Precedents and explorative participation in mathematical discourse

Federica Poli[1,2] and Elena Macchioni[1]

[1] University of Pisa, Italy;

[2] University of Modena and Reggio Emilia, Italy; federica.poli@unimore.it

The paper examines the case of Carlo, a low-achieving student, who participated in a remedial intervention in which he worked with one- and two-dimensional representations of functions in a digital environment. Using a commognitive lens, we analyze two episodes involving task situations that an expert would recognize as similar, but that Carlo approached differently. Our broad aim is to explore how previously learned mathematics can interfere with remedial interventions, such as the one in focus.

Keywords: Digital environments, remedial intervention, commognition, inclusion, low-achievement.

Introduction

In Europe (OECD, 2023), one in three 15-year-old students struggles with basic mathematical proficiency. For instance, although functions are a cornerstone of mathematics learning, students often find it challenging to analyze or describe their behavior through Cartesian graphs, which are not static images but dynamic representations of relationships between variations (Antonini et al., 2020). Several studies (e.g., Baccaglini-Frank, 2021) highlight the potential of digital environments in supporting mathematics learning, providing immediate feedback, and fostering transitions between different representations of mathematical concepts. Moreover, Nachlieli and Tabach (2012), highlight the significance of learners recognizing different mathematical representations as equivalent. These characteristics also seem optimal in the context of remedial interventions.

The study presented is part of the larger research project, DynaMat, which focuses on designing and implementing remedial interventions for learning algebra and functions within digital environments. It targets 10th-grade students in Italy with a persistent history of low achievement in mathematics (for details: https://www.carme.center/progetto-dynamat/). Here we explore a student recalling prior mathematical experiences during a remedial intervention in a digital environment. Such a phenomenon, which we have not found described widely in the literature, seems rather frequent on others of our case studies within the DynaMat project, making the case under focus prototypical.

Theoretical Framework

We have chosen Commognition (Sfard, 2008) as our theoretical framework because it enables us to describe students' participation in terms of their actual possibilities and forms of participation rather than focusing on their deficits. Commognition indeed provides tools for a fine-grained analysis of students' discourse, emphasizing their gradual and continuous (or interrupted) participation in mathematical discourse. This approach prevents assumptions on what learners "can't do", overcoming the duality between "what the student has in their mind" and "what the student actually says." The goal is to better understand difficulties, moving away from a deficit perspective and instead embracing a participationist approach. In this paper, we will see how the proposed activities enabled a student with a long history of difficulties to engage in explorative participation and with agency (episode 1). However, this participation can be hindered when previously learned mathematics interferes with the activity that the student is undertaking (episode 2).

In this paper we will use a specific subset of commognitive analytic tools.

Lavie and colleagues (2019) have distinguished between *task situation* (TS) - when someone feels bound to act (in school, the situation typically refers to assignments that teachers give students); and *task* - the student's interpretation of a *given task situation* relying on *precedent*, a situation similar enough to the current one to justify applying the same approach or actions. The process of identifying precedents relies on a preselection within a *precedent search space*, however, only certain elements will be selected as precedents for a specific TS. This selection occurs through *precedent identifiers* - specific features of the current TS that lead a person to recognize a past TS as a precedent. The same TS can lead to different precedent identifiers, resulting in different interpreted tasks for different students. The pair of interpreted task and procedure is called *routine*, which is a pattern of action. Routines will also be our unit of analysis.

A task situation that is recognized as familiar to one student may instead be unfamiliar to another. In this context, we refer to a TS as *familiar* when the student seems to recognize the TS as similar to one they have previously faced . In this case the student has identified a precedent identifier within the TS, allowing them to retrieve a precedent from their mathematical precedent search space. On the other hand, if the TS is *unfamiliar* - not recognized as familiar - the student does not seem to immediately recall any precedent.

In Commognition, a mathematical object is always a discursive object, defined as a *signifier* with all its *realizations* (Sfard, 2008). For instance, a Cartesian graph or an input-output table can be recognized by an expert as different realizations of the same signifier function. Instead, for students entering mathematical discourse on function, these are initially only *visual mediators* of their discourse. This may also be the case for many low-achieving or failing students.

When participation involves merely executing memorized procedures focused on process, it is defined as *ritualistic*. In contrast, *explorative participation* involves constructing meaningful narratives, focused on product. Explorative and ritualistic participation represent the extremes of a continuum, and a student's discourse typically falls somewhere between these extremes. Processes of de-ritualization, such as *substantiability* (the shift of attention from process to product) and *flexibility* (more procedures can be used to solve the same task situation), can help infer if participation is more ritualistic or explorative (Lavie et al, 2019). Moreover, regarding *agency*, students who participate exploratively are more likely to propose new actions or outcomes (Baccaglini-Frank, 2021).

Carlo is a 10th-grade student who joined the DynaMat project. His case captured our interest because of his seemingly different forms of participation in mathematical discourse across TSs on functions during the intervention. Specifically, his participation seemed to differ significantly in two TSs that seemed to be asking quite similar tasks according to the design of the activities (and therefore in the eyes of experts). These TSs are designed to be not immediately recognizable as tasks previously encountered at school and designed to be accessible, even without a specific procedure, therefore unfamiliar. This paper aims to describe the apparent difference in participation between the two TSs in the light of commognitive constructs outlined in the theoretical framework.

Methods

Carlo participated to the second experimental cycle of DynaMat project between October 2023 and December 2023. All students joined the project voluntarily during the afternoon in the CARME research center. They initially participated in an individual interview, followed by five two-hour activity sessions with digital environments, conducted by a researcher team. The sequence of activities proposed to Carlo involved dynamic representations of functions, known as dynagraphs (Antonini et al., 2020). It begins with a one-dimensional representation of function, as two variables moving along parallel lines (Figure 1a), to promote awareness of covariation. The activities build up to the Cartesian graph by first rotating the line of the dependent variable by 90° relative to the line of the independent variable, so that the variations of the two variables are now perpendicular to each other (Figure 1b). Finally, perpendicular lines are drawn from each variable's position, creating intersections that define each point on the Cartesian graph as the variables change (Figure 1c).

These are the task situations we are going to analyze here:

TS1: The student is given a dynagraph with Cartesian axes and five different Cartesian graphs printed on paper; the tutor asks: "While dragging tick mark A, can you imagine the trajectory of the point (A, B) on the Cartesian plane? Which of the following graphs represents the trajectory of point (A, B)?".

TS2: The student is given a dynagraph with parallel lines, and the tutor asks: "Explore the file and draw below the corresponding graph in the Cartesian plane."

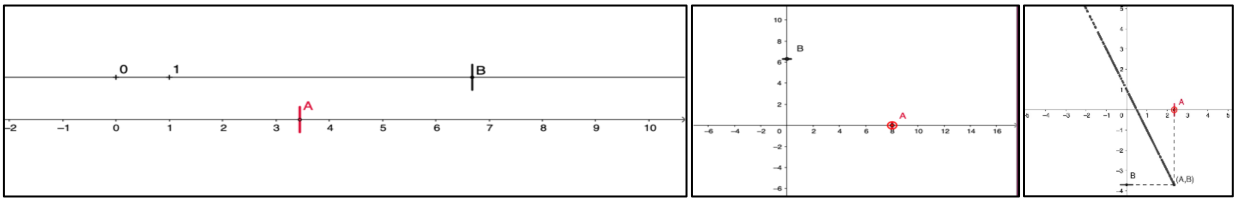

Figure 1: (a) dynagraph with parallel line, (b) dynagraph with Cartesian axes, (c) Cartesian Graph

Both TSs involve dynagraphs, which are usually not introduced in school, and Carlo, in the initial interview, states that he has only studied lines in the Cartesian plane. Moreover, none of the function realizations in these episodes (neither on paper nor in the dynagraph) are lines. For these reasons, we infer that the TSs presented unfamiliar tasks for him. In the TS design process it was essential for the researchers to share the same socio-cultural background as the students.

Both TSs aim to elicit discourse with transitions between multiple realizations of functions. The tutor tried to foster such discourse and she planned to introduce input-output tables. For example she would prompt Carlo by asking: "Could you use tables to write down some pairs of numbers that work?".

Carlo had access to a tablet displaying the dynagraph and to another one for writing or drawing. A tutor would pose the task situations, which could differ from the tasks interpreted by the student. We inferred Carlo's interpreted tasks by observing his procedure.

In the analyses below, both episodes (related to TS1 during session four and TS2 during session five) were transcribed with line numbering starting from 1. Between these two sessions, other activities were carried out.

Data analysis

Episode 1 – Carlo faced with TS1

The tutor reads TS1, which asks to envision the trajectory and identify the corresponding graph, between five different Cartesian graphs printed on paper (TS1). On the other hand, Carlo, after initially interacting with the dynagraph, immediately starts describing it. Thus, the task he interprets is to provide a verbal description of the movement of the tick marks.

3 Carlo: I see that, when I move into negative numbers with A, B rises very, very slowly [moves A to the left towards -∞], while when I lower it, it goes down a little faster [moves A towards zero, approaching from the left].

4 Tutor: Okay, so... when B goes down faster, I didn't understand.

5 Carlo: And then when A... when B goes down more, that is, when A keeps moving towards zero, B goes down faster. Then A, when it goes to 4, on the numbers, as the numbers increase [moves A around 5].

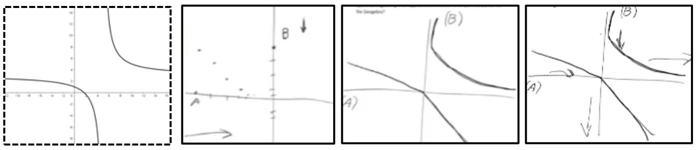

Figure 2: (a) selected Cartesian graph of TS1 ; (b) (c) (d) Carlo faced with TS1

Carlo spontaneously returns to the task situation (showing substantiability) and picks up the graphs chosen to answer the request [Figure 2a]. Now, is interpreted task seems to coincide with TS1.

19 Carlo: This [tracing the graph on the paper with a finger] is the movement of the dot, maybe of the intersection... like, okay, so if like, I don't know, A is here and goes this way, B is here and goes this way [draws the axes, inserts A and B, then the arrows in Figure 2b], this will gradually go like this, this, this, and like this, this, this. Then this goes down, like, I don't know, here, then here, then here, then here, and so on [inserts the ticks on the axes and the points].

20 Tutor: So these points you've drawn are the point (A, B)? The path it followed?

21 Carlo: And then when A reaches zero, B is below, so it forms something like this. This drawing, this is the line that A makes, no, wait. This is the line it makes.

In line 19 Carlo responds to the TS identifying the correct graph between the five different printed graphs, and he justifies his choice. First, he describes the movement of the point (A, B) by looking at the selected graph on paper, and then he starts plotting part of it on the Cartesian graph [Figure 2b]. This utterance shows both flexibility (as multiple procedures can be implemented to respond to the task situation) as well as substantiability (with the focus on the product, the selected graph). So, his participation is toward the explorative side of the spectrum. Then Carlo decides to return to the tablet and summarize his findings, showing agency. Carlo's interpreted task (line 29) seems to be to plot the movement of the point (A, B): he draws Figure 2c, confirming his initial choice (substantiability).

29 Carlo: [draws Figure 2c] It looks like the movement it makes. So this […] because when A increases… goes towards, that is, it grows because it goes towards zero, B decreases, as you can see. B goes this way and A goes this way [inserts the left arrows in Figure 2d]."

31 Carlo: Then when A, when A is in the positive range, B is also positive, and gradually as A moves away, going this way, B decreases [inserts the right arrows in Figure 2d]. Indeed, you can see that it goes lower and lower.

Initially, Carlo refers only to the movements of A and B, which he indicates with arrows in Figure 2b. Then, he makes a first attempt, even if discrete, to dynamically represent some of the positions reached by (A, B). Finally, thanks to the new dragging action, he manages to represent the trajectory as a static object that an expert would recognize as resembling the Cartesian graph of the function (it is not the graph of a function, but it closely approximates the trajectory of the graph shown in Figure 2a). All of Carlo's interpreted tasks are commensurable with those of the expert.

Episode 2 – Carlo faced with TS2

Initially, Carlo hesitated (line 5, "it's a bit difficult"), and the tutor, in accordance with the design, suggested introducing input-output tables as a further realization.

25 Tutor: Maybe it could help if you noted down some values where ticks pass through.

Line 25 changed the task interpreted by Carlo, who began solving the task in an unexpected way.

28 Carlo: Alright, I'll set it to 2.
29 Tutor: Okay, for example, when A is 2, B is on…?
30 Carlo: [He explores the dynagraph and marks the points (2;0) and (0;-2), then draws the ray from the origin passing through the point (2,-2) as shown in Figure 3a] How many points?
31 Tutor: Great, let's test it with other points, perhaps, to see how it behaves?
32 Carlo: At -8?

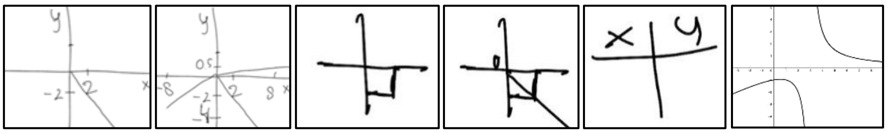

Figure 3: (a) (b) (c) (d) (e) Carlo faced with TS2; (f) expected Cartesian graph of TS2

Carlo marks -8 on the x-axis and -4 on the y-axis, then draws a ray from the origin to (-8, -4). He does the same for the point (8, 0.5), producing Figure 3b instead of the expected Cartesian graph (Figure 3f). The tutor tries to investigate his interpreted task:

37 Tutor: OK, so let's say what you drew what, what are these?
38 Carlo: Uh, various points, and I drew some lines
40 Carlo: All passing through zero
46 Carlo: [point (2;-2)] is an intersection
47 Tutor: Mmm, okay, so how do you find this intersection?
48 Carlo: By using these two numbers and drawing it like this [Figure 3c], you do something similar, and then this, being zero, makes a line like this [Figure 3d]
53 Tutor: Ok... Why did you draw the lines passing through the origin and that point?
54 Carlo: Because with a regular line… I didn't make a table like this x and y [Figure 3e], and I gave a value to x... I mean, I didn't have the equations; I just must look at the graph and take the values
55 Tutor: Okay, why do you make a table when you have the equations instead?
56 Carlo: Yes, I give a value to x, substitute it, that is, two values for x, two values for y, and it works out

By examining Carlo's implemented procedure, it appears that the interpreted task is related to plotting lines on a Cartesian graph, using an input-output table. Precedent identifiers that seem to elicit this task are the tutor’s request to “write down values” (line 25), the formulation of the tutor’s question recalling the input-output procedure (line 29), and the tutor’s suggestion to “test other points” (line 31). This inference is coherent with Carlo’s mention of the input-output table in line 54 and Figure 3e. These precedent identifiers seem to trigger procedures that Carlo had previously memorized (at school), which he describes in detail, but that are not consistent with the TS proposed by the tutor. Since the very beginning, Carlo asks how many points he should consider (line 30) and whether starting with -8 is appropriate (line 32). In other words, he relies on the tutor to ask for validation of the procedure he is implementing. We interpret the tutor's question in line 53 as a request to explore the substantiability (using it to backtrack and address the task). Nonetheless, Carlo again goes through the procedure and compares it with another memorized one. The episode continues with Carlo plotting additional values on the graph and then exploring the behavior of the dynagraph near the vertical asymptote. At the end of the activity, the tutor explicitly asks him to clarify his use of the input-output table, and Carlo provides a detailed description:

99 Carlo: I put [the equation] in explicit form, so I write y equals and… then I change the value of x, perform the operation, and it gives me the result for y, and then with another value, it always gives me another result.

The tutor suggests giving an example for clarification, using the equation $y = x + 1$

105 Carlo: Like, for value 3, it becomes 4. Then another value is 4, and it becomes 5.

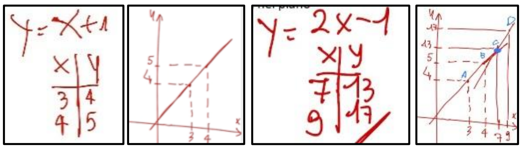

Figure 4: (a) (b) (c) (d) Carlo explains his procedure

Carlo writes Figure 4a and plots these points on a Cartesian plane [Figure 4b]. Then he asks the tutor for another equation to continue. The tutor suggests $\boldsymbol{y = 2x - 1}$**.**

113 Carlo: I put 7, so 7…13, and here 9,18, 17. So 7 and 9 and here [Figure 4c] 13, 17 [Figure 4d]. Uh, I do it like this, this would be like, so I need to connect these two points that are in the same table like this, and then find the common point they have, for example… this is the point they have in common.

Carlo's detailed description of the procedure allows us to infer that the task he interpreted might be about solving linear systems graphically. The procedure is described in lines 99-113 as a series of memorized steps, such as writing the line explicitly, substituting two x-values, finding the corresponding y-values etc. Carlo's focus is on the procedure itself, and hardly at all on its outcome - the point of intersection - which he mentions only once (line 113) when he repeats, upon the tutor’s request, the procedure he had followed (lines 54 and 56). In line 55, the tutor’s request is asking for substantiability; however Carlo’s focus (line 56) remains on the procedure. So, the Carlo's participation in this TS is overall highly ritualistic.

Discussion

Carlo seems to perceive both tasks as unfamiliar with respect to what he was used to doing in school (where he certainly had never used a dynagraph). When facing an unfamiliar TS, he courageously tries to respond, and in both TSs, Carlo recalls a precedent by that guides his discursive actions in the new TS he is currently facing. Let us take a closer look at what happens in each case.

Before addressing TS1, Carlo had already participated in activity sessions at CARME with dynagraphs involving parallel lines, where the TSs were mainly "describe the movement of the tick marks." He had also worked with dynagraphs with perpendicular lines, where the TSs were "imagine/describe/draw the trajectory of the point (A,B) in the Cartesian plane." Our hypothesis is that, when addressing TS1, despite it being unfamiliar, Carlo has sufficient prior precedents developed during the work at CARME, at his disposal. This allows him to recall a precedent consistent with TS1. His participation is therefore explorative, focused on the product, as intended in the design.

In TS2, however, Carlo's interpretation of the task does not align with the designed task (which is what the tutor had in mind). Hence, there appears to be a *commognitive gap* (a mismatch between the tasks interpreted by the participants in the same TS) between Carlo and the tutor. Carlo's procedure, which we only interpreted retrospectively, is similar to a procedure in textbooks for graphically solving systems of linear equations by identifying the intersection point of two lines on a graph. The tutor's intervention, which followed the design of the activities (line 25: “Maybe it could help if you noted down some values”) seems to offer Carlo a precedent identifier for the procedure he applies. Recalling such a precedent leads him to reinterpret (uncanonically) the task and carry out a routine ritualistically. We note that we can infer Carlo's interpreted task because we share his socio-cultural background (including typical school textbooks), and we are aware of the topics covered in Carlo’s class before the experimental activities.

Carlo’s different participation in the two TSs baffled us initially. However, the analyses carried out helped us reinterpret these episodes and make sense of them in a satisfying (at least for now) way. Summing up, in one case, Carlo recalls a precedent that is consistent with the TS and the designed task, probably thanks to his (brief) experience with the tutor and previous activities with dynagraphs. In the other case, he recalls a precedent from his school mathematics experience, which is not consistent with the TS and the designed task; such a precedent interferes with Carlo's success in our remedial intervention, locking him back into ritualistic participation.

Conclusion

Once we identified and described the phenomenon, we noticed that the interference of previous mathematical experiences that occurs as a student tries to act in an unfamiliar TS appears to be a significant factor for many of the students we worked with in our remedial interventions. In this sense, the present study is prototypical and it introduces a new perspective on previously observed phenomena (Funghi et al., 2023): students who, despite exhibiting both the capacity and willingness to create meaningful narratives through exploratory participation, nonetheless revert to a more ritualistic form of participation in certain task situations, in which previously learned mathematics seem to play a crucial role.

Although we cannot know for sure why Carlo's participation varied so significantly, we can advance some hypotheses. We inferred that the tutor's prompt, "find some values," serves as a precedent identifier for the student and is, therefore, particularly significant from our perspective. This intervention occurs after a moment of difficulty for Carlo. We hypothesize that introducing new representations during challenging moments, along with a precedent identifier linked to a well-established ritual, may inhibit his authoring of endorsable narratives, preventing him from engaging in more exploratory participation (as observed in other instances, such as during TS1). This hypothesis encourages further research on the tutor's interventions during remedial interventions, with a specific focus on the timing of these interventions and the stability of the ritual routines.

In the DynaMat project we have only started to explore this direction, and hope to proceed in the future. For now we offer the notion of "interference phenomenon" as a potential dimension to consider when examining the effects of remedial interventions while taking into account the student's prior experiences.

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