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ON MULTIPLICATIVE TIME-DEPENDENT PERTURBATIONS OF SEMIGROUPS AND COSINE FAMILIES GENERATORS

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Abstract. In this work we aim to investigate a second order PDE modelling a vibrating string. Our strategy consists in transforming the PDE problem into a semilinear second order ODE in a suitable infinite dimensional space. Since the tension coefficient of the PDE may vary with time, the linear operator of the ODE depends on time. We therefore provide sufficient conditions guaranteeing that a suitable family of unbounded linear operators generates a fundamental system.

1. **Introduction.** Nonlinear partial differential equations, and specifically second-order partial differential equations evolving in time, play a crucial role in describing a range of problems in physics, biology and many other fields [3, 13, 23]. In this work, we aim to investigate the PDE problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = a(t)\dfrac{\partial^2 u}{\partial \xi^2} + g\left(t, \xi, u, \dfrac{\partial u}{\partial t}\right), & t \in [0,b], \xi \in (0,1) \\ u(t,0) = u(t,1) = 0 \quad t \in [0,b] \end{cases} \tag{1.1}$$

where a and g are given functions. This kind of equation arises in many areas of applied mathematics and in particular in physics [3]. This is, indeed, as a wave equation when considering a vibrating string or a vibrating membrane. In the case of a vibrating string along the $x-$axis, $u(t,x)$ represents the displacement of the string from its equilibrium position at time t and position x. In this scenario, the coefficient $a(t)$ represents properties like tension or stiffness that may vary with time, while the function $g\left(t, \xi, u, \frac{\partial u}{\partial t}\right)$ describes any external forces acting on the string, or nonlinear effects such as damping or nonlinear restoring forces.

Motivated by a variety of applications, many authors have been studying the existence of solutions to partial differential equations as well as to the corresponding boundary value problems. For this reason, a new approach in literature has been appearing in the study of some types of partial differential equations recently. It consists in transforming the partial differential equation that governs the model into an ordinary differential equation in a suitable infinite dimensional space. Following

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this approach, we denote $u(t,\cdot) = x(t)$ and we reformulate equation (1.1) in its abstract form. In particular, we rewrite problem (1.1) as

$$\ddot{x}(t) = \mathbf{A}(t)x(t) + f(t, x(t), \dot{x}(t)), \quad t \in [0, b], \tag{1.2}$$

and we assume suitable conditions on the functions a and g such that, denoted E as the Banach space $L^p([0,1]), D(A)$ as the dense subspace $W^{2,p}([0,1]) \cap W_0^{1,p}([0,1])$ of $E, A : D(A) \to E$ as the linear operator

$$Ay = \ddot{y}, \tag{1.3}$$

then the linear operator $\mathbf{A}(t) : D(A) \to E$

$$\mathbf{A}(t)y = a(t)Ay \tag{1.4}$$

and the function $f : [0,T] \times E \times X \to E$

$$f(t, y) = g(t, \cdot, y, \dot{y}),$$

are well defined, with X to be specified. The existence of a solution of the original equation, subject to a given boundary condition, is then equivalent to the existence of a solution for its abstract formulation.

Many results were obtained in the literature for the ordinary differential equation of type (1.2) in a Banach space E when, for every t, $\mathbf{A}(t)$ is a linear bounded operator in the whole space E, implying that the solutions are classical, i.e. are differentiable with an absolutely continuous derivative and $X = E$. The theory in the case of a constant unbounded operator A dates back to the paper by Sova [20], where a necessary and sufficient condition is provided in order that A generates a cosine family $\{\tilde{C}(t)\}_t$. Then, if the nonlinear term f does not depend on $\dot{x}(t)$, the solution of the associated Cauchy problem is mild, i.e. a continuous function satisfying the variation constants formula. The first paper dealing with the more general equation (1.2), still with A constant, is due to Travis-Webb [21]. Then it is still possible to consider mild solutions for the Cauchy problem, but trivially they need to be continuously differentiable. In [21] it was proved that this holds if and only if $x(0)$ belongs to the subspace $\tilde{X}$ defined as

$$\tilde{X} := \{x \in E : \tilde{C}(\cdot)x \text{ is continuously differentiable }\}. \tag{1.5}$$

The theory when A is not constant, hence generates a fundamental system $\{S(t,s)\}_{t \geq s}$, traces back to Kozak [10, 11]. However, to the best of our knowledge, up to now there are only few sufficient conditions guaranteeing that A generates a fundamental system (see [2, 6]). This is due to the fact that the construction of the solution of a second order equation is obtained reducing it to a first order system. Well-known necessary and sufficient conditions ensure that an unbounded linear operator generates a strongly continuous semigroup $\{T(t)\}_t$. They were firstly proved by Hille-Yosida in the case of a contraction semigroup and then extended by Feller-Miyadera-Phillips in the general case (see, e.g. [22]). On the other hand, only sufficient conditions guaranteeing that a family of unbounded linear operators generate an evolution system $\{U(t,s)\}_{t \geq s}$ were obtained in the literature (see, e.g. [8, 12, 16, 17, 18]) and this affects the theory on fundamental systems and their generators. Moreover, up to our knowledge, the only paper dealing with A not constant and f depending also on $\dot{x}(t)$ is [15], where the existence of a mild solution is obtained under the condition that $x(0)$ belongs to

$$X := \{x \in E : C(\cdot, s)x \text{ is continuously differentiable } \forall\, s \geq 0\}, \tag{1.6}$$

where the operator $C(t,s)$ is related to the fundamental system according to (2.7). Motivated by these reasons, in this paper we provide sufficient conditions guaranteeing that a family of unbounded linear operators of kind (1.4), where $a:[0,b]\to[0,+\infty)$ and A is the generator of a cosine family $\{\tilde{C}(t)\}_t$, generates a fundamental system. This is a novelty, as we point out that the results in [2, 6] do not cover this case. Our result is based on a reduction to the associated first order problem, hence we find an explicit formula for the fundamental system in terms of the evolution system generated by

$$\boldsymbol{\mathcal{A}}(t)=\begin{pmatrix} 0 & \sqrt{a(t)}I \\ \sqrt{a(t)}A & 0 \end{pmatrix}.$$

This allows us to prove that the subspace X in (1.6) coincides with the analogous subspace $\tilde{X}$ defined in (1.5), hence to apply the result in [15], getting a solution of (1.2).

The plan of the paper is the following. After recalling useful definitions and preliminaries in Section 2, we are ready to present our original results in Section 3. In Theorem 3.1 we focus on the first-order problem, proving a sufficient condition ensuring that a suitable family of operators generates an evolution system, whose expression is found explicitly. Then, in Theorem 3.3, requiring more regularity on $a(t)$, we exploit the previous result to provide sufficient conditions to generate a fundamental system, again presenting an explicit formula for it. Eventually, in Theorem 3.6, we show the well-posedness of the Cauchy problem associated to the homogeneous equation $\ddot{x}(t)=\mathbf{A}(t)x(t)$.

2. Preliminaries results.

Let E be a Banach space and $\mathcal{L}(E)$ the Banach space of all bounded linear operators in E. Given two Banach spaces E_1, E_2, π_1 and π_2 denote respectively the natural projection on the first and on the second space, i.e.

$$\pi_1\begin{pmatrix} x \\ y \end{pmatrix}=x, \qquad \pi_2\begin{pmatrix} x \\ y \end{pmatrix}=y. \tag{2.1}$$

In the paper, given $\Omega\subset\mathbb{R}^n$ compact we denote by $C(\Omega)$ the Banach space of continuous functions with norm

$$\|z\|_0=\max_{t\in\Omega}\|z(r)\|$$

and by $L^p(\Omega)$ the Banach space of functions having Lebesgue integrable p-power with norm

$$\|z\|_p=\left(\int_\Omega \|z(r)\|^p\,dr\right)^{\frac{1}{p}}.$$

We briefly introduce the notion of strongly continuous semigroup, group, evolution system, evolution operator and their generators and we refer to [16] and [22] for further details on the theory of semigroups and to [12] for the theory of evolution systems.

Definition 2.1. A family of linear, bounded operators $\{T(t)\}_{t\geq 0}$, with $T(t):E\to E$, is called a C_0-*semigroup* if the following conditions are satisfied:

1. $T(0)=I$;
2. $T(t+s)=T(t)T(s)$ for $t,s\in[0,\infty)$;
3. T is strongly continuous, i.e. the function $t\mapsto T(t)z$ is continuous on $[0,\infty)$, for every $z\in E$.

Definition 2.2. The *infinitesimal generator* of $\{T(t)\}_{t\geq 0}$ is the linear, closed and densely defined operator $\mathcal{A}$ defined by

$$\mathcal{A}z = \lim_{h\to 0^+} \frac{(T(h) - I)\, z}{h},\ z \in D(\mathcal{A})$$

with

$$D(\mathcal{A}) := \left\{ z \in E\ :\ \lim_{h\to 0^+} \frac{(T(h) - I)\, z}{h} \text{ exists } \right\}.$$

As a straightforward consequence of the second property in Definition 2.1 and the boundedness of $T(t)$ for every t, we obtain that

$$T(t)z \in D(\mathcal{A}) \quad t \geq 0, z \in D(\mathcal{A}) \tag{2.2}$$

and

$$\mathcal{A}T(t)z = T(t)\mathcal{A}z \quad t \geq 0, z \in D(\mathcal{A}). \tag{2.3}$$

It is well known that, if $\mathcal{A} \in \mathcal{L}(E)$, then $\mathcal{A}$ generates the C_0-semigroup

$$T(t) = e^{\mathcal{A}t} = \sum_{n=0}^{+\infty} \frac{\mathcal{A}^n t^n}{n!}.$$

Definition 2.3. A family of linear, bounded operators $\{T(t)\}_{t\in\mathbb{R}}$, with $T(t) : E \to E$, is called a C_0-*group* and the linear, closed and densely defined operator $\mathcal{A}$ the *infinitesimal generator* of $\{T(t)\}_{t\in\mathbb{R}}$ if the conditions in Definitions 2.1 and 2.2 are satisfied in $\mathbb{R}$.

Lemma 2.4. *(see* [17, Chapter 1.6] *A is the infinitesimal generator of a strongly continuous group if and only if both A and $-A$ are infinitesimal generators of strongly continuous semigroups respectively denoted by $\{T_+(t)\}_t$ and $\{T_-(t)\}_t$ and $T_-(t) = (T_+(t))^{-1}$ for every $t \geq 0$. Hence A is the infinitesimal generator of the strongly continuous group*

$$T(t) = \begin{cases} T_-(t) & \text{if } t < 0 \\ T_+(t) & \text{if } t \geq 0. \end{cases}$$

Definition 2.5. Let $\Delta := \{(t, s) \in [0, b] \times [0, b]\ :\ 0 \leq s \leq t \leq b\}$ and E be a Banach space. A two parameter family $\{U(t, s)\}_{(t,s)\in\Delta}$, where $U(t, s) : E \to E$ is a bounded linear operator, is called an *evolution system* if the following conditions are satisfied:

1. $U(s, s) = I,\ 0 \leq s \leq b$;
2. $U(t, r)U(r, s) = U(t, s),\ 0 \leq s \leq r \leq t \leq b$;
3. U is strongly continuous, i.e. the map $(t, s) \mapsto U(t, s)z$ is continuous on Δ for every $z \in E$.

To every evolution system we can assign the corresponding *evolution operator* $U : \Delta \to \mathcal{L}(E)$.

Since the evolution operator U is strongly continuous on the compact set Δ, by the uniform boundedness theorem there exists a constant D such that

$$\|U(t, s)\| \leq D, \quad \text{for all } (t, s) \in \Delta. \tag{2.4}$$

Definition 2.6. Let $\{\boldsymbol{\mathcal{A}}(t)\}_t$ be a family of linear not necessarily bounded operator, with $\boldsymbol{\mathcal{A}}(t) : D(\mathcal{A}) \subset E \to E$ and $D(\mathcal{A})$ a dense subset of E not depending on t. We say that $\{\boldsymbol{\mathcal{A}}(t)\}_t$ *generates an evolution operator* if there exists an evolution system $\{U(t, s)\}_{(t,s)\in\Delta}$ with $U : \Delta \to \mathcal{L}(E)$ strongly differentiable in $D(\mathcal{A})$ with respect to t and s, i.e. for every $(t, s) \in \Delta, z \in D(\mathcal{A})$

1. $U(t,s)z \in D(\mathcal{A})$
2. $\dfrac{\partial U(t,s)}{\partial t} z = \boldsymbol{\mathcal{A}}(t)U(t,s)z$
3. $\dfrac{\partial U(t,s)}{\partial s} z = -U(t,s)\boldsymbol{\mathcal{A}}(s)z.$

It is well known that in the case of a constant operator $\boldsymbol{\mathcal{A}}(t) \equiv \mathcal{A}$ generating a C_0-semigroup $\{T(t)\}_{t\geq 0}$, then $\{\boldsymbol{\mathcal{A}}(t)\}_t$ generates the evolution system $U(t,s) = T(t-s)$ as well.

We now introduce the notion of strongly continuous cosine family, fundamental system, fundamental operator and their generators.

Definition 2.7. A one-parameter family $\{\tilde{C}(t)\}_{t\in\mathbb{R}}$ of bounded linear operators mapping the space E into itself is called a *strongly continuous cosine family* if:

1. $\tilde{C}(t+s) + \tilde{C}(s-t) = 2\tilde{C}(s)\tilde{C}(t)$, for all $t,\ s \in \mathbb{R}$;
2. $\tilde{C}(0) = I$;
3. The map $t \to \tilde{C}(t)y$ is continuous in $\mathbb{R}$ for each fixed $y \in E$.

Definition 2.8. The *infinitesimal generator* of $\{\tilde{C}(t)\}_{t\in\mathbb{R}}$ is the linear, closed and densely defined operator A defined by

$$Ay = \frac{d^2}{dt^2}\Big[\tilde{C}(t)y\Big]_{t=0} = 2\lim_{t\to 0^+} \frac{\tilde{C}(t)y - y}{t^2},\ y \in D(A)$$

with

$$D(A) = \left\{ y \in E : \lim_{t\to 0^+} \frac{\tilde{C}(t)y - y}{t^2} \text{ exists } \right\}.$$

Definition 2.9. The one-parameter family $\{\tilde{S}(t)\}_{t\in\mathbb{R}}$ of bounded linear operators mapping the space E into itself defined, for all $t \in \mathbb{R}$ and $y \in E$, by

$$\tilde{S}(t)y = \int_0^t \tilde{C}(s)y\, ds \tag{2.5}$$

is called the *strongly continuous sine family* associated to the cosine family.

Lemma 2.10. *(see [21]) For every $y \in \tilde{X}, t \in \mathbb{R}$*

1. $\tilde{S}(t)y \in D(A)$;
2. $\frac{d}{dt}\tilde{C}(t)y = A\tilde{S}(t)y$.

Lemma 2.11. *(see [9]) A is the generator of the cosine family $\{\tilde{C}(t)\}_{t\in\mathbb{R}}$ if and only if the set $\tilde{X}$ defined in (1.5) endowed with the norm*

$$||y||_{\tilde{X}} = ||y||_E + \max_{t\in[0,1]} ||AS(t)y||_E$$

is a Banach space, where the maximum is achieved according the compactness of $[0,1]$, *and the operator valued function*

$$T(t) = \begin{pmatrix} \tilde{C}(t) & \tilde{S}(t) \\ A\tilde{S}(t) & \tilde{C}(t) \end{pmatrix}$$

is a strongly continuous group of bounded linear operators in $\tilde{X} \times E$ generated by the operator

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix} \tag{2.6}$$

defined on $D(A) \times \tilde{X}$.

Definition 2.12. A two parameter family $\{S(t,s)\}_{t,s\geq 0}$, where $S(t,s): E \to E$ is a bounded linear operator, is called the *fundamental system* generated by a family of linear operators $\{\mathbf{A}(t)\}_t$ defined in the dense and closed subset $D(A)$ of E if

a) for each $y \in E$, the mapping $(t,s) \mapsto S(t,s)y$ is of class C^1;
b) for each $t \geq 0$, $S(t,t) = 0$;
c) for all $t, s \geq 0$ and each $y \in E$,
$$\frac{\partial}{\partial t} S(t,s)\Big|_{t=s} y = y, \quad \frac{\partial}{\partial s} S(t,s)\Big|_{t=s} y = -y;$$
d) for all $t, s \geq 0$, if $y \in D(A)$, then $S(t,s)y \in D(A)$, the mapping $(t,s) \mapsto S(t,s)y$ is of class C^2 and
$$\frac{\partial^2}{\partial t^2} S(t,s)y = \mathbf{A}(t)S(t,s)y,$$
$$\frac{\partial^2}{\partial s^2} S(t,s)y = S(t,s)\mathbf{A}(s)y,$$
$$\frac{\partial^2}{\partial s \partial t} S(t,s)\Big|_{t=s} y = 0;$$
e) for all $t, s \geq 0$, if $y \in D(A)$, then $\frac{\partial}{\partial s} S(t,s)y \in D(A)$, the mapping $(t,s) \mapsto \mathbf{A}(t)\frac{\partial}{\partial s} S(t,s)y$ is continuous, and
$$\frac{\partial^3}{\partial t^2 \partial s} S(t,s)y = \mathbf{A}(t)\frac{\partial}{\partial s} S(t,s)y$$
$$\frac{\partial^3}{\partial s^2 \partial t} S(t,s)y = \frac{\partial}{\partial t} S(t,s)\mathbf{A}(s)y.$$

To every fundamental system we can assign the corresponding *fundamental operator* $S: [0,+\infty) \times [0,+\infty) \to \mathcal{L}(E)$.

Since $S(t,s)$ is of class C^1, we introduce, for each $t, s \geq 0$, the linear and bounded operator
$$C(t,s) := -\frac{\partial}{\partial s} S(t,s). \tag{2.7}$$

3. **Main results.** In this section, we first provide sufficient conditions guaranteeing that a family of operators $\{\boldsymbol{\mathcal{A}}(t)\}_{t\in\Delta}$ generates an evolution system and we find an explicit formula for the evolution system.

Theorem 3.1. *Let $\mathcal{A}: D(\mathcal{A}) \subset E \to E$ be the generator of a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$. Suppose that $a: [0,b] \to [0,+\infty)$ is a continuous function. Then $\boldsymbol{\mathcal{A}}(t) = \sqrt{a(t)}\mathcal{A}$ is the generator of the evolution system*
$$U(t,s) := T\left(\int_s^t \sqrt{a(\tau)}\, d\tau\right). \tag{3.1}$$

Proof. The proof is divided into two steps:

Step 1. $\{U(t,s)\}_{(t,s)\in\Delta}$ in (3.1) is an evolution system.

Step 2. $\{\boldsymbol{\mathcal{A}}(t)\}_t$ generates $\{U(t,s)\}_{t,s\in\Delta}$.

Step 1. We prove that $\{U(t,s)\}_{t,s\in\Delta}$ verifies Definition 2.5.

Claim 1. *$U(t,s)$ is linear and bounded for every $t, s \in \Delta$;*

It follows from the analogous property of $T(t)$ for every $t \geq 0$.

Claim 2. *$U(s,s) = I,\ 0 \le s \le b$;*

The claim easily follows from the fact that $\int_s^s \sqrt{a(r)}\,dr = 0$ for every $s \in [0,b]$ and the first property in Definition 2.1.

Claim 3. *$U(t,r)U(r,s) = U(t,s),\ 0 \le s \le r \le t \le b$;*

According to the second property in Definition 2.1, for every $s, r, t \in [0,b]$, with $s \le r \le t$,

$$\begin{aligned} U(t,r)U(r,s) = &\ T\Big(\int_r^t \sqrt{a(\tau)}\,d\tau\Big)T\Big(\int_s^r \sqrt{a(\tau)}\,d\tau\Big) \\ = &\ T\Big(\int_r^t \sqrt{a(\tau)}\,d\tau + \int_s^r \sqrt{a(\tau)}\,d\tau\Big) = T\Big(\int_s^t \sqrt{a(\tau)}\,d\tau\Big) = U(t,s). \end{aligned}$$

Claim 4. *the map $(t,s) \mapsto U(t,s)y$ is continuous on Δ for every $y \in E$.*

It follows from the third property in Definition 2.1, the absolute continuity of the Lebesgue integral and the continuity of the composition of continuous functions in metric spaces.

Step 2. We prove that $\{U(t,s)\}_{t,s\in\Delta}$ verifies Definition 2.6.

Claim 1. *$U(t,s)y \in D(\mathcal{A})$ for every $(t,s) \in \Delta, y \in D(\mathcal{A})$;*

It follows from the definition of $U(t,s)$ and (2.2).

Claim 2. *$\dfrac{\partial U(t,s)}{\partial t} y = \boldsymbol{\mathcal{A}}(t)U(t,s)y$ for every $(t,s) \in \Delta, y \in D(\mathcal{A})$;*

Fix $(t,s) \in \Delta, y \in D(\mathcal{A})$ and take a positive sequence $\{h_n\}_n$ converging to 0 such that there exists the limit

$$\lim_{n\to+\infty} \frac{U(t+h_n,s)y - U(t,s)y}{h_n}.$$

According to the second property of Definition 2.1,

$$\begin{aligned} &\lim_{n\to+\infty} \frac{U(t+h_n,s)y - U(t,s)y}{h_n} = \\ &\lim_{n\to+\infty} \frac{T\Big(\int_s^{t+h_n} \sqrt{a(\tau)}\,d\tau\Big)y - T\Big(\int_s^t \sqrt{a(\tau)}\,d\tau\Big)y}{h_n} = \\ &\lim_{n\to+\infty} \frac{T\Big(\int_s^t \sqrt{a(\tau)}\,d\tau + \int_t^{t+h_n} \sqrt{a(\tau)}\,d\tau\Big)y - T\Big(\int_s^t \sqrt{a(\tau)}\,d\tau\Big)y}{h_n} = \\ &\lim_{n\to+\infty} \frac{T\Big(\int_t^{t+h_n} \sqrt{a(\tau)}\,d\tau\Big)T\Big(\int_s^t \sqrt{a(\tau)}\,d\tau\Big)y - T\Big(\int_s^t \sqrt{a(\tau)}\,d\tau\Big)y}{h_n} = \\ &\lim_{n\to+\infty} \frac{\Big[T\Big(\int_t^{t+h_n} \sqrt{a(\tau)}\,d\tau\Big) - I\Big]T\Big(\int_s^t \sqrt{a(\tau)}\,d\tau\Big)y}{h_n}. \end{aligned} \tag{3.2}$$

From the continuity of a and the mean value theorem, it follows that, for every n, there exists $\tau_n \in (t, t+h_n)$ such that

$$\int_t^{t+h_n} \sqrt{a(\tau)}\,d\tau = \sqrt{a(\tau_n)}h_n$$

and $a(\tau_n) \to a(t)$ when $n \to +\infty$. If $a(t) \neq 0$, then we may assume w.l.o.g. that for every n, $\sqrt{a(\tau_n)} \neq 0$, hence

$$\int_t^{t+h_n} \sqrt{a(\tau)}\, d\tau \neq 0,$$

and the absolute continuity of the integral and Definition 2.2 imply that, from (3.2), it follows

$$\begin{aligned}
&\lim_{n\to+\infty} \frac{U(t+h_n,s)y - U(t,s)y}{h_n} = \\
&\lim_{n\to+\infty} \frac{\left[T\left(\int_t^{t+h_n} \sqrt{a(\tau)}\, d\tau\right) - I\right] T\left(\int_s^t \sqrt{a(\tau)}\, d\tau\right) y}{h_n} = \\
&\lim_{n\to+\infty} \frac{\int_t^{t+h_n} \sqrt{a(\tau)}\, d\tau}{h_n} \frac{\left[T\left(\int_t^{t+h_n} \sqrt{a(\tau)}\, d\tau\right) - I\right] T\left(\int_s^t \sqrt{a(\tau)}\, d\tau\right) y}{\int_t^{t+h_n} \sqrt{a(\tau)}\, d\tau} = \\
&\lim_{n\to+\infty} \sqrt{a(\tau_n)} \frac{\left[T\left(\int_t^{t+h_n} \sqrt{a(\tau)}\, d\tau\right) - I\right] T\left(\int_s^t \sqrt{a(\tau)}\, d\tau\right) y}{\int_t^{t+h_n} \sqrt{a(\tau)}\, d\tau} = \\
&\sqrt{a(t)}\mathcal{A} T\left(\int_s^t \sqrt{a(\tau)}\, d\tau\right) y = \boldsymbol{\mathcal{A}}(t) U(t,s)y.
\end{aligned}$$

If $a(t) = 0$, according to the continuity of a, we may assume w.l.o.g. that

$$\int_t^{t+h_n} \sqrt{a(\tau)}\, d\tau \neq 0 \quad \forall\, n$$

or

$$\int_t^{t+h_n} \sqrt{a(\tau)}\, d\tau = 0 \quad \forall\, n.$$

In the first case, reasoning as above, it is possible to prove that

$$\lim_{n\to+\infty} \frac{U(t+h_n,s)y - U(t,s)y}{h_n} = \boldsymbol{\mathcal{A}}(t) U(t,s)y.$$

In the second case, according to the first property of Definition 2.1,

$$\begin{aligned}
&\lim_{n\to+\infty} \frac{U(t+h_n,s)y - U(t,s)y}{h_n} = \\
&\lim_{n\to+\infty} \frac{\left[T\left(\int_t^{t+h_n} \sqrt{a(\tau)}\, d\tau\right) - I\right] T\left(\int_s^t \sqrt{a(\tau)}\, d\tau\right) y}{h_n} = \\
&\lim_{n\to+\infty} \frac{\left[T(0) - I\right] T\left(\int_s^t \sqrt{a(\tau)}\, d\tau\right) y}{h_n} = \\
&0 = \sqrt{a(t)}\mathcal{A} U(t,s)y = \boldsymbol{\mathcal{A}}(t) U(t,s)y.
\end{aligned}$$

Therefore, in both cases

$$\lim_{n\to+\infty} \frac{U(t+h_n,s)y - U(t,s)y}{h_n} = \boldsymbol{\mathcal{A}}(t) U(t,s)y.$$

Since it is possible to prove that, if $t > s$, the same conclusion holds for any negative sequence converging to 0, we get that

$$\frac{\partial U(t,s)}{\partial t} y = \boldsymbol{\mathcal{A}}(t) U(t,s) y.$$

Claim 3. $\frac{\partial U(t,s)}{\partial s} y = -U(t,s)\boldsymbol{\mathcal{A}}(s) y$ *for every* $(t,s) \in \Delta, y \in D(\mathcal{A})$.

Similarly as in the proof of Claim 2, since

$$\lim_{h\to 0} \int_{s+h}^{s} \sqrt{a(\tau)}\, d\tau = -\sqrt{a(s)},$$

recalling (2.3), it follows, for every $(t,s) \in \Delta, y \in D(\mathcal{A})$,

$$\frac{\partial U(t,s)}{\partial s} y = -T\left(\int_s^t \sqrt{a(\tau)}\, d\tau\right) \sqrt{a(s)} \mathcal{A} y = -U(t,s) \boldsymbol{\mathcal{A}}(s) y.$$

□

Remark 3.2. Notice that the proofs of Step 1 and Claim 1 of Step 2 of Theorem 3.1 work also under the weaker assumption that the function $\sqrt{a}$ is just Lebesgue integrable.

In [1] the authors consider equation

$$\frac{\partial u}{\partial t} = -a(t,\xi)u + \int_{\xi}^{+\infty} a(t,y) b(t,\xi,y) u(t,y)\, dy, \quad t > 0, \xi > 0$$

which models multiple fragmentation process with time dependent coefficients. Our result covers a slightly different situation with respect to the one considered in [1], since a does not depend on ξ, but $\mathcal{A}$ is quite general, while in [1] $\mathcal{A} = -I$, i.e. it is bounded in the whole space, but the coefficient a depends also on ξ. The authors find an explicit formula for the evolution system, analogous to (3.1) under the assumption that $a \in L^1([0,b], L^\infty([0,1]))$, which implies that $\sqrt{a} \in L^1([0,b], L^\infty([0,1]))$. Notice however that even in the simple case when $\sqrt{a}$ is independent from ξ, its integrability is not sufficient to guarantee that $\{\boldsymbol{\mathcal{A}}(t)\}_t$ generates $\{U(t,s)\}_{t\ge s}$. In fact, consider e.g.

$$a(t) = \begin{cases} 0 \text{ if } & t \le \frac{b}{2} \\ 1 \text{ if } & t > \frac{b}{2}. \end{cases}$$

Then, reasoning like in the proof of Theorem 3.1, for every $y \in D(\mathcal{A}), s \le \frac{b}{2}$ it follows that

$$\begin{aligned} \lim_{h\to 0^+} \frac{U(\frac{b}{2}+h,s)y - U(\frac{b}{2},s)y}{h} &= \lim_{h\to 0^+} \frac{\left[T\left(\int_{\frac{b}{2}}^{\frac{b}{2}+h} d\tau\right) - I\right] T(0) y}{h} \\ &= \lim_{h\to 0^+} \frac{[T(h)-I]U(\frac{b}{2},s)y}{h} = \mathcal{A} U\left(\frac{b}{2}, s\right) y \\ &\ne \sqrt{a(\tfrac{b}{2})} \mathcal{A} U(\tfrac{b}{2},s) y, \end{aligned}$$

thus $\{U(t,s)\}_{t\ge s}$ does not verify Definition 2.6.

In [14] (see also [5]) a sufficient condition for a family of operators $\{\boldsymbol{\mathcal{A}}(t)\}_{t\in I}$, where I is a compact real interval, to generate an evolution system is given. In the case when the domain of $D(\boldsymbol{\mathcal{A}}(t))$ is independent from t, it requires that, denoted by $\{e^{s\boldsymbol{\mathcal{A}}(t)}\}_{s\ge 0}$ the semigroup generated by $\boldsymbol{\mathcal{A}}(t)$, the semigroups pairwise commute, the mapping $t \mapsto \boldsymbol{\mathcal{A}}(t)y$ is continuous for every $y \in D(\mathcal{A})$ and the family $\{\boldsymbol{\mathcal{A}}(t)\}_t$ satisfies a certain stability condition. Then, it is claimed that, when the initial stability is replaced by the stronger Kato stability, i.e. when there exists $M \geq 1, \omega \in \mathbb{R}$, such that for any partition $P = \{t_0, \dots, t_m\}$ of I and any finite subset $\{s_1, \dots s_m\}$ of I,

$$\left\| \prod_{j=0}^{m} e^{s_j \mathcal{A}(t_j)} \right\| \leq M e^{\omega \sum_{j=0}^{m} s_j}, \tag{3.3}$$

the evolution system generated by $\{\boldsymbol{\mathcal{A}}(t)\}_t$ is

$$U(t, s) = e^{\int_s^t \boldsymbol{\mathcal{A}}(r) dr}. \tag{3.4}$$

The assumptions we required in Theorem 3.1 imply those in [14], since, in particular (3.3) holds for $M = M_T$ and $\omega = \omega_T \|a\|_0$, where

$$\|T(t)\| \leq M_T e^{\omega_T t}$$

for every $t \geq 0$. However, equality (3.4) is subject to interpretations, since, according to Definition 2.5, $U(t, s) \in \mathcal{L}(E)$, while, by the definition of Bochner integral, for every $(t, s) \in \Delta, \int_s^t \mathcal{A}(r)dr : D(\mathcal{A}) \to E$, hence the map $e^{\int_s^t \mathcal{A}(r)dr} : I \to \mathcal{L}(E)$. On the other hand, our discussion overcomes this contradictory issue. Indeed, in Theorem 3.1, we are able to present a very simple and direct proof of the thesis and we provide an explicit formula for the evolution system generated by $\{a(t)\mathcal{A}\}_t$, namely (3.1). Namely, we have that for every $(t, s) \in \Delta, \int_s^t a(r)dr \in \mathbb{R}$, hence the operator in its right hand side belongs to $\mathcal{L}(E)$ as $U(t, s)$ does.

We now prove our main theorem, providing sufficient conditions guaranteeing that a family of operators $\{\mathbf{A}(t)\}_t$ generates a fundamental system and finding an explicit formula for the fundamental system itself.

Theorem 3.3. *Let $A : D(A) \subset E \to E$ be the generator of a cosine family $\{\tilde{C}(t)\}_{t \geq 0}$. Suppose that $a : [0, b] \to (0, +\infty)$ is a continuously differentiable function. Denote by $\tilde{X}$ the subspace defined in (1.5).*
Then $\mathbf{A}(t) = a(t)A$ is the generator of the fundamental system

$$S(t, s) := \frac{1}{\sqrt{a(s)}} \left[\tilde{S}\left(\int_s^t \sqrt{a(r)}\, dr \right) y + \int_s^t c(r) \tilde{S}\left(\int_r^t \sqrt{a(\tau)}\, d\tau \right) \left(\frac{\partial S(r, s)}{\partial r} y \right) dr \right], \tag{3.5}$$

where

$$c(t) := -\frac{1}{2} \frac{a'(t)}{a(t)} \tag{3.6}$$

and $\{\tilde{S}(t)\}_t$ is the sine family associated to $\{\tilde{C}(t)\}_t$. Moreover $X = \tilde{X}$.

Proof. Given $s \geq 0, y_s, y'_s \in E$, the Cauchy problem

$$\begin{cases} \ddot{u} = & a(t)Au \\ u(s) = & y_s \\ \dot{u}(s) = & y'_s \end{cases} \tag{3.7}$$

is uniquely solvable if and only if $\{\mathbf{A}(t)\}_t$ generates a fundamental system $\{S(t, s)\}_{t,s \geq 0}$ and the solution of (3.7) is

$$u(t) = C(t, s)y_s + S(t, s)y'_s \tag{3.8}$$

(see [10]). Since the map $(t, s) \mapsto S(t, s)y'_s$ is of class C^1 for every $y'_s \in E$, the mild solution is continuously differentiable if and only if $y_s \in \tilde{X}$ (see [21]), with

$$\dot{u}(t) = \frac{\partial C(t,s)}{\partial t} y_s + \frac{\partial S(t,s)}{\partial t} y'_s. \tag{3.9}$$

Moreover, denoting by $\mathcal{A}$ the operator matrix defined in (2.6), c the function defined in (3.6),

$$\mathcal{B} := \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}, \tag{3.10}$$

and

$$z := \begin{pmatrix} u \\ \frac{\dot{u}}{\sqrt{a}} \end{pmatrix},$$

we have that (3.7) is equivalent to the Cauchy problem

$$\begin{cases} \dot{z} = & [\sqrt{a(t)}\mathcal{A} + c(t)\mathcal{B}]z \\ z(s) = & \begin{pmatrix} y_s \\ \frac{y'_s}{\sqrt{a(s)}} \end{pmatrix} \end{cases} \tag{3.11}$$

(see [2]). The Cauchy problem (3.11) is uniquely solvable if and only if $\sqrt{a(t)}\mathcal{A} + c(t)\mathcal{B}$ generates an evolution system $\{V(t,s)\}_{t,s}$ and the solution of (3.11) is

$$z(t) = V(t,s) \begin{pmatrix} y_s \\ \frac{y'_s}{\sqrt{a(s)}} \end{pmatrix}. \tag{3.12}$$

In particular, comparing (3.8), (3.9) and (3.12), il follows that

$$V(t,s) = \begin{pmatrix} C(t,s) & S(t,s)\sqrt{a(s)} \\ p(t,s) & \frac{\partial S(t,s)}{\partial t} \end{pmatrix}, \tag{3.13}$$

with

$$p(t,s)y_s = \frac{1}{\sqrt{a(s)}} \frac{\partial C(t,s)}{\partial t} y_s$$

if and only if $y_s \in \tilde{X}$. Hence, recalling the definition of natural projection in (2.1),

$$S(t,s)y = \frac{1}{\sqrt{a(s)}} \pi_1 V(t,s) \begin{pmatrix} 0 \\ y \end{pmatrix}. \tag{3.14}$$

The proof is divided into two steps:

Step 1. $\sqrt{a(t)}\mathcal{A} + c(t)\mathcal{B}$ generates the evolution system

$$V(t,s) = U(t,s) + \int_s^t c(r) U(t,r) \mathcal{B} V(r,s)\, dr, \tag{3.15}$$

where $U(t,s)$ is defined in (3.1) and $\{T(t)\}_t$ is the group generated by $\mathcal{A}$.

Step 2. $\{a(t)A\}_t$ generates the fundamental system (3.5) and $X = \tilde{X}$.

Step 1. We prove that $\{V(t,s)\}_{t,s\in\Delta}$is well defined and verifies Definition 2.5. The proof relies on Dyson-Phillips expansions serie method (see [17]).

Notice first of all that, according to Lemma 2.11, $\mathcal{A}$ generates the strongly continuous group $\{T(t)\}_{t\in\mathbb{R}}$ on $F = \tilde{X}\times E$ and $D(\mathcal{A}) = D(A)\times\tilde{X}$, i.e. $\mathcal{A}$ and $-\mathcal{A}$ generate the strongly continuous semigroups respectively $\{T(t)\}_{t\geq 0}$ and $\{T(-t)\}_{t\geq 0}$. Hence, applying Theorem 3.1 both to $\mathcal{A}$ and $-\mathcal{A}$, it is possible to show that $\{\sqrt{a(t)}\mathcal{A}\}_{t\in[0,b]}$ generates the evolution system $\{U(t,s)\}_{(t,s)\in[0,b]\times[0,b]}$ on $F = \tilde{X}\times E$.

Since a is continuously differentiable, c is continuous, hence bounded on the compact set $[0,b]$. Moreover, from the definition of $\mathcal{B}$ in (3.10), it follows that

$$\|\mathcal{B}\| = 1. \tag{3.16}$$

To prove the thesis, we employ the method of successive approximations. For every $(t,s)\in[0,b]\times[0,b]$, consider the sequences of linear and bounded operators $V_n : F\to F$ defined as

$$V_0(t,s) = U(t,s),$$

$$V_n(t,s) = U(t,s) + \int_s^t c(r)U(t,r)\mathcal{B}V_{n-1}(r,s)\,dr \tag{3.17}$$

and $W_n : F\to F$ defined as

$$W_0(t,s) = V_0(t,s),$$

$$W_n(t,s) = V_n(t,s) - V_{n-1}(t,s) = \int_s^t c(r)U(t,r)\mathcal{B}W_{n-1}(r,s)\,dr.$$

Notice, first of all that W_0 is an evolution system according to Theorem 3.1, hence $W_0(t,s)\in\mathcal{L}(F)$ for every $(t,s)\in[0,b]\times[0,b]$ and the map $(t,s)\mapsto W_0(t,s)z$ is continuous for every $z\in F$. Let D be the positive constant defined in (2.4). Reasoning by induction, let us assume that $W_{n-1}(t,s)\in\mathcal{L}(F)$ for every $(t,s)\in[0,b]\times[0,b]$, the map $(t,s)\mapsto W_{n-1}(t,s)z$ is continuous for every $z\in F$ and

$$\|W_{n-1}(t,s)\| \leq \|c\|_0^{n-1}D^n\frac{(t-s)^{n-1}}{(n-1)!} \tag{3.18}$$

for every $(t,s)\in[0,b]\times[0,b], n\in\mathbb{N}$ and fix $(t,s)\in[0,b]\times[0,b], z\in F$. According to the strong continuity of U and W_{n-1} and to the continuity of c and $\mathcal{B}$, we get that the map $r\mapsto c(r)U(t,r)\mathcal{B}W_{n-1}(r,s)z$ is continuous, hence integrable over $[s,t]$. Moreover, by the linearity of the integral and of $W_{n-1}(r,s), U(t,r)$ and $B, W_n(t,s)\in\mathcal{L}(F)$ for every $(t,s)\in[0,b]\times[0,b]$. From (3.18) it follows that, for every $r\in[s,t]$,

$$\|c(r)U(t,r)\mathcal{B}W_{n-1}(r,s)z\| \leq \|c\|_0^n D^{n+1}\frac{(r-s)^{n-1}}{(n-1)!}\|z\| \leq \|c\|_0^n D^{n+1}\frac{b^{n-1}}{(n-1)!}\|z\|, \tag{3.19}$$

where we also exploited (3.16). Consider two sequences $\{t_m\}_m, \{s_m\}_m$ respectively converging to t and s. Without loss of generality we suppose $s_m\leq s\leq t\leq t_m$ for every m. Then, by standard manipulations and exploiting the bound (3.18), we infer

$$\begin{aligned}
&\|W_n(t_m,s_m)z - W_n(t,s)z\| \\
= \ & \left\| \int_{s_m}^{t_m} c(r)U(t_m,r)\mathcal{B}W_{n-1}(r,s_m)z\,dr - \int_s^t c(r)U(t,r)\mathcal{B}W_{n-1}(r,s)z\,dr \right\| \\
\le \ & \|c\|_0 \int_s^t \left\| U(t_m,r)\mathcal{B}W_{n-1}(r,s_m)z - U(t,r)\mathcal{B}W_{n-1}(r,s)z \right\| dr \\
& + \|c\|_0 \int_{s_m}^{s} \left\| U(t_m,r)\mathcal{B}W_{n-1}(r,s_m)z \right\| dr + \|c\|_0 \int_t^{t_m} \left\| U(t_m,r)\mathcal{B}W_{n-1}(r,s_m)z \right\| dr \\
\le \ & \|c\|_0 \int_s^t \left\| U(t_m,r)\Big(\mathcal{B}W_{n-1}(r,s_m)z - \mathcal{B}W_{n-1}(r,s)z \Big) \right\| dr \\
& + \|c\|_0 \int_s^t \left\| U(t_m,r)\Big(\mathcal{B}W_{n-1}(r,s)z \Big) - U(t,r)\Big(\mathcal{B}W_{n-1}(r,s)z \Big) \right\| dr \\
& + \|c\|_0 \int_{s_m}^{s} \left\| U(t_m,r)\mathcal{B}W_{n-1}(r,s_m)z \right\| dr + \|c\|_0 \int_t^{t_m} \left\| U(t_m,r)\mathcal{B}W_{n-1}(r,s_m)z \right\| dr \\
\le \ & D\|c\|_0 \int_s^t \left\| W_{n-1}(r,s_m)z - W_{n-1}(r,s)z \right\| dr \\
& + \|c\|_0 \int_s^t \left\| \Big[U(t_m,r)\Big(\mathcal{B}W_{n-1}(r,s)z \Big) - U(t,r)\Big(\mathcal{B}W_{n-1}(r,s)z \Big) \Big] \right\| dr \\
& + \|c\|_0 \int_{s_m}^{s} \left\| U(t_m,r)\mathcal{B}W_{n-1}(r,s_m)z \right\| dr + \|c\|_0 \int_t^{t_m} \left\| U(t_m,r)\mathcal{B}W_{n-1}(r,s_m)z \right\| dr.
\end{aligned}$$

Since W_{n-1} and U are strongly continuous, by (3.18) and the dominated convergence theorem, we get that

$$\int_s^t \left\| W_{n-1}(r,s_m)z - W_{n-1}(r,s)z \right\| dr \to 0$$

and

$$\int_s^t \left\| \Big[U(t_m,r)\Big(\mathcal{B}W_{n-1}(r,s)z \Big) - U(t,r)\Big(\mathcal{B}W_{n-1}(r,s)z \Big) \Big] \right\| dr \to 0.$$

Moreover, by the absolute continuity of the integral and (3.19), it follows that

$$\int_{s_m}^{s} \left\| U(t_m,r)\mathcal{B}W_{n-1}(r,s_m)z \right\| dr + \int_t^{t_m} \left\| U(t_m,r)\mathcal{B}W_{n-1}(r,s_m)z \right\| dr \to 0.$$

Therefore

$$W_n(t_m,s_m)z \to W_n(t,s)z,$$

i.e. W_n is strongly continuous. Finally

$$\begin{aligned}
\|W_n(t,s)\| \le \ & \int_s^t \|c(r)U(t,r)\mathcal{B}W_{n-1}(r,s)\|\,dr \le \|c\|_0^n D^{n+1} \int_s^t \frac{(r-s)^{n-1}}{(n-1)!}\,dr \\
= \ & \|c\|_0^n D^{n+1} \frac{(t-s)^n}{n!} \le \|c\|_0^n D^{n+1} \frac{b^n}{n!}.
\end{aligned}$$

Hence

$$V_n(t,s) = \sum_{k=0}^{n} W_k(t,s)$$

converges in $\mathcal{L}(F)$ uniformly with respect to $(t,s) \in [0,b] \times [0,b]$. The expression of V_n in (3.17) then implies that

$$V(t,s) = U(t,s) + \int_s^t c(r)U(t,r)\mathcal{B}V(r,s)\,dr.$$

Let us now prove that V is an evolution system and that $\{\sqrt{a(t)}\mathcal{A} + c(t)\mathcal{B}\}_t$ generates V.

The first and the third properties of Definition 2.5 follow from the uniform convergence, since W_n is strongly continuous for every n, hence V_n is strongly continuous and $V_n(t,t) = U(t,t) = I$ for every n and every $t \in [0,b]$.

Let us now show that V and $\{\sqrt{a(t)}\mathcal{A} + c(t)\mathcal{B}\}_t$ satisfy the first and the second properties of Definition 2.6. Fix $z \in D(\mathcal{A}) = D(A) \times \tilde{X}$. Theorem 3.1 implies that $\{U(t,s)\}_{(t,s)}$ is the evolution system generated by $\{\sqrt{a(t)}\mathcal{A}\}_t$, hence from (2.2) and (2.3) it follows that $W_0(t,s)z \in D(\mathcal{A}), \mathcal{A}W_0(t,s)z = W_0(t,s)\mathcal{A}z$ for every $(t,s) \in [0,b] \times [0,b]$ and that $(t,s) \mapsto W_0(t,s)z$ is continuously differentiable with respect to t with

$$\frac{\partial W_0(t,s)z}{\partial t} = \sqrt{a(t)}\mathcal{A}W_0(t,s)z.$$

Assume by induction that $W_{n-1}(t,s)z \in D(\mathcal{A})$ for every $(t,s) \in [0,b] \times [0,b]$ and that $(t,s) \mapsto W_{n-1}(t,s)z$ is continuously differentiable with respect to t with

$$\frac{\partial W_{n-1}(t,s)z}{\partial t} = c(t)\mathcal{B}W_{n-2}(t,s)z + \sqrt{a(t)}\mathcal{A}W_{n-1}(t,s)z.$$

Given $(t,r) \in [0,b] \times [0,b]$, consider a sequence $\{(t_m, r_m)\} \to (t,r)$. Then the bound (3.18), the strong continuity of W_{n-1} and U and the continuity of c imply

$$\begin{aligned}
& \|c(r_m)U(t_m,r_m)\mathcal{B}W_{n-1}(r_m,s)z - c(r)U(t,r)\mathcal{B}W_{n-1}(r,s)z\| \\
\leq\ & \|c(r_m)U(t_m,r_m)\mathcal{B}W_{n-1}(r_m,s)z - c(r_m)U(t_m,r_m)\mathcal{B}W_{n-1}(r,s)z\| \\
& +\|c(r_m)U(t_m,r_m)\mathcal{B}W_{n-1}(r,s)z - c(r_m)U(t,r)\mathcal{B}W_{n-1}(r,s)z\| \\
& +\|c(r_m)U(t,r)\mathcal{B}W_{n-1}(r,s)z - c(r)U(t,r)\mathcal{B}W_{n-1}(r,s)z\| \\
\leq\ & D\|c\|_0\|W_{n-1}(r_m,s)z - W_{n-1}(r,s)z\| \\
& +\|c\|_0\|U(t_m,r_m)[\mathcal{B}W_{n-1}(r,s)z] - U(t,r)[\mathcal{B}W_{n-1}(r,s)z]\| \\
& +\|c\|_0^{n-1}D^{n+1}\|z\|\frac{b^{n-1}}{(n-1)!}|c(r_m) - c(r)| \to 0
\end{aligned}$$

i.e. the map $(t,r) \mapsto h(t,r)$ defined as

$$h(t,r) := c(r)U(t,r)\mathcal{B}W_{n-1}(r,s)z$$

is continuous. Moreover, since $W_{n-1}(t,s)z \in D(\mathcal{A}) = D(A) \times \tilde{X}$, thus

$$\mathcal{B}W_{n-1}(r,s)z = \begin{pmatrix} 0 \\ \pi_2 W_{n-1}(r,s)z \end{pmatrix} \in D(\mathcal{A}). \tag{3.20}$$

Therefore the map $t \mapsto U(t,r)\mathcal{B}W_{n-1}(r,s)z$ is continuously differentiable with respect to t, i.e. h is continuously differentiable with respect to t and

$$\frac{\partial h}{\partial t}(t,r) = c(r)\sqrt{a(t)}\mathcal{A}U(t,r)\mathcal{B}W_{n-1}(r,s)z,$$

according to Theorem 3.1. Arguing as above and using the continuity of a and a' we get that $W_n(t,s)z$ is continuously differentiable with respect to t and, since $U(t,t) = I$ and $\mathcal{A}$ is a closed operator defined on a vector subspace, we get that $W_n(t,s)z \in D(\mathcal{A})$ for every $(t,s) \in [0,b] \times [0,b]$ and

$$\begin{aligned}
\frac{\partial W_n(t,s)z}{\partial t} &= c(t)\mathcal{B}W_{n-1}(t,s)z + \int_s^t c(r)\sqrt{a(t)}\mathcal{A}U(t,r)\mathcal{B}W_{n-1}(r,s)z\,dr \\
&= c(t)\mathcal{B}W_{n-1}(t,s)z + \sqrt{a(t)}\mathcal{A}\int_s^t c(r)U(t,r)\mathcal{B}W_{n-1}(r,s)z\,dr \\
&= c(t)\mathcal{B}W_{n-1}(t,s)z + \sqrt{a(t)}\mathcal{A}W_n(t,s)z.
\end{aligned}$$

Moreover, according to (2.3), (3.10), (3.20) and (2.6),

$$\begin{aligned}\mathcal{A}W_n(t,s)z = & \int_s^t c(r)\mathcal{A}U(t,r)\mathcal{B}W_{n-1}(r,s)z\,dr = \int_s^t c(r)U(t,r)\mathcal{A}\mathcal{B}W_{n-1}(r,s)z\,dr \\ = & \int_s^t c(r)U(t,r)\begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}\begin{pmatrix} 0 \\ \pi_2 W_{n-1}(r,s)z \end{pmatrix} dr \\ = & \int_s^t c(r)U(t,r)\begin{pmatrix} \pi_2 W_{n-1}(r,s)z \\ 0 \end{pmatrix} dr.\end{aligned}$$

Estimate (3.18) then implies that

$$\frac{\partial V_n(t,s)}{\partial t} = \sum_{k=0}^{n} \frac{\partial W_k(t,s)}{\partial t}$$

converges in $\mathcal{L}(F)$ uniformly with respect to $(t,s) \in [0,b] \times [0,b]$, i.e. the map $(t,s) \mapsto V(t,s)z$ is continuously differentiable with respect to $t \in [0,T]$ and

$$\frac{\partial V(t,s)}{\partial t} = \lim_{n\to+\infty} \frac{\partial V_n(t,s)}{\partial t} = c(t)\mathcal{B}V(t,s)z + \sqrt{a(t)}\mathcal{A}V(t,s)z = [\sqrt{a(t)}\mathcal{A} + c(t)\mathcal{B}]V(t,s)z.$$

In particular, this means that $V(t,s)z \in D([\sqrt{a(t)}\mathcal{A} + c(t)\mathcal{B}])$ for every $(t,s) \in [0,b] \times [0,b]$. Similarly, it is possible to prove that

$$\frac{\partial V(t,s)}{\partial s} = -V(t,s)[\sqrt{a(s)}\mathcal{A} + c(s)\mathcal{B}]z,$$

thus also the third property of Definition 2.6 holds. Finally, for every $z \in D(\sqrt{a(t)}\mathcal{A} + c(t)\mathcal{B}), r,s,t \in [0,b]$,

$$\frac{\partial[V(t,s)V(s,r)z]}{\partial s} = -V(t,s)[\sqrt{a(s)}\mathcal{A} + c(s)\mathcal{B}]V(s,r)z + V(t,s)[\sqrt{a(s)}\mathcal{A} + c(s)\mathcal{B}]V(s,r)z = 0,$$

i.e. the map $s \to V(t,s)V(s,r)z$ is constant for every $r,t \in [0,b]$, which implies that

$$V(t,s)V(s,r)z = V(t,t)V(t,r)z = V(t,r)z$$

and also the second property of Definition 2.5 is satisfied, because $D(\sqrt{a(t)}\mathcal{A} + c(t)\mathcal{B})$ is dense in E. Therefore all the properties of Definitions 2.5 and 2.6 are verified, i.e. V is the evolution system generated by $\{\sqrt{a(t)}\mathcal{A} + c(t)\mathcal{B}\}_t$.

Step 2. According to (3.14), (3.13), (3.1) and (3.15) and Lemma 2.11, the fundamental system generated by $\{a(t)A\}_t$ is

$$\begin{aligned}S(t,s)y = & \frac{1}{\sqrt{a(s)}}\pi_1\Big[T\Big(\int_s^t \sqrt{a(r)}\,dr\Big) + \int_s^t c(r)T\Big(\int_r^t \sqrt{a(\tau)}\,d\tau\Big)\mathcal{B}V(r,s)\,dr\Big]\begin{pmatrix} 0 \\ y \end{pmatrix} \\ = & \frac{1}{\sqrt{a(s)}}\Big[\pi_1\begin{pmatrix} \tilde{S}(\int_s^t \sqrt{a(r)}\,dr)y \\ \tilde{C}(\int_s^t \sqrt{a(r)}\,dr)y \end{pmatrix} \\ & + \int_s^t c(r)\pi_1 T\Big(\int_r^t \sqrt{a(\tau)}\,d\tau\Big)\mathcal{B}\begin{pmatrix} S(r,s)\sqrt{a(s)}y \\ \frac{\partial S(r,s)}{\partial r}y \end{pmatrix} dr\Big] \\ = & \frac{1}{\sqrt{a(s)}}\Big[\tilde{S}\Big(\int_s^t \sqrt{a(r)}\,dr\Big)y + \int_s^t c(r)\pi_1 T\Big(\int_r^t \sqrt{a(\tau)}\,d\tau\Big)\begin{pmatrix} 0 \\ \frac{\partial S(r,s)}{\partial r}y \end{pmatrix} dr\Big] \\ = & \frac{1}{\sqrt{a(s)}}\Big[\tilde{S}\Big(\int_s^t \sqrt{a(r)}\,dr\Big)y + \int_s^t c(r)\pi_1 \begin{pmatrix} \tilde{S}\Big(\int_r^t \sqrt{a(\tau)}\,d\tau\Big)(\frac{\partial S(r,s)}{\partial r}y) \\ \tilde{C}\Big(\int_r^t \sqrt{a(\tau)}\,d\tau\Big)(\frac{\partial S(r,s)}{\partial r}y) \end{pmatrix} dr\Big] \\ = & \frac{1}{\sqrt{a(s)}}\Big[\tilde{S}\Big(\int_s^t \sqrt{a(r)}\,dr\Big)y + \int_s^t c(r)\tilde{S}\Big(\int_r^t \sqrt{a(\tau)}\,d\tau\Big)\Big(\frac{\partial S(r,s)}{\partial r}y\Big)\,dr\Big].\end{aligned}$$

Similarly, owing to the definition of $C(t,s)$ in (2.7), it is possibile to prove that

$$\begin{aligned} C(t,s)y = &\ \pi_1\left[T\left(\int_s^t \sqrt{a(r)}\,dr\right) + \int_s^t c(r)T\left(\int_r^t \sqrt{a(\tau)}\,d\tau\right)\mathcal{B}V(r,s)\,dr\right]\begin{pmatrix} y \\ 0 \end{pmatrix} \\ = &\ \left[\tilde{C}\left(\int_s^t \sqrt{a(r)}\,dr\right)y + \int_s^t c(r)\tilde{S}\left(\int_r^t \sqrt{a(\tau)}\,d\tau\right)(p(r,s)y)\,dr\right]. \end{aligned}$$

Recalling the definition of $\tilde{S}$ in (2.5) and using the facts that it is strongly continuously differentiable, $\tilde{S}(0) = 0$ and a is continuous, it follows that, for every $y \in E$,

$$\exists\ \frac{\partial}{\partial t}\int_s^t c(r)\tilde{S}\left(\int_r^t \sqrt{a(\tau)}\,d\tau\right)(p(r,s)y)\,dr = \sqrt{a(t)}\int_s^t c(r)\tilde{C}\left(\int_r^t \sqrt{a(\tau)}\,d\tau\right)(p(r,s)y)\,dr.$$

Thus the map $t \to C(t,s)y$ is continuously differentiable if and only if the map $t \mapsto \tilde{C}\left(\int_s^t \sqrt{a(r)}\,dr\right)y$ is continuously differentiable, i.e. if and only if $y \in \tilde{X}$. □

Remark 3.4. We stress that, arguing as in the proof of Step 1 of Theorem 3.3, it is possibile to prove that if $\mathcal{A}$ is the generator of the strongly continuous semigroup $\{T(t)\}_{t\geq 0}, d$ is a continuous function and $\mathcal{B} : [0, +\infty) \to \mathcal{L}(E)$ is strongly continuous, the family of operators

$$\{d(t)\mathcal{A} + \mathcal{B}(t)\}_t$$

generates the evolution system

$$V(t,s) = U(t,s) + \int_s^t U(t,r)\mathcal{B}(r)V(r,s)\,dr$$

with $U(t,s)$ defined in (3.1). Namely, the first part of the proof works also in this general setting, while the key assumption to prove the second part is that

$$\mathcal{B}(t)U(t,s)z \in D(\mathcal{A}) \tag{3.21}$$

for every $t \geq s \geq 0$ and $z \in D(\mathcal{A})$.

An analogous result was proved in [17]. More precisely, it is considered a family of operators $\{\mathcal{A} + \mathcal{B}(t)\}_t$, where $\mathcal{A}$ generates a strongly continuous semigroup $\{T(t)\}_t$ and $\mathcal{B} : [0, +\infty) \to \mathcal{L}(E)$ is strongly continuously differentiable. Our theorem extends the result obtained in [17] to the case when the coefficient d is not constant and $\mathcal{B}$ is just strongly continuous and satisfying (3.21). Notice however that, as pointed out in [17], the strongly continuous differentiability implies that

$$\int_s^t T(t-r)\mathcal{B}(r)V(r,s)dr \in D(\mathcal{A})$$

for every $t \geq s \geq 0$.

Remark 3.5. In [19] one of the few sufficient conditions guaranteeing that a family of operators generate a fundamental system is given. It states that if A is the generator of a cosine family $\{\tilde{C}(t)\}_t$ and $B(t) : \tilde{X} \to E$ is linear, bounded and strongly continuously differentiable, then the family $\{A + B(t)\}_t$ generates the fundamental system $\{S(t,s)\}_{t,s}$ defined as

$$S(t,s)z = \tilde{S}(t-s)z + \int_s^t \tilde{S}(t-\xi)B(\xi)S(\xi,s)z\,d\xi.$$

This result was then extended in [6], where $B(t)$ is assumed to be bounded in the whole space E. In this case, it is sufficient to require that B is strongly continuous.

We stress that these results cannot be applied to (1.2), except in the trivial case when $a \equiv 1$, because the operator A defined in (1.3) is not bounded.

Another sufficient condition for the existence of the fundamental system has been obtained in [7]. The authors consider a family of operators $\{\mathbf{A}(t) + B(t)\}_t$, with $\mathbf{A}(t) : D(A) \subset V \to H$ and $B(t) : H \to H$, where $V \subset H \subset V^*$ are a Gelfand triple. They assume that $\mathbf{A}(t)$ is self-adjoint and bounded, $B(t)$ is bounded and that both A and B are strongly continuous differentiable.

Noticed that this result can be applied to (1.2) only in quite special cases.

Moreover, no information on X is provided in any of the aforementioned references. Thus, none of the quoted results can be applied to equation (1.2) where the nonlinear term depends also on the first derivative of the solution.

We are now ready to prove that the operator defined in (1.4) generates a fundamental system, hence that the Cauchy problem associated to the homogeneous equation $\ddot{x}(t) = \mathbf{A}(t)x(t)$ is well posed. As consequence of our main theorem, we are also able to identify the subspace X.

Theorem 3.6. *Let E be the Banach space $L^p([0,1]), D(A)$ the dense subspace $W^{2,p}([0,1]) \cap W_0^{1,p}([0,1])$ of $E, A : D(A) \to E$ the linear operator*

$$Ay = \ddot{y}.$$

Suppose that $a : [0,b] \to (0,+\infty)$ is a continuously differentiable function.

Then the linear operator $\mathbf{A}(t) : D(A) \to E$ defined as $\mathbf{A}(t) = a(t)A$ is the generator of a fundamental system and $X = W^{1,p}([0,1]) \cap C_0([0,1])$.

Proof. It is sufficient to apply Theorem 3.3, recalling that A generates a cosine family $\{\tilde{C}(t)\}_t$ with $\tilde{X} = W^{1,p}([0,1]) \cap C_0([0,1])$ (see [4]). □

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