L^P CONTINUITY OF WAVE OPERATORS IN $\mathbb{Z}$

Scipio Cuccagna

Abstract. We recover for discrete Schrödinger operators on the lattice $\mathbb{Z}$, stronger analogues of the results by Weder [W1] and by D'Ancona & Fanelli [DF] on $\mathbb{R}$.

§1 Introduction

We consider the discrete Schrödinger operator

$$(Hu)(n) = -(\Delta u)(n) + q(n)u(n) \tag{1.1}$$

with the discrete Laplacian Δ in $\mathbb{Z}$, $(\Delta u)(n) = u(n+1) + u(n-1) - 2u(n)$ and a potential $q = \{q(n), n \in \mathbb{Z}\}$ with $q(n) \in \mathbb{R}$ for all n. In $\ell^2(\mathbb{Z})$ the spectrum is $\sigma(-\Delta) = [0,4]$. Let for $\langle n \rangle = \sqrt{1+n^2}$

$$\begin{aligned} &\ell^{p,\sigma} = \ell^{p,\sigma}(\mathbb{Z}) = \{u = \{u_n\} : \|u\|^p_{\ell^{p,\sigma}} = \sum_{n\in\mathbb{Z}} \langle n \rangle^{p\sigma} |u(n)|^p < \infty\} \text{ for } p \in [1,\infty) \\ &\ell^{\infty,\sigma} = \ell^{\infty,\sigma}(\mathbb{Z}) = \{u = \{u(n)\} : \|u\|_{\ell^{\infty,\sigma}} = \sup_{n\in\mathbb{Z}} \langle n \rangle^{\sigma} |u(n)| < \infty\}. \end{aligned}$$

We set $\ell^p = \ell^{p,0}$. If $q \in \ell^{1,1}$ then H has at most finitely many eigenvalues, see the Appendix. The eigenvalues are simple and are not contained in $[0,4]$, see for instance Lemma 5.3 [CT]. We denote by $P_c(H)$ the orthogonal projection in ℓ^2 on the space orthogonal to the space generated by the eigenvectors of H. $P_c(H)$ defines a projection in ℓ^p for any $p \in [1,\infty]$, see Lemma 2.6 below. We set $\ell^p_c(H) := P_c(H)\ell^p$. By $q \in \ell^1$, q is a trace class operator. Then, by Pearson's Theorem, see Theorem XI.7[RS], the following two limits exist in ℓ^2, for $w \in \ell^2_c(H)$ and $u \in \ell^2$:

$$Wu = \lim_{t\to+\infty} e^{itH} e^{it\Delta} u \,, \quad Zw = \lim_{t\to+\infty} e^{-it\Delta} e^{-itH} w. \tag{1.2}$$

The operators W and Z intertwine $-\Delta$ acting in ℓ_2 with H acting in $\ell^2_c(H)$. Our main result is the following:

Theorem 1.1. *Consider the operators W initially defined in $\ell^2 \cap \ell^p$ and Z initially defined in $\ell^2(H) \cap \ell^p$.*

(1) Assume H does not have resonances in 0 and 4. Then for $q \in \ell^{1,1}$ the operators extend into isomorphisms $W : \ell^p \to \ell^p_c(H)$ and $Z : \ell^p_c(H) \to \ell^p$ for all $1 < p < \infty$.

(2) Assume H has resonances in 0 and/or 4. Then the above conclusion is true for $q \in \ell^{1,2}$.

(3) Assume that $q \in \ell^{1,2+\sigma}$ with $\sigma > 0$. Then W and Z extend into isomorphisms also for $p = 1, \infty$ exactly when both 0 and 4 are resonances and the transmission coefficient $T(\theta)$, defined for $\theta \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, satisfies $T(0) = T(\pi) = 1$.

Remark 1. W extends into a bounded operator for $p = 1, \infty$ when the sum of the operators (3.1)–(3.4) is bounded and this can happen only for $T(0) = T(\pi) = 1$.

Remark 2. We do not know if Claim 3 holds with $\sigma = 0$.

Remark 3. $\lambda = 0$ or $\lambda = 4$ is a resonance exactly if $Hu = \lambda u$ admits a nonzero solution in ℓ^∞. We say that H is generic if both 0 and 4 are not resonances.

Remark 4. Since $Z = W^*$, by duality it will be enough to consider W.

Theorem 1.1 provides dispersive estimates for solutions of the Klein Gordon equation $u_{tt} + Hu + m^2 u = 0$. In particular in the case of Claim 3, we obtain the optimal $\ell^1 \to \ell^\infty$ estimate, thanks also to [SK] which deals with the $H = -\Delta$ case. The result for $T(0) = 1$ by [W1] proved crucial to us for a nonlinear problem in [C]. There is a close analogy between the theories in $\mathbb{Z}$ and in $\mathbb{R}$. Claims 1 and 2 in Theorem 1.1 are analogous to the result in [DF] for $\mathbb{R}$ while claim 3 is related to analysis in [W1]. Our proof mixes the approach in [W1] with estimates [CT], which in turn is inspired by [GS,DT]. Some effort is spent proving formulas for which we do not know references in the discrete case. The main theme here and in [CT], is that cases $\mathbb{Z}$ and $\mathbb{R}$ are very similar. In particular one can see in [CT] a theory of Jost functions in $\mathbb{Z}$ very similar to the one for $\mathbb{R}$, following the treatment in [DT]. The present paper is inspired by various recent papers on dispersion theory for the group e^{itH}, see [SK,KKK,PS,CT]. In particular the bound $|e^{it\Delta}(n,m)| \le C\langle t\rangle^{-1/3}$ was proved in [SK]. The bound $|P_c(H)e^{itH}(n,m)| \le C\langle t\rangle^{-1/3}$ was proved in [PS] for $q \in \ell^{1,\sigma}(\mathbb{Z})$ with $\sigma > 4$ and for H without resonances. This result was extended by [CT] to $q \in \ell^{1,1}$ for H without resonances and to $q \in \ell^{1,2}$ if 0 or 4 is a resonance. [CT] is able produce for $\mathbb{Z}$ essentially the same argument introduced in [GS] for $\mathbb{R}$, thanks to a a theory of Jost functions in $\mathbb{Z}$ which is basically the same of that for $\mathbb{R}$. Here we recall that [GS] for Schrödinger operators on $\mathbb{R}$ improves an earlier result in [W2]. Theorem 1.1 is the natural transposition to $\mathbb{Z}$, with some improvements, of the theory of wave operators for $\mathbb{R}$ in [W1,GY,DF]. We simplify the argument in [DF] for claims (1) and (2) of Theorem 1.1 and, for claim (3), we use weaker decay hypotheses on the potential than [W1].

We end with some notation. Given an operator A we set $R_A(z) = (A - z)^{-1}$. $\mathcal{S}(\mathbb{Z})$ is the set of functions $f : \mathbb{Z} \to \mathbb{R}$ with $f(n)$ rapidly decreasing as $|n| \nearrow \infty$. For $u \in \ell^2$ we set $F_0[u](\theta) := \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} e^{-in\theta} u(n)$. We set $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. $2\mathbb{Z}$ is the set of even integers; $2\mathbb{Z}+1$ is the set of odd integers. We set

$$\eta(\mu) = \sum_{\nu=\mu}^{\infty} |q(\nu)| \text{ and } \gamma(\mu) = \sum_{\nu=\mu}^{\infty} (\nu - \mu)|q(\nu)|.$$

Given $f \in L^1(\mathbb{T})$ we set $\widehat{f}(\nu) = \int_{-\pi}^{\pi} e^{-i\nu\theta} f(\theta) d\sigma$, with $d\sigma = d\theta/\sqrt{2\pi}$.

§2 Fourier transform associated to H

We recall that the resolvent $R_{-\Delta}(z)$ for $z \in \mathbb{C}\backslash[0,4]$ has kernel

$$R_{-\Delta}(m,n,z) = \frac{-i}{2\sin\theta} e^{-i\theta|n-m|}, \quad m,n \in \mathbb{Z},$$

with θ a solution to $2(1-\cos\theta) = z$ in $D = \{\theta : -\pi \le \Re\theta \le \pi, \Im\theta < 0\}$. In [CT] it is detailed the existence of functions $f_\pm(n,\theta)$ with

$$Hf_\pm(\mu,\theta) = zf_\pm(\mu,\theta) \text{ with } \lim_{\mu\to\pm\infty} \left[f_\pm(\mu,\theta) - e^{\mp i\mu\theta}\right] = 0. \tag{2.1}$$

We have

$$f_\pm(\mu,\theta) = e^{\mp in\theta} - \sum_{\nu=\mu}^{\pm\infty} \frac{\sin(\theta(\mu-\nu))}{\sin\theta} q(\nu) f_\pm(\nu,\theta). \tag{2.2}$$

Define $m_\pm$ by $f_\pm(n,\theta) = e^{\mp in\theta} m_\pm(n,\theta)$. Lemma 5.1 [CT] implies that for fixed n

$$m_\pm(n,\theta) = 1 + \sum_{\nu=1}^{\infty} B_\pm(n,\nu) e^{-i\nu\theta}. \tag{2.3}$$

In Lemma 5.2 [CT] it is proved:

Lemma 2.1. *For $q \in \ell^{1,1}$ and setting $B_+(n,0) = 0$ for all n, we have*

$$B_+(n,2\nu) = \sum_{l=0}^{\nu-1} \sum_{j=n+\nu-l}^{\infty} q(j) B_+(j,2l+1)$$

$$B_+(n,2\nu-1) = \sum_{l=n+\nu}^{\infty} q(l) + \sum_{l=0}^{\nu-1} \sum_{j=n+\nu-l}^{\infty} q(j) B_+(j,2l).$$

We have for $n \ge 0$ the estimate $|B_+(n,\nu)| \le \chi_{[1,\infty)}(\nu) e^{\gamma(0)} \eta(\nu)$. Similarly for $n \le 0$ we have $|B_-(n,\nu)| \le \chi_{[1,\infty)}(\nu) e^{\widetilde{\gamma}(0)} \widetilde{\eta}(\nu)$ with $\widetilde{\gamma}(\mu)$ and $\widetilde{\eta}(\mu)$ defined like $\gamma(\mu)$ and $\eta(\mu)$ but with $q(\nu)$ replaced by $q(-\nu)$.

Lemma 2.1 implies what follows, see the proof of Lemma 5.10 [CT]:

Lemma 2.2. *If $q \in \ell^{1,1+\sigma}$ for $\sigma \geq 0$, then $\|B_\pm(n,\cdot)\|_{\ell^{1,\sigma}} \leq C_\sigma \|q\|_{\ell^{1,1+\sigma}}$ for $\pm n \geq 0$.*

We recall that for two given functions $u(n)$ and $v(n)$ their Wronskian is $[u,v](n) = u(n+1)v(n) - u(n)v(n+1)$. If u and v are solutions of $Hw = zw$ then $[u,v]$ is constant. In particular we set $W(\theta) := [f_+(\theta), f_-(\theta)]$ and $W_1(\theta) := [f_+(\theta), \overline{f}_-(\theta)]$. By an argument in Lemma 5.10 [CT] we have:

Lemma 2.3. *If for $\sigma \geq 0$ we have $q \in \ell^{1,1+\sigma}$, then $W(\theta), W_1(\theta) \in \ell^{1,\sigma}$.*

Lemma 5.4 [CT] states:

Lemma 2.4. *Let $q \in \ell^{1,1}$. For $\theta \in [-\pi,\pi]$ we have $\overline{f_\pm(n,\theta)} = f_\pm(n,-\theta)$ and for $\theta \neq 0, \pm\pi$ we have*

$$f_\mp(n,\theta) = \frac{1}{T(\theta)}\overline{f_\pm(n,\theta)} + \frac{R_\pm(\theta)}{T(\theta)} f_\pm(n,\theta) \tag{1}$$

where $T(\theta)$ and $R_\pm(\theta)$ are defined by (1) and satisfy:

$$[\overline{f_\pm(\theta)}, f_\pm(\theta)] = \pm 2i\sin\theta, \tag{2}$$

$$T(\theta) = \frac{-2i\sin\theta}{W(\theta)}, \quad R_+(\theta) = -\frac{\overline{W}_1(\theta)}{W(\theta)}, \quad R_+(\theta) = -\frac{W_1(\theta)}{W(\theta)} \tag{3}$$

$$\overline{T(\theta)} = T(-\theta)\,,\ \overline{R_\pm(\theta)} = R_\pm(-\theta), \tag{4}$$

$$|T(\theta)|^2 + |R_\pm(\theta)|^2 = 1\,, \quad T(\theta)\overline{R_\pm(\theta)} + R_\mp(\theta)\overline{T(\theta)} = 0. \tag{5}$$

Lemma 5.5 [CT] states:

Lemma 2.5.
(1) For $\theta \in [-\pi,\pi]\backslash\{0,\pm\pi\}$ we have $W(\theta) \neq 0$. We have $|W(\theta)| \geq 2|\sin\theta|$ for all $\theta \in [-\pi,\pi]$ and in the generic case $|W(\theta)| > 0$.
(2) For $j = 0,1$ and $q \in \ell^{1,1+j}$ then $W(\theta)$ and $W_1(\theta)$ are in $C^j[-\pi,\pi]$.
(3) If $q \in \ell^{1,2}$ and $W(\theta_0) = 0$ for a $\theta_0 \in \{0,\pm\pi\}$, then $\dot{W}(\theta_0) \neq 0$. In particular if $q \in \ell^{1,2}$, then $T(\theta) = -2i\sin\theta / W(\theta)$ can be extended continuously in $\mathbb{T}$.

We have the following result:

Lemma 2.6. *Assume that $q \in \ell^{1,1}$ if H is generic and $q \in \ell^{1,2}$ if H has a resonance at 0 or at 4. Then the following statements hold.*
(1) H has finitely many eigenvalues.
(2) If λ is an eigenvalue, then $\dim\ker(H-\lambda) = 1$.
(3) If there are eigenvalues they are in $\mathbb{R}\backslash[0,4]$.
(4) Let $\lambda_1,\ldots,\lambda_n$ be the eigenvalues and $\varphi_1,\ldots,\varphi_n$ corresponding eigenvectors with $\|\varphi_j\|_{\ell^2} = 1$. Then for fixed $C > 0$ and $a > 0$ we have $|\varphi_j(\nu)| \leq Ce^{-a|\nu|}$ for all $j = 1,\ldots,n$ and for all $\nu \in \mathbb{Z}$.

(5) Let $P_d(H) := \sum_j \varphi_j \langle \quad , \varphi_j \rangle$. Then $P_d(H)$ and $P_c(H) := 1 - P_d(H)$ are bounded operators in ℓ^p for all $p \in [1, \infty]$.

Proof. (1) is proved in the Appendix. (2) and (3) are in Lemma 5.3 [CT]. (5) follows from (4). (4) follows from the fact that by the proof in Lemma 5.3 [CT] there are constants $A(\pm, j)$ such that $\varphi_j(\nu) = A(\pm, j) f_\pm(\nu, \theta_j)$, with $\theta_j \in D$ such that $\lambda_j = 2(1 - \cos(\theta_j))$. The fact that $\lambda_j \notin [0, 4]$ implies $\Im(\theta_j) < 0$ for all j.

By Lemmas 5.6-9 [CT] we have

$$\begin{aligned} P_c(H)u &= \frac{1}{2\pi i} \int_0^4 \left[R_H^+(\lambda) - R_H^-(\lambda) \right] u d\lambda = \\ &= \frac{1}{2\pi i} \sum_{\nu \in \mathbb{Z}} \int_{-\pi}^{\pi} K(n, \nu, \theta) d\theta u(\nu) \text{ with} \end{aligned} \tag{2.4}$$

$$\begin{aligned} K(n, \nu, \theta) &= f_-(n, \theta) f_+(\nu, \theta) \frac{\sin(\theta)}{W(\theta)} \text{ for } \nu > n \\ K(n, \nu, \theta) &= f_+(n, \theta) f_-(\nu, \theta) \frac{\sin(\theta)}{W(\theta)} \text{ for } \nu \le n. \end{aligned} \tag{2.5}$$

Consider now plane waves defined as follows:

Definition 2.7. *We consider the following functions:*

$$\begin{aligned} \psi(\nu, \theta) &= \frac{1}{\sqrt{2\pi}} T(\theta) e^{-i\nu\theta} m_+(\nu, \theta) \text{ for } \theta \ge 0 \\ \psi(\nu, \theta) &= \frac{1}{\sqrt{2\pi}} T(-\theta) e^{-i\nu\theta} m_-(\nu, -\theta) \text{ for } \theta < 0\ . \end{aligned}$$

Lemma 2.8. *The kernel $P_c(H)(\mu, \nu)$ of $P_c(H)$ can be expressed as*

$$P_c(H)(\mu, \nu) = \int_{-\pi}^{\pi} \overline{\psi(\mu, \theta)} \psi(\nu, \theta) d\theta. \tag{1}$$

Proof. We assume $\mu \ge \nu$. By (2.4-5)

$$P_c(H)(\mu, \nu) = \frac{1}{2\pi i} \int_0^{\pi} \left[\frac{f_-(\nu, \theta) f_+(\mu, \theta)}{W(\theta)} - \frac{f_-(\nu, -\theta) f_+(\mu, -\theta)}{W(-\theta)} \right] \sin(\theta)\, d\theta.$$

We have by Lemma 2.4

$$\overline{f_\pm(n,\theta)} = f_\pm(n,-\theta)\,,\ \overline{T(\theta)} = T(-\theta)\,,\ \overline{R_\pm(\theta)} = R_\pm(-\theta),$$
$$f_-(\nu,-\theta) = T(\theta)f_+(\nu,\theta) - R_-(\theta)f_-(\nu,\theta),$$
$$f_+(\mu,\theta) = \overline{T(\theta)f_-(\mu,\theta) - R_+(\theta)f_+(\mu,\theta)}.$$

Substituting the last two lines in the square bracket in the integral,

$$\begin{aligned}[\cdots] = {} & \frac{\overline{T(\theta)f_-(\mu,\theta)}f_-(\nu,\theta)}{W(\theta)} - \frac{T(\theta)f_+(\nu,\theta)f_+(\mu,-\theta)}{W(-\theta)} \\ & - \overline{f_+(\mu,\theta)}f_-(\nu,\theta)\left[\frac{\overline{R_+(\theta)}}{W(\theta)} - \frac{R_-(\theta)}{W(-\theta)}\right].\end{aligned} \tag{2}$$

The last line is zero by (5) Lemma 2.4 and by

$$-i\sin(\theta)\left[\frac{\overline{R_+(\theta)}}{W(\theta)} - \frac{R_-(\theta)}{W(-\theta)}\right] = (T\overline{R_+} + \overline{T}R_-)(\theta) = 0.$$

We have by $T(\theta) = -i\sin(\theta)/W(\theta)$

$$\text{rhs}(2) = \frac{1}{2\pi}|T(\theta)|^2\overline{f_+(\mu,\theta)}f_+(\nu,\theta) + \frac{1}{2\pi}|T(\theta)|^2\overline{f_-(\mu,\theta)}f_-(\nu,\theta).$$

This yields formula (1) for $\mu \geq \nu$. For $\mu < \nu$ the argument is similar.

Lemma 2.9. *Let $F[u](\theta) := \sum_n \psi(n,\theta)u(n)$. Then:*
(1) $F : \ell^2_c(H) \to L^2(\mathbb{T})$ is an isometric isomorphism.
(2) $F^*[f](n) := \int_{-\pi}^{\pi} \overline{\psi(n,\theta)}f(\theta)d\theta$ is the inverse of F.
(3) $F[Hu](\theta) = 2(1-\cos\theta)F[u](\theta)$.

$F[u](\theta)$ is a generalization of Fourier series expansions $F[u_0](\theta)$. Lemma 2.9 is a consequence of Lemma 2.8 except for the fact that we could have $F(\ell^2_c(H)) \subsetneqq L^2(\mathbb{T})$. The fact $F(\ell^2_c(H)) = L^2(\mathbb{T})$ follows from $F_0(\ell^2) = L^2(\mathbb{T})$, from the fact that W and Z in (1.2) are isomorphisms between ℓ^2 and $\ell^2_c(H)$ and from Lemma 2.10 below. In the next section the following formula will be important:

Lemma 2.10. *For the operator in (1.2) we have $W = F^*F_0$.*

We have, for $u, v \in \mathcal{S}(\mathbb{Z})$ and $v \in L^2_c(H)$

$$\langle Wu, v\rangle_{\ell^2} - \langle u, v\rangle_{\ell^2} = i\lim_{\epsilon\searrow 0}\int_0^\infty \langle e^{itH}qe^{it\Delta}u, v\rangle_{\ell^2}e^{-\epsilon t}dt.$$

We have for $L^2 = L^2(\mathbb{T})$
$\langle e^{itH} q e^{it\Delta} u, v\rangle_{\ell^2} = \langle e^{i2t(1-\cos\theta)} F[qe^{it\Delta}u], F[v]\rangle_{L^2} = \langle F[qe^{it(\Delta+2(1-\cos\theta)}u], F[v]\rangle_{L^2}$.
Then

$$i\int_0^\infty \langle e^{itH} q e^{it\Delta} u, v\rangle_{\ell^2} e^{-\epsilon t} dt = \langle F[qR_{-\Delta}(2-2\cos\theta + i\epsilon)u], F[v]\rangle_{L^2}$$

and

$$\begin{aligned}
&\langle Wu, v\rangle_{\ell^2} - \langle u, v\rangle_{\ell^2} = \\
&= \int_{-\pi}^{\pi} d\theta\, \overline{F[v]}(\theta) \sum_{\nu\in\mathbb{Z}} \psi(\nu,\theta) q(\nu) (R^+_{-\Delta}(2-2\cos\theta)u)(\nu) =
\end{aligned}$$

$$\int_{-\pi}^{\pi} d\theta\, \overline{F[v]}(\theta) \sum_{\nu'\in\mathbb{Z}} u(\nu') \frac{-i}{2\sin|\theta|} \sum_{\nu\in\mathbb{Z}} e^{-i|\theta|\ |\nu-\nu'|} q(\nu)\psi(\nu,\theta). \tag{1}$$

We claim we have

$$\psi(\mu,\theta) = e^{-i\mu\theta}/\sqrt{2\pi} + \frac{i}{2\sin\theta} \sum_{\nu\in\mathbb{Z}} e^{-i\theta\ |\nu-\mu|} q(\nu)\psi(\nu,\theta) \text{ for } \theta > 0 \tag{2}$$

$$\psi(\mu,\theta) = e^{-i\mu\theta}/\sqrt{2\pi} - \frac{i}{2\sin\theta} \sum_{\nu\in\mathbb{Z}} e^{i\theta\ |\nu-\mu|} q(\nu)\psi(\nu,\theta) \text{ for } \theta < 0. \tag{3}$$

Assuming (2)–(3)

$$\begin{aligned}
\langle Wu, v\rangle_{\ell^2} - \langle u, v\rangle_{\ell^2} &= \int_{-\pi}^{\pi} \sum_{\nu'\in\mathbb{Z}} d\theta\, \overline{F[v]}(\theta) u(\nu') \left[e^{-i\nu'\theta}/\sqrt{2\pi} - \psi(\nu',\theta)\right] \\
= \int_{-\pi}^{\pi} d\theta\, \overline{F[v]}(\theta) \left[F_0[u](\theta) - F[u](\theta)\right] &= \langle F^* F_0 u, v\rangle_{\ell^2} - \langle u, v\rangle_{\ell^2}.
\end{aligned}$$

This yields $W = F^* F_0$. Now we focus on (2) and (3). For $\theta > 0$ it is possible to rewrite (2.2) as follows, for some constant $A(\theta)$,

$$f_+(\mu,\theta) = e^{-i\mu\theta} A(\theta) - R^+_{-\Delta}(2-2\cos\theta) q f_+(\cdot,\theta)(\mu). \tag{4}$$

Using (2.2) for f_- we obtain $-2i\sin(\theta)A(\theta) = [f_+(\theta), f_-(\mu,\theta)]$. Hence $A(\theta) = 1/T(\theta)$. So multiplying (4) by $T(\theta)/\sqrt{2\pi}$ we obtain (2). We have for $\theta < 0$

$$f_-(\mu,\theta) = e^{i\mu\theta} B(\theta) - R^-_{-\Delta}(2-2\cos\theta) q f_-(\cdot,\theta)(\mu) \tag{5}$$

for some constant $B(\theta)$. One checks that $-2i\sin(\theta)B(\theta) = [f_+(\theta), f_-(\mu,\theta)]$. Hence $B(\theta) = 1/T(\theta)$. So multiplying (5) by $T(\theta)/\sqrt{2\pi}$ we obtain

$$\frac{T(\theta)}{\sqrt{2\pi}} f_-(\mu,\theta) = \frac{e^{i\mu\theta}}{\sqrt{2\pi}} - R^-_{-\Delta}(2-2\cos\theta) q \frac{T(\theta)}{\sqrt{2\pi}} f_-(\cdot,\theta)(\mu).$$

Taking complex conjugate we obtain (3).

§3 Bounds on W

It is not restrictive to consider $\chi_{[0,\infty]}(n)Wu(n)$ instead of $Wu(n)$. Indeed the proof for $\chi_{(-\infty,0)}(n)Wu(n)$ is similar. Claims 1 and 2 in Theorem 1.1 are a consequences of Lemma 3.1 below. We follow [W1], exploiting at some crucial points results proved in [CT] and inspired by [GS]. We set $n_\pm(\mu,\theta) := m_\pm(\mu,\theta) - 1$.

Lemma 3.1. *Let $q \in \ell^{1,1}$ in the generic case and $q \in \ell^{1,2}$ in the non generic case. Then $\|\chi_{[0,\infty]}Wu\|_{\ell^p} \le C_p\|u\|_{\ell^p} \quad \forall\, p \in (1,\infty)$.*

Proof. Recall $F_0^*[n_\pm(\mu,\cdot)](\nu) = B_\pm(\mu,\nu)$. Furthermore in Lemma 5.10 [CT] it is proved that $F_0^*[T] \in \ell^1$. One can prove similarly that also $F_0^*[R_\pm] \in \ell^1$. For $d\sigma = d\theta/\sqrt{2\pi}$ and by $\overline{m_\pm}(\mu,\theta) = m_\pm(\mu,-\theta)$, $\overline{T}(\theta) = T(-\theta)$, we consider

$$Wf(\mu) = \int_{-\pi}^{\pi} \overline{\psi(\mu,\theta)} F_0[f](\theta)d\theta = \int_0^{\pi} T(-\theta)e^{i\mu\theta}m_+(\mu,-\theta)F_0[f](\theta)d\sigma$$
$$+ \int_{-\pi}^{0} T(\theta)e^{i\mu\theta}m_-(\mu,\theta)F_0[f](\theta)d\sigma.$$

We consider only $\mu \ge 0$. We substitute $n_\pm(\mu,\theta) := m_\pm(\mu,\theta)-1$ and $T(\theta)m_-(\mu,\theta) = m_+(\mu,-\theta) + e^{-2i\mu\theta}R_+(\theta)m_+(\mu,\theta)$ obtaining

$$\chi_{[0,\infty]}(\mu)Wf(\mu) = \int_{-\pi}^{\pi} e^{i\mu\theta}T(-\theta)\frac{1+\mathrm{sign}(\theta)}{2}F_0[f](\theta)d\sigma$$
$$+ \int_{-\pi}^{\pi} e^{i\mu\theta}\frac{1-\mathrm{sign}(\theta)}{2}F_0[f](\theta)d\sigma + \int_{-\pi}^{\pi} e^{-i\mu\theta}R_+(\theta)\frac{1-\mathrm{sign}(\theta)}{2}F_0[f](\theta)d\sigma$$
$$+ \int_{-\pi}^{\pi} e^{i\mu\theta}T(-\theta)n_+(\mu,-\theta)\frac{1+\mathrm{sign}(\theta)}{2}F_0[f](\theta)d\sigma$$
$$+ \int_{-\pi}^{\pi} e^{i\mu\theta}n_+(\mu,-\theta)\frac{1-\mathrm{sign}(\theta)}{2}F_0[f](\theta)d\sigma$$
$$+ \int_{-\pi}^{\pi} e^{-i\mu\theta}R_+(\theta)n_+(\mu,\theta)\frac{1-\mathrm{sign}(\theta)}{2}F_0[f](\theta)d\sigma.$$

We have $\chi_{[0,\infty]}(\mu)Wf(\mu) = \widetilde{W}_1f(\mu)+\widetilde{W}_2f(\mu)$ where, for $W_j = 2\sqrt{2\pi}\widetilde{W}_j$ for $j = 1,2$:

$$W_1f(\mu) = \int_{-\pi}^{\pi} e^{i\mu\theta}T(-\theta)F_0[f](\theta)d\theta + \sqrt{2\pi}f + \int_{-\pi}^{\pi} e^{-i\mu\theta}R_+(\theta)F_0[f](\theta)d\theta$$
$$+ \int_{-\pi}^{\pi} e^{i\mu\theta}\left(T(-\theta)+1\right)n_+(\mu,-\theta)F_0[f](\theta)d\theta + \int_{-\pi}^{\pi} e^{-i\mu\theta}R_+(\theta)n_+(\mu,\theta)F_0[f](\theta)d\theta;$$

$$W_2 f(\mu) = \int_{-\pi}^{\pi} e^{i\mu\theta} \left(T(-\theta) - 1\right) m_+(\mu, -\theta) \mathrm{sign}(\theta) F_0[f](\theta) d\theta -$$
$$- \int_{-\pi}^{\pi} e^{-i\mu\theta} R_+(\theta) m_+(\mu, \theta) \mathrm{sign}(\theta) F_0[f](\theta) d\theta.$$

W_1 is bounded for $p \in [1, \infty]$. Indeed for example,

$$\begin{aligned} &\left\| \chi_{[0,\infty)}(\cdot) F_0^* \left[R_+(\theta) n_+(\mu, \theta) F_0[f](\theta) \right] (-\cdot) \right\|_{\ell^p} \le \\ &\left\| \chi_{[0,\infty)}(\cdot) \left(\left| F_0^* \left[R_+ \right] \right| * \chi_{[1,\infty)} e^{\gamma(0)} \eta * |f| \right) (-\cdot) \right\|_{\ell^p} \\ &\le e^{\gamma(0)} \gamma(0) \| F_0^* \left[R_+ \right] \|_{\ell^1} \, \| f \|_{\ell^p}, \end{aligned}$$

where we have used $|B_+(\mu, \nu)| \le \chi_{[1,\infty)}(\nu) e^{\gamma(0)} \eta(\nu)$ for $\mu \ge 0$. Other terms of W_1 can be treated similarly. By the same argument W_2 is bounded for $p \in (1, \infty)$. For W_2 we cannot include $p = 1, \infty$ because $\mathrm{sign}(\theta)$ is the symbol of the Calderon-Zygmund operator

$$\mathcal{H} v(\nu) = \int_{-\pi}^{\pi} e^{i\nu\theta} F_0[v](\theta) \, d\sigma = \frac{2i}{\pi} \sum_{\nu' \in \nu + 2\mathbb{Z} + 1} \frac{v(\nu')}{\nu - \nu'}$$

which is unbounded in ℓ^1 and in ℓ^∞. So the proof of Lemma 3.1 is completed.

Consider now $W_2 f(\mu) = \chi_{[0,\infty]}(\mu) W_2 f(\mu)$

Lemma 3.2. *Let $q \in \ell^{1,2+\sigma}$ with $\sigma > 0$. Then W_2 extends into a bounded operator also for $p = 1, \infty$ exactly when both 0 and 4 are resonances and the transmission coefficient $T(\theta)$ defined in $\mathbb{T}$ satisfies $T(0) = T(\pi) = 1$.*

Proof. We consider a partition of unity $1 = \chi + (1 - \chi)$ on $\mathbb{T}$ with χ even, $\chi = 1$ near 0 and $\chi = 0$ near π. Correspondingly we have $W_2 = U_1 + U_2$ with U_1 written below and U_2 given by the same formula with χ replaced by $1 - \chi$. We focus on U_1. We have $U_1 = U_{11} + U_{12}$ with for $\mu \ge 0$

$$\begin{aligned} &U_{11} f(\mu) = U_{111} f(\mu) + U_{112} f(\mu) \\ &U_{111} f(\mu) = m_+(\mu, 0) \int_{-\pi}^{\pi} e^{i\mu\theta} \left(T(-\theta) - T(0)\right) \mathrm{sign}(\theta) \chi(\theta) F_0[f](\theta) d\theta \\ &- m_+(\mu, 0) \int_{-\pi}^{\pi} e^{-i\mu\theta} \left(R_+(\theta) - R_+(0)\right) \mathrm{sign}(\theta) F_0[f](\theta) d\theta \\ &U_{112} f(\mu) = \int_{-\pi}^{\pi} e^{i\mu\theta} \left(T(-\theta) - 1\right) \left(n_+(\mu, -\theta) - n_+(\mu, 0)\right) \mathrm{sign}(\theta) \chi(\theta) F_0[f](\theta) d\theta \\ &- \int_{-\pi}^{\pi} e^{-i\mu\theta} R_+(\theta) \left(n_+(\mu, \theta) - n_+(\mu, 0)\right) \mathrm{sign}(\theta) \chi(\theta) F_0[f](\theta) d\theta \end{aligned}$$

and
(3.1)

$$
\begin{aligned}
&U_{12}f(\mu) = \chi_{[0,\infty)}(\mu)\,(T(0)-1)\,m_+(\mu,0)\int_{-\pi}^{\pi} e^{i\mu\theta}\mathrm{sign}(\theta)\chi(\theta)F_0[f](\theta)\\
&-\chi_{[0,\infty)}(\mu)R_+(0)m_+(\mu,0)\int_{-\pi}^{\pi} e^{-i\mu\theta}\mathrm{sign}(\theta)\chi(\theta)F_0[f](\theta)d\theta\\
&=\chi_{[0,\infty)}(\mu)\,(T(0)-1)\,m_+(\mu,0)(\mathcal{H}f)(-\mu)-\chi_{[0,\infty)}(\mu)R_+(0)m_+(\mu,0)(\mathcal{H}f)(\mu).
\end{aligned}
$$

We have:

Lemma 3.3. *$U_{12}\in B(L^p,L^p)$ for all $p\in[1,\infty]$ if and only if*

$$T(0)-1+R_+(0)=0. \tag{1}$$

Proof. We have $m_+(\mu,0)\to 1$ for $\mu\nearrow\infty$ if $q\in\ell^{1,1}$. We have $(\mathcal{H}f)(-\mu)=(\mathcal{H}f(-\,\cdot))(\mu)$. Set $\widehat{\chi}=F_0^*(\chi)$. Then $U_{12}\in B(L^p,L^p)$ for $p=1,\infty$ exactly if

$$\chi_{\mathbb{N}}(\mu)\,(T(0)-1+R_+(0))\,\mathcal{H}(\widehat{\chi}*f)(\mu)\in\ell^p \text{ for all } f \text{ even in } \ell^p \tag{2}$$

$$\chi_{\mathbb{N}}(\mu)\,(T(0)-1-R_+(0))\,\mathcal{H}(\widehat{\chi}*f)(\mu)\in\ell^p \text{ for all } f \text{ odd in } \ell^p. \tag{3}$$

We show that (2) requires (1). We have $\widehat{\chi}*\chi_{\{0\}}=\widehat{\chi}$ and

$$(\mathcal{H}\widehat{\chi})(\mu)=\frac{2i}{\pi\mu}\sum_{\nu\in\mu+2\mathbb{Z}+1}\widehat{\chi}(\nu)-\frac{2i}{\pi}\sum_{\nu\in\mu+2\mathbb{Z}+1}\left[\frac{1}{\mu}-\frac{1}{\mu-\nu}\right]\widehat{\chi}(\nu).$$

The second term on the right is in $\ell^1([1,\infty)$ but the first is $i\frac{\sqrt{2}}{\sqrt{\pi}\mu}$, which is not in $\ell^1([1,\infty)$. Hence we need equality (1). So (2) requires (1). We now show that (3) occurs always. It is enough to prove $\mathcal{H}f\in\ell^p$ for all f odd. We have

$$\sum_{\nu\in\mu+2\mathbb{Z}+1}\frac{1}{\mu-\nu}f(\nu)=2\sum_{\nu\in\mu+2\mathbb{Z}+1}^{\nu>0}\frac{\nu}{\mu^2-\nu^2}f(\nu).$$

So

$$\|\mathcal{H}f\|_{\ell^1}\lesssim\sum_{\nu>0}|f(\nu)|\sum_{\mu\in\nu+2\mathbb{Z}+1}\frac{\nu}{|\mu^2-\nu^2|}\le C\|f\|_{\ell^1}$$

for a fixed $C<\infty$.

Our next step is to show in Lemma 3.4 that $U_{111}\in B(L^p,L^p)$ for all $p\in[1,\infty]$. In Lemma 3.5 that $U_{112}\in B(L^p,L^p)$ for all $p\in[1,\infty]$. Hence $U_1\in B(L^p,L^p)$ for all $p\in[1,\infty]$ exactly if $U_{12}\in B(L^p,L^p)$ for all $p\in[1,\infty]$.

Lemma 3.4. *Let $q \in \ell^{1,2+\sigma}$ with $\sigma > 0$. Then $U_{111} \in B(L^p, L^p)$ for all $p \in [1, \infty]$.*

Proof. If for $g = (R_+(\theta) - R_+(0)) \operatorname{sign}(\theta)\chi(\theta)$ and $f = (T(\theta) - T(0)) \operatorname{sign}(\theta)\chi(\theta)$ we have $F_0^* f$ and $F_0^* g \in \ell^1$, then by $|m_+(\mu, 0)| \le C$ for all $\mu \ge 0$, we get Lemma 3.3. Here consider only $F_0^* f$ only, since the proof for $F_0^* g$ is similar. We have for $\widetilde{\chi}(\theta)$ another even smooth cutoff function in $\mathbb{T}$ with $\widetilde{\chi} = 1$ on the support of χ and $\widetilde{\chi} = 0$ near π,

$$\chi(\theta)T(\theta) = -2i\frac{\chi(\theta)\sin(\theta)}{\widetilde{\chi}(\theta)W(\theta)}.$$

By Lemma 2.3 we have $F_0^* W \in \ell^{1,1+\sigma}$. By the argument in Lemma 5.10 [CT] we have $F_0^* \left[\frac{W(\theta)}{\sin(\theta)}\right] \in \ell^{1,\sigma}$. Then $F_0^* [\chi(\theta)T(\theta)] \in \ell^{1,\sigma}$ by Wiener's Lemma: case $\sigma = 0$ is stated in 11.6 [R]; for $\sigma > 0$ one can provide $\ell^{1,\sigma}$ with a structure of commutative Banach algebra (changing the norm to an equivalent one, 10.2 [R]) and then repeat the argument in 11.6 [R].

Consider now $A(\theta) = (T(\theta) - T(0))\chi(\theta)$. We have $F_0^* [A] \in \ell^{1,\sigma}$ and $A(0) = A(\pi) = 0$. We have

$$\widehat{f}(\nu) = \frac{2i}{\pi} \sum_{\mu \in \nu + 2\mathbb{Z} + 1} \frac{1}{\nu - \mu} \widehat{A}(\mu).$$

We consider

$$\sum_{\nu \in \mathbb{Z}} |\widehat{f}(\nu)| \le I + II + III$$

with

$$I = \sum_{\nu \in \mathbb{Z}} \left| \sum_{|\mu| \le |\nu|/2, \mu \in \nu + 2\mathbb{Z} + 1} \frac{\widehat{A}(\mu)}{\nu - \mu} \right| ,$$

$$II = \sum_{\nu \in \mathbb{Z}} \sum_{|\nu|/2 \le |\mu| \le 2|\nu|} \frac{|\widehat{A}(\mu)|}{\langle \nu - \mu \rangle} , \; III = \sum_{\nu \in \mathbb{Z}} \sum_{|\mu| \ge 2|\nu|} \frac{|\widehat{A}(\mu)|}{\langle \nu - \mu \rangle}.$$

We see immediately that

$$III \lesssim \|\widehat{A}\|_{\ell^{1,\sigma}} \sum_{\nu \in \mathbb{Z}} \langle \nu \rangle^{-1-\sigma} < \infty.$$

We have

$$II \lesssim \sum_{\mu \in \mathbb{Z}} \langle \mu \rangle^{\sigma} |\widehat{A}(\mu)| \sum_{|\nu| \le 2|\mu|} \langle \nu - \mu \rangle^{-1} \langle \mu \rangle^{-\sigma} \lesssim \sum_{\mu \in \mathbb{Z}} \langle \mu \rangle^{\sigma} |\widehat{A}(\mu)| < \infty.$$

We write

$$\sum_{|\mu| \le |\nu|/2, \mu \in \nu + 2\mathbb{Z} + 1} \frac{\widehat{A}(\mu)}{\nu - \mu} = \sum_{|\mu| \le |\nu|/2, \mu \in \nu + 2\mathbb{Z} + 1} \frac{\widehat{A}(\mu)}{\nu} + \sum_{|\mu| \le |\nu|/2, \mu \in \nu + 2\mathbb{Z} + 1} \frac{\mu}{(\nu - \mu)\nu} \widehat{A}(\mu) \; .$$

Notice

$$\sum_{\nu\in\mathbb{Z}}\sum_{|\mu|\le|\nu|/2}\frac{|\mu\widehat{A}(\mu)|}{\langle\nu-\mu\rangle\langle\nu\rangle}\lesssim\sum_{\mu\in\mathbb{Z}}|\mu\widehat{A}(\mu)|\sum_{|\nu|\ge 2|\mu|}\langle\nu\rangle^{-2}\lesssim\|\widehat{A}\|_{\ell^1}<\infty.$$

The fact that $A(0)=0$ implies $\sum\widehat{A}(\mu)=0$. The fact that $A(\pi)=0$ implies $\sum(-1)^{\mu}\widehat{A}(\mu)=0$. Hence

$$\sum_{\mu\in 2\mathbb{Z}}\widehat{A}(\mu)=\sum_{\mu\in 2\mathbb{Z}+1}\widehat{A}(\mu)=0.$$

This implies that

$$\sum_{|\mu|\le|\nu|/2,\mu\in\nu+2\mathbb{Z}+1}\widehat{A}(\mu)=-\sum_{|\mu|>|\nu|/2,\mu\in\nu+2\mathbb{Z}+1}\widehat{A}(\mu).$$

Then

$$\sum_{\nu\in\mathbb{Z}\setminus\{0\}}\left|\sum_{|\mu|\le|\nu|/2,\mu\in\nu+2\mathbb{Z}+1}\frac{\widehat{A}(\mu)}{\nu}\right|=\sum_{\nu\in\mathbb{Z}\setminus\{0\}}\left|\sum_{|\mu|>|\nu|/2,\mu\in\nu+2\mathbb{Z}+1}\frac{\widehat{A}(\mu)}{\nu}\right|.$$

This can be bounded with the same argument of III. Hence we have shown $\widehat{f}\in\ell^1$.

Lemma 3.5. *Let $q\in\ell^{1,1+\sigma}$ with $\sigma>0$. Then $U_{112}\in B(L^p,L^p)$ for all $p\in[1,\infty]$.*

Proof. The proof is similar to the previous one. Let $g(\mu,\theta)=A(\mu,\theta)\mathrm{sign}(\theta)$ with $A(\mu,\theta)=(n_+(\mu,\theta)-n_+(\mu,0))\,\chi(\theta)$. Set $\widehat{g}(\mu,\cdot)=F^*[g(\mu,\cdot)]$ and $\widehat{A}(\mu,\cdot)=F^*[A(\mu,\cdot)]$. It is enough to show that there exists $b(\nu)$ in ℓ^1 such that $|\widehat{g}(\mu,\nu)|\le b(\nu)$ for all $\mu\ge 0$ and all $\nu\in\mathbb{Z}$. Notice that $F^*[n_+(\mu,\cdot)-n_+(\mu,0)](\nu)=\chi_{(0,\infty)}(\nu)B_+(\mu,\nu)$ for $\nu\ne 0$ and $=-n_+(\mu,0)$ for $\nu=0$. By Lemma 2.1 we have $|B_+(\mu,\nu)|\le e^{\gamma(0)}\chi_{(0,\infty)}(\nu)\eta(\nu)$. Hence $|\widehat{A}(\mu,\nu)|\le h(\nu)$ for all $\mu\ge 0$ and all $\nu\in\mathbb{Z}$, with $h\in\ell^{1,\sigma}$.

We have

$$\widehat{g}(\mu,\nu)=\frac{2i}{\pi}\sum_{\nu'-\nu\in 2\mathbb{Z}+1}\frac{1}{\nu-\nu'}\widehat{A}(\mu,\nu')=\frac{2i}{\pi}(I+II+III)$$

with

$$I=\sum_{|\nu'|\le|\nu|/2,\nu'\in\nu+2\mathbb{Z}+1}\frac{\widehat{A}(\mu,\nu')}{\nu-\nu'},$$

$$II=\sum_{|\nu|/2<|\nu'|\le 2|\nu|,\nu'\in\nu+2\mathbb{Z}+1}\frac{\widehat{A}(\mu,\nu')}{\nu-\nu'},$$

$$III=\sum_{|\nu'|>2|\nu|,\nu'\in\nu+2\mathbb{Z}+1}\frac{\widehat{A}(\mu,\nu')}{\nu-\nu'}.$$

We have

$$|III(\mu,\nu)| \lesssim \|h\|_{\ell^{1,\sigma}} \langle \nu \rangle^{-1-\sigma}.$$

We have

$$|II(\mu,\nu)| \lesssim \alpha(\nu) := \sum_{|\nu|/2<|\nu'|\le 2|\nu|} \frac{|h(\nu')|}{\langle \nu-\nu' \rangle}.$$

We write

$$\sum_{|\nu'|\le|\nu|/2, \nu'\in\nu+2\mathbb{Z}+1} \frac{\widehat{A}(\mu,\nu')}{\nu-\nu'} = I_1 + I_2$$

$$I_1 = \frac{1}{\nu} \sum_{|\nu'|\le|\nu|/2, \nu'\in\nu+2\mathbb{Z}+1} \widehat{A}(\mu,\nu') , \quad I_2 = \sum_{|\nu'|\le|\nu|/2, \nu'\in\nu+2\mathbb{Z}+1} \frac{\nu'}{(\nu-\nu')\nu} \widehat{A}(\mu,\nu').$$

We have

$$I_1(\mu,\nu) = -\frac{1}{\nu} \sum_{|\nu'|>|\nu|/2, \nu'\in\nu+2\mathbb{Z}+1} \widehat{A}(\mu,\nu')$$

and so

$$|I_1(\mu,\nu)| \lesssim \|h\|_{\ell^{1,\sigma}} \langle \nu \rangle^{-1-\sigma}.$$

Finally

$$|I_2(\mu,\nu)| \lesssim \beta(\nu) := \sum_{|\nu'|\le|\nu|/2,} \frac{\langle \nu' \rangle}{\langle \nu-\nu' \rangle \langle \nu \rangle} h(\nu')$$

Then there is a function $b(\nu)$ in ℓ^1 such that $|\widehat{g}(\mu,\nu)| \le b(\nu)$ of the form $b(\nu) = C(\alpha(\nu) + \beta(\nu) + \langle \nu \rangle^{-1-\sigma})$.

By repeating the previous arguments one has:

Lemma 3.6. *For $q \in \ell^{1,2+\sigma}$ with $\sigma > 0$ the operator W extends into a bounded operator in ℓ^p for $p = 1, \infty$ when operators (3.1)–(3.4) are bounded. Here (3.1) has been defined above while (3.2)–(3.4) are defined as follows, for $\chi + \chi_1$ a smooth partition of unity in $\mathbb{T}$ with $\chi = 1$ near 0 and $\chi = 0$ near π:*

$$\begin{aligned} V_2 f(\mu) &= \chi_{[0,\infty)}(\mu)\, (T(\pi)-1)\, m_+(\mu,0) \int_{-\pi}^{\pi} e^{i\mu\theta} sign(\theta) \chi_1(\theta) F_0[f](\theta) \\ &- \chi_{[0,\infty)}(\mu) R_+(\pi) m_+(\mu,0) \int_{-\pi}^{\pi} e^{-i\mu\theta} sign(\theta) \chi_1(\theta) F_0[f](\theta) d\theta. \end{aligned} \tag{3.2}$$

$$\begin{aligned} V_3 f(\mu) &= \chi_{(-\infty,0)}(\mu)\, (1-T(0))\, m_-(\mu,0) \int_{-\pi}^{\pi} e^{i\mu\theta} sign(\theta) \chi(\theta) F_0[f](\theta) \\ &+ \chi_{(-\infty,0)}(\mu) R_-(0) m_-(\mu,0) \int_{-\pi}^{\pi} e^{-i\mu\theta} sign(\theta) \chi(\theta) F_0[f](\theta) d\theta. \end{aligned} \tag{3.3}$$

$$V_4 f(\mu) = \chi_{(-\infty,0)}(\mu)\,(1-T(0))\,m_-(\mu,0)\int_{-\pi}^{\pi} e^{i\mu\theta} sign(\theta)\chi_1(\theta)F_0[f](\theta)$$
$$+\,\chi_{(-\infty,0)}(\mu)R_-(0)m_-(\mu,0)\int_{-\pi}^{\pi} e^{-i\mu\theta} sign(\theta)\chi_1(\theta)F_0[f](\theta)d\theta. \tag{3.4}$$

We have:

Lemma 3.7. *$W \in B(\ell^p,\ell^p)$ for $p = 1,\infty$ exactly when $T(0) = T(\pi) = 1$.*

Proof. If $T(0) = T(\pi) = 1$ we have $V_j = 0$ for all j. Then $W \in B(\ell^p,\ell^p)$ for $p = 1,\infty$. Viceversa $W \in B(\ell^1,\ell^1)$ implies $V_j \in B(\ell^1,\ell^1)$ for all j. If $V_3 \in B(\ell^1,\ell^1)$ then, proceeding as in Lemma 3.3,

$$1 - T(0) - R_-(0) = 1 - T(0) + R_+(0) = 0.$$

This together with (1) in Lemma 3.3 implies $T(0) = 1$. The implication $T(\pi) = 1$ is obtained similarly.

§A Appendix: finite number of eigenvalues

We will prove:

Lemma A.1. *If $q \in \ell^{1,1}$ the total number of eigenvalues of H is $\le 4 + \|\nu q(\nu)\|_{\ell^1}$.*

Let $q_-(\nu) = \min(0, q(\nu))$. We recall that if we have $(-\Delta + q)u = \lambda u$, then if we define v by $v(\nu) = (-1)^\nu u(\nu)$ we have $(-\Delta - q)v = (4-\lambda)v$. Hence Lemma 6.1 is a consequence of:

Lemma A.2. *If $q \in \ell^{1,1}$ the total number of eigenvalues of H inside $(-\infty,0)$ is $\le 2 + \|\nu q_-(\nu)\|_{\ell^1}$.*

Proof. For $\lambda \le 0$ we set $u(\nu,\lambda) = f_+(\nu,\theta)$, where $\lambda = 2(1-\cos(\theta))$. Notice that $u(\nu,\lambda) \in \mathbb{R}$. We denote by $X(\lambda)$ the set of those ν such that either $u(\nu,\lambda) = 0$ or $u(\nu,\lambda)u(\nu+1,\lambda) < 0$. We denote by $N(\lambda)$ the cardinality of $X(\lambda)$. Notice that by the min-max principle the operator $\widetilde{H} = -\Delta - q_-$ has at least as many negative eigenvalues as H. So, to prove our Lemma 6.2 it is not restrictive to assume $q(\nu) = q_-(\nu) = -|q(\nu)|$ for all ν in Lemma A.3 below. We have:

Lemma 6.3. *We have $N(0) \le 2 + \|\nu q_-(\nu)\|_{\ell^1}$.*

Proof. We assume $N(0) > 1$. Let $\nu_0, \nu_1 \in X(0)$ be two consecutive elements, with $\nu_0 < \nu_1$. For $u(\nu) = u(\nu,0)$ we have

$$u(\nu) = u(\nu_0) + (u(\nu_0+1) - u(\nu_0))(\nu - \nu_0) - \sum_{j=\nu_0}^{\nu-1} (j-\nu_0)|q(j)|u(j).$$

It is not restrictive below to assume $A := u(\nu_0+1) - u(\nu_0) > 0$. Then $u(\nu_1+1) < 0$ or $u(\nu_1) = 0$. In the first case, we have

$$0 > u(\nu_0+1) - u(\nu_1+1) = A(\nu_1 - \nu_0)\left(1 - \sum_{j=\nu_0}^{\nu_1} (j - \nu_0)|q(j)|\right).$$

This implies

$$(1) \qquad \sum_{j=\nu_0+1}^{\nu_1} (j-\nu_0)|q(j)| \geq 1. \text{ By a similar argument } \sum_{j=\nu_0}^{\nu_1-1} (\nu_1 - j)|q(j)| \geq 1.$$

(1) holds also if $u(\nu_1) = 0$. So for $\nu_0 < \nu_1 < ... < \nu_n$ consecutive elements in $X(0)$,

$$\text{we have } \sum_{j=\nu_0+1}^{\nu_n} (j-\nu_0)|q(j)| \geq n \text{ and } \sum_{j=\nu_0}^{\nu_n-1} (\nu_n - j)|q(j)| \geq n.$$

Then $q \in \ell^{1,1}$ implies $N(0) < \infty$. If $X(0)$ is formed by

$$\nu_0 < ... < \nu_n (< 0 \leq) \mu_0 < ... < \mu_m$$

then

$$n \leq \sum_{j=\nu_0}^{\nu_n-1} (\nu_n - j)|q(j)| \leq \sum_{j=\nu_0}^{\nu_n-1} |j||q(j)|$$

and

$$m \leq \sum_{j=\mu_0+1}^{\mu_m} (j-\mu_0)|q(j)| \leq \sum_{j=\mu_0+1}^{\mu_m} |j||q(j)|.$$

So $n + m \leq \|\nu q(\nu)\|_{\ell^1}$. Then $N(0) \leq 2 + \|\nu q(\nu)\|_{\ell^1}$. This yields Lemma 6.2.

Notice that

$$\langle Hu, u\rangle = \sum_{\nu\in\mathbb{Z}} |u(\nu+1) - u(\nu)|^2 + \sum_{\nu\in\mathbb{Z}} q(\nu)|u(\nu)|^2.$$

If H has negative eigenvalues, there is a minimal one λ_0. Then we have $u(\nu, \lambda_0) = |u(\nu, \lambda_0)| > 0$ for all ν by the min-max principle and by the fact that $u(\nu, \lambda_0) = e^{i\nu\theta} m_+(\nu, \theta_0)$ where $m_+(\nu, \theta) \to 1$ for $|\nu| \nearrow \infty$ by (1) Lemma 5.1 [CT]. Notice that by this argument it is easy to conclude that $N(\lambda) < \infty$ for any $\lambda < 0$.

Next we have the following discrete version of the Sturm oscillation theorem, see Lemma 4.4 [T].

Lemma A.4. *$N(\lambda)$ is increasing for $\lambda \leq 0$.*

Lemmas A.4 and A.3 yield Lemma A.2.

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DISMI University of Modena and Reggio Emilia, via Amendola 2, Padiglione Morselli, Reggio Emilia 42100 Italy

E-mail address: cuccagna.scipio@unimore.it