.7}$$

As in the proof of Lemma 2.2 we obtain

$$\tilde{u}(x) \ge \frac{n}{L}\tilde{f}^*\left(\frac{L}{n}(x-x_0)\right) + \inf_{\partial B_R} \varphi(x) - \sup_{\partial B_R} \frac{n}{L}\tilde{f}^*\left(\frac{L}{n}(x-x_0)\right) \ge \frac{n}{L} f_+^*\left(\frac{L}{n}(x-x_0)\right) - \|\varphi\|_{L^\infty(B_R)} - \sup_{\partial B_R} \frac{n}{L} f_-^*\left(\frac{L}{n}(x-x_0)\right).$$

Similar computation yields the inequality from above. $\square$

3. A priori estimates

In this section we prove a result that is in the same flavor of [8, Lemma 6.2]; we underline that here we deal also with lower order terms. For this reason in the proof we will highlight only the main technical points that arise from the different structure of the functional.

Theorem 3.1. *Suppose that f satisfies the growth assumptions* (F1)–(F4) *with the parameters α, β, μ related by* (1.2). *In addition, assume that f is of class $C^2(\mathbb{R}^n)$ and for every $M > 0$ there exists a positive constant $\ell = \ell(M)$ such that*

$$\ell\,|\lambda|^2 \le \sum_{i,j=1}^n f_{\xi_i\xi_j}(\xi)\,\lambda_i\,\lambda_j \qquad \forall \lambda, \xi \in \mathbb{R}^n,\ |\xi| \le M. \tag{3.1}$$

Assume moreover that g fulfills assumptions (G1).

Let $u \in W_{loc}^{1,\infty}(\Omega)$ be a local minimizer of the functional (1.1). *Then for every $R > 0$ sufficiently small and $0 < \rho < R$ there exists a positive constant C depending on C_1, C_2, α, β, μ, $h_1(t_0)$, L, such that*

$$\|Du\|_{L^\infty(B_\rho;\mathbb{R}^n)} \le C\left\{\frac{1}{(R-\rho)^n}\int_{B_R}\{1 + f(Du)\}\,dx\right\}^\theta \tag{3.2}$$

where $\theta = \frac{(2-\mu)\alpha}{2-\mu-\alpha(n\beta-\mu)}$.

Remark 3.2. It is worth noticing that Theorem 3.1 is also valid when assumption (F2) is replaced by

(F2)′ $t \mapsto t^{\mu} h_2(t)$ is decreasing and $t \mapsto t h_2(t)$ is increasing.

Similarly the following Lemma is analogous to Lemma 6.1 in [8], where we use (F2) while in [8] it is assumed (F2)′. The proofs are omitted since they require only minor changes.

For the sake of simplicity we assume, in the remaining part of the section, that $t_0 = 1$.

Lemma 3.3. *Let us assume that* (F2) *and* (F3) *hold. Then for every* $\gamma \geq 0$ *there exists a constant* $C_3 = C_3(C_1, h_1(1)) > 0$ *independent of* γ*, such that, for every* $t \geq 0$

$$C_3 \left[1 + h_2(1+t)^{\frac{1}{2^*}} \frac{(1+t)^{\frac{\gamma}{2}+1-\beta}}{\left(\frac{\gamma}{2}+1-\beta\right)^2} \right] \leq 1 + \int_0^t (1+s)^{\frac{\gamma-2}{2}} s \sqrt{h_1(1+s)}\, ds. \tag{3.3}$$

Now we proceed with the proof of Theorem 3.1.

Proof. Since the local minimizer u is in $W^{1,\infty}_{loc}(\Omega)$, it satisfies the Euler equation: for every open set Ω' compactly contained in Ω we have

$$\int_\Omega \sum_{i=1}^n f_{\xi_i}(Du)\, \varphi_{x_i} + g_u(x,u)\, \varphi\, dx = 0 \qquad \forall \varphi \in W^{1,2}_0(\Omega').$$

Moreover, by the techniques of the difference quotient (see for example [28, Theorem 1.1′, Ch. II]), $u \in W^{2,2}_{\mathrm{loc}}(\Omega)$, then the second variation holds:

$$\int_\Omega \sum_{i,j=1}^n f_{\xi_i \xi_j}(Du) u_{x_j x_k} \varphi_{x_i} - g_u(x,u) \varphi_{x_k}\, dx = 0, \quad \forall k = 1, \dots, n \tag{3.4}$$

$\forall \varphi \in W^{1,2}_0(\Omega')$.

For fixed $k = 1, \dots, n$ let $\eta \in C^1_0(\Omega')$ be equal to 1 in B_ρ, with support contained in B_R, such that $|D\eta| \leq \frac{2}{(R-\rho)}$, and consider

$$\varphi = \eta^2\, u_{x_k}\, \Phi((|Du|-1)_+)$$

with Φ non negative, increasing, locally Lipschitz continuous on $[0, +\infty)$, such that $\Phi(0) = 0$. Here $(a)_+$ denotes the positive part of $a \in \mathbb{R}$; in the following we denote $\Phi((|Du|-1)_+) = \Phi(|Du|-1)_+$. Then a.e. in Ω

$$\varphi_{x_i} = 2\eta\, \eta_{x_i} u_{x_k} \Phi(|Du|-1)_+ + \eta^2 u_{x_i x_k} \Phi(|Du|-1)_+ + \eta^2 u_{x_k} \Phi'(|Du|-1)_+ [(|Du|-1)_+]_{x_i}.$$

Proceeding along the lines of [15], see also [8], we therefore deduce that

$$\begin{aligned}
0 = & \int_\Omega 2\eta \Phi(|Du|-1)_+ \sum_{i,j=1}^n \eta_{x_i} u_{x_k} f_{\xi_i \xi_j}(Du) u_{x_j x_k}\, dx \\
& - \int_\Omega 2\eta \Phi(|Du|-1)_+ \eta_{x_k} u_{x_k} g_u(x,u)\, dx \\
& + \int_\Omega \eta^2 \Phi(|Du|-1)_+ \sum_{i,j=1}^n f_{\xi_i \xi_j}(Du) u_{x_j x_k} u_{x_i x_k}\, dx \\
& - \int_\Omega \eta^2 \Phi(|Du|-1)_+ u_{x_k x_k} g_u(x,u)\, dx \\
& + \int_\Omega \eta^2 \Phi'(|Du|-1)_+ \sum_{i,j=1}^n f_{\xi_i \xi_j}(Du) u_{x_j x_k} u_{x_k} [(|Du|-1)_+]_{x_i}\, dx \\
& - \int_\Omega \eta^2 \Phi'(|Du|-1)_+ g_u(x,u) u_{x_k} [(|Du|-1)_+]_{x_k}\, dx \\
= & \; I_{1k} - I_{2k} + I_{3k} - I_{4k} + I_{5k} - I_{6k}.
\end{aligned}$$

We now sum the previous equation with respect to k from 1 to n, and we denote by I_1–I_6 the corresponding integrals.

First of all we have

$$I_3 + I_5 \leq |I_1| + |I_2| + |I_4| + |I_6|. \tag{3.5}$$

We start by estimating $|I_1|$ by using the Cauchy–Schwarz inequality and the Young inequality so that

$$
\begin{aligned}
|I_1| &= \left| \int_\Omega 2\eta\Phi(|Du|-1)_+ \sum_{i,j,k=1}^n f_{\xi_i\xi_j}(Du) u_{x_j x_k} \eta_{x_i} u_{x_k}\, dx \right| \\
&\le \int_\Omega 2\Phi(|Du|-1)_+ \left(\eta^2 \sum_{i,j,k=1}^n f_{\xi_i\xi_j}(Du) u_{x_i x_k} u_{x_j x_k} \right)^{\frac12} \left(\sum_{i,j,k=1}^n f_{\xi_i\xi_j}(Du) \eta_{x_i} u_{x_k} \eta_{x_j} u_{x_k} \right)^{\frac12} dx \\
&\le \frac12 \int_\Omega \eta^2 \Phi(|Du|-1)_+ \sum_{i,j,k=1}^n f_{\xi_i\xi_j}(Du) u_{x_i x_k} u_{x_j x_k}\, dx + 2\int_\Omega \Phi(|Du|-1)_+ \sum_{i,j,k=1}^n f_{\xi_i\xi_j}(Du) \eta_{x_i} u_{x_k} \eta_{x_j} u_{x_k}\, dx \\
&\le \frac12 \int_\Omega \eta^2 \Phi(|Du|-1)_+ \sum_{i,j,k=1}^n f_{\xi_i\xi_j}(Du) u_{x_i x_k} u_{x_j x_k}\, dx + 2\int_\Omega |D\eta|^2\, \Phi(|Du|-1)_+ h_2(1+(|Du|-1)_+)\,(1+(|Du|-1)_+)^2\, dx.
\end{aligned}
$$

On the other hand, by (F2), recalling that, for $t \ge 1$, $h_2(t) \ge h_1(t) \ge\ h_1(1)/t$, we deduce

$$
\begin{aligned}
|I_2| &= \left| \int_\Omega 2\eta\Phi(|Du|-1)_+ \sum_{k=1}^n \eta_{x_k} u_{x_k} g_u(x,u)\, dx \right| \\
&\le L \int_\Omega (\eta^2 + |D\eta|^2)|Du|\, \Phi(|Du|-1)_+\, dx \\
&\le \frac{L}{h_1(1)} \int_\Omega (\eta^2 + |D\eta|^2)\Phi(|Du|-1)_+ h_2(|Du|)\, |Du|^2\, dx \\
&= \frac{L}{h_1(1)} \int_\Omega (\eta^2 + |D\eta|^2)\Phi(|Du|-1)_+ h_2(1+(|Du|-1)_+)\,(1+(|Du|-1)_+)^2\, dx.
\end{aligned}
$$

Now we estimate the term $|I_4|$. Taking into account that, for $t \ge 1$,

$$
h_1(t)\, h_2(t)\, t^2 \ge h_1(t)^2 t^2 \ge h_1(1)^2 \tag{3.6}
$$

we have

$$
\begin{aligned}
|I_4| &= \left| \int_\Omega \eta^2 \Phi(|Du|-1)_+ \sum_{k=1}^n u_{x_k x_k} g_u(x,u)\, dx \right| \\
&\le \frac{nL}{h_1(1)} \int_\Omega \left[\eta^2 |D^2u|^2 \Phi(|Du|-1)_+ h_1(1+(|Du|-1)_+)\right]^{\frac12} \left[\eta^2 \Phi(|Du|-1)_+ h_2(1+(|Du|-1)_+)(1+(|Du|-1)_+)^2\right]^{\frac12}\, dx \\
&\le \varepsilon \int_\Omega \eta^2 \Phi(|Du|-1)_+ h_1(1+(|Du|-1)_+)|D^2u|^2\, dx + \frac{n^2L^2}{4h_1(1)^2\varepsilon} \int_\Omega \eta^2 \Phi(|Du|-1)_+ h_2(1+(|Du|-1)_+)(1+(|Du|-1)_+)^2\, dx,
\end{aligned}
$$

where ε is a positive parameter that will be suitably chosen later.

We then estimate $|I_6|$ as follows

$$
\begin{aligned}
|I_6| &= \left| \int_\Omega \eta^2 \Phi'(|Du|-1)_+ g_u(x,u) \sum_{k=1}^n u_{x_k} [(|Du|-1)_+]_{x_k}\, dx \right| \le\ L \int_\Omega \eta^2 \Phi'(|Du|-1)_+ (1+(|Du|-1)_+)|D(|Du|-1)_+|\, dx \\
&\overset{(3.6)}{\le} \frac{L}{h_1(1)} \int_\Omega \eta^2 \Phi'(|Du|-1)_+ (1+(|Du|-1)_+)|D(|Du|-1)_+| \sqrt{h_1(1+(|Du|-1)_+)\, h_2(1+(|Du|-1)_+)}(1+(|Du|-1)_+)\, dx \\
&\le \frac{L}{h_1(1)} \int_\Omega \left[\ \eta^2 |D(|Du|-1)_+|^2 \Phi'(|Du|-1)_+ (1+(|Du|-1)_+) h_1(1+(|Du|-1)_+)\ \right]^{\frac12} \\
&\quad \times \left[\ \eta^2 \Phi'(|Du|-1)_+ (1+(|Du|-1)_+) h_2(1+(|Du|-1)_+)(1+(|Du|-1)_+)^2\ \right]^{\frac12}\, dx \\
&\le \varepsilon \int_\Omega \eta^2 \Phi'(|Du|-1)_+ (1+(|Du|-1)_+) h_1(1+(|Du|-1)_+)|D(|Du|-1)_+|^2\, dx \\
&\quad + \frac{L^2}{4h(1)^2\varepsilon} \int_\Omega \eta^2 \Phi'(|Du|-1)_+ (1+(|Du|-1)_+) h_2(1+(|Du|-1)_+)(1+(|Du|-1)_+)^2\, dx.
\end{aligned}
$$

Finally, since a.e. in Ω

$$
[(|Du|-1)_+]_{x_i} = \begin{cases} (|Du|)_{x_i} = \frac{1}{|Du|} \sum_k u_{x_i x_k} u_{x_k} & \text{if } |Du| > 1, \\ 0 & \text{if } |Du| \le 1, \end{cases}
$$

we obtain

$$I_5=\int_\Omega \eta^2\Phi'(|Du|-1)_+\sum_{i,j,k=1}^n f_{\xi_i\xi_j}(Du)u_{x_jx_k}u_{x_k}[(|Du|-1)_+]_{x_i}\,dx$$
$$=\int_\Omega \eta^2\Phi'(|Du|-1)_+|Du|\sum_{i,j=1}^n f_{\xi_i\xi_j}(Du)[(|Du-1|)_+]_{x_j}[(|Du|-1)_+]_{x_i}\,dx.$$

Inserting the estimates obtained in (3.5), we deduce

$$\begin{aligned}
&\int_\Omega \eta^2\Phi(|Du|-1)_+\sum_{i,j,k=1}^n f_{\xi_i\xi_j}(Du)u_{x_jx_k}u_{x_ix_k}\,dx+\int_\Omega \eta^2\Phi'(|Du|-1)_+|Du|\sum_{i,j=1}^n f_{\xi_i\xi_j}(Du)[(|Du-1|)_+]_{x_j}[(|Du|-1)_+]_{x_i}\,dx\\
&\le \frac12\int_\Omega \eta^2\Phi(|Du|-1)_+\sum_{i,j,k=1}^n f_{\xi_i\xi_j}(Du)u_{x_ix_k}u_{x_jx_k}\,dx+2\int_\Omega |D\eta|^2\,\Phi(|Du|-1)_+h_2(1+(|Du|-1)_+)\,(1+(|Du|-1)_+)^2\,dx\\
&+\frac{L}{h_1(1)}\int_\Omega(\eta^2+|D\eta|^2)\Phi(|Du|-1)_+h_2(1+(|Du|-1)_+)\,(1+(|Du|-1)_+)^2\,dx+\varepsilon\int_\Omega \eta^2\Phi(|Du|-1)_+h_1(1+(|Du|-1)_+)|D^2u|^2\,dx\\
&+\frac{L^2}{4h(1)^2\varepsilon}\int_\Omega \eta^2\Phi(|Du|-1)_+h_2(1+(|Du|-1)_+)(1+(|Du|-1)_+)^2\,dx\\
&+\varepsilon\int_\Omega \eta^2\Phi'(|Du|-1)_+(1+(|Du|-1)_+)h_1(1+(|Du|-1)_+)|D(|Du|-1)_+|^2\,dx\\
&+\frac{L^2}{4h(1)^2\varepsilon}\int_\Omega \eta^2\Phi'(|Du|-1)_+(1+(|Du|-1)_+)h_2(1+(|Du|-1)_+)(1+(|Du|-1)_+)^2\,dx
\end{aligned}$$

Absorbing the first term in the right side of the inequality by the left hand side and rearranging the terms in the right hand side, we deduce

$$\begin{aligned}
&\frac12\int_\Omega \eta^2\Phi(|Du|-1)_+\sum_{i,j,k=1}^n f_{\xi_i\xi_j}(Du)u_{x_jx_k}u_{x_ix_k}\,dx+\int_\Omega \eta^2\Phi'(|Du|-1)_+|Du|\sum_{i,j=1}^n f_{\xi_i\xi_j}(Du)[(|Du|-1)_+]_{x_j}[(|Du|-1)_+]_{x_i}\,dx\\
&\le \varepsilon\int_\Omega \eta^2\Phi(|Du|-1)_+h_1(1+(|Du|-1)_+)|D^2u|^2\,dx\\
&+\varepsilon\int_\Omega \eta^2\Phi'(|Du|-1)_+(1+(|Du|-1)_+)h_1(1+(|Du|-1)_+)|D(|Du|-1)_+|^2\,dx\\
&+\frac{C}{\varepsilon}\int_\Omega(\eta^2+|D\eta|^2)\Phi(|Du|-1)_+h_2(1+(|Du|-1)_+)(1+(|Du|-1)_+)^2\,dx\\
&+\frac{C}{\varepsilon}\int_\Omega(\eta^2+|D\eta|^2)\Phi'(|Du|-1)_+(1+(|Du|-1)_+)\,h_2(1+(|Du|-1)_+)(1+(|Du|-1)_+)^2\,dx
\end{aligned}$$

where C is a constant depending only on $n, L, h_1(1)$. In the sequel we denote by C a constant depending only on $n, L, h_1(1)$, not necessarily the same constant. Using the ellipticity condition in (F1) and the inequality $|D(|Du|-1)_+|^2 \le |D^2u|^2$, choosing ε sufficiently small, we then obtain

$$\begin{aligned}
&\int_\Omega \eta^2\Phi(|Du|-1)_+h_1(1+(|Du|-1)_+)\,|D(|Du|-1)_+|^2\,dx\\
&\le \int_\Omega \eta^2[\Phi(|Du|-1)_++|Du|\Phi'(|Du|-1)_+]h_1(1+(|Du|-1)_+)\,|D(|Du|-1)_+|^2\,dx\\
&\le C\int_\Omega(\eta^2+|D\eta|^2)\Phi(|Du|-1)_+h_2(1+(|Du|-1)_+)(1+(|Du|-1)_+)^2\,dx\\
&+C\int_\Omega(\eta^2+|D\eta|^2)\Phi'(|Du|-1)_+(1+(|Du|-1)_+)h_2(1+(|Du|-1)_+)(1+(|Du|-1)_+)^2\,dx
\end{aligned}\tag{3.7}$$

Let us define

$$G(t)=1+\int_0^t\sqrt{\Phi(s)\,h_1(1+s)}\,ds\qquad\forall\, t\ge 0.\tag{3.8}$$

By Jensen's inequality and the monotonicity of Φ, since $t\mapsto th_1(t)$ is increasing,

$$G(t)=1+\int_0^t\sqrt{\Phi(s)(1+s)h_1(1+s)\frac{1}{1+s}}\,ds\le 1+\sqrt{\Phi(t)(1+t)h_1(1+t)}\int_0^t\frac{1}{\sqrt{1+s}}\,ds\le 1+2\sqrt{\Phi(t)(1+t)h_1(1+t)}\,\sqrt{1+t},$$

hence, recalling that $h_1\le h_2$

$$[G(t)]^2\le 8\left[1+\Phi(t)(1+t)^2h_1(1+t)\right]\le 8\left[1+\Phi(t)(1+t)^2h_2(1+t)\right].$$

On the other hand

$$\begin{aligned}
&|D[\eta\, G(|Du|-1)_+]|^2\le 2\,|D\eta|^2[G((|Du|-1)_+)]^2+2\eta^2[G'((|Du|-1)_+)]^2\,|D((|Du|-1)_+)|^2\\
&\le 16\,|D\eta|^2\,\left[1+\Phi(|Du|-1)_+\,h_2(1+(|Du|-1)_+)(1+(|Du|-1)_+)^2\right]+2\,\eta^2\,\Phi(|Du|-1)_+h_1(1+(|Du|-1)_+)\,|D(|Du|-1)_+|^2.
\end{aligned}$$

Since $\Phi(|Du(x)|-1)_+ = 0$ when $|Du(x)| \le 1$, by (3.7) we get

$$\begin{aligned} \int_\Omega |D(\eta\, G((|Du|-1)_+))|^2\,dx \le C \int_\Omega (\eta^2 + |D\eta|^2)\left[1 + \Phi(|Du|-1)_+ h_2(1+(|Du|-1)_+)(1+(|Du|-1)_+)^2\right]\,dx \\ + C\int_\Omega (\eta^2+|D\eta|^2)\Phi'(|Du|-1)_+(1+(|Du|-1)_+)h_2(1+(|Du|-1)_+)(1+(|Du|-1)_+)^2\,dx. \end{aligned} \tag{3.9}$$

Let us assume

$$\Phi(t) = (1+t)^{\gamma-2}t^2 \qquad \gamma \ge 0 \tag{3.10}$$

from which we deduce

$$\Phi'(t) = (1+t)^{\gamma-3}t(\gamma t+2) \le (\gamma+2)(1+t)^{\gamma-2}t.$$

With these assumptions (3.9) reads

$$\int_\Omega |D(\eta\, G((|Du|-1)_+))|^2\,dx \le C(\gamma+2)\int_\Omega (\eta^2+|D\eta|^2)\left[1 + (1+(|Du|-1)_+)^{\gamma+2}\, h_2(1+(|Du|-1)_+)\right]\,dx. \tag{3.11}$$

By the Sobolev inequality, there exists a constant c_S such that

$$\left\{\int_\Omega [\eta\, G((|Du|-1)_+)]^{2^*}\,dx\right\}^{2/2^*} \le c_S \int_\Omega |D(\eta(G(|Du|-1)_+))|^2\,dx \tag{3.12}$$

where $2^* = \frac{2n}{n-2}$ if $n>2$ and a number greater than $\frac{2}{1-\beta}$ if $n=2$. We apply (3.3) with the choice $t = (|Du|-1)_+$

$$G((|Du|-1)_+) = 1 + \int_0^{(|Du|-1)_+} (1+s)^{\frac{\gamma-2}{2}}\, s\sqrt{h_1(1+s)}\,ds \ge C_3\left[1 + h_2(1+(|Du|-1)_+)^{\frac{1}{2^*}}\,\frac{(1+(|Du|-1)_+)^{\frac{\gamma}{2}+1-\beta}}{\left(\frac{\gamma}{2}+1-\beta\right)^2}\right]$$

thus by (3.11) we obtain that there exists $c = c(C_3) > 0$ such that, for all $\gamma \ge 0$,

$$\begin{aligned} &\left\{\int_\Omega \eta^{2^*}(1+(1+(|Du|-1)_+)^{(\gamma+2-2\beta)\frac{2^*}{2}}\, h_2(1+(|Du|-1)_+))\,dx\right\}^{\frac{2}{2^*}} \\ &\quad\le c\left(\frac{\gamma}{2}+1-\beta\right)^4(\gamma+2)\int_\Omega (\eta^2+|D\eta|^2)(1+(1+(|Du|-1)_+))^{\gamma+2}h_2(1+(|Du|-1)_+)\,dx \\ &\quad\le c\,(\gamma+2)^5\int_\Omega (\eta^2+|D\eta|^2)\left[1+(1+(|Du|-1)_+)^{\gamma+2}\, h_2(1+(|Du|-1)_+)\right]\,dx \end{aligned} \tag{3.13}$$

where we used once more (3.9) and (3.12). From this point onward, we follow the proofs of [8, Lemma 6.2] and [8, Lemma 6.3]. The previous inequality, indeed, is the analogous of (6.10) in [8]. Now, by the same iteration process, we obtain that, for $0<\rho<R$, $\overline{B}_R \subset \Omega$, there exists a positive constant C depending only on $n, L, h_1(1)$, such that

$$\|1+(|Du|-1)_+\|_{L^\infty(B_\rho)}^{2-n\beta} \le \frac{C}{(R-\rho)^n}\int_{B_R}(1+(|Du|-1)_+)^2 h_2(1+(|Du|-1)_+)\,dx. \tag{3.14}$$

As in [8, Lemma 6.3], set

$$V = (1+(|Du|-1)_+)^2 h_2((|Du|-1)_+).$$

By (F2) and (3.14) there exists $C_\mu > 0$ such that

$$\|V\|_{L^\infty(B_\rho)}^{\frac{2-n\beta}{2-\mu}} \le \frac{C_\mu}{(R-\rho)^n}\int_{B_R} V(x)\,dx.$$

Moreover, since by (1.2)

$$\frac{2-\mu}{2-n\beta}\left(1-\frac{1}{\alpha}\right) < 1,$$

from (F4) we can deduce (3.2). □

4. Proof of Theorem 1.1

The proof of the theorem is divided in three steps. We start by considering, as in [8], suitable approximations of the functional; in the second step we consider minimizers of these approximating functionals with regular boundary conditions that, in particular, satisfy (BSC). This allows us to use the a priori estimates of Section 3. The coercivity of the functional is a crucial property to perform the passage to the limit in Step 3. We remark that the superlinearity of f follows from assumption (F2): in fact it implies that $h_1(t) \ge h_1(t_0)t^{-1}$ for $t \ge t_0$, then there exists $m>0$ such that

$$f(\xi) \ge m|\xi|\log|\xi| \tag{4.1}$$

for ξ sufficiently large (see Lemma 7.2 in [8]).

STEP 1: APPROXIMATION. In [8] (see the proof of Theorem 2.1) it has been proved that there exists a sequence $f_k \in C^2(\mathbb{R}^n)$ of locally uniformly convex functions such that

1. f_k satisfies (F1)-(F4) with $2h_2$ instead of h_2, constants C_1 and C_2 independent of k;
2. f_k uniformly converges to f on compact sets;
3. for every $\delta > 0$ and for every k sufficiently large

$$f(\xi) \le \begin{cases} f_k(\xi) + \delta & \text{if } |\xi| \le t_0 + 2 \\ f_k(\xi) & \text{if } |\xi| > t_0 + 2; \end{cases} \tag{4.2}$$

4. for every $\xi \in \mathbb{R}^n$

$$f(\xi) - 1 \le f_k(\xi) \le f(\xi) + |\xi| + 1.$$

We observe that, thanks to (4.1), the functions f_k are superlinear.

Let u_ε a mollification of u on B_R. Let $v_{k,\varepsilon}$ be the minimizer of

$$\int_{B_R} f_k(Dv) + g(x, v)\, dx \tag{4.3}$$

such that $v = u_\varepsilon$ on ∂B_R. We observe that $u_\varepsilon \in C^\infty(B_R)$ and hence (see [29,30]) it fulfills the (BSC) on ∂B_R.

STEP 2: BOUNDEDNESS OF THE GRADIENTS. In this step we are going to prove that $v_{k,\varepsilon} \in W^{1,\infty}(B_R)$ for R sufficiently small.

First of all we recall that, since u_ε satisfies the (BSC) on B_R, for every $z \in \partial B_R$, there exists κ_z^- and κ_z^+ such that

$$\kappa_z^-(x - z) + u_\varepsilon(z) \le u_\varepsilon(x) \le \kappa_z^+(x - z) + u_\varepsilon(z) \qquad \forall x \in \partial\Omega, \tag{4.4}$$

see [29,30]. Our aim is to construct lower and upper Lipschitz barriers for the boundary datum. We follow the ideas used in [11,16], remarking the fact that here we are in a slightly different set of assumptions.

We recall that by Proposition 2.1 we have that f_k^* is defined in $\mathbb{R}^n$ and superlinear.

Let us fix $z \in \partial B_R$ and let κ_z^- as in the left hand side of (4.4). We consider the set

$$\left\{ \frac{n}{L} f_k^* \left(\frac{L}{n} x \right) - \kappa_z^- \cdot x - c \le 0 \right\} = \Omega_{\kappa_z^-, c}.$$

We observe that for c sufficiently large $\Omega_{\kappa_z^-,c}$ is not empty and convex. The superlinearity of f_k^* implies that it is bounded and the fact that f_k^* is finite for every $x \in \mathbb{R}^n$ implies that

$$\lim_{c \to +\infty} \min\{|x| \;:\; x \in \partial\Omega_{\kappa_z^-,c}\} = +\infty. \tag{4.5}$$

Moreover, by Proposition 2.1(v) it follows that, for c sufficiently large, $\partial\Omega_{\kappa_z^-,c}$ is C^2 so that we can perform the same computations as in Step 2 of the proof of Theorem 4.5 in [11] and we can show that the principal curvatures of $\partial\Omega_{\kappa_z^-,c}$ at every point x are less or equal than

$$\frac{|D^2 f_k^*(\frac{L}{n}x)|}{|Df_k^*(\frac{L}{n}x)|} \le \frac{1}{h_1(|Df_k^*(\frac{L}{n}x)|)|Df_k^*(\frac{L}{n}x)|}, \tag{4.6}$$

where we have also used assumption (F1).

Now we fix $\frac{L}{n}x \in \mathbb{R}^n \setminus B_{s_0}$, where s_0 is given by Proposition 2.1(v), and we define

$$\varphi(t) = f_k^* \left(t \frac{x}{|x|} \right) \qquad \text{for } t \ge s_0;$$

then we obtain

$$\varphi'(t) = Df_k^* \left(t \frac{x}{|x|} \right) \cdot \frac{x}{|x|} \qquad \text{for } t \ge s_0$$

and, by using once more Proposition 2.1(v) and assumption (F1)

$$\varphi''(t) = D^2 f_k^* \left(t \frac{x}{|x|} \right) \frac{x}{|x|} \cdot \frac{x}{|x|} \ge \frac{1}{h_2(t)} \qquad \text{for } t \ge s_0.$$

It follows that, for $t = \frac{L}{n}|x|$, there exists a non negative constant C such that

$$\left| Df_k^* \left(\frac{L}{n} x \right) \right| \ge \varphi' \left(\frac{L}{n}|x| \right) \ge \int_{s_0}^{\frac{L}{n}|x|} \frac{1}{h_2(\tau)}\, d\tau + C$$

and the last term goes to $+\infty$ as $|x| \to +\infty$. Assumption (F2) implies that there exists $\bar{t}$ such that $h_1(t)t \ge \delta > 0$ for every $t \ge \bar{t}$. It follows that, if c is sufficiently large the principal curvatures of $\Omega_{\kappa_z^-,c}$ are less or equal to $\frac{1}{\delta}$.

Let now $R < \delta$ and ν be the normal vector to ∂B_R in z. Thus, there exists $x_z \in \partial\Omega_{\kappa_z^-,c}$ such that its normal vector is exactly ν. Let us consider the function

$$v_z(x) = \frac{n}{L} f_k^*\left(\frac{L}{n}(x-(z-x_z))\right) + u_\varepsilon(z) - \frac{n}{L} f_k^*\left(\frac{L}{n}x_z\right)$$

We define

$$\tilde{\Omega}_{\kappa_z^-,c} = \left\{ v_z(x) - \kappa_z^- \cdot (x-(z-x_z)) - u_\varepsilon(z) + \frac{n}{L} f_k^*\left(\frac{L}{n}x_z\right) - c \le 0 \right\} \tag{4.7}$$

and obviously $\tilde{\Omega}_{\kappa_z^-,c} = \Omega_{\kappa_z^-,c} + (z - x_z)$ so that the curvature of $\partial\tilde{\Omega}_{\kappa_z^-,c}$ in z is the same of $\partial\Omega_{\kappa_z^-,c}$ in x_z.

Since $R < \delta$, we have that $B_R \subset \tilde{\Omega}_{\kappa_z^-,c}$ and $z \in \partial B_R \cap \partial\tilde{\Omega}_{\kappa_z^-,c}$. Moreover we remark that

$$\tilde{\Omega}_{\kappa_z^-,c} = \{ v_z(x) \le \kappa_z^- \cdot (x-z) + u_\varepsilon(z) \} \tag{4.8}$$

so that we can apply the comparison principle in [16, Theorem 2.4] and in [11, Theorem 2.4] between the minimizer $v_{k,\varepsilon}$ and the function v_z and conclude that $v_{k,\varepsilon}(x) \ge v_z(x)$ a.e. in B_R. The construction of the lower barrier is completed considering every $z \in \partial B_R$ and defining

$$\ell^-(x) = \sup_{z\in\partial B_R} v_z(x). \tag{4.9}$$

Repeating an analogous construction we can construct also the upper barrier ℓ^+.

Remarking that $\ell^\pm$ are Lipschitz continuous in B_R and arguing as in [23, Theorem 5.2] and in [11, Theorem 4.6], we conclude the proof of this step.

STEP 3: PASSAGE TO THE LIMIT. We can apply Lemma 2.4 with $f_-(\xi) = f(\xi) - 1$ and $f_+(\xi) = f(\xi) + |\xi| + 1$ to deduce that there exists a constant $\tilde{M}$ such that $\|v_{k,\varepsilon}\|_{L^\infty(B_R)} \le \tilde{M}$ for every k and ε. Hence assumptions (G1) and (G2) imply that there exists a constant $\tilde{K}$ such that $\|g(x,u_\varepsilon)\|_{L^1(B_R)} \le \tilde{K}$ and $\|g(x,v_{k,\varepsilon})\|_{L^1(B_R)} \le \tilde{K}$. Step 2 implies that $v_{k,\varepsilon} \in W^{1,\infty}(B_R)$ and from Theorem 3.1 we get the estimate

$$\|Dv_{k,\varepsilon}\|_{L^\infty(B_\rho;\mathbb{R}^n)} \le C\left(\frac{1}{(R-\rho)^n}\int_{B_R}\{1+f_k(Dv_{k,\varepsilon})\}\,dx\right)^\theta$$

where the constant C does not depend on k and ε. Adding and subtracting $g(x,v_{k,\varepsilon})$ and using the minimality of $v_{k,\varepsilon}$ we obtain

$$\begin{aligned}\|Dv_{k,\varepsilon}\|_{L^\infty(B_\rho;\mathbb{R}^n)} \le& C\left(\frac{1}{(R-\rho)^n}\int_{B_R}\{1+f_k(Dv_{k,\varepsilon})+g(x,v_{k,\varepsilon})\}\,dx\,\frac{1}{(R-\rho)^n}\int_{B_R}g(x,v_{k,\varepsilon})\,dx\right)^\theta\\ \le& C\left(\frac{1}{(R-\rho)^n}\int_{B_R}\{1+f_k(Du_\varepsilon)+g(x,u_\varepsilon)\}\,dx+\frac{\tilde{K}}{(R-\rho)^n}\right)^\theta.\end{aligned}$$

Therefore

$$\limsup_{k\to+\infty}\|Dv_{k,\varepsilon}\|_{L^\infty(B_\rho;\mathbb{R}^n)} \le M_\varepsilon$$

where

$$M_\varepsilon = C\left[\frac{1}{(R-\rho)^n}\left(\int_{B_R}\{1+f(Du_\varepsilon)+g(x,u_\varepsilon)\}\,dx+\tilde{K}\right)\right]^\theta.$$

The sequence $v_{\varepsilon,k}$ is bounded in $W^{1,\infty}(B_\rho)$ uniformly with respect to k, then there exists a subsequence $k_j \to \infty$, such that $\{v_{\varepsilon,k_j}\}$ is weakly* convergent in $W^{1,\infty}(B_\rho)$. Now we fix a sequence $\rho_j \to R$ and, by a diagonalization argument, we extract a subsequence, that we still denote by $\{v_{\varepsilon,k_j}\}$, weakly* converging to $\bar{v}_\varepsilon$ in $W^{1,\infty}(B_\rho)$ for every $\rho < R$. Recall that $\{v_{\varepsilon,k_j}\} \subset u_\varepsilon + W_0^{1,1}(B_R)$. Moreover for every $\rho < R$

$$\|D\bar{v}_\varepsilon\|_{L^\infty(B_\rho;\mathbb{R}^n)} \le M_\varepsilon. \tag{4.10}$$

The next step is to prove that, up to subsequences, v_{ε,k_j} weakly converges to $\bar{v}_\varepsilon$ in $W^{1,1}(B_R)$ so that $\bar{v}_\varepsilon \in u_\varepsilon + W_0^{1,1}(B_R)$. Indeed by the minimality of v_{ε,k_j}, as $j\to\infty$ we have

$$\begin{aligned}\int_{B_R} f(Dv_{\varepsilon,k_j})\,dx \le& \int_{B_R} 1+f_{k_j}(Dv_{\varepsilon,k_j})+g(x,v_{\varepsilon,k_j})\,dx - \int_{B_R} g(x,v_{\varepsilon,k_j})\,dx\\ \le& \int_{B_R} f_{k_j}(Du_\varepsilon)+g(x,u_\varepsilon)\,dx + (\tilde{K}+1) \;\to\; \int_{B_R} f(Du_\varepsilon)+g(x,u_\varepsilon)\,dx+(\tilde{K}+1).\end{aligned}$$

The superlinearity of f and de la Vallée-Poussin Theorem imply that we can choose the sequence k_j such that $Dv_{\varepsilon,k_j} \rightharpoonup D\bar{v}_\varepsilon$ in $L^1(B_R;\mathbb{R}^n)$ and then $(v_{\varepsilon,k_j} - u_\varepsilon) \rightharpoonup (\bar{v}_\varepsilon - u_\varepsilon) \in W_0^{1,1}(B_R)$.

On the other hand, by the minimality of v_{ε,k_j}

$$\begin{aligned}\int_{B_R} f(Dv_{\varepsilon,k_j})+g(x,v_{\varepsilon,k_j})\,dx =& \int_{B_R} f_{k_j}(Dv_{\varepsilon,k_j})+g(x,v_{\varepsilon,k_j})\,dx + \int_{B_R}(f(Dv_{\varepsilon,k_j})-f_{k_j}(Dv_{\varepsilon,k_j}))\,dx\\ \le& \int_{B_R} f_{k_j}(Du_\varepsilon)+g(x,u_\varepsilon)\,dx + \int_{B_R}(f(Dv_{\varepsilon,k_j})-f_{k_j}(Dv_{\varepsilon,k_j}))\,dx.\end{aligned}$$

By (4.2) for every δ there exists $\bar{k}$ such that for every $k_j > \bar{k}$

$$\int_{B_R} f(Dv_{\varepsilon,k_j}) + g(x, v_{\varepsilon,k_j})\,dx \le \int_{B_R} f_{k_j}(Du_\varepsilon) + g(x,u_\varepsilon)\,dx + \delta|B_R|.$$

By lower semicontinuity in $W^{1,1}(B_R)$, passing to the limit for $j \to \infty$, we get

$$\begin{aligned}\int_{B_R} f(D\bar{v}_\varepsilon) + g(x,\bar{v}_\varepsilon)\,dx &\le \liminf_{j\to\infty} \int_{B_R} f(Dv_{\varepsilon,k_j}) + g(x, v_{\varepsilon,k_j})\,dx \\ &\le \lim_{j\to\infty} \int_{B_R} f_{k_j}(Du_\varepsilon) + g(x,u_\varepsilon)\,dx + \delta|B_R| = \int_{B_R} f(Du_\varepsilon) + g(x,u_\varepsilon)\,dx + \delta|B_R|\end{aligned}$$

for every $\delta > 0$ and then for $\delta \to 0$

$$\int_{B_R} f(D\bar{v}_\varepsilon) + g(x,\bar{v}_\varepsilon)\,dx \le \int_{B_R} f(Du_\varepsilon) + g(x,u_\varepsilon)\,dx. \tag{4.11}$$

We observe that, thanks to Jensen's inequality and the Dominated Convergence Theorem (see [15] and [8, Lemma 7.1]),

$$\lim_{\varepsilon\to 0} \int_{B_R} f(Du_\varepsilon) + g(x,u_\varepsilon)\,dx = \int_{B_R} f(Du) + g(x,u)\,dx \tag{4.12}$$

and hence the right hand side of (4.11) is uniformly bounded w.r.t. ε. We apply once more de la Vallée-Poussin Theorem to extract a sequence $\varepsilon_j \to 0$ such that $\bar{v}_{\varepsilon_j} - u_{\varepsilon_j} \rightharpoonup \bar{v} - u$ in $W_0^{1,1}(B_R)$. By the lower semicontinuity of the functional, (4.11) and (4.12)

$$\begin{aligned}\int_{B_R} f(D\bar{v}) + g(x,\bar{v})\,dx &\le \liminf_{j\to\infty} \int_{B_R} f(D\bar{v}_{\varepsilon_j}) + g(x,\bar{v}_{\varepsilon_j})\,dx \\ &\le \lim_{j\to\infty} \int_{B_R} f(Du_{\varepsilon_j}) + g(x,u_{\varepsilon_j})\,dx = \int_{B_R} f(Du) + g(x,u)\,dx.\end{aligned} \tag{4.13}$$

Then $\bar{v}$ is another minimizer for (1.1) with $\Omega = B_R$. Moreover from (4.10) we can also assume that $\{\bar{v}_{\varepsilon_j}\}_j$ is weakly* convergent to $\bar{v}$ in $W^{1,\infty}(B_\rho)$ for every $0 < \rho < R$. Therefore, thanks to (4.10) and (4.12), we have that for every $0 < \rho < R$

$$\begin{aligned}\|D\bar{v}\|_{L^\infty(B_\rho;\mathbb{R}^n)} &\le \liminf_{j\to\infty} \|D\bar{v}_{\varepsilon_j}\|_{L^\infty(B_\rho;\mathbb{R}^n)} \\ &\le \lim_{j\to\infty} C\left\{\frac{1}{(R-\rho)^n}\left(\int_{B_R} 1 + f(Du_{\varepsilon_j}) + g(x,u_{\varepsilon_j})\,dx + \tilde{K}\right)\right\}^\theta \\ &= C\left\{\frac{1}{(R-\rho)^n}\left(\int_{B_R} f(Du) + g(x,u)\,dx + \kappa\right)\right\}^\theta,\end{aligned} \tag{4.14}$$

where $\kappa = \tilde{K} + |B_R|$.

Since $\bar{v}$ and u are two different minimizers of F in B_R and $f(\xi)$ is strictly convex for $|\xi| > t_0$, by proceeding as in [7] it is possible to prove that the set

$$E_0 := \left\{x \in B_R : \left|\frac{Du(x) + D\bar{v}(x)}{2}\right| > t_0\right\} \cap \{Du \ne D\bar{v}\}.$$

has zero measure. Therefore

$$\|Du\|_{L^\infty(B_\rho;\mathbb{R}^n)} \le \|Du + D\bar{v}\|_{L^\infty(B_\rho;\mathbb{R}^n)} + \|D\bar{v}\|_{L^\infty(B_\rho;\mathbb{R}^n)} \le 2t_0 + \|D\bar{v}\|_{L^\infty(B_\rho;\mathbb{R}^n)}.$$

5. Some additional results

We conclude by presenting some additional results related to Theorem 1.1. We start by the following theorem which is a slightly more general version of Theorem 1.1, where the global Lipschitz continuity of g, namely assumption (G1), is replaced by the following local Lipschitzianity

(G1)′ for every $M > 0$, there exists $L(M)$ such that $|g(x,\eta_1) - g(x,\eta_2)| \le L(M)|\eta_1 - \eta_2|$ for a.e. in $x \in \Omega$ and for every $\eta_1, \eta_2 \in [-M, M]$.

In this case it turns out that the constant C in (1.3) depends also on $\|u\|_{L^\infty(B_R)}$.

Theorem 5.1. *Let $u \in W^{1,1}_{loc}(\Omega) \cap L^\infty_{loc}(\Omega)$ be a local minimizer of the functional (1.1). Suppose that f satisfies the growth assumptions* (F1)–(F4)*, with the parameters α, β, μ related by the condition (1.2). Assume moreover that g fulfills assumptions* (G1)′-(G2)-(G3)-(G4)*.*

Then u is locally Lipschitz continuous in Ω and it satisfies estimate (1.3) as in Theorem 1.1 where in this case the constant C depends also on $\|u\|_{L^\infty(B_R)}$.

Proof. The result is a straightforward consequence of Theorem 1.1: in fact it is sufficient to consider $\|u\|_{L^\infty(B_R)}$ with B_R instead of Ω and to observe that g satisfies (G1)′ with $M = \|u\|_{L^\infty(B_R)}$. □

The next theorem is obtained by considering functionals of type (1.1) in the space $u_0 + W_0^{1,1}(\Omega)$, where u_0 is a fixed boundary datum.

Theorem 5.2. *Let $u_0 \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ and u be a minimizer of the functional (1.1) in the class $u_0 + W_0^{1,1}(\Omega)$. Suppose that f satisfies the growth assumptions* (F1)–(F4)*, with the parameters α, β, μ related by the condition (1.2). Assume moreover that g fulfills assumptions* (G1)′-(G2)-(G3)-(G4).

Then u is locally Lipschitz continuous in Ω and it satisfies estimate (1.3) as in Theorem 1.1 where, this time, the constants C and κ depend also on $\|u_0\|_{L^\infty(\Omega)}$.

Proof. We apply Lemma 2.2 to get an L^∞ bound for the minimizer and we proceed then as in the previous theorem. □

Remark 1. We notice that any function $g(x,u)$ such that $g(x,u) = g(u)$ satisfying assumption (G3), fulfills assumption (G1)′, (G2) and (G4), therefore the only assumption required, in this case, is the convexity.

On the other hand, assumption (G1)′ allows us to consider also significant cases for applications. For instance we can deal with functionals modeling the elastoplastic torsion, where

$$g(x,u) = (\lambda u - a(x))u$$

with $a(x) \in W^{1,\infty}(\Omega)$ and $\lambda > 0$, or the reconstruction of an image u from a degraded data $a(x)$, where

$$g(x,u) = |a(x) - \lambda u|^2, \quad a(x) \in C^1(\bar{\Omega}),\ \lambda \in \mathbb{R}.$$

We conclude this section by considering the case of radially symmetric Lagrangian $f(\xi) = h(|\xi|)$, for a given function h. In this case, as we already remarked, condition (1.2) is always satisfied.

Theorem 5.3. *Let $u \in W^{1,1}_{loc}(\Omega) \cap L^\infty_{loc}(\Omega)$ be a local minimizer of the functional (1.1). Suppose that $f(\xi) = h(|\xi|)$ where h is non negative, convex, increasing, superlinear and $h \in C([0,+\infty)) \cap C^2([t_0,+\infty))$ for a suitable $t_0 > 0$. We also assume that there exist $\mu \in [0,1]$, $\beta \in \left(0, \frac{2}{n}\right)$ and a positive constant C such that, for every $t \geq t_0$*

(i) $h''(t) \leq \frac{h'(t)}{t}$;
(ii) $t \mapsto h'(t)t^{\mu-1}$ *is decreasing;*
(iii) $h''(t) \geq \frac{c}{t^{\mu \frac{2}{2^*} + 2\beta}}$,

where, as before, $2^* = \frac{2n}{n-2}$ if $n \geq 3$ while in the case $n = 2$ it must be replaced with any fixed positive number greater than $\frac{2}{1-\beta}$. Assume moreover that g fulfills assumptions (G1)-(G2)-(G3)-(G4).

Then u is locally Lipschitz continuous in Ω and it satisfies estimate (1.3) as in Theorem 1.1.

Proof. First of all we notice that, following the same notation of assumptions (F1)–(F4) and recalling (i) and [26, equation (3.3)], we have that (F1) holds with

$$h_1(t) = h''(t) \quad \text{and} \quad h_2(t) = \frac{h'(t)}{t}.$$

We remark that, as for Lemma 3.3, also Theorem 3.1 still holds assuming, in (F2), $t \mapsto th_2(t)$ is increasing instead of $t \mapsto th_1(t)$ is increasing. The convexity of h implies $t \mapsto th_2(t)$ is increasing and the first condition in (F2) is satisfied by (ii).

On the other hand, from assumption (ii), we infer the existence of a constant $C > 0$ such that

$$h'(t) \leq \frac{C}{t^{\mu-1}}.$$

Therefore

$$[h_2(t)]^{\frac{2}{2^*}} = \left[\frac{h'(t)}{t}\right]^{\frac{2}{2^*}} \leq \left[\frac{C}{t^\mu}\right]^{\frac{2}{2^*}} \leq C^{\frac{2}{2^*}} t^{2\beta} \frac{1}{t^{\mu + \frac{2}{2^*} + 2\beta}} \overset{\text{(iii)}}{\leq} \frac{C^{\frac{2}{2^*}}}{c} t^{2\beta} h''(t) = C_1 t^{2\beta} h_1(t)$$

and (F3) holds. Finally, it is sufficient to show that (F4) holds for $\alpha = 1$.

We have

$$\begin{aligned} h(t) - h(t_0) &= \int_{t_0}^t h'(s)ds = \int_{t_0}^t \frac{h'(s)}{s^{\mu-1}} s^{\mu-1} ds \overset{\text{(ii)}}{\geq} h'(t)t^{\mu-1} \int_{t_0}^t s^{1-\mu} ds \\ &= \frac{1}{2-\mu} h'(t)t - \frac{1}{2-\mu} h'(t)t^{\mu-1} t_0^{2-\mu} \overset{\text{(ii)}}{\geq} \frac{1}{2-\mu} h'(t)t - \frac{1}{2-\mu} h'(t_0)t_0 \end{aligned}$$

therefore

$$h'(t)t \leq (2-\mu)h(t) - (2-\mu)h(t_0) + h'(t_0)t_0 \leq C[h(t) + 1].$$

Summing up, all the assumptions of Theorem 3.1 hold; in particular, being $\alpha = 1$, (1.2) is always satisfied, being equivalent to ask that $\beta < \frac{2}{n}$ so that the a priori estimate holds true. It remains to discuss the proof of Theorem 1.1 in our setting.

Also in this case we approximate the function h with a sequence of functions h_k satisfying (i), (ii) and (iii) with a constant c independent of k. We need to remark that the functions $f_k(\xi) = h_k(|\xi|)$ belong to $C^2(\mathbb{R}^n)$ and are locally uniformly convex in $\mathbb{R}^n$.

From now on the proof follows the same ideas of the proof of Theorem 1.1. We only underline that the computation of the curvatures of the boundary of the set

$$\Omega_{\kappa_z^-,c} = \left\{ \frac{n}{L} h_k^* \left(\frac{L}{n} |x| \right) - \kappa_z^- \cdot x - c \leq 0 \right\}$$

can be estimated, as in Step 2 of the proof of [16, Theorem 4.3], by

$$\frac{|(h_k^*)''(\frac{L}{n}|x|)|}{|(h_k^*)'(\frac{L}{n}|x|)|^3} = \frac{1}{h_k''(|(h_k^*)'(\frac{L}{n}|x|)|)|(h_k^*)'(\frac{L}{n}|x|)|^3}. \tag{5.1}$$

At this point, (iii) and the fact that $\mu \frac{2}{2^*} + 2\beta < 3$, yields that

$$\lim_{t \to +\infty} h_k''(t) t^3 = +\infty$$

which allows us to infer the existence of $\bar{t}$ such that $h_k''(t)t^3 \geq \delta > 0$ for every $t \geq \bar{t}$. It follows that if c is sufficiently large, the principal curvatures of $\Omega_{\kappa_z^-,c}$ are less or equal to $\frac{1}{\delta}$ and therefore it is now possible then to conclude as in Step 3 of Theorem 1.1. □

Data availability

No datasets were generated or analyzed during the current study.

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