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Nontrivial, nonnegative periodic solutions of a system of singular-degenerate parabolic equations with nonlocal terms

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We study the existence of nontrivial, nonnegative periodic solutions for systems of singular-degenerate parabolic equations with nonlocal terms and satisfying Dirichlet boundary conditions. The method employed in this paper is based on the Leray–Schauder topological degree theory. However, verifying the conditions under which such a theory applies is more involved due to the presence of the singularity. The system can be regarded as a possible model of the interactions of two biological species sharing the same isolated territory, and our results give conditions that ensure the coexistence of the two species.

Keywords: Singular-degenerate parabolic equations; periodic solutions; *a priori* bounds; topological degree theory.

Mathematics Subject Classification 2010: 35K65, 35B10, 47H11

1. Introduction

In this paper we consider a periodic system of singular-degenerate parabolic equations with delayed nonlocal terms and Dirichlet boundary conditions of the form

$$\begin{cases} u_t - \operatorname{div}(|\nabla u^m|^{p-2}\nabla u^m) = \left(a(x,t) - \displaystyle\int_\Omega K_1(\xi,t)u^2(\xi,t-\tau_1)d\xi \right. \\ \qquad\qquad \left. + \displaystyle\int_\Omega K_2(\xi,t)v^2(\xi,t-\tau_2)d\xi\right) u^{p-1} \quad \text{in } Q_T, \\ v_t - \operatorname{div}(|\nabla v^n|^{q-2}\nabla v^n) = \left(b(x,t) + \displaystyle\int_\Omega K_3(\xi,t)u^2(\xi,t-\tau_3)d\xi \right. \\ \qquad\qquad \left. - \displaystyle\int_\Omega K_4(\xi,t)v^2(\xi,t-\tau_4)d\xi\right) v^{q-1} \quad \text{in } Q_T, \\ u(x,t) = v(x,t) = 0 \quad \text{for } (x,t) \in \partial\Omega \times (0,T), \\ u(\cdot,0) = u(\cdot,T) \quad \text{and} \quad v(\cdot,0) = v(\cdot,T) \end{cases} \tag{1.1}$$

and we look for continuous weak solutions. Here Ω is an open bounded domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, satisfying the property of positive geometric density, see [39], $Q_T := \Omega \times (0,T)$, $T > 0$, $\tau_i \in (0,+\infty)$, the functions K_i, a, and b belong to $L^\infty(Q_T)$. The exponents p and q belong to the interval $(1,2)$, $m > p$, $n > q$ and $s^m = |s|^{m-1}s$. Setting $Au := \operatorname{div}(|\nabla u^m|^{p-2}\nabla u^m)$ and $l := (m-1)(p-1)$, the operator Au becomes $m^{p-1}\operatorname{div}(|u|^l|\nabla u|^{p-2}\nabla u)$, which is the operator considered by Ivanov in [32–34]. According to the classification proposed in these papers, we say that the first equation in (1.1) is of

(1) *slow diffusion* type if $m > \frac{1}{p-1}$,
(2) *normal diffusion* type if $m = \frac{1}{p-1}$,
(3) *fast diffusion* type if $m < \frac{1}{p-1}$.

Of course, analogous definition in terms of n and q can be given for the second equation in (1.1).

The aim of this paper is to extend the results of [25, 26], concerning the existence of nonnegative, nontrivial periodic solutions, to a system of singular-degenerate parabolic equations. To the best of our knowledge, this is the first result for the case when $1 < p, q < 2$, $m > p$ and $n > q$, also in the case of a single equation. We recall that the cases $p, q > 2$, $m, n > 1$ and $p = q = 2$, $m, n > 1$ were treated respectively in [25, 26], see also [21, 22] for a system of anisotropic $(p(x), q(x))$-Laplacian parabolic equations, with $p(x), q(x) > 2$ in $\overline{\Omega}$, and $m = n = 1$. In the very recent paper [61], the authors replace the nonlocal terms of (1.1) by $\int_\Omega K_i(\xi,t)u(\xi,t)d\xi$, for $i = 1,3$, and $\int_\Omega K_i(\xi,t)v(\xi,t)d\xi$, for $i = 2,4$. By means of local conditions, different from those proposed in [25, 26], the authors obtain the coexistence of the two species via a similar topological approach when $p, q \geq 2$, $m, n \geq 1$ and, thus, only the slow and normal diffusion occurs, i.e. $m(p-1) \geq 1, n(q-1) \geq 1$. More precisely, models for the interaction between two biological species sharing the same isolated territory, with the interactions represented by means of the kernels $K_i, i = 1, 2, 3, 4$, were considered in related systems of doubly degenerate parabolic equations in [25, 61] and in systems of porous medium equations in [26]. On the other hand, some previous biological models found in the literature, see, e.g., [1, 2, 52, 51], involve the p-Laplacian with $p > 1$ (and $m = 1$). Furthermore, we observe that the equations of the system we consider treat all the possible types of diffusion: slow, normal and fast, while in [25, 26, 61] only the slow and normal diffusions were presented. In fact, as it can be easily checked, if $p \in (1, \frac{1+\sqrt{5}}{2})$ one can have all the three types of diffusion, while if $p \in [\frac{1+\sqrt{5}}{2}, 2)$ only the slow diffusion is possible under the condition $m > p$.

In the case of the fast diffusion and superlinear growth in u, v of the right-hand sides, the solutions may blow up or vanish in some finite time depending on the initial conditions as illustrated in [11, 40, 41, 46] and the references therein for the simple equation obtained from the first of system (1.1) by letting $p = 2$, $a > 0$ constant and all the kernels $K_i \equiv 0$ (observe that in the case when $p = q = 2$ no restrictions on m, n are required, see Remark 2.2). If $\Omega = \mathbb{R}^N$ for such equation we have that the solution blows up for any initial condition in the case when the superlinear growth in u is less than a certain critical exponent, see [46], and the same occurs for doubly degenerate parabolic equations, see [47]. If the growth is linear or sublinear we do not have blow up of any solution, see [41], hence solutions exist for all $t \geq 0$ and in the linear case, depending on the initial condition, they may vanish in finite time or become unbounded as $t \to +\infty$ and, thus, the considered initial conditions cannot give rise to a periodic solution.

The choice of the sublinear exponents $p-1$ and $q-1$, respectively, for u and v in (1.1) is mainly technical since it depends on the topological method employed in the paper, which is based on *a priori* bounds of the solutions. Indeed, this choice enable us to establish the required *a priori* bounds on the solutions of the approximating problem (2.1) in a uniform way with respect to the perturbation parameter $\epsilon > 0$. We remark that, if the diffusion is slow and $\Omega \subset \mathbb{R}^N$ is a bounded and open domain, then we can allow a superlinear growth in u, v in order to have both global existence solutions and periodic solutions, together with their L^∞-estimates, see [61, 8, 13, 14, 45, 48, 49, 57, 62–64] and the references therein. Due to the singularity of the p, q-Laplacian, the way of proving the *a priori* bounds deeply differs from that employed in [25, 26]. Moreover, in order to pass from the L^2- to the L^∞-estimates, in Lemma 2.2 we have readapted Moser's technique to the case when $1 < p, q < 2$. Moreover we have to impose the technical restriction $m > p$ and $n > q$ in order to get the gradient estimates in Lemma 2.4.

Due to their importance in different physical and other natural sciences such as non-Newtonian fluid mechanics, flow in porous medium, nonlinear elasticity, glaciology, population biology, etc., degenerate and singular parabolic equations have been the subject of extensive research in the last 25 years, with particular emphasis on the study of regularity for nonnegative weak solutions. We mention here, among many others, the papers [32–34, 53] and the monographs [17, 54]. In particular, we refer to the very recent monograph [18] for a comprehensive treatment of the Harnack inequality for nonnegative solutions to p-Laplacian and porous medium equations. Moreover, the monograph [18] provides an historical presentation of the achievements in this research field and many references to the applications concerning the topics mentioned above.

The regularity results for the singular p-Laplacian are crucial for the application of the topological degree approach used in this paper. Similar topological methods are also employed to a great extent for the existence of nonnegative periodic solutions of degenerate and doubly degenerate parabolic equations, see [61, 45, 48, 62–64, 3, 9, 20, 30, 31, 38, 42, 44, 55, 56, 59, 60, 58, 67, 68, 65]. Nonlocal models to study aggregation in biological systems with degenerate diffusion are proposed in several papers, see [12, 43] and the references therein.

Moreover, we recall that the interest in studying the existence of periodic solutions for degenerate and nondegenerate parabolic equations modeling biological and physical phenomena relies in the consideration that the periodic behavior of certain biological and physical nonnegative quantities is the most natural and desirable one, see, e.g., [25, 26, 21, 22, 61, 57, 3, 31, 42, 56, 67, 68, 5–7, 29, 35, 50]. We also recall the related problems faced in [23, 24] also for higher-order operators, and in [19] for $p = 2$ and $N = 1$.

The paper is organized as follows. The goal of Sec. 2 is the proof of a coexistence result based on the explicit knowledge of suitable *a priori* bounds on the L^2-norms of the solutions. The search for such bounds is carried on in Sec. 3. The reason to split the argument in this way lies in the fact that our main coexistence conditions, namely Assumption 2, are applicable regardless of any other assumption on the terms of the equations. On the other hand, *a priori* bounds for the periodic solutions are more easily obtained when we focus on specific situations like the competitive (i.e. $K_2, K_3 \leq 0$) and cooperative (i.e. $K_2, K_3 \geq 0$) cases and those in which K_1, K_4 are bounded away from zero or not and other restrictions on the exponents of the left-hand sides are imposed.

More precisely, in order to deal with the singular-degenerate system (1.1), in Sec. 2 we introduce an approximating system (2.1) of nondegenerate-singular equations depending on a small parameter $\varepsilon > 0$. Such equations satisfy structure conditions which, for any $\varepsilon > 0$, allow the use of well-known regularity results, i.e. Hölder continuity, from, e.g., [33, 34]; we will use this regularity to show that the map which associates to any couple of functions $(f, g) \in L^\infty(Q_T) \times L^\infty(Q_T)$ the solution of the regularized system is a compact map from $L^\infty(Q_T) \times L^\infty(Q_T) \to L^\infty(Q_T) \times L^\infty(Q_T)$, see Lemma 2.1. Then, for $\varepsilon > 0$, the problem of showing the existence of a nonnegative solution $(u_\varepsilon, v_\varepsilon)$ to (2.1) is equivalent to showing the existence of a nonnegative fixed point of such a solution map. The way we do this in Proposition 2.2 is based on the classical tools of the Leray–Schauder topological degree: first, we establish uniform (with respect to $\varepsilon > 0$) *a priori* bounds, in this specific case in $L^\infty(Q_T) \times L^\infty(Q_T)$, for all possible nonnegative solutions of (2.1). Then, by the homotopy invariance of the topological degree, Proposition 2.2 guarantees the existence of a solution $(u_\varepsilon, v_\varepsilon)$ of (2.1) in a large ball $B_R \subset L^\infty(Q_T) \times L^\infty(Q_T)$. Moreover, by means of suitable conditions on the first positive eigenvalue of the p-Laplacian and on some estimates on the gradient of convenient powers of u_ε and v_ε established in Lemma 2.4, we are able to prove that $\|u_\varepsilon\|_{L^\infty}$ and $\|v_\varepsilon\|_{L^\infty}$ are bounded away from zero uniformly for $\varepsilon > 0$ small enough, see Proposition 2.3. To conclude, by using the uniform bounds of $(u_\varepsilon, v_\varepsilon)$ in $L^\infty(Q_T) \times L^\infty(Q_T)$ and the consequent uniform Hölder continuity of $(u_\varepsilon, v_\varepsilon)$ in $\overline{Q_T}$, we can pass to the limit as $\varepsilon \to 0$ and show in Theorem 2.1 that $(u_\varepsilon, v_\varepsilon)$, by passing to a subsequence if necessary, converges to a solution (u, v) of (1.1) with $u \neq 0$ and $v \neq 0$.

In Sec. 3, we give conditions on the kernels $K_i, i = 1, 2, 3, 4$, of the nonlocal terms that suffice for the existence of uniform *a priori* bounds in $L^2(Q_T) \times L^2(Q_T)$ for the solutions $(u_\varepsilon, v_\varepsilon)$ of (2.1). By Lemma 2.2 these *a priori* bounds imply uniform *a priori* bounds of $(u_\varepsilon, v_\varepsilon)$ in $L^\infty(Q_T) \times L^\infty(Q_T)$ and so, from now on, we can proceed as outlined in Sec. 2 in order to apply Theorem 2.1. In terms of the biological interpretations, system (1.1) is a model of the interactions of two biological species, with density u and v respectively, disliking crowding, i.e. $m, n > 1$, see [50, 28, 27], and whose diffusion involves, as in [1, 2, 52, 51], the singular p-Laplacian, i.e. $1 < p < 2$. The nonlocal terms $\int_\Omega K_i(\xi, t)u^2(\xi, t - \tau_i)d\xi$ and $\int_\Omega K_i(\xi, t)v^2(\xi, t - \tau_i)d\xi$ evaluate a weighted fraction of individuals that actually interact at time $t > 0$. Nonlocal terms in biological models were first introduced in [16, 15]. The delayed densities u, v at time $t - \tau_i$, that appear in the nonlocal terms, take into account the time needed to an individual to become adult, and, thus to interact and to compete. The conditions on $K_i, i = 1, 2, 3, 4$, have the meaning of competitive systems if $K_i \leq 0, i = 2, 3$, or of cooperative systems if $K_i \geq 0, i = 2, 3$; on the other hand, we always assume that $K_i \geq 0, i = 1, 4$, to take into account the intra-species competition. The term on the right-hand side of each equation in (1.1) denotes the actual increasing rate of the population at $(x, t) \in Q_T$. Related results are presented in the coexistence Theorem 3.1, which considers the coercive case, i.e. $K_i \geq \underline{k_i} > 0, i = 1, 4$, and its consequences: Corollaries 3.1 and 3.2 for the coercive-cooperative and coercive-competitive cases, respectively. In the noncoercive case we prove Theorem 3.2 for competitive systems when the diffusion is slow or normal for both the equations, and Theorems 3.3 and 3.4 under a stronger assumption on m, p, n, q, but without any conditions on the sign of K_2 and K_3. Observe that these results concerning the slow and normal diffusion are relevant for the considered biological model, in fact the slow and normal diffusion are more realistic for the biological models as pointed out in [33, 51, 64, 50, 27]. Finally, in Sec. 4, for a generalization of system (1.1) which consists in having any power $\alpha \geq 1$ of u and v in the nonlocal terms, we obtain, only in the competitive case, the coexistence Theorem 4.1 and the related Theorem 4.2 for the coercive case and Theorem 4.3 for the noncoercive case. Note that such a generalization of system (1.1) is a completely new contribution with respect to [25, 26].

2. The Approximating Problem

Throughout the paper we will make the following assumptions.

Assumption 1.

(1) The exponents p, q, m, n are such that $p, q \in (1, 2)$, $m > p$ and $n > q$.
(2) The delays $\tau_i \in (0, +\infty)$, $i = 1, 2, 3, 4$.
(3) The functions a, b and K_i, $i = 1, 2, 3, 4$, belong to $L^\infty(Q_T)$ and are extended to $\Omega \times \mathbb{R}$ by T-periodicity. Moreover, a, b and K_i, $i = 1, 4$, are nonnegative functions and there are constants $\underline{k}_i, \overline{k}_i \geq 0$, $i = 2, 3$, such that

$$-\underline{k}_i \leq K_i(x,t) \leq \overline{k}_i \quad \text{for } i = 2, 3,$$

for a.a. $(x, t) \in Q_T$.

We now recall the definition of weak solution to (1.1).

Definition 2.1. A pair of functions (u, v) is said to be a weak solution of (1.1) if $u, v \in C(\overline{Q}_T)$, $u^m \in L^p((0,T); W_0^{1,p}(\Omega))$, $v^n \in L^q((0,T); W_0^{1,q}(\Omega))$ and (u, v) satisfies

$$\iint_{Q_T} \left(-u\frac{\partial \phi}{\partial t} + |\nabla u^m|^{p-2}\nabla u^m \nabla \phi - a u^{p-1}\phi \right.$$
$$\left. + u^{p-1}\phi \int_\Omega [K_1(\xi,t)u^2(\xi, t-\tau_1) - K_2(\xi,t)v^2(\xi,t-\tau_2)]d\xi \right) dxdt = 0$$

and

$$\iint_{Q_T} \left(-v\frac{\partial \phi}{\partial t} + |\nabla v^n|^{q-2}\nabla v^n \nabla \phi - b v^{q-1}\phi \right.$$
$$\left. + v^{q-1}\phi \int_\Omega [-K_3(\xi,t)u^2(\xi, t-\tau_3) + K_4(\xi,t)v^2(\xi,t-\tau_4)]d\xi \right) dxdt = 0,$$

for any $\phi \in C^1(\overline{Q}_T)$ such that $\phi(x, T) = \phi(x, 0)$ for any $x \in \Omega$ and $\phi(x, t) = 0$ for any $(x, t) \in \partial\Omega \times [0, T]$.

Here and in the following we assume that the functions $t \mapsto u(\cdot, t)$ and $t \mapsto v(\cdot, t)$ are extended from $[0, T]$ to $\mathbb{R}$ by T-periodicity so that (u, v) is a solution defined for all $t \in \mathbb{R}^+$.

In order to study system (1.1) we now consider the following nondegenerate-singular approximating p, q-Laplacian system

$$
\begin{cases}
\dfrac{\partial u}{\partial t} - \operatorname{div}((\varepsilon + m^{p-1}u^{(m-1)(p-1)})|\nabla u|^{p-2}\nabla u) \\
\quad = \left(a(x,t) - \displaystyle\int_\Omega K_1(\xi,t)u^2(\xi,t-\tau_1)d\xi \right. \\
\qquad \left. + \displaystyle\int_\Omega K_2(\xi,t)v^2(\xi,t-\tau_2)d\xi\right) u^{p-1} \quad \text{in } Q_T, \\
\dfrac{\partial v}{\partial t} - \operatorname{div}((\varepsilon + n^{q-1}v^{(n-1)(q-1)})|\nabla v|^{q-2}\nabla v) \\
\quad = \left(b(x,t) + \displaystyle\int_\Omega K_3(\xi,t)u^2(\xi,t-\tau_3)d\xi \right. \\
\qquad \left. - \displaystyle\int_\Omega K_4(\xi,t)v^2(\xi,t-\tau_4)d\xi\right) v^{q-1} \quad \text{in } Q_T, \\
u(\cdot,t)|_{\partial\Omega} = v(\cdot,t)|_{\partial\Omega} = 0 \quad \text{for a.a. } t \in (0,T), \\
u(\cdot,0) = u(\cdot,T) \quad \text{and} \quad v(\cdot,0) = v(\cdot,T),
\end{cases} \tag{2.1}
$$

where $\varepsilon > 0$. A solution (u, v) of (1.1) will be then obtained as the limit, for $\varepsilon \to 0$, of the solutions $(u_\varepsilon, v_\varepsilon)$ of (2.1) with $u_\varepsilon, v_\varepsilon \geq 0$. For this we give the following definition.

Definition 2.2. A couple of functions $(u_\varepsilon, v_\varepsilon)$ is said to be a generalized (weak) solution of (2.1) if

$$u_\varepsilon \in L^p(0,T;W_0^{1,p}(\Omega)) \cap C(\overline{Q}_T), \quad v_\varepsilon \in L^q(0,T;W_0^{1,q}(\Omega)) \cap C(\overline{Q}_T)$$

and $(u_\varepsilon, v_\varepsilon)$ satisfies

$$
\begin{aligned}
\iint_{Q_T} \bigg(&-u\frac{\partial \phi}{\partial t} + \varepsilon|\nabla u|^{p-2}\nabla u\nabla\phi + |\nabla u^m|^{p-2}\nabla u^m\nabla\phi - au_\varepsilon^{p-1}\phi \\
&+ u^{p-1}\phi\int_\Omega [K_1(\xi,t)u^2(\xi,t-\tau_1) - K_2(\xi,t)v^2(\xi,t-\tau_2)]d\xi\bigg)\,dxdt = 0
\end{aligned}
$$

and

$$
\begin{aligned}
\iint_{Q_T} \bigg(&-v\frac{\partial \phi}{\partial t} + \varepsilon|\nabla v|^{q-2}\nabla v\nabla\phi + |\nabla v^n|^{q-2}\nabla v^n\nabla\phi - bv^{q-1}\phi \\
&+ v^{q-1}\phi\int_\Omega [K_4(\xi,t)v^2(\xi,t-\tau_4) - K_3(\xi,t)u^2(\xi,t-\tau_3)]d\xi\bigg)\,dxdt = 0
\end{aligned}
$$

for any $\phi \in C^1(\overline{Q}_T)$ such that $\phi(x,T) = \phi(x,0)$ for any $x \in \Omega$ and $\phi(x,t) = 0$ for any $(x,t) \in \partial\Omega \times [0,T]$.

To deal with the existence of T-periodic solutions $(u_\varepsilon, v_\varepsilon)$ of system (2.1), with $u_\varepsilon, v_\varepsilon \geq 0$ in Q_T, we introduce, for any $\varepsilon > 0$, the map $G_\varepsilon : [0,1] \times L^\infty(Q_T) \times L^\infty(Q_T) \to X$, where $X := L^p(0,T; W_0^{1,p}(\Omega) \cap L^2(\Omega)) \times L^q(0,T; W_0^{1,q}(\Omega) \cap L^2(\Omega))$, as follows:

$$(\sigma, f, g) \mapsto (u_\varepsilon, v_\varepsilon) = G_\varepsilon(\sigma, f, g)$$

if and only if $(u_\varepsilon, v_\varepsilon)$ solves the following uncoupled problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \varepsilon \operatorname{div}(|\nabla u|^{p-2}\nabla u) - \operatorname{div}(|\nabla(\sigma u^m)|^{p-2}\nabla(\sigma u^m)) = f \quad \text{in } Q_T, \\ \dfrac{\partial v}{\partial t} - \varepsilon \operatorname{div}(|\nabla v|^{q-2}\nabla v) - \operatorname{div}(|\nabla(\sigma v^n)|^{q-2}\nabla(\sigma v^n)) = g \quad \text{in } Q_T, \\ u(\cdot,t)|_{\partial\Omega} = v(\cdot,t)|_{\partial\Omega} = 0 \quad \text{for a.a. } t \in (0,T), \\ u(\cdot,0) = u(\cdot,T) \quad \text{and} \quad v(\cdot,0) = v(\cdot,T). \end{cases} \tag{2.2}$$

For any fixed $\sigma \in [0,1]$ the operator $A : \mathcal{X} = L^p(0,T; W_0^{1,p}(\Omega) \cap L^2(\Omega)) \to \mathcal{X}'$, $u \mapsto \varepsilon\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \operatorname{div}(|\nabla(\sigma u^m)|^{p-2}\nabla(\sigma u^m))$, is hemicontinuous, strictly monotone (and hence pseudomonotone), coercive and bounded. Thus, by [66, Theorem 32.D], the map G_ε is well defined. Now, consider

$$f(\alpha,\beta) := \left(a - \int_\Omega K_1(\xi,\cdot)\,\alpha^2(\xi,\cdot - \tau_1)d\xi + \int_\Omega K_2(\xi,\cdot)\,\beta^2(\xi,\cdot - \tau_2)d\xi\right)\alpha^{p-1}$$

and

$$g(\alpha,\beta) := \left(b + \int_\Omega K_3(\xi,\cdot)\,\alpha^2(\xi,\cdot - \tau_3)d\xi - \int_\Omega K_4(\xi,\cdot)\,\beta^2(\xi,\cdot - \tau_4)d\xi\right)\beta^{q-1},$$

where α and β belong to $L^\infty(Q_T)$. Clearly, if the nonnegative functions $u_\varepsilon, v_\varepsilon \in L^\infty(Q_T)$ are such that $(u_\varepsilon, v_\varepsilon) = G_\varepsilon(1, f(u_\varepsilon, v_\varepsilon), g(u_\varepsilon, v_\varepsilon))$, then $(u_\varepsilon, v_\varepsilon)$ is also a solution of (2.1) (with u_ε and $v_\varepsilon \geq 0$) in Q_T. Hence, the existence of a nonnegative solution of (2.1) is equivalent to the existence of a fixed point (α,β) of the map $(\alpha,\beta) \mapsto G_\varepsilon(1, f(\alpha,\beta), g(\alpha,\beta))$ with α and $\beta \geq 0$.

Let $T_\varepsilon(\sigma,\alpha,\beta) := G_\varepsilon(\sigma, f(\alpha,\beta), g(\alpha,\beta))$. By [33, Theorem 1.1; 34, Theorem 1.3] we have the Hölder estimate in the interior of Q_T of the solutions to (2.2). Moreover, the property of positive geometric density of $\partial\Omega$, the fact that the Dirichlet data is Hölder continuous and the periodicity condition ensure that one can obtain the Hölder estimate up to the parabolic boundary of Q_T, see the comments to [33, Theorem 1.1; 34, Theorem 7.1].

We will need the following result, which was proved in [26, Lemma 2.1] for the $p, q = 2$, $m, n > 1$, but whose proof is the same, so we omit it.

Lemma 2.1. *Let $(\alpha,\beta) \in L^\infty(Q_T) \times L^\infty(Q_T)$ and let $\varepsilon > 0$. Then $T_\varepsilon : [0,1] \times L^\infty(Q_T) \times L^\infty(Q_T) \to L^\infty(Q_T) \times L^\infty(Q_T), (\sigma,\alpha,\beta) \mapsto T_\varepsilon(\sigma,\alpha,\beta) = (u_\varepsilon, v_\varepsilon)$ is compact. Moreover $u_\varepsilon, v_\varepsilon \in C(\overline{Q}_T)$.*

Our aim is to prove the existence of T-periodic solutions $u_\varepsilon, v_\varepsilon \in C(\overline{Q}_T)$ of problem (2.1) with u_ε, $v_\varepsilon > 0$ in Q_T, for all $\varepsilon > 0$ small enough, as positive fixed points of the map $(\alpha, \beta) \mapsto T_\varepsilon(1, \alpha, \beta)$. As a first step we prove the following result.

Proposition 2.1. *If the nontrivial pair* $(u_\varepsilon, v_\varepsilon)$ *solves*

$$(u, v) = G_\varepsilon(\sigma, f(u^+, v^+) + (1 - \sigma), g(u^+, v^+) + (1 - \sigma)) \tag{2.3}$$

for some $\sigma \in [0, 1]$, *then*

$$u_\varepsilon(x, t) \geq 0 \quad \textit{and} \quad v_\varepsilon(x, t) \geq 0 \quad \textit{for any } (x, t) \in Q_T.$$

Moreover, if $u_\varepsilon \neq 0$ *or* $v_\varepsilon \neq 0$, *then* $u_\varepsilon > 0$ *or* $v_\varepsilon > 0$ *in* Q_T, *respectively.*

Proof. Assume that $(u_\varepsilon, v_\varepsilon)$ solves (2.3) with $u_\varepsilon \neq 0$ for some $\sigma \in [0, 1]$. We first prove that $u_\varepsilon \geq 0$. Multiplying the first equation of (2.2), where $f(\alpha, \beta)$ is replaced by $f(u_\varepsilon^+, v_\varepsilon^+) + (1 - \sigma)$, by $u_\varepsilon^- := \min\{0, u_\varepsilon\}$, integrating over Q_T and passing to the limit in the Steklov averages $(u_\varepsilon)_h \in H^1(Q_{T-\delta})$, $\delta, h > 0$, in a standard way [39, p. 85], we obtain

$$\begin{aligned} &\varepsilon \iint_{Q_T} |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \nabla u_\varepsilon^- + \iint_{Q_T} |\nabla(\sigma u_\varepsilon^m)|^{p-2} \nabla(\sigma u_\varepsilon^m) \nabla u_\varepsilon^- \\ &\quad = \iint_{Q_T} (1 - \sigma) u_\varepsilon^- \leq 0, \end{aligned}$$

by the T-periodicity of u_ε and taking into account that $u_\varepsilon^+ u_\varepsilon^- = 0$. Hence we obtain

$$\varepsilon \iint_{Q_T} |\nabla u_\varepsilon^-|^p \leq \varepsilon \iint_{Q_T} |\nabla u_\varepsilon|^{p-2} \nabla u \nabla u_\varepsilon^- + \iint_{Q_T} |\nabla(\sigma u_\varepsilon^m)|^{p-2} \nabla(\sigma u_\varepsilon^m) \nabla u_\varepsilon^- \leq 0.$$

Thus

$$\iint_{Q_T} |\nabla u_\varepsilon^-|^p = 0.$$

The Poincaré inequality gives

$$0 \leq \int_\Omega |u_\varepsilon^-|^p dx \leq \frac{1}{\mu_p} \int_\Omega |\nabla u_\varepsilon^-|^p dx \quad \text{for all } t,$$

where μ_p is the first positive eigenvalue of the problem

$$\begin{cases} -\operatorname{div}(|\nabla z|^{p-2} \nabla z) = \mu |z|^{p-2} z, & x \in \Omega, \\ z = 0, & x \in \partial\Omega, \end{cases}$$

(see, for example, [37]). Integrating over $(0, T)$, we have

$$0 \leq \iint_{Q_T} |u_\varepsilon^-|^p \leq \frac{1}{\mu_p} \iint_{Q_T} |\nabla u_\varepsilon^-|^p = 0,$$

which, together with the boundary conditions and the fact that $u_\varepsilon^- \in C(\overline{Q}_T)$, implies $u_\varepsilon^-(x, t) = 0$ for all $(x, t) \in Q_T$.

Now we prove that $u_\varepsilon > 0$ in Q_T. Since u_ε is nontrivial, there exists $(x_0, t_0) \in \Omega \times (0, T]$ such that $u_\varepsilon(x_0, t_0) > 0$. Hence $u_\varepsilon(x, t) > 0$ for all $(x, t) \in Q_T$ (see [10, p. 3; 36]). In the same way, one can prove that $v_\varepsilon \neq 0$ implies $v_\varepsilon(x, t) > 0$ for all $(x, t) \in Q_T$. $\square$

The next lemma is crucial to prove Proposition 2.2 below.

Lemma 2.2. *Let $K > 0$ and assume that u is a nonnegative T-periodic continuous function such that $x \mapsto u(x, t) \in W_0^{1,p}(\Omega)$ for all $t \in [0, T]$ and which satisfies*

$$u_t - \varepsilon \mathrm{div}(|\nabla u|^{p-2}\nabla u) - \mathrm{div}(|\nabla u^m|^{p-2}\nabla u^m) \leq K u^{p-1} \quad in\ Q_T.$$

Then there exists $R > 0$ such that

$$\|u\|_{L^\infty} \leq R \quad for\ all\ \varepsilon > 0.$$

Proof. We follow Moser's technique to show the stated *a priori* bounds. Multiplying

$$u_t - \varepsilon\, \mathrm{div}(|\nabla u|^{p-2}\nabla u) - \mathrm{div}(|\nabla u^m|^{p-2}\nabla u^m) \leq K u^{p-1}$$

by u^{s+1}, with $s \geq 0$, integrating over Ω and passing to the limit as $h \to 0$ in the Steklov averages $u_h \in H^1(Q_{T-\delta})$, $\delta, h > 0$, we have

$$\begin{aligned} &\frac{1}{s+2}\frac{d}{dt}\|u(t)\|_{L^{s+2}(\Omega)}^{s+2} + \int_\Omega (\varepsilon|\nabla u|^{p-2}\nabla u + |\nabla u^m|^{p-2}\nabla u^m)\nabla u^{s+1} dx \\ &\quad \leq K\|u(t)\|_{L^{s+p}(\Omega)}^{s+p} \leq C_{|\Omega|}\|u(t)\|_{L^{s+2}(\Omega)}^{s+p} \end{aligned}$$

and thus

$$\begin{aligned} &\frac{d}{dt}\|u(t)\|_{L^{s+2}(\Omega)}^{s+2} + (s+1)(s+2)m^{p-1}\int_\Omega u^{(m-1)(p-1)+s}|\nabla u|^p dx \\ &\quad \leq (s+2)C_{|\Omega|}\|u(t)\|_{L^{s+2}(\Omega)}^{s+p}, \end{aligned}$$

where $C_{|\Omega|} := \sup_{s \geq 0} K|\Omega|^{1-\frac{s+p}{s+2}}$. Since $m, p > 1$, it follows

$$\begin{aligned} &\frac{d}{dt}\|u(t)\|_{L^{s+2}(\Omega)}^{s+2} + \frac{s+2}{[m(p-1)+s+1]^p}\int_\Omega |\nabla u^{\frac{m(p-1)+s+1}{p}}|^p dx \\ &\quad \leq \frac{d}{dt}\|u(t)\|_{L^{s+2}(\Omega)}^{s+2} + (s+2)m^{p-1}\left(\frac{p}{m(p-1)+s+1}\right)^p \int_\Omega |\nabla u^{\frac{m(p-1)+s+1}{p}}|^p dx \\ &\quad \leq \frac{d}{dt}\|u(t)\|_{L^{s+2}(\Omega)}^{s+2} + (s+1)(s+2)m^{p-1}\int_\Omega u^{(m-1)(p-1)+s}|\nabla u|^p dx \\ &\quad \leq C_{|\Omega|}(s+2)\|u(t)\|_{L^{s+2}(\Omega)}^{s+p}. \end{aligned} \tag{2.4}$$

For ε fixed and $k = 1, 2, \ldots$, setting

$$s_k := 2p^k + \frac{p^k - p}{p-1} + m - 1, \quad \alpha_k := \frac{p(s_k+2)}{m(p-1)+s_k+1}, \quad w_k := u^{\frac{m(p-1)+s_k+1}{p}},$$

we obtain by (2.4)

$$\frac{d}{dt}\|w_k(t)\|^{\alpha_k}_{L^{\alpha_k}(\Omega)} + \frac{(s_k+2)}{[m(p-1)+s_k+1]^p}\|\nabla w_k(t)\|^p_{L^p(\Omega)} \leq C_{|\Omega|}(s_k+2)\|w_k(t)\|^{\alpha_k\frac{s_k+p}{s_k+2}}_{L^{\alpha_k}(\Omega)}. \tag{2.5}$$

Now, let us fix $s > p$ such that the continuous Sobolev immersion $W^{1,p}_0(\Omega) \subset L^s(\Omega)$ holds and observe that since $s_k \to +\infty$, as $k \to +\infty$, there exists k_0 such that $\alpha_k \in (1,s)$ for all $k \geq k_0$. By the interpolation and the Sobolev inequalities, it results

$$\|w_k(t)\|_{L^{\alpha_k}(\Omega)} \leq \|w_k(t)\|^{\theta_k}_{L^1(\Omega)}\|w_k(t)\|^{1-\theta_k}_{L^s(\Omega)} \leq C\|w_k(t)\|^{\theta_k}_{L^1(\Omega)}\|\nabla w_k(t)\|^{1-\theta_k}_{L^p(\Omega)}$$

for all $k \geq k_0$. Here $\theta_k = (s-\alpha_k)/[\alpha_k(s-1)]$ and C is a positive constant. Using the fact that $\|w_k(t)\|_{L^1(\Omega)} = \|w_{k-1}(t)\|^{\alpha_{k-1}}_{L^{\alpha_{k-1}}(\Omega)}$ and defining $x_k := \sup_{t\in\mathbb{R}} \|w_k(t)\|_{L^{\alpha_k}(\Omega)}$, one has

$$\begin{aligned}\|w_k(t)\|^{\frac{p}{1-\theta_k}}_{L^{\alpha_k}(\Omega)} &\leq C\|w_{k-1}(t)\|^{p\alpha_{k-1}\frac{\theta_k}{1-\theta_k}}_{L^{\alpha_{k-1}}(\Omega)}\|\nabla w_k(t)\|^p_{L^p(\Omega)}\\ &\leq Cx_{k-1}^{p\alpha_{k-1}\frac{\theta_k}{1-\theta_k}}\|\nabla w_k(t)\|^p_{L^p(\Omega)}\end{aligned}$$

for all $k \geq k_0$. Thus, by (2.5),

$$\begin{aligned}&\frac{d}{dt}\|w_k(t)\|^{\alpha_k}_{L^{\alpha_k}(\Omega)}\\ &\quad\leq C_{|\Omega|}(s_k+2)\|w_k(t)\|^{\alpha_k\frac{s_k+p}{s_k+2}}_{L^{\alpha_k}(\Omega)} - \frac{(s_k+2)\|w_k(t)\|^{\frac{p}{1-\theta_k}}_{L^{\alpha_k}(\Omega)}}{C[m(p-1)+s_k+1]^p}x_{k-1}^{p\alpha_{k-1}\frac{\theta_k}{\theta_k-1}}\\ &\quad= \left(C_{|\Omega|} - \frac{\|w_k(t)\|^{\frac{p}{1-\theta_k}-\alpha_k\frac{s_k+p}{s_k+2}}_{L^{\alpha_k}(\Omega)}}{C[m(p-1)+s_k+1]^p}x_{k-1}^{p\alpha_{k-1}\frac{\theta_k}{\theta_k-1}}\right)(s_k+2)\|w_k(t)\|^{\alpha_k\frac{s_k+p}{s_k+2}}_{L^{\alpha_k}(\Omega)}\end{aligned} \tag{2.6}$$

for all $k \geq k_0$. By Lemma 2.3 and (2.6), one has

$$\|w_k(t)\|_{L^{\alpha_k}(\Omega)} \leq (C_{|\Omega|}M_k x_{k-1}^{p\alpha_{k-1}\frac{\theta_k}{1-\theta_k}})^{\eta_k} \tag{2.7}$$

for all $k \geq k_0$, where $\eta_k := \frac{(1-\theta_k)(s_k+2)}{p(s_k+2)-\alpha_k(s_k+p)(1-\theta_k)}$ and $M_k := C[m(p-1)+s_k+1]^p$. By definition of x_k and (2.7), we get

$$x_k \leq (C_{|\Omega|}M_k)^{\eta_k}x_{k-1}^{\nu_k}$$

for all $k \geq k_0$, with $\nu_k := p\alpha_{k-1}\theta_k(s_k+2)/[p(s_k+2)-\alpha_k(s_k+p)(1-\theta_k)]$.

If $x_{k_n} \leq 1$ along a sequence $k_n \to +\infty$, then one has $\|u\|_{L^\infty} \leq 1$ by the very definition of x_k and the lemma is proved. On the other hand, assume $x_k > 1$ for all $k \geq k_0$ and observe that there exists $\overline{k}_0$ such that, for all $k \geq \overline{k}_0$,

$$\eta_k \leq 1/(p\theta) \quad \text{and} \quad \nu_k \leq \alpha_{k-1}.$$

Here $\theta := (s-p)/[p(s-1)]$. Without loss of generality, assume $k_0 = \max\{\overline{k}_0, k_0\}$. Then, there exists a positive constant A such that

$$\begin{aligned} x_k &\leq (C_{|\Omega|}C)^{\eta_k}[m(p-1)+s_k+1]^{p\eta_k} x_{k-1}^{\nu_k} \\ &\leq (C_{|\Omega|}C)^{\eta_k}\left(mp + \frac{2p^{k+1}}{p-1}\right)^{p\eta_k} x_{k-1}^{\nu_k} \leq Ap^{\frac{k+1}{\theta}} x_{k-1}^{\alpha_{k-1}} \end{aligned}$$

for all $k \geq k_0$. Thus

$$\begin{aligned} \log x_k &\leq \log A + \frac{k+1}{\theta}\log p + \alpha_{k-1}\log x_{k-1} \\ &\leq \left(1 + \sum_{i=1}^{k-k_0-1}\prod_{j=1}^{i}\alpha_{k-j}\right)\log A + \left(k+1+\sum_{i=k_0+2}^{k} i \prod_{j=1}^{k+1-i}\alpha_{k-j}\right)\frac{\log p}{\theta} \\ &\quad + \left(\prod_{j=1}^{k-k_0}\alpha_{k-j}\right)\log x_{k_0}. \end{aligned} \tag{2.8}$$

Since

$$\alpha_k = 1 + \frac{1-m(p-1)}{m(p-1)+s_k+1} \leq 1 + \frac{|1-m(p-1)|}{2p^k},$$

for $i \leq k$, we have that

$$\begin{aligned} \log\left(\frac{1}{p^i}\prod_{j=1}^{i}\alpha_{k-j}\right) &= \sum_{j=1}^{i}\log\frac{\alpha_{k-j}}{p} \\ &\leq \sum_{j=1}^{i}\log\left(1+\frac{|1-m(p-1)|}{2p^{k-j+1}}\right) \\ &\leq \frac{|1-m(p-1)|}{2p^k}\sum_{j=1}^{i}p^{j-1} \\ &\leq \frac{|1-m(p-1)|}{2(p-1)}, \end{aligned}$$

which means that

$$\prod_{j=1}^{i}\alpha_{k-j} \leq Mp^i \quad \text{with } M = \exp\frac{|1-m(p-1)|}{2(p-1)}.$$

Therefore, from (2.8) we obtain

$$\begin{aligned}\frac{\log x_k}{M} &\le \log A \sum_{i=0}^{k-k_0-1} p^i + \frac{\log p}{\theta} \sum_{i=k_0+2}^{k+1} i p^{k+1-i} + p^{k-k_0} \log x_{k_0} \\ &\le \frac{\log p}{\theta} \frac{p^{k-k_0}}{(p-1)^2}[k_0(p-1)+2p-1] + \log A \frac{1-p^{k-k_0}}{1-p} + p^{k-k_0} \log x_{k_0}.\end{aligned} \tag{2.9}$$

In fact, taking $x = \frac{1}{p}$ in

$$x \frac{d}{dx} \sum_{i=0}^{k+1} x^i = x \frac{d}{dx} \left(\frac{1-x^{k+2}}{1-x} \right),$$

it results

$$\begin{aligned}\sum_{i=k_0+2}^{k+1} i p^{k+1-i} &= \frac{p^{k+3}}{(p-1)^2} \left[\frac{1}{p^{k+2}} \left(\frac{k+1}{p} - k - 2 \right) - \frac{1}{p^{k_0+2}} \left(\frac{k_0+1}{p} - k_0 - 2 \right) \right] \\ &\le \frac{p^{k+3}}{(p-1)^2} \frac{1}{p^{k_0+2}} \left(k_0 + 2 - \frac{k_0+1}{p} \right) \\ &= \frac{p^{k-k_0}}{(p-1)^2}[k_0(p-1)+2p-1].\end{aligned}$$

Then, by (2.9), it follows

$$x_k^{1/M} \le A^{\frac{1-p^{k-k_0}}{1-p}} p^{\frac{p^{k-k_0}}{\theta(p-1)^2}[k_0(p-1)+2p-1]} x_{k_0}^{p^{k-k_0}}.$$

Since $x_k = \sup_{t\in\mathbb{R}} \|u(t)\|_{s_k+2}^{\frac{m(p-1)+s_k+1}{p}}$, we obtain

$$\begin{aligned}\|u(t)\|_{L^\infty(\Omega)} &\le \lim_{k\to\infty} \|u(t)\|_{s_k+2} \\ &\le \limsup_{k\to\infty} \{ A^{\frac{p}{m(p-1)+s_k+1} \frac{1-p^{k-k_0}}{1-p}} x_{k_0}^{\frac{p^{k-k_0+1}}{m(p-1)+s_k+1}} p^{\frac{p^{k-k_0+1}(k_0(p-1)+2p-1)}{\theta(p-1)^2(m(p-1)+s_k+1)}} \}^M \\ &=: R \quad \forall\, t \in \mathbb{R},\end{aligned}$$

where R is a positive constant. Hence $\sup_{t\in\mathbb{R}} \|u(t)\|_{L^\infty(\Omega)} \le R$. It remains to prove that R is independent of ε as claimed. To this aim it is sufficient to prove that there exists $C > 0$, independent of ε, such that $x_{k_0} \le C$. In fact, by (2.5) with $s_0 := s_{k_0}$, it follows

$$\frac{d}{dt}\|u(t)\|_{L^{s_0+2}(\Omega)}^{s_0+2} + \frac{(s_0+2)\int_\Omega |\nabla u^{\frac{m(p-1)+s_0+1}{p}}|^p dx}{[m(p-1)+s_0+1]^p} \le C_{|\Omega|}(s_0+2)\|u(t)\|_{L^{s_0+2}(\Omega)}^{s_0+p}. \tag{2.10}$$

Moreover, we have

$$\|u(t)\|_{L^{s_0+2}(\Omega)}^{m(p-1)+s_0+1} \le C_1 \left[\int_\Omega (u^{\frac{m(p-1)+s_0+1}{p}})^{p\frac{s_0+2}{s_0+p}}\, dx\right]^{\frac{s_0+p}{s_0+2}}$$

by Hölder's inequality with $r = \frac{m(p-1)+s_0+1}{s_0+p}$. Now, without loss of generality, we can assume that k_0 is chosen large enough such that the continuous Sobolev immersion $W_0^{1,p}(\Omega) \subset L^{p\frac{s_0+2}{s_0+p}}(\Omega)$ holds and, hence, we deduce that

$$\|u(t)\|_{L^{s_0+2}(\Omega)}^{m(p-1)+s_0+1} \le C_2 \big\|\nabla u^{\frac{m(p-1)+s_0+1}{p}}\big\|_{L^p(\Omega)}^p,$$

for a positive constant C_2. Thus, using (2.10), one has

$$\begin{aligned}
&\frac{d}{dt}\|u(t)\|_{L^{s_0+2}(\Omega)}^{s_0+2} + \frac{s_0+2}{C_2[m(p-1)+s_0+1]^p}\|u(t)\|_{L^{s_0+2}(\Omega)}^{m(p-1)+s_0+1} \\
&\quad \le \frac{d}{dt}\|u(t)\|_{L^{s_0+2}(\Omega)}^{s_0+2} + \frac{s_0+2}{[m(p-1)+s_0+1]^p}\big\|\nabla u^{\frac{m(p-1)+s_0+1}{p}}\big\|_{L^p(\Omega)}^p \\
&\quad \le C_{|\Omega|}(s_0+2)\|u(t)\|_{L^{s_0+2}(\Omega)}^{s_0+p}.
\end{aligned}$$

Hence

$$\frac{d}{dt}\|u(t)\|_{L^{s_0+2}(\Omega)}^{s_0+2} \le \|u(t)\|_{L^{s_0+2}(\Omega)}^{s_0+p}(C_{|\Omega|}(s_0+2) - M\|u(t)\|_{L^{s_0+2}(\Omega)}^{(m-1)(p-1)}),$$

where $M := \frac{s_0+2}{C_2[m(p-1)+s_0+1]^p}$. This inequality and Lemma 2.3 below imply that

$$\|u(t)\|_{L^{s_0+2}(\Omega)} \le \{C_2 C_{|\Omega|}[m(p-1)+s_0+1]^p\}^{\frac{1}{(m-1)(p-1)}} \quad \forall\, t \in \mathbb{R}.$$

Thus, there exists $C > 0$, independent of ε, such that

$$x_{k_0} = \sup_{t\in\mathbb{R}} \|u(t)\|_{L^{s_0+2}(\Omega)}^{\frac{m(p-1)+s_0+1}{p}} \le C,$$

and this concludes the proof. □

Lemma 2.3. *Let $f : \mathbb{R} \to (0,+\infty)$ be a differentiable and T-periodic function; suppose that there exist positive constants s, α, β, γ such that*

$$f'(t) \le f^s(t)(\beta - \gamma f^\alpha(t)),$$

for all $t \in \mathbb{R}$. Then $\beta - \gamma f^\alpha(t) \ge 0$ for all $t \in \mathbb{R}$.

Proof. By the periodicity and continuity of f, let t_0 be any point in which f attains its maximum. Then we have:

$$\beta - \gamma f^\alpha(t) \ge \beta - \gamma f^\alpha(t_0) \ge \frac{f'(t_0)}{f^s(t_0)} = 0 \quad \forall\, t \in \mathbb{R}.$$

□

Next, we show that the map $I - G_\varepsilon : \{1\} \times L^\infty(Q_T) \times L^\infty(Q_T) \to L^\infty(Q_T) \times L^\infty(Q_T)$ has the Leray–Schauder topological degree different from zero in the intersection of a sufficiently large ball centered at the origin with the cone of nonnegative functions.

From now on we make the following assumption.

Assumption 2. There exist two positive constants C_1, C_2 such that

(1) for all $\varepsilon > 0$ and all solution pairs $(u_\varepsilon, v_\varepsilon)$ of

$$(u, v) = G_\varepsilon(1, f(u^+, v^+), g(u^+, v^+)), \tag{2.11}$$

it results

$$\|u_\varepsilon\|^2_{L^2} \le C_1 \quad \text{and} \quad \|v_\varepsilon\|^2_{L^2} \le C_2, \tag{2.12}$$

(2) we have:

$$\min\left\{\frac{1}{T}\iint_{Q_T} ae_p^p - \frac{\underline{k}_2 C_2}{T}, \frac{1}{T}\iint_{Q_T} be_q^q - \frac{\underline{k}_3 C_1}{T}\right\} > 0,$$

where $\underline{k}_2, \underline{k}_3$ are as in Assumption 1(3), and, for any $r > 1$, e_r stands for the positive eigenvector associated to the first eigenvalue μ_r of $-\Delta_r$ with Dirichlet boundary conditions and normalized in such a way that $\|e_r\|_{L^r(\Omega)} = 1$.

Remark 2.1. The last assumptions, and in particular statement (2), will grant the periodic coexistence, that is the existence of a T-periodic solution couple (u, v) with u, v both nonnegative and nontrivial (see Theorem 2.1). Generally speaking, the *explicit* knowledge of the *a priori* bounds C_1, C_2 is required to check the validity of Assumption 2(2). This is the task we will devote ourselves to in the next section. A notable exception is the case in which K_2, K_3 are not negative (namely, the cooperative case), since one can choose $\underline{k}_2 = \underline{k}_3 = 0$ and, thus, Assumption 2(2) is readily satisfied regardless of the values of C_1, C_2, if neither of the coefficients a, b is trivial.

The next result shows how we can pass from an L^2-estimate to an L^∞-estimate.

Proposition 2.2. *There is a constant $R > 0$ such that*

$$\|u_\varepsilon\|_{L^\infty}, \|v_\varepsilon\|_{L^\infty} < R$$

for all solution pairs $(u_\varepsilon, v_\varepsilon)$ of (2.11) with $\varepsilon > 0$ sufficiently small. In particular, one has that

$$\deg((u, v) - G_\varepsilon(1, f(u^+, v^+), g(u^+, v^+)), B_R, 0) = 1.$$

Proof. Assume $u_\varepsilon \ne 0$, thus $u_\varepsilon > 0$ and $v_\varepsilon \ge 0$ in Q_T by Proposition 2.1. Multiplying by u_ε the first equation of (2.1), integrating over Ω and using the Steklov averages $(u_\varepsilon)_h \in H^1(Q_{T-\delta})$, $\delta, h > 0$, see [39, p. 85], we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (u_\varepsilon)_h^2 dx + \varepsilon\int_\Omega |\nabla(u_\varepsilon)_h|^p dx + m^{p-1}\int_\Omega (u_\varepsilon)_h^{(m-1)(p-1)}|\nabla(u_\varepsilon)_h|^p dx$$

$$\leq \left(\|a\|_{L^\infty} + \int_\Omega K_2(\xi,t)(v_\varepsilon)_h^2(\xi,t-\tau_2)d\xi\right)\int_\Omega (u_\varepsilon)_h^p dx$$

$$\leq |\Omega|^{1-\frac{p}{2}}\left(\|a\|_{L^\infty} + \int_\Omega K_2(\xi,t)(v_\varepsilon)_h^2(\xi,t-\tau_2)d\xi\right)\left(\int_\Omega (u_\varepsilon)_h^2 dx\right)^{\frac{p}{2}}.$$

Thus

$$\frac{\frac{1}{2}\frac{d}{dt}\int_\Omega (u_\varepsilon)_h^2 dx + \varepsilon\int_\Omega |\nabla(u_\varepsilon)_h|^p dx + m^{p-1}\int_\Omega (u_\varepsilon)_h^{(m-1)(p-1)}|\nabla(u_\varepsilon)_h|^p dx}{\left(\int_\Omega (u_\varepsilon)_h^2 dx\right)^{\frac{p}{2}}}$$
$$\leq |\Omega|^{1-\frac{p}{2}}\left(\|a\|_{L^\infty} + \|K_2\|_{L^\infty}\int_\Omega (v_\varepsilon)_h^2(\xi,t-\tau_2)d\xi\right). \tag{2.13}$$

Since $t \mapsto \|u(t)\|_{L^2(\Omega)}$ is continuous in $[0,T]$, there exist t_1 and t_2 in $[0,T]$ such that

$$\int_\Omega u_\varepsilon^2(x,t_1)dx = \min_{t\in[0,T]}\int_\Omega u_\varepsilon^2(x,t)dx$$

and

$$\int_\Omega u_\varepsilon^2(x,t_2)dx = \max_{t\in[0,T]}\int_\Omega u_\varepsilon^2(x,t)dx.$$

Without loss of generality, by periodicity, we can assume that $t_1 < t_2$. Then, integrating (2.13) between t_1 and t_2 and passing to the limit as $h \to 0$, we find

$$\int_{t_1}^{t_2}\left(\int_\Omega u_\varepsilon^2 dx\right)^{-\frac{p}{2}}\frac{d}{dt}\left(\int_\Omega u_\varepsilon^2 dx\right)dt$$
$$\leq 2|\Omega|^{1-\frac{p}{2}}\int_0^T\left(\|a\|_{L^\infty} + \|K_2\|_{L^\infty}\int_\Omega v_\varepsilon^2(\xi,t-\tau_2)d\xi\right)dt.$$

Thus

$$\left(\int_\Omega u_\varepsilon^2(x,t_2)dx\right)^{\frac{2-p}{2}} - \left(\int_\Omega u_\varepsilon^2(x,t_1)dx\right)^{\frac{2-p}{2}} \leq C(T\|a\|_{L^\infty} + \|K_2\|_{L^\infty}C_2),$$

where $C := (2-p)|\Omega|^{1-\frac{p}{2}}$. Hence

$$\left(\int_\Omega u_\varepsilon^2(x,t_2)dx\right)^{\frac{2-p}{2}} \leq \left(\int_\Omega u_\varepsilon^2(x,t_1)dx\right)^{\frac{2-p}{2}} + C(T\|a\|_{L^\infty} + \|K_2\|_{L^\infty}C_2),$$

or, equivalently,

$$\max_{t\in[0,T]}\int_\Omega u_\varepsilon^2(x,t)dx$$
$$\leq \left\{\left(\min_{t\in[0,T]}\int_\Omega u_\varepsilon^2(x,t)dx\right)^{\frac{2-p}{2}} + C(T\|a\|_{L^\infty} + \|K_2\|_{L^\infty}C_2)\right\}^{\frac{2}{2-p}}.$$

This implies that there exists a constant $\gamma > 0$, independent of ε, such that

$$\max_{t\in[0,T]}\int_\Omega u_\varepsilon^2(x,t)dx \leq \gamma.$$

Otherwise, for all $\gamma > 0$ there would exist $\varepsilon > 0$ such that the corresponding solution u_ε satisfies

$$\begin{aligned}\gamma &< \max_{t\in[0,T]} \int_\Omega u_\varepsilon^2(x,t)dx \\ &\le \left\{ \left(\min_{t\in[0,T]} \int_\Omega u_\varepsilon^2(x,t)dx \right)^{\frac{2-p}{2}} + C(T\|a\|_{L^\infty} + \|K_2\|_{L^\infty} C_2) \right\}^{\frac{2}{2-p}}.\end{aligned}$$

Using the fact that $\frac{2}{2-p} > 1$ and integrating the previous inequality on $[0,T]$ for sufficiently large γ, one would have

$$\gamma T \le \iint_{Q_T} u_\varepsilon^2 + CT(T\|a\|_{L^\infty} + \|K_2\|_{L^\infty} C_2),$$

that is u_ε is unbounded in $L^2(Q_T)$, in contradiction with Assumption 2(1). Of course, an analogous inequality holds for v_ε.

Now, we have

$$\begin{aligned}\frac{\partial u_\varepsilon}{\partial t} &- \varepsilon \operatorname{div}(|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon) - \operatorname{div}(|\nabla u_\varepsilon^m|^{p-2}\nabla u_\varepsilon) \\ &\le \left(\|a\|_{L^\infty} + \|K_2\|_{L^\infty} \max_{t\in[0,T]} \int_\Omega v_\varepsilon^2(x,t)dx \right) u_\varepsilon^{p-1} \\ &\le (\|a\|_{L^\infty} + \|K_2\|_{L^\infty}\gamma) u_\varepsilon^{p-1},\end{aligned} \tag{2.14}$$

i.e.

$$\frac{\partial u_\varepsilon}{\partial t} - \varepsilon \operatorname{div}(|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon) - \operatorname{div}(\nabla u_\varepsilon^m |\nabla u_\varepsilon^m|^{p-2}) \le K u_\varepsilon^{p-1} \quad \text{in } Q_T,$$

where $K := \|a\|_{L^\infty} + \|K_2\|_{L^\infty}\gamma$. By Lemma 2.2 we conclude that $\|u_\varepsilon\|_{L^\infty} \le R_1$ for some $R_1 > 0$ independent of ε. Analogously, $\|v_\varepsilon\|_{L^\infty} \le R_2$ for some constant $R_2 > 0$. Therefore it is enough to choose $R > \max\{R_1, R_2\}$.

The previous calculations also show that any solution pair of

$$(u,v) = G_\varepsilon(1, \rho f(u^+, v^+), \rho g(u^+, v^+))$$

with $\rho \in [0,1]$ satisfies the same inequality (2.14). Therefore, the homotopy invariance property of the Leray–Schauder degree implies that

$$\begin{aligned}&\deg((u,v) - T_\varepsilon(1, u^+, v^+), B_R, 0) \\ &\quad = \deg((u,v) - G_\varepsilon(1, \rho f(u^+, v^+), \rho g(u^+, v^+)), B_R, 0)\end{aligned}$$

for any $\rho \in [0,1]$. If we take $\rho = 0$, using the fact that G_ε at $\rho = 0$ is the zero map, we obtain

$$\deg((u,v) - T_\varepsilon(1, u^+, v^+), B_R, 0) = \deg((u,v), B_R, 0) = 1. \qquad \square$$

In order to prove that the solutions $(u_\varepsilon, v_\varepsilon)$ of (2.1) we are going to find are not bifurcating from the trivial solution $(0,0)$, the next estimate will be crucial.

Lemma 2.4. *Let $s > 0$ be such that*

$$s < \min\left\{\frac{(p-1)(m-p)}{p}, \frac{(q-1)(n-q)}{q}\right\}.$$

Then, there exist two positive constants M_1 and M_2 such that

$$\left\|\nabla u_\varepsilon^{\frac{(p-1)(m-1)-s}{p-1}}\right\|_{L^p} \le M_1 \quad \text{and} \quad \left\|\nabla v_\varepsilon^{\frac{(q-1)(n-1)-s}{q-1}}\right\|_{L^q} \le M_2,$$

for all solution pairs $(u_\varepsilon, v_\varepsilon)$ of (2.11) and $\varepsilon > 0$ sufficiently small.

Proof. Let $\gamma := \frac{(p-1)(m-p)-ps}{p-1} > 0$. Multiplying the equation

$$\begin{aligned}&\frac{\partial u_\varepsilon}{\partial t} - \varepsilon \operatorname{div}(|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon) - \operatorname{div}(|\nabla u_\varepsilon^m|^{p-2}\nabla u_\varepsilon^m)\\ &\quad = \left(a(x,t) - \int_\Omega K_1(\xi,t)u_\varepsilon^2(\xi,t-\tau_1)d\xi + \int_\Omega K_2(\xi,t)v_\varepsilon^2(\xi,t-\tau_2)d\xi\right) u_\varepsilon^{p-1}\end{aligned}$$

by u_ε^γ, integrating over Q_T and passing to the limit in the Steklovy averages, by the T-periodicity of u_ε we obtain

$$\begin{aligned}&\varepsilon \iint_{Q_T} |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \nabla u_\varepsilon^\gamma + \iint_{Q_T} |\nabla u_\varepsilon^m|^{p-2}\nabla u_\varepsilon^m \nabla u_\varepsilon^\gamma\\ &\quad = \iint_{Q_T} a u_\varepsilon^{\gamma+p-1} - \int_0^T \left(\int_\Omega u_\varepsilon^{\gamma+p-1}(x,t)dx\right)\left(\int_\Omega K_1(\xi,t)u_\varepsilon^2(\xi,t-\tau_1)d\xi\right) dt\\ &\qquad + \int_0^T \left(\int_\Omega u_\varepsilon^{\gamma+p-1}(x,t)dx\right)\left(\int_\Omega K_2(\xi,t)v_\varepsilon^2(\xi,t-\tau_2)d\xi\right) dt.\end{aligned}$$

Now, since

$$\iint_{Q_T} |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \nabla u_\varepsilon^\gamma = \gamma \iint_{Q_T} u_\varepsilon^{\gamma-1}|\nabla u_\varepsilon|^p \ge 0,$$

then

$$\begin{aligned}&\gamma m^{p-1}\left(\frac{p-1}{(p-1)(m-1)-s}\right)^p \left\|\nabla u_\varepsilon^{\frac{(p-1)(m-1)-s}{p-1}}\right\|_{L^p}^p\\ &\quad = \gamma m^{p-1} \iint_{Q_T} u_\varepsilon^{(p-1)(m-1)+\gamma-1}|\nabla u_\varepsilon|^p\\ &\quad \le \varepsilon \iint_{Q_T} |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \nabla u_\varepsilon^\gamma + \iint_{Q_T} |\nabla u_\varepsilon^m|^{p-2}\nabla u_\varepsilon^m \nabla u_\varepsilon^\gamma\\ &\quad \le \|a\|_{L^\infty} \iint_{Q_T} u_\varepsilon^{\gamma+p-1} - \int_0^T \left(\int_\Omega u_\varepsilon^{\gamma+p-1}(x,t)dx\right)\\ &\qquad \times \left(\int_\Omega K_1(\xi,t)u_\varepsilon^2(\xi,t-\tau_1)d\xi\right) dt\\ &\qquad + \int_0^T \left(\int_\Omega u_\varepsilon^{\gamma+p-1}(x,t)dx\right)\left(\int_\Omega K_2(\xi,t)v_\varepsilon^2(\xi,t-\tau_2)d\xi\right) dt. \qquad (2.15)\end{aligned}$$

Moreover, by the Hölder inequality with $\beta := p\frac{(p-1)(m-1)-s}{(p-1)(m-1)-ps} > 1$ and the Poincaré inequality, one has

$$\begin{aligned}\iint_{Q_T} u_\varepsilon^{\gamma+p-1} &\leq |Q_T|^{\frac{1}{\beta'}} \left(\iint_{Q_T} u_\varepsilon^{p\frac{(p-1)(m-1)-s}{p-1}}\right)^{\frac{1}{\beta}} \\ &= |Q_T|^{\frac{1}{\beta'}} \left\|u_\varepsilon^{\frac{(p-1)(m-1)-s}{p-1}}\right\|_{L^p}^{\frac{p}{\beta}} \\ &\leq |Q_T|^{\frac{1}{\beta'}} \left(\frac{1}{\mu_p}\right)^{\frac{1}{\beta}} \left\|\nabla u_\varepsilon^{\frac{(p-1)(m-1)-s}{p-1}}\right\|_{L^p}^{\frac{p}{\beta}}. \end{aligned} \tag{2.16}$$

Here β' is such that $\frac{1}{\beta} + \frac{1}{\beta'} = 1$. Thus (2.15) and (2.16) imply

$$\begin{aligned}&\gamma m^{p-1} \left(\frac{p-1}{(p-1)(m-1)-s}\right)^p \left\|\nabla u_\varepsilon^{\frac{(p-1)(m-1)-s}{p-1}}\right\|_{L^p}^p \\ &\quad \leq \|a\|_{L^\infty} |Q_T|^{\frac{1}{\beta'}} \left(\frac{1}{\mu_p}\right)^{\frac{1}{\beta}} \left\|\nabla u_\varepsilon^{\frac{(p-1)(m-1)-s}{p-1}}\right\|_{L^p}^{\frac{p}{\beta}} \\ &\qquad - \int_0^T \left(\int_\Omega u_\varepsilon^{\gamma+p-1}(x,t)dx\right)\left(\int_\Omega K_1(\xi,t)u_\varepsilon^2(\xi,t-\tau_1)d\xi\right) dt \\ &\qquad + \int_0^T \left(\int_\Omega u_\varepsilon^{\gamma+p-1}(x,t)dx\right)\left(\int_\Omega K_2(\xi,t)v_\varepsilon^2(\xi,t-\tau_2)d\xi\right) dt. \end{aligned} \tag{2.17}$$

By assumptions, there are constants $\overline{k}_i \geq 0$, $i = 2, 3$, such that $K_i(x,t) \leq \overline{k}_i$ for a.a. $(x,t) \in Q_T$. Thus, by (2.16), (2.17) and Proposition 2.2, we have

$$\begin{aligned}&\gamma m^{p-1} \left(\frac{p-1}{(p-1)(m-1)-s}\right)^p \left\|\nabla u_\varepsilon^{\frac{(p-1)(m-1)-s}{p-1}}\right\|_{L^p}^p \\ &\quad \leq \|a\|_{L^\infty} |Q_T|^{\frac{1}{\beta'}} \left(\frac{1}{\mu_p}\right)^{\frac{1}{\beta}} \left\|\nabla u_\varepsilon^{\frac{(p-1)(m-1)-s}{p-1}}\right\|_{L^p}^{\frac{p}{\beta}} + \overline{k}_2|\Omega|R^2 \iint_{Q_T} u_\varepsilon^{\gamma+p-1} \\ &\quad \leq (\|a\|_{L^\infty} + \overline{k}_2|\Omega|R^2)|Q_T|^{\frac{1}{\beta'}} \left(\frac{1}{\mu_p}\right)^{\frac{1}{\beta}} \left\|\nabla u_\varepsilon^{\frac{(p-1)(m-1)-s}{p-1}}\right\|_{L^p}^{\frac{p}{\beta}}. \end{aligned}$$

In particular,

$$\left\|\nabla u_\varepsilon^{\frac{(p-1)(m-1)-s}{p-1}}\right\|_{L^p} \leq M_1,$$

where

$$M_1 := \left(\frac{(\|a\|_{L^\infty} + \overline{k}_2|\Omega|R^2)|Q_T|^{\frac{1}{\beta'}}(\frac{1}{\mu_p})^{\frac{1}{\beta}}[(p-1)(m-1)-s]^p}{[m(p-1)]^{p-1}[(p-1)(m-p)-ps]}\right)^{\frac{\beta}{p(\beta-1)}}.$$

Analogously, one can prove that

$$\left\|\nabla v_\varepsilon^{\frac{(q-1)(n-1)-s}{q-1}}\right\|_{L^q} \le M_2,$$

where

$$M_2 := \left(\frac{(\|b\|_{L^\infty} + \overline{k}_3|\Omega|R^2)|Q_T|^{\frac{1}{\delta'}}(\frac{1}{\mu_q})^{\frac{1}{\delta}}[(q-1)(n-1)-s]^q}{[n(q-1)]^{q-1}[(q-1)(n-q)-qs]} \right)^{\frac{\delta}{q(\delta-1)}}.$$

Here δ and δ' are such that $\delta := q\frac{(q-1)(n-1)-s}{(q-1)(n-1)-qs}$ and $\frac{1}{\delta} + \frac{1}{\delta'} = 1$. $\square$

Remark 2.2. Observe that in the case when $p = q = 2$ *a priori* bounds for $\|\nabla u_\varepsilon^m\|_{L^2}, \|\nabla v_\varepsilon^n\|_{L^2}$ have been obtained in [25] for sufficiently small $\epsilon > 0$ under the conditions that $m, n > 1$, i.e. in the case of slow diffusion. Under the same condition $m, n > 1$ *a priori* bounds for $\|\nabla u_\varepsilon^m\|_{L^p}, \|\nabla v_\varepsilon^n\|_{L^q}$ have been obtained in [26] when $p, q > 2$, which again corresponds to the case of slow diffusion. Therefore the assumptions $m > p$ and $n > q$ are required in Lemma 2.4 only for the singular case $p, q \in (1, 2)$, which, as already noticed, allows the fast diffusion if $p, q \in (1, \frac{1+\sqrt{5}}{2})$. Finally, observe that, if $p, q > 1$, we have the fast diffusion when $m < 1/(p-1)$ and $n < 1/(q-1)$: to the best of our knowledge, this case is not treated in the existing literature devoted to this problem.

The following result guarantees that the foreseen solutions $(u_\varepsilon, v_\varepsilon)$ of (2.1) are not bifurcating from the trivial solution $(0, 0)$ as ε ranges in $(0, \varepsilon_0)$, where ε_0 is such that

$$\theta(C_1, C_2) := \min\left\{ \frac{1}{T}\iint_{Q_T} ae_p^p - \varepsilon_0\mu_p - \frac{\underline{k}_2 C_2}{T}, \frac{1}{T}\iint_{Q_T} be_q^q - \varepsilon_0\mu_q - \frac{\underline{k}_3 C_1}{T} \right\} > 0, \tag{2.18}$$

where μ_p, μ_q, e_p, e_q, $\underline{k}_2$ and $\underline{k}_3$ are as in Assumption 2.

To this aim let

$$r_0 := \min\left\{ \left(\frac{\iint_{Q_T} ae_p^p - \varepsilon_0 T\mu_p}{D_1} \right)^{\frac{1}{2}}, \left(\frac{\iint_{Q_T} ae_p^p - \varepsilon_0 T\mu_p}{D_1} \right)^{\frac{1}{s}}, \left(\frac{\iint_{Q_T} be_q^q - \varepsilon_0 T\mu_q}{D_2} \right)^{\frac{1}{2}}, \left(\frac{\iint_{Q_T} be_q^q - \varepsilon_0 T\mu_q}{D_2} \right)^{\frac{1}{s}} \right\},$$

where

$$D_1 := \|K_1\|_{L^1} + \|K_2\|_{L^1} + p\|e_p^{p-1}\|_{L^\infty}\|\nabla e_p\|_{L^\infty} \left(\frac{m(p-1)M_1}{(p-1)(m-1)-s} \right)^{p-1} |Q_T|^{\frac{1}{p}},$$

$$D_2 := \|K_3\|_{L^1} + \|K_4\|_{L^1} + q\|e_q^{q-1}\|_{L^\infty}\|\nabla e_q\|_{L^\infty} \left(\frac{n(q-1)M_2}{(q-1)(n-1)-s} \right)^{q-1} |Q_T|^{\frac{1}{q}}$$

and M_1, M_2, s are as in Lemma 2.4. By (2.18), r_0 is obviously positive.

Proposition 2.3. *For all solution pairs* $(u_\varepsilon, v_\varepsilon)$ *of* (2.11) *and all* $\varepsilon \in (0, \varepsilon_0)$, *it results*

$$\max\{\|u_\varepsilon\|_{L^\infty}, \|v_\varepsilon\|_{L^\infty}\} \geq r_0.$$

Moreover $\deg((u,v) - T_\varepsilon(1, u^+, v^+), B_r, 0) = 0$ *for all* $r \in (0, r_0)$.

Proof. By contradiction, assume that for some $r \in (0, r_0)$ there exists a pair $(u_\varepsilon, v_\varepsilon) \neq (0,0)$ such that $(u_\varepsilon, v_\varepsilon) = G_\varepsilon(1, f(u_\varepsilon^+, v_\varepsilon^+), g(u_\varepsilon^+, v_\varepsilon^+))$ with $\|u_\varepsilon\|_{L^\infty} \leq r$ and $\|v_\varepsilon\|_{L^\infty} \leq r$. Assume that $u_\varepsilon \neq 0$ and take $\phi \in C_c^\infty(\Omega)$. Since by Proposition 2.1 we have $u_\varepsilon > 0$ in Q_T, we can multiply the equation

$$\begin{aligned}\frac{\partial u_\varepsilon}{\partial t} &- \varepsilon \operatorname{div}(|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon) - \operatorname{div}(|\nabla u_\varepsilon^m|^{p-2}\nabla u_\varepsilon^m) \\ &= \left(a - \int_\Omega K_1(\xi,t)u_\varepsilon^2(\xi, t-\tau_1)d\xi + \int_\Omega K_2(\xi,t)v_\varepsilon^2(\xi,t-\tau_2)d\xi\right)u_\varepsilon^{p-1}\end{aligned}$$

by $\frac{\phi^p}{u_\varepsilon^{p-1}}$, integrate over Q_T and pass to the limit in the Steklov averages in order to obtain

$$\begin{aligned}-\varepsilon \iint_{Q_T} \frac{\phi^p}{u_\varepsilon^{p-1}} \operatorname{div}(|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon) &- \iint_{Q_T} \frac{\phi^p}{u_\varepsilon^{p-1}} \operatorname{div}(|\nabla u_\varepsilon^m|^{p-2}\nabla u_\varepsilon^m) \\ = \iint_{Q_T} \phi^p a &- \iint_{Q_T} \phi^p(x)\left(\int_\Omega K_1(\xi,t)u_\varepsilon^2(\xi,t-\tau_1)d\xi\right)dxdt \\ &+ \iint_{Q_T} \phi^p(x)\left(\int_\Omega K_2(\xi,t)v_\varepsilon^2(\xi,t-\tau_2)d\xi\right)dxdt, \end{aligned} \tag{2.19}$$

by the T-periodicity of u_ε. By the generalized Picone's identity due to Allegretto–Huan, see [4], one has

$$\begin{aligned}-\varepsilon &\iint_{Q_T} \frac{\phi^p}{u_\varepsilon^{p-1}} \operatorname{div}(|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon) \\ &= \varepsilon \iint_{Q_T} |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \nabla\left(\frac{\phi^p}{u_\varepsilon^{p-1}}\right) \leq \varepsilon \iint_{Q_T} |\nabla\phi|^p. \end{aligned} \tag{2.20}$$

Indeed, we have that

$$\begin{aligned}|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \nabla\left(\frac{\phi^p}{u_\varepsilon^{p-1}}\right) &\leq p|\nabla\phi|\left(\frac{\phi}{u_\varepsilon}|\nabla u_\varepsilon|\right)^{p-1} - (p-1)\left(\frac{\phi}{u_\varepsilon}|\nabla u_\varepsilon|\right)^p \\ &= \left(\frac{\phi}{u_\varepsilon}|\nabla u_\varepsilon|\right)^p + p\left(\frac{\phi}{u_\varepsilon}|\nabla u_\varepsilon|\right)^{p-1}\left(|\nabla\phi| - \frac{\phi}{u_\varepsilon}|\nabla u_\varepsilon|\right) \\ &\leq |\nabla\phi|^p,\end{aligned}$$

since the function $\mathbb{R} \ni \xi \mapsto |\xi|^p$ is convex for $p > 1$.

Moreover,

$$
\begin{aligned}
&-\iint_{Q_T} \frac{\phi^p}{u_\varepsilon^{p-1}} \operatorname{div}(\nabla u_\varepsilon^m |\nabla u_\varepsilon^m|^{p-2}) \\
&\quad = \iint_{Q_T} \nabla \left(\frac{\phi^p}{u_\varepsilon^{p-1}} \right) \nabla u_\varepsilon^m |\nabla u_\varepsilon^m|^{p-2} \\
&\quad = m^{p-1} \iint_{Q_T} u_\varepsilon^{(m-1)(p-1)} |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \frac{u_\varepsilon^{p-1} \nabla \phi^p - \phi^p \nabla u_\varepsilon^{p-1}}{u_\varepsilon^{2(p-1)}} \\
&\quad = p m^{p-1} \iint_{Q_T} \phi^{p-1} u_\varepsilon^{(p-1)(m-2)} |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \nabla \phi \\
&\qquad - m^{p-1}(p-1) \iint_{Q_T} \phi^p u_\varepsilon^{(m-2)(p-1)-1} |\nabla u_\varepsilon|^p \\
&\quad \le p \|\phi^{p-1}\|_{L^\infty} \|\nabla \phi\|_{L^\infty} m^{p-1} \iint_{Q_T} |\nabla u_\varepsilon|^{p-1} u_\varepsilon^{(p-1)(m-2)}. \qquad (2.21)
\end{aligned}
$$

By (2.19)–(2.21), it follows

$$
\begin{aligned}
&\iint_{Q_T} \phi^p a - \iint_{Q_T} \phi^p(x) \left(\int_\Omega K_1(\xi,t) u_\varepsilon^2(\xi, t-\tau_1) d\xi \right) dxdt \\
&\qquad + \iint_{Q_T} \phi^p(x) \left(\int_\Omega K_2(\xi,t) v_\varepsilon^2(\xi, t-\tau_2) d\xi \right) dxdt \\
&\quad \le p \|\phi^{p-1}\|_{L^\infty} \|\nabla \phi\|_{L^\infty} m^{p-1} \iint_{Q_T} |\nabla u_\varepsilon|^{p-1} u_\varepsilon^{(p-1)(m-2)} + \varepsilon \iint_{Q_T} |\nabla \phi|^p.
\end{aligned}
$$

Taking $\phi(x) = \phi_j(x) \to e_p(x)$ in $C_0^1(\Omega)$ as $j \to +\infty$ and since $\varepsilon < \varepsilon_0$, one has

$$
\begin{aligned}
&\iint_{Q_T} e_p^p a - \iint_{Q_T} e_p^p(x) \left(\int_\Omega K_1(\xi,t) u_\varepsilon^2(\xi, t-\tau_1) d\xi \right) dxdt \\
&\qquad + \iint_{Q_T} e_p^p(x) \left(\int_\Omega K_2(\xi,t) v_\varepsilon^2(\xi, t-\tau_2) d\xi \right) dxdt \\
&\quad \le p \|e_p^{p-1}\|_{L^\infty} \|\nabla e_p\|_{L^\infty} m^{p-1} \iint_{Q_T} |\nabla u_\varepsilon|^{p-1} u_\varepsilon^{(p-1)(m-2)} + \varepsilon_0 \iint_{Q_T} |\nabla e_p|^p.
\end{aligned}
$$

Taking into account that $\|e_p\|_{L^p(\Omega)} = 1$, the previous inequality implies

$$
\begin{aligned}
&\iint_{Q_T} e_p^p a - \varepsilon_0 T \mu_p \\
&\quad \le \iint_{Q_T} K_1(\xi,t) u_\varepsilon^2(\xi, t-\tau_1) d\xi dt - \iint_{Q_T} K_2(\xi,t) v_\varepsilon^2(\xi, t-\tau_2) d\xi dt \\
&\qquad + p \|e_p^{p-1}\|_{L^\infty} \|\nabla e_p\|_{L^\infty} m^{p-1} \iint_{Q_T} |\nabla u_\varepsilon|^{p-1} u_\varepsilon^{(p-1)(m-2)}. \qquad (2.22)
\end{aligned}
$$

Now we estimate the term $\iint_{Q_T} |\nabla u_\varepsilon|^{p-1} u_\varepsilon^{(p-1)(m-2)}$. Since $\|u_\varepsilon\|_{L^\infty} \leq r$, using the Hölder inequality, one has

$$\begin{aligned}
\iint_{Q_T} |\nabla u_\varepsilon|^{p-1} u_\varepsilon^{(p-1)(m-2)} &\leq r^s \iint_{Q_T} |\nabla u_\varepsilon|^{p-1} u_\varepsilon^{(p-1)(m-2)-s} \\
&= r^s \left(\frac{p-1}{(p-1)(m-1)-s}\right)^{p-1} \iint_{Q_T} |\nabla u_\varepsilon^{\frac{(p-1)(m-1)-s}{p-1}}|^{p-1} \\
&\leq r^s \left(\frac{p-1}{(p-1)(m-1)-s}\right)^{p-1} |Q_T|^{\frac{1}{p}} \left\|\nabla u_\varepsilon^{\frac{(p-1)(m-1)-s}{p-1}}\right\|_{L^p}^{p-1}. \qquad (2.23)
\end{aligned}$$

Observe that $(p-1)(m-1)-s > 0$, since, by assumption, $s < (p-1)(m-p)/p < (p-1)(m-1)$. By Lemma 2.4, (2.22) and (2.23), it follows

$$\begin{aligned}
&\iint_{Q_T} e_p^p a - \varepsilon_0 T \mu_p \leq (\|K_1\|_{L^1} \\
&\quad + \|K_2\|_{L^1}) r^2 + p\|e_p^{p-1}\|_{L^\infty} \|\nabla e_p\|_{L^\infty} \left(\frac{m(p-1)M_1}{(p-1)(m-1)-s}\right)^{p-1} |Q_T|^{\frac{1}{p}} r^s \\
&\leq (\|K_1\|_{L^1} + \|K_2\|_{L^1} + C) \max\{r^2, r^s\},
\end{aligned}$$

where $C := p\|e_p^{p-1}\|_{L^\infty} \|\nabla e_p\|_{L^\infty} (\frac{m(p-1)M_1}{(p-1)(m-1)-s})^{p-1} |Q_T|^{\frac{1}{p}}$.

Thus, if $\max\{r^2, r^s\} = r^2$, then

$$r_0 \leq \left(\frac{\iint_{Q_T} e_p^p a - \varepsilon_0 \mu_p T}{\|K_1\|_{L^1} + \|K_2\|_{L^1} + C}\right)^{\frac{1}{2}} \leq r,$$

which is a contradiction; analogously if $\max\{r^2, r^s\} = r^s$. The same argument applies if $v_\varepsilon \neq 0$. Fix any $r \in (0, r_0)$. The result above shows that

$$(u, v) \neq G_\varepsilon(\sigma, f(u^+, v^+) + (1-\sigma), g(u^+, v^+) + (1-\sigma)),$$

for all $(u, v) \in \partial B_r$ and all $\sigma \in [0, 1]$. From the homotopy invariance of the Leray–Schauder degree, we have

$$\begin{aligned}
&\deg((u, v) - T_\varepsilon(1, u^+, v^+), B_r, 0) \\
&\quad = \deg((u, v) - G_\varepsilon(0, f(u^+, v^+) + 1, g(u^+, v^+) + 1), B_r, 0).
\end{aligned}$$

The right-hand side is zero since the equation

$$(u, v) = G_\varepsilon(0, f(u^+, v^+) + 1, g(u^+, v^+) + 1)$$

admits neither trivial nor trivial solution in B_r, since $r < r_0$. □

The next result is our general coexistence result for (1.1).

Theorem 2.1. *Problem* (1.1) *has a T-periodic nonnegative solution (u, v) with both nontrivial u, v.*

Proof. By Propositions 2.2 and 2.3 and the excision property of the topological degree, there are $R > r > 0$, independent of ε, such that

$$\deg((u_\varepsilon, v_\varepsilon) - G_\varepsilon(1, f(u_\varepsilon^+, v_\varepsilon^+), g(u_\varepsilon^+, v_\varepsilon^+)), B_R \backslash \overline{B}_r, 0) = 1,$$

for any $\varepsilon \in (0, \varepsilon_0)$.

Let us fix any $\varepsilon \in (0, \varepsilon_0)$. There is $\sigma_0 = \sigma_0(\varepsilon) \in (0, 1)$ such that still

$$\deg((u_\varepsilon, v_\varepsilon) - G_\varepsilon(\sigma, f(u_\varepsilon^+, v_\varepsilon^+) + (1-\sigma), g(u_\varepsilon^+, v_\varepsilon^+) + (1-\sigma)), B_R \backslash \overline{B}_r, 0) = 1$$

for all $\sigma \in [\sigma_0, 1]$, by the continuity of Leray–Schauder degree. This implies that the set of solution triples $(\sigma, u_\varepsilon, v_\varepsilon) \in [0, 1] \times (B_R \backslash \overline{B}_r)$ such that

$$(u_\varepsilon, v_\varepsilon) = G_{\varepsilon\eta}(\sigma, f(u_\varepsilon^+, v_\varepsilon^+) + (1-\sigma), g(u_\varepsilon^+, v_\varepsilon^+) + (1-\sigma)) \tag{2.24}$$

contains a continuum $\mathcal{S}_\varepsilon$ with the property that

$$\mathcal{S}_\varepsilon \cap [\{\sigma\} \times (B_R \backslash \overline{B}_r)] \neq \emptyset \quad \text{for all } \sigma \in [\sigma_0, 1].$$

Now, all the pairs $(u_\varepsilon, v_\varepsilon)$ such that $(1, u_\varepsilon, v_\varepsilon) \in \mathcal{S}_\varepsilon$ are T-periodic solutions of (2.1) with $(u_\varepsilon, v_\varepsilon) \neq (0, 0)$ and, hence, satisfy (2.12). Since the L^2-norm is continuous with respect to the L^∞-norm and $\mathcal{S}_\varepsilon$ is a continuum, for every $\nu > 0$ there is $\sigma_\nu \in [\sigma_0, 1)$ such that

$$\|u_\varepsilon\|_{L^2}^2 \leq C_1 + \nu \quad \text{and} \quad \|v_\varepsilon\|_{L^2}^2 \leq C_2 + \nu$$

for all $(u_\varepsilon, v_\varepsilon)$ with $(\sigma, u_\varepsilon, v_\varepsilon) \in \mathcal{S}_\varepsilon$ and $\sigma \in [\sigma_\nu, 1]$. Observe that, if $(\sigma, u_\varepsilon, v_\varepsilon) \in \mathcal{S}_\varepsilon$ for $\sigma < 1$, then u_ε and v_ε are *positive* solutions of (2.24). Moreover, if ν is sufficiently small, then we still have $\theta(C_1 + \nu, C_2 + \nu) > 0$.

Now, setting

$$K_p := \left[\|K_1\|_{L^1} + p\|e_p^{p-1}\|_{L^\infty} \|\nabla e_p\|_{L^\infty} \left(\frac{m(p-1)M_1}{(p-1)(m-1)-s} \right)^{p-1} |Q_T|^{\frac{1}{p}} \right],$$

$$K_q := \left[\|K_4\|_{L^1} + q\|e_q^{q-1}\|_{L^\infty} \|\nabla e_q\|_{L^\infty} \left(\frac{n(q-1)M_2}{(q-1)(n-1)-s} \right)^{q-1} |Q_T|^{\frac{1}{q}} \right],$$

we can prove that, if ν is sufficiently small, then

$$\|u_\varepsilon\|_{L^\infty}, \|v_\varepsilon\|_{L^\infty} \geq \min \left\{ \left(\frac{T\theta(C_1+\nu, C_2+\nu)}{K_p} \right)^{\frac{1}{2}}, \left(\frac{T\theta(C_1+\nu, C_2+\nu)}{K_p} \right)^{\frac{1}{s}}, \right.$$
$$\left. \left(\frac{T\theta(C_1+\nu, C_2+\nu)}{K_q} \right)^{\frac{1}{2}}, \left(\frac{T\theta(C_1+\nu, C_2+\nu)}{K_q} \right)^{\frac{1}{s}} \right\} =: \lambda_\nu$$

for all $u_\varepsilon, v_\varepsilon$ such that $(\sigma, u_\varepsilon, v_\varepsilon) \in \mathcal{S}_\varepsilon$ and $\sigma \in [\sigma_\nu, 1)$. Indeed, let $(u_\varepsilon, v_\varepsilon)$ be a solution of (2.24). Arguing by contradiction, assume that $\|u_\varepsilon\|_{L^\infty} < \lambda_\nu$ and proceeding as in the proof of Proposition 2.3 (recall that $u_\varepsilon > 0$ since $(u_\varepsilon, v_\varepsilon)$ solves (2.24) with $\sigma < 1$) we obtain the inequality

$$\iint_{Q_T} e_p^p a - \varepsilon_0 \mu_p T < \max\{\lambda_\nu^2, \lambda_\nu^s\} K_p + \underline{k}_2(C_2 + \nu).$$

Thus, if $\max\{\lambda_\nu^2, \lambda_\nu^s\} = \lambda_\nu^2$, using the definition of θ, one has

$$T\theta(C_1 + \nu, C_2 + \nu) \le \iint_{Q_T} e_p^p a - \varepsilon_0 \mu_p T - \underline{k}_2(C_2 + \nu) < \lambda_\nu^2 K_p,$$

that is

$$\left(\frac{T\theta(C_1 + \nu, C_2 + \nu)}{K_p} \right)^{\frac{1}{2}} < \lambda_\nu,$$

which is a contradiction with the definition of λ_ν; analogously if $\max\{\lambda_\nu^2, \lambda_\nu^s\} = \lambda_\nu^s$. The same argument shows that $\|v_\varepsilon\|_{L^\infty} \ge \lambda_\nu$.

Now, if we let $\sigma \to 1$ and $\nu \to 0$, we obtain that (2.1) has at least a solution $(u_\varepsilon, v_\varepsilon)$ such that $\|u_\varepsilon\|_{L^\infty}, \|v_\varepsilon\|_{L^\infty} \ge \lambda_0$, since $\mathcal{S}_\varepsilon$ is a continuum and $\lambda_\nu \to \lambda_0$ as $\nu \to 0$.

Finally, we show that a solution (u, v) of (1.1) with both nontrivial $u, v \ge 0$ is obtained as a limit of $(u_\varepsilon, v_\varepsilon)$ as $\varepsilon \to 0$, since λ_0 is independent of ε.

Since $u_\varepsilon, v_\varepsilon$ are Hölder continuous in $\overline{Q}_T$, bounded in $C(\overline{Q}_T)$ uniformly in $\varepsilon > 0$ and the structure conditions of [33, 34] are satisfied for the equations of system (2.1), whenever $\varepsilon \in (0, \varepsilon_0)$, [33, Theorem 1.1; 34, Theorem 1.3] apply to conclude that the inequality

$$|u_\varepsilon(x_1, t_1) - u_\varepsilon(x_2, t_2))| \le \Gamma(|x_1 - x_2|^\beta + |t_1 - t_2|^{\frac{\beta}{p}})$$

holds for any $(x_1, t_1), (x_2, t_2) \in \overline{Q}_T$, where the constants $\Gamma > 0$ and $\beta \in (0, 1)$ are independent of $\|u_\varepsilon\|_{L^\infty}$. The same inequality holds for v_ε. Therefore, by the Ascoli–Arzelà theorem, a subsequence of $(u_\varepsilon, v_\varepsilon)$ converges uniformly in $\overline{Q}_T$ to a pair (u, v) satisfying

$$\lambda_0 \le \|u\|_{L^\infty}, \|v\|_{L^\infty} \le R.$$

Moreover, from (2.14) we have that u_ε satisfies the inequality

$$\frac{\partial u_\varepsilon}{\partial t} - \varepsilon \operatorname{div}(|\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon) - \operatorname{div}(|\nabla u_\varepsilon^m|^{p-2} \nabla u_\varepsilon^m) \le K u_\varepsilon^{p-1} \quad \text{in } Q_T, \tag{2.25}$$

where K is a positive constant independent of ε. Multiplying (2.25) by u_ε^m, integrating over Q_T and passing to the limit in the Steklov averages $(u_\varepsilon)_h$, one has

$$\begin{aligned}\iint_{Q_T} |\nabla u_\varepsilon^m|^p &\le \varepsilon m \iint_{Q_T} u^{m-1} |\nabla u_\varepsilon|^p + \iint_{Q_T} |\nabla u_\varepsilon^m|^p \\ &= \varepsilon \iint_{Q_T} |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \nabla u_\varepsilon^m + \iint_{Q_T} \nabla u_\varepsilon^m \nabla u_\varepsilon^m |\nabla u_\varepsilon^m|^{p-2}\end{aligned}$$

$$\leq K \iint_{Q_T} u_\varepsilon^{p+m-1}$$
$$\leq M, \tag{2.26}$$

by the T-periodicity of u_ε, its nonnegativity and its boundedness in $L^\infty(Q_T)$. Here M is positive and independent of ε. An analogous estimate holds for v_ε. Thus the sequences $u_\varepsilon^m, v_\varepsilon^n$ are uniformly bounded in $L^p(0,T;W_0^{1,p}(\Omega))$ and in $L^q(0,T;W_0^{1,q}(\Omega))$, respectively. Thus, up to subsequence if necessary, $(u_\varepsilon^m, v_\varepsilon^n)$ converges weakly in $L^p(0,T;W_0^{1,p}(\Omega)) \times L^q(0,T;W_0^{1,q}(\Omega))$ and strongly in $C(\overline{Q}_T) \times C(\overline{Q}_T)$ to (u^m, v^n). In particular $(u^m, v^n) \in L^p(0,T;W_0^{1,p}(\Omega)) \times L^q(0,T;W_0^{1,q}(\Omega))$. We finally claim that the pair (u,v) satisfies the identities

$$0 = \iint_{Q_T} \left\{ -u\frac{\partial \phi}{\partial t} + |\nabla u^m|^{p-2}\nabla u^m \cdot \nabla\phi - au^{p-1}\phi \right.$$
$$\left. + u^{p-1}\phi \int_\Omega [K_1(\xi,t)u^2(\xi,t-\tau_1) - K_2(\xi,t)v^2(\xi,t-\tau_2)]d\xi \right\} dxdt$$

and

$$0 = \iint_{Q_T} \left\{ -v\frac{\partial \phi}{\partial t} + |\nabla v^n|^{q-2}\nabla v^n \cdot \nabla\phi - bv^{q-1}\phi \right.$$
$$\left. + v^{q-1}\phi \int_\Omega [-K_3(\xi,t)u^2(\xi,t-\tau_3) + K_4(\xi,t)v^2(\xi,t-\tau_4)]d\xi \right\} dxdt,$$

for any $\phi \in C^1(\overline{Q}_T)$ such that $\phi(x,T) = \phi(x,0)$ for any $x \in \Omega$ and $\phi(x,t) = 0$ for any $(x,t) \in \partial\Omega \times [0,T]$, that is (u,v) is a generalized solution of (1.1). The approach for doing this is standard, in the sequel we write it in detail for the reader's convenience. First of all, observe that

$$\lim_{\varepsilon \to 0} \varepsilon \iint_{Q_T} |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \nabla\phi = 0 \tag{2.27}$$

for all test functions ϕ. In fact, multiplying the equation

$$\frac{\partial u_\varepsilon}{\partial t} - \varepsilon\,\mathrm{div}(|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon) - \mathrm{div}(|\nabla u_\varepsilon^m|^{p-2}\nabla u_\varepsilon^m)$$
$$= \left(a(x,t) - \int_\Omega K_1(\xi,t)u_\varepsilon^2(\xi,t-\tau_1)d\xi + \int_\Omega K_2(\xi,t)v_\varepsilon^2(\xi,t-\tau_2)d\xi \right) u_\varepsilon^{p-1}$$

by u_ε, integrating over Q_T, using the T-periodicity of u_ε and its nonnegativity and passing, as $h \to 0$, to the limit in the Steklov averages $(u_\varepsilon)_h$, we obtain

$$\|\sqrt[p]{\varepsilon}\nabla u_\varepsilon\|_{L^p}^p = \varepsilon \iint_{Q_T} |\nabla u_\varepsilon|^p$$
$$\leq \varepsilon \iint_{Q_T} |\nabla u_\varepsilon|^p + m^{p-1} \iint_{Q_T} u_\varepsilon^{(m-1)(p-1)}|\nabla u_\varepsilon|^p$$

$$\begin{aligned}&\le \varepsilon \iint_{Q_T} |\nabla u_\varepsilon|^p + \iint_{Q_T} |\nabla u_\varepsilon^m|^{p-2}\nabla u_\varepsilon^m \nabla u_\varepsilon \\ &\le C,\end{aligned}$$

where $C := (\|a\|_{L^\infty} + \|K_2\|_{L^\infty} R^2)|Q_T|^{1-\frac{p}{2}} C_1^{\frac{p}{2}}$ (recall that, by assumption, being $p < 2$, $\|u_\varepsilon\|_{L^p} \le |Q_T|^{\frac{1}{p}-\frac{1}{2}} \|u_\varepsilon\|_{L^2} \le |Q_T|^{\frac{1}{p}-\frac{1}{2}} \sqrt{C_1}$). Thus, by the Hölder inequality,

$$\begin{aligned}\left|\varepsilon \iint_{Q_T} |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \nabla\phi\right| &\le \iint_{Q_T} \varepsilon^{\frac{1}{p'}} |\nabla u_\varepsilon|^{p-1} \varepsilon^{\frac{1}{p}} |\nabla\phi| \\ &\le \|\sqrt[p]{\varepsilon}\nabla u_\varepsilon\|_{L^p}^{\frac{p}{p'}} \varepsilon^{\frac{1}{p}} \|\nabla\phi\|_{L^p} \\ &\le \sqrt[p]{\varepsilon}\ \sqrt[p']{C} \|\nabla\phi\|_{L^p} \to 0\end{aligned}$$

as $\varepsilon \to 0$, for all test functions ϕ.

In what follows we will prove that

$$\lim_{\varepsilon\to 0} \iint_{Q_T} |\nabla u_\varepsilon^m|^{p-2}\nabla u_\varepsilon^m \cdot \nabla\phi = \iint_{Q_T} |\nabla u^m|^{p-2}\nabla u^m \cdot \nabla\phi, \tag{2.28}$$

for all test functions ϕ. To this aim, observe that $|\nabla u_\varepsilon^m|^{p-2}\nabla u_\varepsilon^m$ is bounded in $(L^{\frac{p}{p-1}}(Q_T))^N$. In fact,

$$\iint_{Q_T} \left||\nabla u_\varepsilon^m|^{p-2}\nabla u_\varepsilon^m\right|^{\frac{p}{p-1}} = \iint_{Q_T} |\nabla u_\varepsilon^m|^p \le M,$$

as proved in (2.26). Thus there exists $H \in (L^{\frac{p}{p-1}}(Q_T))^N$ such that $|\nabla u_\varepsilon^m|^{p-2}\nabla u_\varepsilon^m$ weakly converges to H in $(L^{\frac{p}{p-1}}(Q_T))^N$ as $\varepsilon \to 0$. Now, using (2.27), it is easy to prove that

$$\begin{aligned}0 = \iint_{Q_T} \Big\{ &-u\frac{\partial\phi}{\partial t} + H \cdot \nabla\phi - au^{p-1}\phi \\ &+ u^{p-1}\phi \int_\Omega [K_1(\xi,t)u^2(\xi,t-\tau_1) - K_2(\xi,t)v^2(\xi,t-\tau_2)]d\xi \Big\} dxdt \end{aligned}\tag{2.29}$$

for any $\phi \in C^1(\overline{Q}_T)$ such that $\phi(x,T) = \phi(x,0)$ for any $x \in \Omega$ and $\phi(x,t) = 0$ for any $(x,t) \in \partial\Omega \times [0,T]$ (and, by density, for any T-periodic $\phi \in L^p(0,T;W_0^{1,p}(\Omega)) \cap C(\overline{Q_T})$). For this it remains to prove that for every $\phi \in C^1(\overline{Q}_T)$

$$\iint_{Q_T} |\nabla u^m|^{p-2}\nabla u^m \cdot \nabla\phi = \iint_{Q_T} H \cdot \nabla\phi. \tag{2.30}$$

Consider the vector function $H(Y) := |Y|^{p-2}Y$. Then

$$H'(Y) = |Y|^{p-2}I + (p-2)|Y|^{p-4} Y \otimes Y$$

is a positive definite matrix and, taken $w \in L^p(0,T;W_0^{1,p}(\Omega))$, there exists a vector Y such that

$$0 \le \langle H'(Y)(\nabla u_\varepsilon^m - \nabla w), \nabla u_\varepsilon^m - \nabla w\rangle = \langle H(\nabla u_\varepsilon^m) - H(\nabla w), \nabla u_\varepsilon^m - \nabla w\rangle.$$

The previous inequality implies

$$\iint_{Q_T} \{|\nabla u_\varepsilon^m|^{p-2}\nabla u_\varepsilon^m - |\nabla w|^{p-2}\nabla w\} \cdot \nabla(u_\varepsilon^m - w) \geq 0,$$

for all $w \in L^p(0,T;W_0^{1,p}(\Omega))$, that is

$$\iint_{Q_T} |\nabla u_\varepsilon^m|^p - \iint_{Q_T} |\nabla u_\varepsilon^m|^{p-2}\nabla u_\varepsilon^m \cdot \nabla w - \iint_{Q_T} |\nabla w|^{p-2}\nabla w \cdot \nabla(u_\varepsilon^m - w) \geq 0,$$

for all $w \in L^p(0,T;W_0^{1,p}(\Omega))$. As in (2.26), one has

$$\begin{aligned}\iint_{Q_T} |\nabla u_\varepsilon^m|^p &\leq \varepsilon m \iint_{Q_T} u^{m-1}|\nabla u_\varepsilon|^p + \iint_{Q_T} |\nabla u_\varepsilon^m|^p \\ &\leq \iint_{Q_T} \left[a - \int_\Omega K_1(\xi,t)u_\varepsilon^2(\xi,t-\tau_1)d\xi \right. \\ &\quad \left. + \int_\Omega K_2(\xi,t)v_\varepsilon^2(\xi,t-\tau_2)d\xi\right] u_\varepsilon^{p+m-1}dxdt.\end{aligned}$$

Thus, from the previous two inequalities, we obtain

$$\begin{aligned}&\iint_{Q_T} |\nabla u_\varepsilon^m|^{p-2}\nabla u_\varepsilon^m \cdot \nabla w + \iint_{Q_T} |\nabla w|^{p-2}\nabla w \cdot \nabla(u_\varepsilon^m - w) \\ &\quad\leq \iint_{Q_T} \left[a - \int_\Omega K_1(\xi,t)u_\varepsilon^2(\xi,t-\tau_1)d\xi \right. \\ &\qquad \left. + \int_\Omega K_2(\xi,t)v_\varepsilon^2(\xi,t-\tau_2)d\xi\right] u_\varepsilon^{p+m-1}dxdt.\end{aligned}$$

Letting $\varepsilon \to 0$ and using (2.26), we have

$$\begin{aligned}&\iint_{Q_T} [H \cdot \nabla w + |\nabla w|^{p-2}\nabla w \cdot \nabla(u^m - w)] \\ &\quad\leq \iint_{Q_T} \left[a - \int_\Omega K_1(\xi,t)u^2(\xi,t-\tau_1)d\xi \right. \\ &\qquad \left. + \int_\Omega K_2(\xi,t)v^2(\xi,t-\tau_2)d\xi\right] u^{p+m-1}dxdt. \end{aligned} \tag{2.31}$$

Observe that, being $p > 1$, ∇u_ε^m is also bounded in $L^1(Q_T)$.

On the other hand, by density we can take $u^m = \phi$ in (2.29) and obtain

$$\begin{aligned}\iint_{Q_T} H \cdot \nabla u^m = \iint_{Q_T} &\left[a - \int_\Omega K_1(\xi,t)u^2(\xi,t-\tau_1)d\xi \right. \\ &\left. + \int_\Omega K_2(\xi,t)v^2(\xi,t-\tau_2)d\xi\right] u^{p+m-1}dxdt.\end{aligned}$$

This equality, together with (2.31), implies

$$0 \le \iint_{Q_T} (H - |\nabla w|^{p-2}\nabla w) \cdot \nabla(u^m - w). \tag{2.32}$$

Taking $w := u^m - \lambda\phi$, with $\lambda > 0$ and $\phi \in C^1(\overline{Q}_T)$, we get

$$0 \le \iint_{Q_T} (H - |\nabla(u^m - \lambda\phi)|^{p-2}\nabla(u^m - \lambda\phi)) \cdot \nabla\phi.$$

Letting $\lambda \to 0$ yields

$$0 \le \iint_{Q_T} (H - |\nabla u^m|^{p-2}\nabla u^m) \cdot \nabla\phi.$$

If in (2.32) we take $w := u^m + \lambda\phi$, with $\lambda > 0$, $\phi \in C^1(\overline{Q}_T)$ and letting again $\lambda \to 0$, then

$$\iint_{Q_T} (H - |\nabla u^m|^{p-2}\nabla u^m) \cdot \nabla\phi \le 0.$$

Thus (2.30) holds and (2.28) is proved. □

Obviously, the previous result holds also for a single equation. In particular, we have the following corollary.

Corollary 2.1. *Consider the problem*

$$\begin{cases} u_t - \mathrm{div}(|\nabla u^m|^{p-2}\nabla u^m) = \left(a(x,t) - \displaystyle\int_\Omega K(\xi,t)u^2(\xi,t-\tau)d\xi\right) u^{p-1} & \text{in } Q_T, \\ u(x,t) = 0 \quad \text{for } (x,t) \in \partial\Omega \times (0,T), \\ u(\cdot,0) = u(\cdot,T), \end{cases} \tag{2.33}$$

and assume that

(1) *the exponents p, m are such that $p \in (1,2)$ and $m > p$,*
(2) *the delay $\tau \in (0,+\infty)$,*
(3) *the functions a and K belong to $L^\infty(Q_T)$, are extended to $\Omega \times \mathbb{R}$ by T-periodicity and are nonnegative for a.a. $(x,t) \in Q_T$,*
(4) *there exists a positive constant C such that for all $\varepsilon > 0$ and all the nonnegative solutions u_ε of*

$$u = G_\varepsilon(1, f(u^+)),$$

it results

$$\|u_\varepsilon\|^2_{L^2} \le C.$$

Then problem (2.33) has a T-periodic nonnegative and nontrivial solution.

Here $G_\varepsilon(1, f(u^+))$ is defined as before.

3. *A Priori* Bounds in $L^2(Q_T)$

In this section we apply Theorem 2.1 by looking for explicit *a priori* bounds in $L^2(Q_T)$ for the solutions of the approximating problem (2.1) in different situations. More precisely, under different assumptions on the kernels $K_i, i = 1, 2, 3, 4$, which model the interactions between the quantities u, v, we determine the constants C_1, C_2 of (2.12) in an explicit form. For this we consider two main different cases. In the first one, which we call the "coercive case", we assume that $K_i(x,t) \geq \underline{k}_i > 0$ a.a. in Q_T for $i = 1, 4$. In the second one, the "noncoercive case", we allow the nonnegative functions K_1, K_4 to vanish on sets with positive measure. We distinguish also between cooperative and competitive situations by imposing sign conditions on K_2, K_3 having in mind the biological interpretation of model (1.1).

3.1. *The coercive case*

Theorem 3.1. *Assume that*

(1) *Assumption* 1 *is satisfied,*
(2) *there are constants* $\underline{k}_i > 0, i = 1, 4,$ *such that*

$$K_i(x,t) \geq \underline{k}_i \quad \textit{for } i = 1, 4,$$

for a.a. $(x,t) \in Q_T$, *and* $\underline{k}_1\underline{k}_4 > \overline{k}_2\overline{k}_3$, *where* $\overline{k}_2, \overline{k}_3$ *are as in Assumption* 1,
(3) *Assumption* 2(2) *is satisfied with*

$$\begin{aligned} C_1 &= \frac{T(\underline{k}_4\|a\|_{L^\infty} + \overline{k}_2\|b\|_{L^\infty})}{\underline{k}_1\underline{k}_4 - \overline{k}_2\overline{k}_3}, \\ C_2 &= \frac{T(\overline{k}_3\|a\|_{L^\infty} + \underline{k}_1\|b\|_{L^\infty})}{\underline{k}_1\underline{k}_4 - \overline{k}_2\overline{k}_3}. \end{aligned} \tag{3.1}$$

Then problem (1.1) *has a nonnegative* T*-periodic solution* (u, v) *with nontrivial* u, v.

Proof. We just need to show that $\|u_\varepsilon\|^2_{L^2} \leq C_1$ and $\|v_\varepsilon\|^2_{L^2} \leq C_2$ for any solution $(u_\varepsilon, v_\varepsilon)$ of (2.11). Then, assume $u_\varepsilon \not\equiv 0$, thus $u_\varepsilon > 0$ and $v_\varepsilon \geq 0$ in Q_T by Proposition 2.1. Multiplying the inequality

$$\begin{aligned} &\frac{\partial u_\varepsilon}{\partial t} - \varepsilon\,\mathrm{div}(|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon) - \mathrm{div}(|\nabla u_\varepsilon^m|^{p-2}\nabla u_\varepsilon^m) \\ &\quad \leq \left[\|a\|_{L^\infty} - \int_\Omega K_1(\xi,t)u_\varepsilon^2(\xi,t-\tau_1)d\xi + \int_\Omega K_2(\xi,t)v_\varepsilon^2(\xi,t-\tau_2)d\xi\right] u_\varepsilon^{p-1} \end{aligned}$$

by u_ε, integrating over Ω and using the Steklov averages $(u_\varepsilon)_h \in H^1(Q_{T-\delta}), \delta,$ $h > 0$, we obtain

$$\begin{aligned} &\frac{\frac{1}{2}\frac{d}{dt}\int_\Omega (u_\varepsilon)_h^2 + \varepsilon\int_\Omega |\nabla(u_\varepsilon)_h|^p + m^{p-1}\int_\Omega (u_\varepsilon)_h^{(m-1)(p-1)}|\nabla(u_\varepsilon)_h|^p}{\int_\Omega (u_\varepsilon)_h^p} \\ &\quad \leq \left(\|a\|_{L^\infty} - \int_\Omega K_1(\xi,t)(u_\varepsilon)_h^2(\xi,t-\tau_1)d\xi + \int_\Omega K_2(\xi,t)(v_\varepsilon)_h^2(\xi,t-\tau_2)d\xi\right). \end{aligned}$$

Integrating the previous inequality over $[0, T]$, and passing to the limit as $h \to 0$, by the T-periodicity of u_ε, we have that

$$0 < (T\|a\|_{L^\infty} - \underline{k}_1\|u_\varepsilon\|_{L^2}^2 + \overline{k}_2\|v_\varepsilon\|_{L^2}^2). \tag{3.2}$$

The same procedure, when it is applied to the second equation of (2.1), leads to

$$0 < (T\|b\|_{L^\infty} - \underline{k}_4\|v_\varepsilon\|_{L^2}^2 + \overline{k}_3\|u_\varepsilon\|_{L^2}^2). \tag{3.3}$$

Hence, if $u_\varepsilon \not\equiv 0$ and if $v_\varepsilon \not\equiv 0$, by the positiveness of the right-hand sides of (3.2) and (3.3), we have

$$\left(1 - \frac{\overline{k}_2\overline{k}_3}{\underline{k}_1\underline{k}_4}\right)\|u_\varepsilon\|_{L^2}^2 < \frac{T}{\underline{k}_1}\left(\|a\|_{L^\infty} + \frac{\overline{k}_2}{\underline{k}_4}\|b\|_{L^\infty}\right),$$
$$\left(1 - \frac{\overline{k}_2\overline{k}_3}{\underline{k}_1\underline{k}_4}\right)\|v_\varepsilon\|_{L^2}^2 < \frac{T}{\underline{k}_4}\left(\|b\|_{L^\infty} + \frac{\overline{k}_3}{\underline{k}_1}\|a\|_{L^\infty}\right),$$

for any $\varepsilon \in (0, \varepsilon_0)$ and the desired bounds follow since $\overline{k}_2\overline{k}_3 < \underline{k}_1\underline{k}_4$. Obviously, if $v_\varepsilon \equiv 0$, then

$$\|u_\varepsilon\|_{L^2}^2 \le \frac{T}{\underline{k}_1}\|a\|_{L^\infty} \le C_1,$$

or if $u_\varepsilon \equiv 0$, then

$$\|v_\varepsilon\|_{L^2}^2 \le \frac{T}{\underline{k}_4}\|b\|_{L^\infty} \le C_2. \qquad \square$$

As an immediate consequence of the previous result we obtain the following corollaries for the coercive-cooperative (see Remark 2.1) and the coercive-competitive cases, respectively.

Corollary 3.1. *Assume that*

(1) *Assumption* 1 *is satisfied with nontrivial coefficients* a, b,
(2) $0 \le K_i(x, t)$ *for* $i = 2, 3$, *for a.a.* $(x, t) \in Q_T$,
(3) *there are constants* $\underline{k}_i > 0, i = 1, 4,$ *such that*

$$K_i(x, t) \ge \underline{k}_i \quad \textit{for } i = 1, 4,$$

for a.a. $(x, t) \in Q_T$, *and* $\underline{k}_1\underline{k}_4 > \overline{k}_2\overline{k}_3$, *where* $\overline{k}_2, \overline{k}_3$ *are as in Assumption* 1.

Then problem (1.1) *has a nonnegative* T*-periodic solution* (u, v).

Corollary 3.2. *Assume that*

(1) *Assumption* 1 *is satisfied,*
(2) $K_i(x, t) \le 0$ *for* $i = 2, 3$, *for a.a.* $(x, t) \in Q_T$,
(3) *there are constants* $\underline{k}_i > 0, i = 1, 4,$ *such that*

$$K_i(x, t) \ge \underline{k}_i \quad \textit{for } i = 1, 4,$$

for a.a. $(x, t) \in Q_T$,

(4) *Assumption* 2(2) *is satisfied with*

$$C_1 = \frac{T}{\underline{k}_1}\|a\|_{L^\infty} \quad \text{and} \quad C_2 = \frac{T}{\underline{k}_4}\|b\|_{L^\infty}.$$

Then problem (1.1) *has a nonnegative T-periodic solution* (u, v).

We observe that the condition $\overline{k}_2\overline{k}_3 < \underline{k}_1\underline{k}_4$ of Theorem 3.1 is crucial to establish the *a priori* L^2-bounds on the solution pairs $(u_\varepsilon, v_\varepsilon)$ of (2.1). Roughly speaking this condition guarantees that the terms in the equations that contribute to the growth of the respective species do not prevail globally on those limiting the growth.

On the other hand, when the strict positivity of the functions K_1 and K_4 is relaxed, obtaining the needed *a priori* bounds becomes more difficult (at least with our approach). In fact, we are able to obtain simple *a priori* bounds in the noncoercive case when the system is competitive, provided that $\min\{n(q-1), m(p-1)\} \geq 1$, i.e. when each equation of (1.1) is of slow or normal diffusion type. Otherwise, we have to impose one more technical restriction, i.e. $\min\{m\frac{p-1}{p+1}, n\frac{q-1}{q+1}\} \geq 1$ to obtain a result like Theorem 3.1 with no sign condition on the functions K_2 and K_3.

Obviously, Theorem 3.1 holds also for a single equation. In particular, we have the following corollary.

Corollary 3.3. *Consider the problem* (2.33) *and assume that*

(1) *the exponents p, m are such that $p \in (1, 2)$ and $m > p$,*
(2) *the delay $\tau \in (0, +\infty)$,*
(3) *the functions a and K belong to $L^\infty(Q_T)$, are extended to $\Omega \times \mathbb{R}$ by T-periodicity and are nonnegative for a.a. $(x, t) \in Q_T$ and there exists a constant $\underline{k} > 0$ such that*

$$K(x, t) \geq \underline{k}$$

for a.a. $(x, t) \in Q_T$,

(4) *hypothesis* (4) *of Corollary* 2.1 *is satisfied with*

$$C = \frac{T}{\underline{k}}\|a\|_{L^\infty}.$$

Then problem (2.33) has a T-periodic nonnegative and nontrivial solution.

3.2. *The noncoercive case*: *The competitive system*

Theorem 3.2. *Assume that*

(1) *Assumption* 1 *is satisfied,*
(2) $\overline{k}_2 = \overline{k}_3 = 0$, *that is*

$$K_i(x, t) \leq 0 \quad \text{for } i = 2, 3,$$

for a.a. $(x, t) \in Q_T$,

(3) *m, n, p and q are such that $m \geq \frac{1}{p-1}$ and $n \geq \frac{1}{q-1}$, i.e. both the equations of system* (1.1) *have slow or normal diffusion,*
(4) *Assumption* 2(2) *is satisfied with*

$$C_1 = \frac{|Q_T|^{\frac{m(p-1)-1}{(p-1)(m-1)}}}{\mu_p^{\frac{2}{(p-1)(m-1)}}} \left(\frac{|\Omega|^{1-\frac{p}{2}}(m(p-1)+1)^p \|a\|_{L^\infty}}{m^{p-1}p^p} \right)^{\frac{2}{(p-1)(m-1)}},$$

$$C_2 = \frac{|Q_T|^{\frac{n(q-1)-1}{(q-1)(n-1)}}}{\mu_q^{\frac{2}{(q-1)(n-1)}}} \left(\frac{|\Omega|^{1-\frac{q}{2}}(n(q-1)+1)^q \|b\|_{L^\infty}}{n^{q-1}q^q} \right)^{\frac{2}{(q-1)(n-1)}}.$$

Then problem (1.1) has a T-periodic nonnegative solution (u, v) with nontrivial u, v.

Proof. As a first step we find the bound for the nonnegative solutions u_ε. Multiplying the first equation of (2.11) by u_ε, integrating over Q_T and passing to the limit, as $h \to 0$, in the Steklov averages $(u_\varepsilon)_h$, we obtain

$$\begin{aligned}
&\left(\frac{m^{\frac{p-1}{p}} p}{m(p-1)+1} \right)^p \iint_{Q_T} |\nabla u_\varepsilon^{\frac{m(p-1)+1}{p}}|^p \\
&\quad \leq \varepsilon \iint_{Q_T} |\nabla u_\varepsilon|^p + \iint_{Q_T} |\nabla u_\varepsilon^m|^{p-2} \nabla u_\varepsilon^m \nabla u_\varepsilon \\
&\quad \leq \|a\|_{L^\infty} \|u_\varepsilon\|_{L^p}^p \\
&\quad \leq |\Omega|^{1-\frac{p}{2}} \|a\|_{L^\infty} \|u_\varepsilon\|_{L^2}^p,
\end{aligned} \tag{3.4}$$

by the T-periodicity of u_ε and the nonpositivity of the function K_2. Using the Hölder inequality with $r := \frac{m(p-1)+1}{2}$, and the Poincaré inequality, one has:

$$\begin{aligned}
\iint_{Q_T} u_\varepsilon^2 &\leq |Q_T|^{\frac{m(p-1)-1}{m(p-1)+1}} \left(\iint_{Q_T} u_\varepsilon^{m(p-1)+1} \right)^{\frac{2}{m(p-1)+1}} \\
&= |Q_T|^{\frac{m(p-1)-1}{m(p-1)+1}} \|u_\varepsilon^{\frac{m(p-1)+1}{p}}\|_{L^p}^{\frac{2p}{m(p-1)+1}} \\
&\leq |Q_T|^{\frac{m(p-1)-1}{m(p-1)+1}} \left(\frac{1}{\sqrt[p]{\mu_p}} \|\nabla u_\varepsilon^{\frac{m(p-1)+1}{p}}\|_{L^p} \right)^{\frac{2p}{m(p-1)+1}}.
\end{aligned}$$

Thus by (3.4) we get

$$\begin{aligned}
\|u_\varepsilon\|_{L^2}^2 &\leq |Q_T|^{\frac{m(p-1)-1}{m(p-1)+1}} \left(\frac{1}{\sqrt[p]{\mu_p}} \|\nabla u_\varepsilon^{\frac{m(p-1)+1}{p}}\|_{L^p} \right)^{\frac{2p}{m(p-1)+1}} \\
&\leq \frac{|Q_T|^{\frac{m(p-1)-1}{m(p-1)+1}}}{\mu_p^{\frac{2}{m(p-1)+1}}} \left(\frac{|\Omega|^{1-\frac{p}{2}}(m(p-1)+1)^p \|a\|_{L^\infty}}{m^{p-1}p^p} \right)^{\frac{2}{m(p-1)+1}} \|u_\varepsilon\|_{L^2}^{\frac{2p}{m(p-1)+1}}.
\end{aligned}$$

This implies

$$\|u_\varepsilon\|_{L^2}^2 \le \frac{|Q_T|^{\frac{m(p-1)-1}{(p-1)(m-1)}}}{\mu_p^{\frac{2}{(p-1)(m-1)}}} \left(\frac{|\Omega|^{1-\frac{p}{2}}(m(p-1)+1)^p\|a\|_{L^\infty}}{m^{p-1}p^p}\right)^{\frac{2}{(p-1)(m-1)}}.$$

In an analogous way we obtain that

$$\|v_\varepsilon\|_{L^2}^2 \le \frac{|Q_T|^{\frac{n(q-1)-1}{(q-1)(n-1)}}}{\mu_q^{\frac{2}{(q-1)(n-1)}}} \left(\frac{|\Omega|^{1-\frac{q}{2}}(n(q-1)+1)^q\|b\|_{L^\infty}}{n^{q-1}q^q}\right)^{\frac{2}{(q-1)(n-1)}},$$

if v_ε is a solution of the second equation of (2.11). □

The previous result still holds for a single equation.

Corollary 3.4. *Consider problem* (2.33) *and assume that*

(1) *the exponents* p, m *are such that* $p \in (1,2)$ *and* $m \ge \frac{1}{p-1}$,
(2) *the delay* $\tau \in (0,+\infty)$,
(3) *the functions* a *and* K *belong to* $L^\infty(Q_T)$, *are extended to* $\Omega \times \mathbb{R}$ *by* T*-periodicity and are nonnegative for a.a.* $(x,t) \in Q_T$,
(4) *hypothesis* (4) *of Corollary* 2.1 *is satisfied with*

$$C = \frac{|Q_T|^{\frac{m(p-1)-1}{(p-1)(m-1)}}}{\mu_p^{\frac{2}{(p-1)(m-1)}}} \left(\frac{|\Omega|^{1-\frac{p}{2}}(m(p-1)+1)^p\|a\|_{L^\infty}}{m^{p-1}p^p}\right)^{\frac{2}{(p-1)(m-1)}}.$$

Then problem (2.33) *has a* T*-periodic nonnegative and nontrivial solution.*

3.3. *The noncoercive case*: $\min\{m\frac{p-1}{p+1}, n\frac{q-1}{q+1}\} \ge 1$

In the case that $\min\{m\frac{p-1}{p+1}, n\frac{q-1}{q+1}\} \ge 1$, we are able to find explicit bounds (although complicated) without any assumption on the sign of the functions K_2, K_3, as shown in the next result.

Theorem 3.3. *Assume*

(1) $\min\{m\frac{p-1}{p+1}, n\frac{q-1}{q+1}\} > 1$,
(2) $K_i(x,t) \ge 0, i = 1,4$ *and* $K_i(x,t) \le \overline{k}_i, i = 2,3$ *for a.a.* $(x,t) \in Q_T$ *and for some positive constants* $\overline{k}_i, i = 2,3$,
(3) *Assumption* 2(2) *is satisfied with*

$$C_1 = \sqrt{T}\left\{\frac{(q-1)(n-1)(p-1)(m-1)}{(q-1)(n-1)(p-1)(m-1)-4}\alpha_p + \beta_p^{\frac{(q-1)(n-1)(p-1)(m-1)}{(q-1)(n-1)(p-1)(m-1)-4}}\right\}^{1/2},$$

$$C_2 = \sqrt{T}\left\{\frac{(q-1)(n-1)(p-1)(m-1)}{(q-1)(n-1)(p-1)(m-1)-4}\alpha_q + \beta_q^{\frac{(q-1)(n-1)(p-1)(m-1)}{(q-1)(n-1)(p-1)(m-1)-4}}\right\}^{1/2}, \tag{3.5}$$

where

$$\begin{aligned}
\alpha_p &:= C_p^{\frac{(p-1)(m-1)+2}{(p-1)(m-1)}} (2T\|a\|^2_{L^\infty})^{\frac{2}{(p-1)(m-1)}} \\
&\quad + C_p^{\frac{(p-1)(m-1)+2}{(p-1)(m-1)}} (2\overline{k}_2^2 C_q^{\frac{(q-1)(n-1)+2}{(q-1)(n-1)}})^{\frac{2}{(p-1)(m-1)}} (2T\|b\|^2_{L^\infty})^{\frac{4}{(q-1)(n-1)(p-1)(m-1)}}, \\
\alpha_q &:= C_q^{\frac{(q-1)(n-1)+2}{(q-1)(n-1)}} (2T\|b\|^2_{L^\infty})^{\frac{2}{(q-1)(n-1)}} \\
&\quad + C_q^{\frac{(q-1)(n-1)+2}{(q-1)(n-1)}} (2\overline{k}_3^2 C_p^{\frac{(p-1)(m-1)+2}{(p-1)(m-1)}})^{\frac{2}{(q-1)(n-1)}} (2T\|a\|^2_{L^\infty})^{\frac{4}{(q-1)(n-1)(p-1)(m-1)}}, \\
\beta_p &:= C_p^{\frac{(p-1)(m-1)+2}{(p-1)(m-1)}} (2\overline{k}_2^2 C_q^{\frac{(q-1)(n-1)+2}{(q-1)(n-1)}})^{\frac{2}{(p-1)(m-1)}} (2\overline{k}_3^2)^{\frac{4}{(q-1)(n-1)(p-1)(m-1)}}, \\
\beta_q &:= C_q^{\frac{(q-1)(n-1)+2}{(q-1)(n-1)}} (2\overline{k}_3^2 C_p^{\frac{(p-1)(m-1)+2}{(p-1)(m-1)}})^{\frac{2}{(q-1)(n-1)}} (2\overline{k}_2^2)^{\frac{4}{(q-1)(n-1)(p-1)(m-1)}}.
\end{aligned}$$

Here

$$C_p := \left(\frac{[(p-1)(m-1)+2]^p|\Omega|^{\frac{(p-1)(m-1)}{2}}}{p^p m^{p-1}(3-p)\mu_p}\right)^{\frac{4}{(p-1)(m-1)+2}} T^{\frac{(p-1)(m-1)-2}{(p-1)(m-1)+2}} \tag{3.6}$$

and

$$C_q := \left(\frac{[(q-1)(n-1)+2]^q|\Omega|^{\frac{(q-1)(n-1)}{2}}}{q^q n^{q-1}(3-q)\mu_q}\right)^{\frac{4}{(q-1)(n-1)+2}} T^{\frac{(q-1)(n-1)-2}{(q-1)(n-1)+2}}. \tag{3.7}$$

Then problem (1.1) *has a nonnegative T-periodic solution* (u, v) *with nontrivial* u, v.

Proof. Let $(u_\varepsilon, v_\varepsilon)$ be a solution of (2.11). We have, by the Poincaré inequality and the Hölder inequality with $r := \frac{(p-1)(m-1)+2}{2}$,

$$\begin{aligned}
\left(\int_\Omega u_\varepsilon^2\right)^{\frac{(p-1)(m-1)+2}{2}} &\le |\Omega|^{\frac{(p-1)(m-1)}{2}} \int_\Omega u_\varepsilon^{(p-1)(m-1)+2} \\
&\le \frac{1}{\mu_p}|\Omega|^{\frac{(p-1)(m-1)}{2}} \int_\Omega \left|\nabla u_\varepsilon^{\frac{(p-1)(m-1)+2}{p}}\right|^p.
\end{aligned}$$

Integrating over $[0, T]$, we have:

$$\int_0^T \left(\int_\Omega u_\varepsilon^2\right)^{\frac{(p-1)(m-1)+2}{2}} \le \frac{1}{\mu_p}|\Omega|^{\frac{(p-1)(m-1)}{2}} \iint_{Q_T} \left|\nabla u_\varepsilon^{\frac{(p-1)(m-1)+2}{p}}\right|^p. \tag{3.8}$$

Now, we estimate the right-hand side of (3.8). Multiplying the first equation of (2.11) by u_ε^{3-p}, integrating over Q_T and passing to the limit in the Steklov averages

$(u_\varepsilon)_h$, we obtain by the T-periodicity of u_ε

$$m^{p-1}(3-p)\left[\frac{p}{(p-1)(m-1)+2}\right]^p \iint_{Q_T} |\nabla u_\varepsilon^{\frac{(p-1)(m-1)+2}{p}}|^p$$
$$\le \int_0^T \left[\|a\|_{L^\infty} + \overline{k}_2 \int_\Omega v_\varepsilon^2(\xi, t-\tau_2)d\xi\right]\left(\int_\Omega u_\varepsilon^2\right) dt$$
$$\le \left[\int_0^T \left(\|a\|_{L^\infty} + \overline{k}_2 \int_\Omega v_\varepsilon^2(\xi, t-\tau_2)d\xi\right)^2 dt\right]^{\frac{1}{2}} \left[\int_0^T \left(\int_\Omega u_\varepsilon^2\right)^2\right]^{\frac{1}{2}}.$$

Thus

$$\iint_{Q_T} |\nabla u_\varepsilon^{\frac{(p-1)(m-1)+2}{p}}|^p$$
$$\le M_p \left[\int_0^T \left(\|a\|_{L^\infty} + \overline{k}_2 \int_\Omega v_\varepsilon^2(\xi, t-\tau_2)d\xi\right)^2 dt\right]^{\frac{1}{2}} \left[\int_0^T \left(\int_\Omega u_\varepsilon^2\right)^2\right]^{\frac{1}{2}}, \tag{3.9}$$

where $M_p := \frac{1}{m^{p-1}(3-p)}[\frac{(p-1)(m-1)+2}{p}]^p$. By the Hölder inequality with $s := \frac{(p-1)(m-1)+2}{4}$ (observe that $s \ge 1$ by the assumption on m and p) and by (3.8), (3.9), it follows

$$\int_0^T \left(\int_\Omega u_\varepsilon^2\right)^2 \le T^{\frac{(p-1)(m-1)-2}{(p-1)(m-1)+2}} \left(\int_0^T \left(\int_\Omega u_\varepsilon^2\right)^{\frac{(p-1)(m-1)+2}{2}}\right)^{\frac{4}{(p-1)(m-1)+2}}$$
$$\le C_p \left\{\left[\int_0^T \left(\|a\|_{L^\infty} + \overline{k}_2 \int_\Omega v_\varepsilon^2(\xi, t-\tau_2)d\xi\right)^2 dt\right] \times \left[\int_0^T \left(\int_\Omega u_\varepsilon^2\right)^2\right]\right\}^{\frac{2}{(p-1)(m-1)+2}},$$

where C_p is the constant defined in (3.6). Therefore, setting $U = \int_0^T (\int_\Omega u_\varepsilon^2)^2$, $V = \int_0^T (\int_\Omega v_\varepsilon^2)^2$, and using the assumption $m > \frac{p+1}{p-1}$, the last inequality implies

$$U \le C_p^{\frac{(p-1)(m-1)+2}{(p-1)(m-1)}} \left[\int_0^T \left(\|a\|_{L^\infty} + \overline{k}_2 \int_\Omega v_\varepsilon^2\right)^2\right]^{\frac{2}{(p-1)(m-1)}}$$
$$\le C_p^{\frac{(p-1)(m-1)+2}{(p-1)(m-1)}} [2T\|a\|_{L^\infty}^2 + 2\overline{k}_2^2 V]^{\frac{2}{(p-1)(m-1)}}$$
$$\le C_p^{\frac{(p-1)(m-1)+2}{(p-1)(m-1)}} [(2T\|a\|_{L^\infty}^2)^{\frac{2}{(p-1)(m-1)}} + (2\overline{k}_2^2 V)^{\frac{2}{(p-1)(m-1)}}].$$

In an analogous way, we can show that

$$V \le C_q^{\frac{(q-1)(n-1)+2}{(q-1)(n-1)}} [(2T\|b\|_{L^\infty}^2)^{\frac{2}{(q-1)(n-1)}} + (2\overline{k}_3^2 U)^{\frac{2}{(q-1)(n-1)}}],$$

where C_q is the constant introduced in (3.7). Hence, it results

$$\begin{aligned} U &\leq C_p^{\frac{(p-1)(m-1)+2}{(p-1)(m-1)}} [(2T\|a\|^2_{L^\infty})^{\frac{2}{(p-1)(m-1)}} + (2\overline{k}_2^2 V)^{\frac{2}{(p-1)(m-1)}}] \\ &\leq C_p^{\frac{(p-1)(m-1)+2}{(p-1)(m-1)}} (2T\|a\|^2_{L^\infty})^{\frac{2}{(p-1)(m-1)}} \\ &\quad + C_p^{\frac{(p-1)(m-1)+2}{(p-1)(m-1)}} (2\overline{k}_2^2 C_q^{\frac{(q-1)(n-1)+2}{(q-1)(n-1)}})^{\frac{2}{(p-1)(m-1)}} (2T\|b\|^2_{L^\infty})^{\frac{4}{(q-1)(n-1)(p-1)(m-1)}} \\ &\quad + C_p^{\frac{(p-1)(m-1)+2}{(p-1)(m-1)}} (2\overline{k}_2^2 C_q^{\frac{(q-1)(n-1)+2}{(q-1)(n-1)}})^{\frac{2}{(p-1)(m-1)}} (2\overline{k}_3^2 U)^{\frac{4}{(q-1)(n-1)(p-1)(m-1)}}. \end{aligned}$$

The last inequality has the form:

$$U \leq \alpha + \beta U^{\frac{4}{(q-1)(n-1)(p-1)(m-1)}}, \tag{3.10}$$

with $\alpha, \beta > 0$. Since $\min\{m\frac{p-1}{p+1}, n\frac{q-1}{q+1}\} > 1$ the function $f(U) := \alpha + \beta U^{\frac{4}{(q-1)(n-1)(p-1)(m-1)}}$ is strictly concave, and then

$$U \leq f(U) \leq f(U_0) + f'(U_0)(U - U_0), \tag{3.11}$$

where $U_0 := \beta^{\frac{(q-1)(n-1)(p-1)(m-1)}{(q-1)(n-1)(p-1)(m-1)-4}}$. Using the fact that $f(U_0) = \alpha + U_0$ and (3.11), one has

$$U \leq \frac{(q-1)(n-1)(p-1)(m-1)}{(q-1)(n-1)(p-1)(m-1)-4}\alpha + \beta^{\frac{(q-1)(n-1)(p-1)(m-1)}{(q-1)(n-1)(p-1)(m-1)-4}}.$$

A final application of Hölder's inequality shows that $\|u_\varepsilon\|^2_{L^2} \leq T^{1/2}U^{1/2} = C_1$. The argument for v_ε proceeds in a similar way. □

As a consequence of Theorem 3.3 one has the next corollaries for the cooperative and competitive cases, respectively.

Corollary 3.5. *Assume that*

(1) $\min\{m\frac{p-1}{p+1}, n\frac{q-1}{q+1}\} > 1$,
(2) $K_i(x,t) \geq 0$ *for* $i = 1, 4$, *for a.a.* $(x,t) \in Q_T$, *and there are positive constants* $\overline{k}_2, \overline{k}_3$ *such that*

$$0 \leq K_i(x,t) \leq \overline{k}_i \quad \text{for } i = 2, 3,$$

for a.a. $(x,t) \in Q_T$,
(3) *Assumption* 2(2) *is satisfied with* C_1 *and* C_2 *as in* (3.5).

Then problem (1.1) *has a nonnegative* T*-periodic solution* (u,v) *with nontrivial* u, v.

Corollary 3.6. *Assume that*

(1) $\min\{m\frac{p-1}{p+1}, n\frac{q-1}{q+1}\} > 1$,

(2) *$K_i(x,t) \geq 0$ for $i = 1, 4$, for a.a. $(x,t) \in Q_T$, and there are nonnegative constants $\underline{k}_2, \underline{k}_3$ such that*

$$-\underline{k}_i \leq K_i(x,t) \leq 0 \quad \textit{for } i = 2, 3,$$

for a.a. $(x,t) \in Q_T$,

(3) *Assumption 2(2) is satisfied with*

$$C_1 = \left(TC_p^{\frac{(p-1)(m-1)+2}{(p-1)(m-1)}} (2T\|a\|_{L^\infty}^2)^{\frac{2}{(p-1)(m-1)}}\right)^{\frac{1}{2}},$$

$$C_2 = \left(TC_q^{\frac{(q-1)(n-1)+2}{(q-1)(n-1)}} (2T\|b\|_{L^\infty})^2)^{\frac{2}{(q-1)(n-1)}}\right)^{\frac{1}{2}},$$

where C_p and C_q are as in (3.6) and in (3.7).

Then problem (1.1) has a nonnegative T-periodic solution (u, v).

The proof of Theorem 3.3 suggests the following result when $\min\{m\frac{p-1}{p+1}, n\frac{q-1}{q+1}\} = 1$.

Theorem 3.4. *Suppose that assumptions (2) and (3) of Theorem 3.3 hold true. If, in addition,*

$$\min\left\{m\frac{p-1}{p+1}, n\frac{q-1}{q+1}\right\} = 1$$

and

$$C_p^{\frac{(p-1)(m-1)+2}{(p-1)(m-1)}} \left(2\overline{k}_2^2 C_q^{\frac{(q-1)(n-1)+2}{(q-1)(n-1)}}\right)^{\frac{2}{(p-1)(m-1)}} \left(2\overline{k}_3^2\right)^{\frac{4}{(q-1)(n-1)(p-1)(m-1)}} < 1, \quad (3.12)$$

then problem (1.1) has a nonnegative T-periodic solution (u, v).

Proof. First note that, if, for instance, $n\frac{q-1}{q+1} = 1$, then $(q-1)(n-1) = 2$, so that the expression in (3.12) can be simplified. Now the proof proceeds as the one of Theorem 3.3 up to inequality (3.10), which now reads

$$U \leq \alpha + \beta U,$$

where β is the left-hand side of (3.12). Since $\beta < 1$, we obtain the desired upper bound on U. $\square$

Remark 3.1. Observe that the technique used to prove Theorem 2.1 (or Corollary 2.1), and the *a priori* estimates in $L^2(Q_T)$ can be adapted to prove analogous results if we consider system (1.1) with $p, q \geq 2$, that is if we consider a double degeneracy (or a single degenerate equation) as in [25], but with a p-linear term in the right-hand side.

4. A Generalization in the Competitive Case

The techniques used in the previous sections allow us to prove the existence of a T-periodic nonnegative solution (u, v) with nontrivial u, v for the following system:

$$\begin{cases} u_t - \operatorname{div}(|\nabla u^m|^{p-2}\nabla u^m) = \left(a - \int_\Omega K_1(\xi,t)u^\alpha(\xi,t-\tau_1)d\xi \right. \\ \qquad\qquad \left. + \int_\Omega K_2(\xi,t)v^\alpha(\xi,t-\tau_2)d\xi\right) u^{p-1} \quad \text{in } Q_T, \\ v_t - \operatorname{div}(|\nabla v^n|^{q-2}\nabla v^n) = \left(b + \int_\Omega K_3(\xi,t)u^\alpha(\xi,t-\tau_3)d\xi \right. \\ \qquad\qquad \left. - \int_\Omega K_4(\xi,t)v^\alpha(\xi,t-\tau_4)d\xi\right) v^{q-1} \quad \text{in } Q_T, \\ u(x,t) = v(x,t) = 0 \quad \text{for } (x,t) \in \partial\Omega \times (0,T), \\ u(\cdot,0) = u(\cdot,T) \quad \text{and} \quad v(\cdot,0) = v(\cdot,T), \end{cases} \tag{4.1}$$

where $\alpha \geq 1$, $K_i(t,x) \leq 0$ $(i = 2,3)$, and m, n, p, q, $\tau_i (i = 1,2,3,4)$, a, b and K_i $(i = 1,4)$, are as in Assumption 1.

4.1. *The coexistence theorem*

As before, one can prove that Lemma 2.1 and Proposition 2.1 still hold for the associated nondegenerate singular p-Laplacian problem

$$\begin{cases} u_t - \operatorname{div}(\varepsilon|\nabla u|^{p-2}\nabla u + |\nabla u^m|^{p-2}\nabla u^m) \\ \quad = \left(a - \int_\Omega K_1(\xi,t)u^\alpha(\xi,t-\tau_1)d\xi \right. \\ \qquad \left. + \int_\Omega K_2(\xi,t)v^\alpha(\xi,t-\tau_2)d\xi\right) u^{p-1} \qquad \text{in } Q_T, \\ v_t - \operatorname{div}(\varepsilon|\nabla v|^{q-2}\nabla v + |\nabla v^n|^{q-2}\nabla v^n) \\ \quad = \left(b + \int_\Omega K_3(\xi,t)u^\alpha(\xi,t-\tau_3)d\xi \right. \\ \qquad \left. - \int_\Omega K_4(\xi,t)v^\alpha(\xi,t-\tau_4)d\xi\right) v^{q-1} \qquad \text{in } Q_T, \\ u(\cdot,t)|_{\partial\Omega} = v(\cdot,t)|_{\partial\Omega} = 0 \quad \text{for a.a. } t \in (0,T), \\ u(\cdot,0) = u(\cdot,T) \quad \text{and} \quad v(\cdot,0) = v(\cdot,T), \end{cases} \tag{4.2}$$

where $\varepsilon > 0$ is small enough. Moreover the next result holds.

Proposition 4.1. *There is a constant $R > 0$ such that*

$$\|u_\varepsilon\|_{L^\infty}, \|v_\varepsilon\|_{L^\infty} < R$$

for all solution pairs $(u_\varepsilon, v_\varepsilon)$ *of*

$$(u,v) = G_\varepsilon(1, f(u^+,v^+), g(u^+,v^+)),$$

and all $\varepsilon > 0$. *In particular, one has that*

$$\deg((u,v) - G_\varepsilon(1, f(u^+,v^+), g(u^+,v^+)), B_R, 0) = 1.$$

Here G_ε *is defined as in Sec.* 2 *and*:

$$\begin{aligned} f(u^+,v^+) &:= \left(a - \int_\Omega K_1(\xi,\cdot)\,(u^+)^\alpha(\xi,\cdot-\tau_1)d\xi \right.\\ &\quad \left. + \int_\Omega K_2(\xi,\cdot)\,(v^+)^\alpha(\xi,\cdot-\tau_2)d\xi\right)(u^+)^{p-1} \\ g(u^+,v^+) &:= \left(b + \int_\Omega K_3(\xi,\cdot)\,(u^+)^\alpha(\xi,\cdot-\tau_3)d\xi \right.\\ &\quad \left. - \int_\Omega K_4(\xi,\cdot)\,(v^+)^\alpha(\xi,\cdot-\tau_4)d\xi\right)(v^+)^{q-1}. \end{aligned}$$

Proof. By the first equation of (4.2), we have

$$\frac{\partial u_\varepsilon}{\partial t} - \varepsilon\,\mathrm{div}(|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon) - \mathrm{div}(\nabla u_\varepsilon^m|\nabla u_\varepsilon^m|^{p-2}) \le K u_\varepsilon^{p-1},$$

where $K := \|a\|_{L^\infty}$. By Lemma 2.2, using the Steklov averages $(u_\varepsilon)_h \in H^1(Q_{T-\delta})$, $\delta, h > 0$, and the fact that $(u_\varepsilon)_h$ converges to u_ε in $L^\infty(Q_T)$, we conclude that $\|u_\varepsilon\|_{L^\infty} \le R_1$ for some $R_1 > 0$ independent of ε. Analogously, $\|v_\varepsilon\|_{L^\infty} \le R_2$ for some constant $R_2 > 0$. Therefore it is enough to choose $R > \max\{R_1, R_2\}$.

The second part of the proposition follows as in the proof of Proposition 2.2. $\square$

From now on we make the following assumption.

Assumption 3. The functions a and b are such that

$$\min\left\{\frac{1}{T}\iint_{Q_T} a e_p^p - \frac{\underline{k}_2 C_2}{T}, \frac{1}{T}\iint_{Q_T} b e_q^q - \frac{\underline{k}_3 C_1}{T}\right\} > 0.$$

Here $\underline{k}_2, \underline{k}_3$ are as in Assumption 1(3), μ_p, μ_q, e_p and e_q are as in Sec. 2.

As before, take ε in $(0, \varepsilon_0)$, where ε_0 is such that

$$\theta(C_1, C_2) := \min\left\{\frac{1}{T}\iint_{Q_T} a e_p^p - \varepsilon_0\mu_p - \frac{\underline{k}_2 C_2}{T}, \frac{1}{T}\iint_{Q_T} b e_q^q - \varepsilon_0\mu_q - \frac{\underline{k}_3 C_1}{T}\right\} > 0.$$

Then, Lemma 2.4 and Proposition 2.3 still hold with

$$M_1 := \left(\frac{|Q_T|^{\frac{1}{\beta'}}(\frac{1}{\mu_p})^{\frac{1}{\beta}}[(p-1)(m-1)-s]^p\|a\|_{L^\infty}}{[m(p-1)]^{p-1}[(p-1)(m-p)-ps]}\right)^{\frac{\beta}{p(\beta-1)}},$$

$$M_2 := \left(\frac{|Q_T|^{\frac{1}{\delta'}} (\frac{1}{\mu_q})^{\frac{1}{\delta}} [(q-1)(n-1)-s]^q \|b\|_{L^\infty}}{[n(q-1)]^{q-1}[(q-1)(n-q)-qs]} \right)^{\frac{\delta}{q(\delta-1)}}$$

and

$$r_0 := \min\left\{ \left(\frac{\iint_{Q_T} ae_p^p - \varepsilon_0 T\mu_p}{D_1} \right)^{\frac{1}{\alpha}}, \left(\frac{\iint_{Q_T} ae_p^p - \varepsilon_0 T\mu_p}{D_1} \right)^{\frac{1}{s}}, \right.$$

$$\left. \left(\frac{\iint_{Q_T} be_q^q - \varepsilon_0 T\mu_q}{D_2} \right)^{\frac{1}{\alpha}}, \left(\frac{\iint_{Q_T} be_q^q - \varepsilon_0 T\mu_q}{D_2} \right)^{\frac{1}{s}} \right\},$$

where β, β', δ, δ', D_1 and D_2 are as in Sec. 2. Proceeding as in Theorem 2.1, one can prove that the next coexistence result holds.

Theorem 4.1. *Assume that there exist two positive constants C_1, C_2 such that for all $\varepsilon > 0$ and all solution pairs $(u_\varepsilon, v_\varepsilon)$ of (4.2) it results*

$$\|u_\varepsilon\|_{L^\alpha}^\alpha \le C_1 \quad \text{and} \quad \|v_\varepsilon\|_{L^\alpha}^\alpha \le C_2.$$

Then, problem (4.1) has a T-periodic nonnegative solution (u, v) with nontrivial u, v.

Obviously, the previous result holds also for a single equation. In particular, we have the following corollary.

Corollary 4.1. *Consider the problem*

$$\begin{cases} u_t - \operatorname{div}(|\nabla u^m|^{p-2}\nabla u^m) \\ \qquad = \left(a - \int_\Omega K(\xi,t)u^\alpha(\xi,t-\tau)d\xi \right) u^{p-1}, \quad \text{in } Q_T, \\ u(x,t) = 0 \quad \text{for } (x,t) \in \partial\Omega \times (0,T), \\ u(\cdot,0) = u(\cdot,T), \end{cases} \tag{4.3}$$

and assume that

(1) *the exponents p, m are such that $p \in (1,2)$ and $m > p$,*
(2) *the delay $\tau \in (0, +\infty)$,*
(3) *the functions a and K belong to $L^\infty(Q_T)$, are extended to $\Omega \times \mathbb{R}$ by T-periodicity and are nonnegative for a.a. $(x,t) \in Q_T$,*
(4) *there exists a positive constant C such that for all $\varepsilon > 0$ and all solutions u_ε of*

$$u = G_\varepsilon(1, f(u^+)),$$

it results

$$\|u_\varepsilon\|_{L^\alpha}^\alpha \le C.$$

Then problem (4.3) has a T-periodic nonnegative and nontrivial solution.

4.2. *A priori bounds in $L^\alpha(Q_T)$: The coercive and the noncoercive cases*

In this subsection we apply Theorem 4.1 by looking for explicit *a priori* bounds in $L^\alpha(Q_T)$ for the solutions of (4.2) in the "coercive case" and in the "noncoercive case".

Theorem 4.2. *Assume that*

(1) *Assumption* 1 *is satisfied,*
(2) *there exist constants* $\underline{k}_i > 0$, $i = 1, 4$, *such that*

$$K_i(x,t) \geq \underline{k}_i \quad \textit{for } i = 1, 4,$$

for a.a. $(x,t) \in Q_T$,
(3) $K_i(x,t) \leq 0$ *for* $i = 2, 3$, *for a.a.* $(x,t) \in Q_T$,
(4) *Assumption* 2(2) *is satisfied with*

$$C_1 = \frac{T}{\underline{k}_1}\|a\|_{L^\infty} \quad \textit{and} \quad C_2 = \frac{T}{\underline{k}_4}\|b\|_{L^\infty}.$$

Then problem (1.1) *has a nonnegative T-periodic solution* (u, v) *with nontrivial* u, v.

Proof. We just need to show that $\|u_\varepsilon\|^\alpha_{L^\alpha} \leq C_1$ and $\|v_\varepsilon\|^\alpha_{L^\alpha} \leq C_2$ for any solution $(u_\varepsilon, v_\varepsilon)$ of (4.2). Proceeding as in Theorem 3.1, one has

$$0 < (T\|a\|_{L^\infty} - \underline{k}_1\|u_\varepsilon\|^\alpha_{L^\alpha}).$$

The same procedure, when applied to the second equation of (2.1), leads to

$$0 < (T\|b\|_{L^\infty} - \underline{k}_4\|v_\varepsilon\|^\alpha_{L^\alpha}),$$

Hence, we have

$$\|u_\varepsilon\|^\alpha_{L^\alpha} \leq \frac{T\|a\|_{L^\infty}}{\underline{k}_1} \quad \text{and} \quad \|v_\varepsilon\|^\alpha_{L^\alpha} \leq \frac{T\|b\|_{L^\infty}}{\underline{k}_4}.$$

□

Theorem 4.3. *Assume that*

(1) *Assumption* 1 *is satisfied,*
(2) $K_i(x,t) \leq 0$ *for* $i = 2, 3$, *for a.a.* $(x,t) \in Q_T$,
(3) *Assumption* 2(2) *is satisfied with*

$$C_1 = \frac{|Q_T|}{\mu_p^{\frac{\alpha}{m(p-1)}}}\left(\frac{(m(p-1)+\alpha)^p\|a\|_{L^\infty}}{\alpha m^{p-1}p^p}\right)^{\frac{\alpha}{m(p-1)}},$$

$$C_2 = \frac{|Q_T|}{\mu_q^{\frac{\alpha}{n(q-1)}}}\left(\frac{(n(q-1)+\alpha)^q\|b\|_{L^\infty}}{\alpha n^{q-1}q^q}\right)^{\frac{\alpha}{n(q-1)}}.$$

Here $\overline{k}_2, \overline{k}_3$, *are as in Assumption* 1. *Then problem* (1.1) *has a* T*-periodic nonnegative solution* (u, v) *with nontrivial* u, v.

Proof. We just need to show that $\|u_\varepsilon\|_{L^\alpha}^\alpha \leq C_1$ and $\|v_\varepsilon\|_{L^\alpha}^\alpha \leq C_2$ for any solution $(u_\varepsilon, v_\varepsilon)$ of (4.2). Multiplying the first equation of (4.2) by $u_\varepsilon^{\alpha-p+1}$, integrating over Q_T and passing to the limit, as $h \to 0$, in the Steklov averages $(u_\varepsilon)_h$, we obtain, as in Proposition 2.2,

$$
\begin{aligned}
m^{p-1}\alpha\left(\frac{p}{m(p-1)+\alpha}\right)^p &\iint_{Q_T} |\nabla u_\varepsilon^{\frac{m(p-1)+\alpha}{p}}|^p \\
&\leq \varepsilon\alpha \iint_{Q_T} u^{\alpha-1}|\nabla u_\varepsilon|^p + \iint_{Q_T} |\nabla u_\varepsilon^m|^{p-2}\nabla u_\varepsilon^m \nabla u_\varepsilon^\alpha \\
&\leq \|a\|_{L^\infty}\|u_\varepsilon\|_{L^\alpha}^\alpha,
\end{aligned}
$$

by the T-periodicity of u_ε and the nonpositivity of the function K_2. Using the Hölder inequality, with $r := \frac{m(p-1)+\alpha}{\alpha}$, and the Poincaré inequality, one has:

$$
\begin{aligned}
\iint_{Q_T} u_\varepsilon^\alpha &\leq |Q_T|^{\frac{m(p-1)}{m(p-1)+\alpha}} \left(\iint_{Q_T} u_\varepsilon^{m(p-1)+\alpha}\right)^{\frac{\alpha}{m(p-1)+\alpha}} \\
&= |Q_T|^{\frac{m(p-1)}{m(p-1)+\alpha}} \|u_\varepsilon^{\frac{m(p-1)+\alpha}{p}}\|_{L^p}^{\frac{\alpha p}{m(p-1)+\alpha}} \\
&\leq |Q_T|^{\frac{m(p-1)}{m(p-1)+\alpha}} \left(\frac{1}{\sqrt[p]{\mu_p}}\left\|\nabla u_\varepsilon^{\frac{m(p-1)+\alpha}{p}}\right\|_{L^p}\right)^{\frac{\alpha p}{m(p-1)+\alpha}}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\|u_\varepsilon\|_{L^\alpha}^\alpha &\leq |Q_T|^{\frac{m(p-1)}{m(p-1)+\alpha}} \left(\frac{1}{\sqrt[p]{\mu_p}}\left\|\nabla u_\varepsilon^{\frac{m(p-1)+\alpha}{p}}\right\|_{L^p}\right)^{\frac{\alpha p}{m(p-1)+\alpha}} \\
&\leq \frac{|Q_T|^{\frac{m(p-1)}{m(p-1)+\alpha}}}{\mu_p^{\frac{\alpha}{m(p-1)+\alpha}}} \left(\frac{(m(p-1)+\alpha)^p\|a\|_{L^\infty}}{\alpha m^{p-1}p^p}\right)^{\frac{\alpha}{m(p-1)+\alpha}} \|u_\varepsilon\|_{L^\alpha}^{\frac{\alpha^2}{m(p-1)+\alpha}}.
\end{aligned}
$$

This implies

$$
\|u_\varepsilon\|_{L^\alpha}^\alpha \leq \frac{|Q_T|}{\mu_p^{\frac{\alpha}{m(p-1)}}} \left(\frac{(m(p-1)+\alpha)^p\|a\|_{L^\infty}}{\alpha m^{p-1}p^p}\right)^{\frac{\alpha}{m(p-1)}}.
$$

In an analogous way, we obtain that

$$
\|v_\varepsilon\|_{L^\alpha}^\alpha \leq \frac{|Q_T|}{\mu_q^{\frac{\alpha}{n(q-1)}}} \left(\frac{(n(q-1)+\alpha)^q\|b\|_{L^\infty}}{\alpha n^{q-1}q^q}\right)^{\frac{\alpha}{n(q-1)}},
$$

if v_ε is a solution of the second equation of (4.2). $\square$

An immediate consequence of Theorems 4.2 and 4.3 is the following existence result for a single equation.

Corollary 4.2. *Consider the problem* (4.3) *and assume that*

(1) *the exponents* p, m *are such that* $p \in (1,2)$ *and* $m > p$,
(2) *the delay* $\tau \in (0, +\infty)$,
(3) *the functions* a *and* K *belong to* $L^\infty(Q_T)$, *are extended to* $\Omega \times \mathbb{R}$ *by* T*-periodicity and are nonnegative for a.a.* $(x,t) \in Q_T$,
(4) *hypothesis* (4) *of Corollary* 4.1 *is satisfied with*

$$C = \frac{T}{\underline{k}} \|a\|_{L^\infty}$$

if $K(t,x) \geq \underline{k} > 0$, *or*

$$C = \frac{|Q_T|}{\mu_p^{\frac{\alpha}{m(p-1)}}} \left(\frac{(m(p-1)+\alpha)^p \|a\|_{L^\infty}}{\alpha m^{p-1} p^p} \right)^{\frac{\alpha}{m(p-1)}}$$

if $K(t,x) \geq 0$.

Then problem (4.3) *has a* T*-periodic nonnegative and nontrivial solution.*

Proof. We just need to show that $\|u_\varepsilon\|_{L^\alpha}^\alpha \leq C$ for a positive constant C. Proceeding as in Theorem 4.2 if $K(t,x) \geq \underline{k} > 0$ or in Theorem 4.3 if $K(t,x) \geq 0$, one has that $\|u_\varepsilon\|_{L^\alpha}^\alpha \leq \frac{T}{\underline{k}_1}\|a\|_{L^\infty}$ or $\|u_\varepsilon\|_{L^\alpha}^\alpha \leq \frac{|Q_T|}{\mu_p^{\frac{\alpha}{m(p-1)}}} (\frac{(m(p-1)+\alpha)^p \|a\|_{L^\infty}}{\alpha m^{p-1} p^p})^{\frac{\alpha}{m(p-1)}}$, respectively. The thesis follows by Corollary 4.1. □

Remark 4.1. We underline the fact that the generalization presented in this section can be extended to a single degenerate equation or to a double degenerate system, namely when $p, q \geq 2$, as considered in [25].

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