

## PROFINITE GROUPS WITH AN AUTOMORPHISM WHOSE FIXED POINTS ARE RIGHT ENGEL

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ABSTRACT. An element  $g$  of a group  $G$  is said to be right Engel if for every  $x \in G$  there is a number  $n = n(g, x)$  such that  $[g, {}_n x] = 1$ . We prove that if a profinite group  $G$  admits a coprime automorphism  $\varphi$  of prime order such that every fixed point of  $\varphi$  is a right Engel element, then  $G$  is locally nilpotent.

### 1. INTRODUCTION

Let  $G$  be a profinite group and let  $\varphi$  be a (continuous) automorphism of  $G$  of finite order. We say for short that  $\varphi$  is a *coprime automorphism* of  $G$  if its order is coprime to the orders of elements of  $G$  (understood as Steinitz numbers), in other words, if  $G$  is an inverse limit of finite groups of order coprime to the order of  $\varphi$ . Coprime automorphisms of profinite groups have many properties similar to the properties of coprime automorphisms of finite groups. In particular, if  $\varphi$  is a coprime automorphism of  $G$ , then for any (closed) normal  $\varphi$ -invariant subgroup  $N$  the fixed points of the induced automorphism (which we denote by the same letter) in  $G/N$  are images of the fixed points in  $G$ , that is,  $C_{G/N}(\varphi) = C_G(\varphi)N/N$ . Therefore, if  $\varphi$  is a coprime automorphism of prime order  $q$  such that  $C_G(\varphi) = 1$ , Thompson's theorem [18] implies that  $G$  is pronilpotent, and Higman's theorem [7] implies that  $G$  is nilpotent of class bounded in terms of  $q$ .

In this paper we consider profinite groups admitting a coprime automorphism of prime order all of whose fixed points are right Engel elements. Recall that the  $n$ -Engel word  $[y, {}_n x]$  is defined recursively by  $[y, {}_0 x] = y$  and  $[y, {}_{i+1} x] = [[y, {}_i x], x]$ . An element  $g$  of a group  $G$  is said to be right Engel if for any  $x \in G$  there is an integer  $n = n(g, x)$  such that  $[g, {}_n x] = 1$ . If all elements of a group are right Engel (therefore also left Engel), then the group is called an Engel group. By a theorem of Wilson and Zelmanov [20] based on Zelmanov's results [21–23] on Engel Lie algebras, an Engel profinite group is locally nilpotent. Recall that a group is

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said to be locally nilpotent if every finite subset generates a nilpotent subgroup. In our main result the right Engel condition is imposed on the fixed points of a coprime automorphism of prime order.

**Theorem 1.1.** *Suppose that  $\varphi$  is a coprime automorphism of prime order of a profinite group  $G$ . If every element of  $C_G(\varphi)$  is a right Engel element of  $G$ , then  $G$  is locally nilpotent.*

The proof of Theorem 1.1 begins with the observation that a group  $G$  satisfying the hypothesis is pronilpotent. Indeed, right Engel elements of a finite group are contained in the hypercentre by the well-known theorem of Baer [1]. Therefore every finite quotient of  $G$  by a  $\varphi$ -invariant open normal subgroup is nilpotent by Thompson's theorem [18], since  $\varphi$  acts fixed-point-freely on the quotient by the hypercentre. Assuming in addition that  $G$  is finitely generated, it remains to prove that all Sylow  $p$ -subgroups  $S_p$  of  $G$  are nilpotent with a uniform upper bound for the nilpotency class. This is achieved in two stages. First a bound for the nilpotency class of  $S_p$  depending on  $p$  is obtained for all  $p$ . Then a bound independent of  $p$  is obtained for all sufficiently large primes  $p$ . At both stages we apply Lie ring methods, and the crucial tool is Zelmanov's theorem [21–23] on Lie algebras and some of its consequences. Other important ingredients include criteria for a pro- $p$  group to be  $p$ -adic analytic in terms of the associated Lie algebra due to Lazard [11] and in terms of bounds for the rank due to Lubotzky and Mann [13] and a theorem of Bahturin and Zaicev [2] on Lie algebras admitting a group of automorphisms whose fixed-point subalgebra is PI.

## 2. PRELIMINARIES

**Lie rings and algebras.** Products in Lie rings and algebras are called commutators. We use simple commutator notation for left-normed commutators  $[x_1, \dots, x_k] = [\dots[x_1, x_2], \dots, x_k]$ , and the shorthand for Engel commutators  $[x, {}_n y] = [x, y, y, \dots, y]$  with  $y$  occurring  $n$  times. An element  $a$  of a Lie ring or a Lie algebra  $L$  is said to be ad-nilpotent if there exists a positive integer  $n$  such that  $[x, {}_n a] = 0$  for all  $x \in L$ . If  $n$  is the least integer with this property, then we say that  $a$  is ad-nilpotent of index  $n$ .

The next theorem is a deep result of Zelmanov [21–23].

**Theorem 2.1.** *Let  $L$  be a Lie algebra over a field and suppose that  $L$  satisfies a polynomial identity. If  $L$  can be generated by a finite set  $X$  such that every commutator in elements of  $X$  is ad-nilpotent, then  $L$  is nilpotent.*

An important criterion for a Lie algebra to satisfy a polynomial identity is provided by the next theorem, which was proved by Bahturin and Zaicev for soluble group of automorphisms [2] (and later extended by Linchenko to the general case [12]). We use the centralizer notation for the fixed point subring  $C_L(A)$  of a group of automorphisms  $A$  of  $L$ .

**Theorem 2.2.** *Let  $L$  be a Lie algebra over a field  $K$ . Assume that a finite group  $A$  acts on  $L$  by automorphisms in such a manner that  $C_L(A)$  satisfies a polynomial identity. Assume further that the characteristic of  $K$  is either 0 or coprime with the order of  $A$ . Then  $L$  satisfies a polynomial identity.*

Both Theorems 2.1 and 2.2 admit respective quantitative versions (see for example [16]). For our purposes, we shall need the following result for Lie rings proved in [17, Proposition 2.6], which combines both versions. As usual,  $\gamma_i(L)$  denotes the  $i$ -th term of the lower central series of  $L$ .

**Proposition 2.3.** *Let  $L$  be a Lie ring and let  $A$  be a finite group of automorphisms of  $L$  such that  $C_L(A)$  satisfies a polynomial identity  $f \equiv 0$ . Suppose that  $L$  is generated by an  $A$ -invariant set of  $m$  elements such that every commutator in these elements is ad-nilpotent of index at most  $n$ . Then there exist positive integers  $e$  and  $c$  depending only on  $|A|$ ,  $f$ ,  $m$ , and  $n$  such that  $e\gamma_c(L) = 0$ .*

We also quote the following useful result proved in [10, Lemma 5] (although it was stated for Lie algebras in [10], the proof is the same for Lie rings).

**Lemma 2.4.** *Let  $L$  be a Lie ring and let  $M$  be a subring of  $L$  generated by  $m$  elements such that all commutators in these elements are ad-nilpotent in  $L$  of index at most  $n$ . If  $M$  is nilpotent of class  $c$ , then for some number  $\varepsilon = \varepsilon(m, n, c)$  bounded in terms of  $m, n, c$  we have  $[L, \underbrace{M, M, \dots, M}_\varepsilon] = 0$ .*

**Associated Lie rings and algebras.** We now remind the reader of one of the ways of associating a Lie ring with a group. A series of subgroups of a group  $G$

$$(2.1) \quad G = G_1 \supseteq G_2 \supseteq \dots$$

is called a *filtration* (or an  *$N$ -series* or a *strongly central series*) if

$$(2.2) \quad [G_i, G_j] \subseteq G_{i+j} \quad \text{for all } i, j.$$

For any filtration (2.1) we can define an associated Lie ring  $L(G)$  with additive group

$$L(G) = \bigoplus_i G_i/G_{i+1},$$

where the factors  $L_i = G_i/G_{i+1}$  are additively written. The Lie product is defined on homogeneous elements  $xG_{i+1} \in L_i, yG_{j+1} \in L_j$  via the group commutators by

$$[xG_{i+1}, yG_{j+1}] = [x, y]G_{i+j+1} \in L_{i+j}$$

and extended to arbitrary elements of  $L(G)$  by linearity. Condition (2.2) ensures that this Lie product is well-defined, and group commutator identities imply that  $L(G)$  with these operations is a Lie ring. If all factors  $G_i/G_{i+1}$  of a filtration (2.1) have prime exponent  $p$ , then  $L(G)$  can be viewed as a Lie algebra over the field of  $p$  elements  $\mathbb{F}_p$ . If all terms of (2.1) are invariant under an automorphism  $\varphi$  of the group  $G$ , then  $\varphi$  naturally induces an automorphism of  $L(G)$ .

We shall normally indicate which filtration is used for constructing an associated Lie ring. One example of a filtration (2.1) is given by the lower central series, the terms of which are denoted by  $\gamma_1(G) = G$  and  $\gamma_{i+1}(G) = [\gamma_i(G), G]$ . It is worth noting that the corresponding associated Lie ring  $L(G)$  is generated by the homogeneous component  $L_1 = G/\gamma_2(G)$ .

Another example for a fixed prime number  $p$  is the *Zassenhaus  $p$ -filtration* (also called the  *$p$ -dimension series*), which is defined by

$$G_i = \langle g^{p^k} \mid g \in \gamma_j(G), \quad jp^k \geq i \rangle.$$

The factors of this filtration are elementary abelian  $p$ -groups, so the corresponding associated Lie ring  $D_p(G)$  is a Lie algebra over  $\mathbb{F}_p$ . We denote by  $L_p(G)$  the subalgebra generated by the first factor  $G/G_2$ . It is well known that the homogeneous components of  $D_p(G)$  of degree  $s$  coincide with the homogeneous components of  $L_p(G)$  for all  $s$  that are not divisible by  $p$ . In particular,  $L_p(G)$  is nilpotent if and only if  $D_p(G)$  is nilpotent. (Sometimes, the notation  $L_p(G)$  is used for  $D_p(G)$ .)

A group  $G$  is said to satisfy a *coset identity* if there is a group word  $w(x_1, \dots, x_m)$  and cosets  $a_1H, \dots, a_mH$  of a subgroup  $H \leq G$  of finite index such that  $w(a_1h, \dots, a_mh) = 1$  for any  $h \in H$ . Wilson and Zelmanov [20] proved that if a group  $G$  satisfies a coset identity, then the Lie algebra  $L_p(G)$  constructed with respect to the Zassenhaus  $p$ -filtration satisfies a polynomial identity. In fact, the proof of Theorem 1 in [20] can be slightly modified to become valid for any filtration (2.1) with abelian factors of prime exponent  $p$  and the corresponding associated Lie algebra.

**Profinite groups.** We always consider a profinite group as a topological group. A subgroup of a topological group will always mean a closed subgroup, all homomorphisms are continuous, and quotients are by closed normal subgroups. This also applies to taking commutator subgroups, normal closures, subgroups generated by subsets, etc. We say that a subgroup is generated by a subset  $X$  if it is generated by  $X$  as a topological group. Note that if  $\varphi$  is a continuous automorphism of a topological group  $G$ , then the fixed-point subgroup  $C_G(\varphi)$  is closed.

Recall that a pronilpotent group is a pro-(finite nilpotent) group, that is, an inverse limit of finite nilpotent groups. For a prime  $p$ , a pro- $p$  group is an inverse limit of finite  $p$ -groups. The Frattini subgroup  $P'P^p$  of a pro- $p$  group  $P$  is the product of the derived subgroup  $P'$  and the subgroup generated by all  $p$ -th powers of elements of  $P$ . A subset generates  $P$  (as a topological group) if and only if its image generates the elementary abelian quotient  $P/(P'P^p)$ . See, for example, [19] for these and other properties of profinite groups.

### 3. LOCAL NILPOTENCY OF SYLOW $p$ -SUBGROUPS

In this section we prove the local nilpotency of a pro- $p$  group satisfying the hypotheses of the main Theorem 1.1. We shall use without special references the fact that fixed points  $C_{G/N}(\varphi)$  of an automorphism  $\varphi$  of finite coprime order in a quotient by a  $\varphi$ -invariant normal open subgroup  $N$  are covered by the fixed points in the group:  $C_{G/N}(\varphi) = C_G(\varphi)N/N$ .

**Theorem 3.1.** *Let  $p$  be a prime and suppose that a finitely generated pro- $p$  group  $G$  admits an automorphism  $\varphi$  of prime order  $q \neq p$ . If every element of  $C_G(\varphi)$  is a right Engel element of  $G$ , then  $G$  is nilpotent.*

We begin with constructing a normal subgroup with nilpotent quotient that will be the main focus of the proof. Recall that  $h(q)$  is a function bounding the nilpotency class of a nilpotent group admitting a fixed-point-free automorphism of prime order  $q$  by Higman's theorem [7].

**Lemma 3.2.** *There is a finite set  $S \subseteq C_G(\varphi)$  of fixed points of  $\varphi$  such that the quotient  $G/H$  by its normal closure  $H = \langle S^G \rangle$  is nilpotent of class  $h(q)$ .*

*Proof.* In the nilpotent quotient  $G/\gamma_{h(q)+2}(G)$  of the finitely generated group  $G$  every subgroup is finitely generated. Therefore there is a finite set  $S$  of elements of

$C_G(\varphi)$  whose images cover all fixed points of  $\varphi$  in  $G/\gamma_{h(q)+2}(G)$ . Let  $H = \langle S^G \rangle$  be the normal closure of  $S$ . Then the quotient of  $G$  by  $H\gamma_{h(q)+2}(G)$  is nilpotent of class  $h(q)$  by Higman’s theorem, which means that  $\gamma_{h(q)+1}(G) \leq H\gamma_{h(q)+2}(G)$ . Since the group  $G/H$  is pronilpotent, it follows that  $\gamma_{h(q)+1}(G) \leq H$ , as required.  $\square$

We fix the notation for the subgroup  $H = \langle S^G \rangle$  and the finite set  $S \subseteq C_G(\varphi)$  given by Lemma 3.2. We aim at an application of Zelmanov’s Theorem 2.1 to the associated Lie algebra of  $H$ , verifying the requisite conditions in a number of steps. The first step is to show that the quotient  $G/H'$  is nilpotent, which is achieved by the following lemma.

**Lemma 3.3.** *Suppose that  $L$  is a finitely generated pro- $p$  group,  $M$  is an abelian normal subgroup equal to the normal closure  $M = \langle T^L \rangle$  of a finite set  $T$  consisting of right Engel elements of  $L$ , and  $L/M$  is nilpotent. Then  $L$  is nilpotent.*

*Proof.* We proceed by induction on the nilpotency class of  $L/C_L(M)$ . The base of induction is the case where  $L/C_L(M) = 1$ , that is,  $L = C_L(M)$ ; then  $L$  is obviously nilpotent.

Let  $T = \{t_1, \dots, t_k\}$ . Let  $Z$  be the inverse image of the centre  $Z(L/C_L(M))$  of  $L/C_L(M)$ . We claim that  $Z$  is nilpotent. For any fixed  $z \in Z$  there are positive integers  $n_i$  such that  $[t_i, n_i z] = 1$ . Set  $n = \max_i n_i$ ; then  $[t_i, n z] = 1$  for all  $i$ . Moreover, for any  $g \in L$  we have  $[t_i^g, n z] = [t_i, n z]^g = 1$  since  $[z, g] \in C_L(M)$ . Since  $M = \langle T^L \rangle$  is abelian, this implies that  $[m, n z] = 1$  for any finite product  $m$  of the elements  $t_i^g, g \in G$ . Since these finite products form a dense subset of  $M$ , we obtain

$$(3.1) \quad [m, n z] = 1 \quad \text{for any } m \in M.$$

Since  $L/M$  is nilpotent and finitely generated,  $Z/M$  is nilpotent and finitely generated. Together with (3.1) this implies that  $Z$  is nilpotent. Indeed, let  $Z = \langle M, z_1, \dots, z_s \rangle$ . Any sufficiently long simple commutator in the elements of  $M$  and  $z_1, \dots, z_s$  has an initial segment that belongs to  $M$  because  $Z/M$  is nilpotent. Since  $Z/C_Z(M)$  is abelian, the remaining elements (which can all be assumed to be among the  $z_i$ ) can be arbitrarily rearranged without changing the value of the commutator. If the commutator is sufficiently long, one of the  $z_i$  will appear sufficiently many times in a row making the commutator trivial by (3.1).

We now consider  $L/Z'$ , denoting by the bar the corresponding images of subgroups and elements. Clearly,  $\bar{L}, \bar{M}$ , and  $\bar{T}$  satisfy the hypotheses of the lemma. But now  $\bar{Z} \leq C_{\bar{L}}(\bar{M})$ , so the nilpotency class of  $\bar{L}/C_{\bar{L}}(\bar{M})$  is less than that of  $L/C_L(M)$  (unless  $Z = L$  when the proof is complete). By the induction hypothesis,  $L/Z'$  is nilpotent. Together with the nilpotency of  $Z$  proved above, this implies that  $L$  is nilpotent by Hall’s theorem [5].  $\square$

**Lemma 3.4.** *The subgroup  $H$  is generated by finitely many right Engel elements.*

*Proof.* By Lemma 3.3 applied with  $L = G/H'$ ,  $M = H/H'$ , and  $T = S$ , the quotient  $G/H'$  is nilpotent. Then  $H/H'$  is finitely generated as a subgroup of a finitely generated nilpotent group. The Frattini quotient  $H/(H'H^p)$  is a finite elementary abelian  $p$ -group. Since  $H$  is generated by a set of right Engel elements, conjugates of elements of  $C_G(\varphi)$ , we can choose a finite subset of these elements whose images generate  $H/(H'H^p)$ . Then this finite set also generates the pro- $p$  group  $H$ .  $\square$

Let  $L_p(H)$  be the associated Lie algebra of  $H$  over  $\mathbb{F}_p$  constructed with respect to the Zassenhaus  $p$ -filtration of  $H$ .

**Proposition 3.5.** *The Lie algebra  $L_p(H)$  is nilpotent.*

*Proof.* This will follow from Zelmanov’s Theorem 2.1 if we show that  $L_p(H)$  satisfies a polynomial identity and is generated by finitely many elements such that all commutators in these elements are ad-nilpotent.

**Lemma 3.6.** *The Lie algebra  $L_p(H)$  satisfies a polynomial identity.*

*Proof.* Note that  $H$  is a  $\varphi$ -invariant subgroup, since  $H = \langle S^G \rangle$  for  $S \subseteq C_G(\varphi)$ . As a profinite Engel group,  $C_H(\varphi) = H \cap C_G(\varphi)$  is locally nilpotent by the Wilson–Zelmanov theorem [20]. It follows that  $C_H(\varphi)$  satisfies a coset identity on cosets of an open subgroup of  $C_H(\varphi)$ . For example, in the group  $C_H(\varphi) \times C_H(\varphi)$  the subsets

$$E_i = \{(x, y) \in C_H(\varphi) \times C_H(\varphi) \mid [x, {}_i y] = 1\}$$

are closed in the product topology, and

$$C_H(\varphi) \times C_H(\varphi) = \bigcup_{i=1}^{\infty} E_i.$$

Hence by the Baire category theorem [8, Theorem 34], one of these subsets  $E_n$  contains an open subset of  $C_H(\varphi) \times C_H(\varphi)$ , which means that there are cosets  $x_0 K_1, y_0 K_2$  of open subgroups  $K_1, K_2 \leq C_H(\varphi)$  such that  $[x, {}_n y] = 1$  for all  $x \in x_0 K_1$  and  $y \in y_0 K_2$ , and therefore for all  $x \in x_0(K_1 \cap K_2)$  and  $y \in y_0(K_1 \cap K_2)$ . Thus,  $C_H(\varphi)$  satisfies a coset identity.

The intersections  $C_i = C_H(\varphi) \cap H_i$  with the terms  $H_i$  of the Zassenhaus  $p$ -filtration for  $H$  form a filtration of  $C_H(\varphi)$ , since obviously  $[C_i, C_j] \leq C_{i+j}$ . Let  $\hat{L}_p(C_H(\varphi))$  be the Lie algebra constructed for  $C_H(\varphi)$  with respect to the filtration  $\{C_i\}$ . Since  $\varphi$  is a coprime automorphism, the fixed-point subalgebra  $C_{L_p(H)}(\varphi)$  is isomorphic to  $\hat{L}_p(C_H(\varphi))$ . We apply a version of the Wilson–Zelmanov result [20, Theorem 1] by which a coset identity on  $C_H(\varphi)$  implies that  $\hat{L}_p(C_H(\varphi))$  satisfies some polynomial identity. Indeed, the proof of Theorem 1 in [20] only uses the filtration property  $[F_i, F_j] \leq F_{i+j}$  for showing that the homogeneous Lie polynomial constructed from a coset identity on a group  $F$  is an identity of the Lie algebra constructed with respect to the filtration  $\{F_i\}$ .

Thus, the fixed-point subalgebra  $C_{L_p(H)}(\varphi)$  satisfies a polynomial identity. Hence the Lie algebra  $L_p(H)$  also satisfies a polynomial identity by the Bahturin–Zaicev Theorem 2.2. □

**Lemma 3.7.** *The Lie algebra  $L_p(H)$  is generated by finitely many elements such that all commutators in these elements are ad-nilpotent.*

*Proof.* By Lemma 3.4 the group  $H$  is generated by finitely many right Engel elements, say,  $h_1, \dots, h_m$ . Their images  $\bar{h}_1, \dots, \bar{h}_m$  in the first factor  $H/H_2$  of the Zassenhaus  $p$ -filtration of  $H$  generate the Lie algebra  $L_p(H)$ . Let  $\bar{c}$  be some commutator in these generators  $\bar{h}_i$ , and let  $c$  be the same group commutator in the elements  $h_i$ . For every  $j$ , since  $[h_j, {}_{k_j} c] = 1$  for some  $k_j = k_j(c)$ , we also have  $[\bar{h}_j, {}_{k_j} \bar{c}] = 0$  in  $L_p(H)$ . We choose a positive integer  $s$  such that  $p^s \geq \max\{k_1, \dots, k_m\}$ . Then  $[\bar{h}_j, {}_{p^s} \bar{c}] = 0$  for all  $j$ . In characteristic  $p$  this implies that

$$(3.2) \quad [z, {}_{p^s} \bar{c}] = 0$$

for any commutator  $z$  in the  $\bar{h}_i$ . This easily follows by induction on the weight of  $z$  from the formula

$$[[u, v], p^s w] = [[u, p^s w], v] + [u, [v, p^s w]],$$

which holds in any Lie algebra of characteristic  $p$ . This formula follows from the Leibnitz formula

$$[[u, v], n w] = \sum_{i=0}^n \binom{n}{i} [[u, i w], [v, n-i w]]$$

(where  $[a, 0b] = a$ ), because the binomial coefficient  $\binom{p^s}{i}$  is divisible by  $p$  unless  $i = 0$  or  $i = p^s$ .

Since any element of  $L_p(H)$  is a linear combination of commutators in the  $\bar{h}_i$ , equation (3.2) by linearity implies that  $\bar{c}$  is ad-nilpotent of index at most  $p^s$ .  $\square$

We can now finish the proof of Proposition 3.5. Lemmas 3.6 and 3.7 show that  $L_p(H)$  satisfies the hypotheses of Zelmanov’s Theorem 2.1, by which  $L_p(H)$  is nilpotent.  $\square$

*Proof of Theorem 3.1.* By Lemma 3.2 the quotient  $G/H$  is nilpotent. Being finitely generated,  $G/H$  is a group of finite rank. Here, the rank of a pro- $p$  group is the supremum of the minimum number of (topological) generators over all open subgroups.

The nilpotency of the Lie algebra  $L_p(H)$  of the finitely generated pro- $p$  group  $H$  established in Proposition 3.5 implies that  $H$  is a  $p$ -adic analytic group. This result goes back to Lazard [11]; see also [15, Corollary D]. By the Lubotzky–Mann theorem [13], being a  $p$ -adic analytic group is equivalent to being a pro- $p$  group of finite rank. Thus, both  $H$  and  $G/H$  have finite rank, and therefore the whole pro- $p$  group  $G$  has finite rank. Hence  $G$  is a  $p$ -adic analytic group and therefore a linear group. By Gruenberg’s theorem [4], right Engel elements of a linear group are contained in the hypercentre. Since  $H$  is generated by right Engel elements, we obtain that  $H$  is contained in the hypercentre of  $G$ , and since  $G/H$  is nilpotent, the whole group  $G$  is hypercentral. Being also finitely generated,  $G$  is nilpotent (see [14, 12.2.4]).  $\square$

#### 4. UNIFORM BOUND FOR THE NILPOTENCY CLASS

In the main Theorem 1.1, we need to prove that if a finitely generated profinite group  $G$  admits a coprime automorphism  $\varphi$  of prime order  $q$ , all of whose fixed points are right Engel in  $G$ , then  $G$  is nilpotent. We already know that  $G$  is pronilpotent and every Sylow  $p$ -subgroup of  $G$  is nilpotent by Theorem 3.1. This would imply the nilpotency of  $G$  if we had a uniform bound for the nilpotency class of Sylow  $p$ -subgroups independent of  $p$ . However, the nilpotency class furnished by the proof of Theorem 3.1 depends on  $p$ .

In this section we prove that for large enough primes  $p$  the nilpotency classes of Sylow  $p$ -subgroups of  $G$  are uniformly bounded above in terms of certain parameters of the group  $G$ . Together with bounds depending on  $p$  given by Theorem 3.1, this will complete the proof of the nilpotency of  $G$ . In the proof, we do not specify the conditions on  $p$  beforehand, but proceed with our arguments noting along the way that our conclusions hold for all large enough primes  $p$ .

One of the aforementioned parameters is the finite number of generators of  $G$ , say,  $d$ . Clearly, every Sylow  $p$ -subgroup of  $G$  can also be generated by  $d$  elements, being a homomorphic image of  $G$  by the Cartesian product of all other Sylow subgroups.

**Lemma 4.1.** *There are positive integers  $n$  and  $N_1$  such that for every  $p > N_1$  all fixed points of  $\varphi$  in the Sylow  $p$ -subgroup  $P$  of  $G$  are right  $n$ -Engel elements of  $P$ .*

*Proof.* In the group  $C_G(\varphi) \times G$ , the subsets

$$E_i = \{(x, y) \in C_G(\varphi) \times G \mid [x, {}_i y] = 1\}$$

are closed in the product topology. By hypothesis,

$$\bigcup_i E_i = C_G(\varphi) \times G.$$

Hence, by the Baire category theorem [8, Theorem 34], one of these subsets  $E_n$  contains an open subset of  $C_G(\varphi) \times G$ , which means that there are cosets  $x_0K$  and  $y_0L$  of open subgroups  $K \leq C_G(\varphi)$  and  $L \leq G$  such that  $[x, {}_n y] = 1$  for all  $x \in x_0K$  and  $y \in y_0L$ . Since the indices  $|C_G(\varphi) : K|$  and  $|G : L|$  are finite, for all large enough primes  $p > N_1$  the Sylow  $p$ -subgroups of  $C_G(\varphi)$  and  $G$  are contained in the subgroups  $K$  and  $L$ , respectively. Then for every prime  $p > N_1$ , in the Sylow  $p$ -subgroup  $P$  the centralizer  $C_P(\varphi)$  consists of right  $n$ -Engel elements of  $P$ .  $\square$

**Lemma 4.2.** *There are positive integers  $c$  and  $N_2$  such that for every  $p > N_2$  in the Sylow  $p$ -subgroup  $P$  the fixed-point subgroup  $C_P(\varphi)$  is nilpotent of class  $c$ .*

*Proof.* By Lemma 4.1, for  $p > N_1$  in the Sylow  $p$ -subgroup  $P$  the subgroup  $C_P(\varphi)$  is an  $n$ -Engel group. By a theorem of Burns and Medvedev [3], then  $C_P(\varphi)$  has a normal subgroup  $N_p$  of exponent  $e(n)$  such that the quotient  $C_P(\varphi)/N_p$  is nilpotent of class  $c(n)$  for some numbers  $e(n)$  and  $c(n)$  depending only on  $n$ . Clearly,  $N_p = 1$  for all large enough primes  $p > N_2 \geq N_1$ . Thus, for every prime  $p > N_2$  the subgroup  $C_P(\varphi)$  is nilpotent of class  $c = c(n)$ .  $\square$

The following proposition will complete the proof of the main Theorem 1.1 in view of Lemmas 4.1 and 4.2.

**Proposition 4.3.** *There are functions  $N_3(d, q, n, c)$  and  $f(d, q, n, c)$  of four positive integer variables  $d, q, n, c$  with the following property. Let  $p$  be a prime, and suppose that  $P$  is a  $d$ -generated pro- $p$  group admitting an automorphism  $\varphi$  of prime order  $q \neq p$  such that  $C_P(\varphi)$  is nilpotent of class  $c$  and consists of right  $n$ -Engel elements of  $P$ . If  $p > N_3(d, q, n, c)$ , then the group  $P$  is nilpotent of class at most  $f(d, q, n, c)$ .*

*Proof.* It is sufficient to obtain a bound for the nilpotency class in terms of  $d, q, n, c$  for every finite quotient  $T$  of  $P$  by a  $\varphi$ -invariant open normal subgroup. Consider the associated Lie ring  $L(T)$  constructed with respect to the filtration consisting of the terms  $\gamma_i(T)$  of the lower central series of  $T$ :

$$L(T) = \bigoplus \gamma_i(T)/\gamma_{i+1}(T).$$

As is well known, this Lie ring is nilpotent of exactly the same nilpotency class as  $T$  (see, for example, [9, Theorem 6.9]). Therefore it is sufficient to obtain a required bound for the nilpotency class of  $L(T)$ . We set  $L = L(T)$  for brevity. Let  $\tilde{L} = L \otimes_{\mathbb{Z}} \mathbb{Z}[\omega]$  be the Lie ring obtained by extending the ground ring by a primitive  $q$ -th root of unity  $\omega$ . We regard  $L$  as  $L \otimes 1$  embedded in  $\tilde{L}$ . The automorphism

of  $L$  and of  $\tilde{L}$  induced by  $\varphi$  is denoted by the same letter. Since the order of the automorphism  $\varphi$  is coprime to the orders of elements of the additive group of  $\tilde{L}$ , which is a  $p$ -group, we have the decomposition into analogues of eigenspaces

$$\tilde{L} = \bigoplus_{i=0}^{q-1} L_j, \quad \text{where } L_j = \{x \in \tilde{L} \mid x^\varphi = \omega^j x\}.$$

For clarity we call the additive subgroups  $L_j$  *eigenspaces* and their elements *eigenvectors*. This decomposition can also be viewed as a  $(\mathbb{Z}/q\mathbb{Z})$ -grading of  $\tilde{L}$ , since

$$[L_i, L_j] \subseteq L_{i+j \pmod{q}}.$$

Note that  $L_0 = C_L(\varphi) \otimes_{\mathbb{Z}} \mathbb{Z}[\omega]$ .

**Lemma 4.4.** *The fixed-point subring  $C_{\tilde{L}}(\varphi)$  is nilpotent of class at most  $c$ .*

*Proof.* Since  $\varphi$  is a coprime automorphism of  $T$ , we have

$$C_L(\varphi) = \bigoplus_i (C_T(\varphi) \cap \gamma_i(T))\gamma_{i+1}(T)/\gamma_{i+1}(T).$$

Since the fixed-point subgroup  $C_T(\varphi)$  is nilpotent of class  $c$ , the definition of the Lie products implies that the same is true for  $C_L(\varphi)$  and therefore also for  $C_{\tilde{L}}(\varphi) = C_L(\varphi) \otimes_{\mathbb{Z}} \mathbb{Z}[\omega]$ . □

Our main aim is to enable an application to  $\tilde{L}$  of the effective version of Zelmanov’s theorem given by Proposition 2.3. For that we need a  $\varphi$ -invariant set of generators of  $\tilde{L}$  such that all commutators in these generators are ad-nilpotent of bounded index.

Let  $L_{(k)} = \gamma_k(T)/\gamma_{k+1}(T)$  denote the homogeneous component of weight  $k$  of the Lie ring  $L$ , and let  $\tilde{L}_{(k)} = L_{(k)} \otimes_{\mathbb{Z}} \mathbb{Z}[\omega]$ . For clarity we say that elements of  $\tilde{L}_{(k)}$  or  $L_{(k)}$  are *homogeneous*. The component  $L_{(1)}$  generates the Lie ring  $L$ , and  $\tilde{L}_{(1)}$  generates  $\tilde{L}$ . If elements  $t_1, \dots, t_d$  generate the group  $T$ , then their images  $\bar{t}_1, \dots, \bar{t}_d$  in  $L_{(1)} = T/\gamma_2(T)$  generate the Lie ring  $L$ , as well as  $\tilde{L}$  (over the extended ground ring). Writing  $\bar{t}_i = \sum_{j=0}^{q-1} t_{ij}$ , where  $t_{ij} \in \tilde{L}_{(1)} \cap L_j$  we obtain a  $\varphi$ -invariant set of generators of  $\tilde{L}$ :

$$\{\omega^k t_{ij} \mid i = 1, \dots, d; j = 0, \dots, q - 1; k = 0, \dots, q - 1\}.$$

We claim that for  $p > n$  all commutators in these generators are ad-nilpotent of index bounded in terms of  $q, n, c$ .

We set for brevity  $\tilde{L}_{(v)k} = \tilde{L}_{(v)} \cap L_k$  for any weight  $v$ . A commutator of weight  $v$  in the eigenvectors  $t_{ij}$  is an eigenvector belonging to  $\tilde{L}_{(v)k}$ , where  $k$  is the modulo  $q$  sum of the second indices of the  $t_{ij}$  involved. We actually prove that for  $p > n$  any homogeneous eigenvector  $l_k \in \tilde{L}_{(v)k}$  is ad-nilpotent of index  $s$  bounded in terms of  $q, n, c$ . It is clearly sufficient to show that  $[x_j, {}_s l_k] = 0$  for any homogeneous eigenvector  $x_j \in \tilde{L}_{(u)j}$  for any weights  $u, v$  and any indices  $j, k \in \{0, 1, \dots, q - 1\}$ . (Here we use indices  $j, k$  for elements  $x_j, l_k$  only to indicate the eigenspaces they belong to.) First we consider the case where  $j = 0$ .

**Lemma 4.5.** *If  $p > n$ , then for any weights  $u, v$ , for any eigenvector  $x_0 \in \tilde{L}_{(u)0}$ , and for any homogeneous element  $l \in \tilde{L}_{(v)}$  we have  $[x_0, {}_n l] = 0$ .*

*Proof.* Since  $\varphi$  is an automorphism of coprime order, for  $x_0 \in \tilde{L}_{(u)0}$  there are elements  $y_i \in C_T(\varphi) \cap \gamma_u(T)$  such that  $x_0 = \sum_{i=0}^{q-2} \omega^i \bar{y}_i$ , where  $\bar{y}_i$  is the image of  $y_i$  in  $\gamma_u(T)/\gamma_{u+1}(T)$  (here the indices of the  $y_i$  are used for numbering). For any  $\bar{h} \in L_{(v)}$ , there is an element  $h \in T \cap \gamma_v(T)$  such that  $\bar{h}$  is the image of  $h$  in  $L_{(v)} = \gamma_v(T)/\gamma_{v+1}(T)$ . Since  $[y_i, {}_n h] = 1$  in the group  $T$  by the hypothesis of Proposition 4.3, we have  $[\bar{y}_i, {}_n \bar{h}] = 0$  in  $L$  for every  $i$ . Hence, by linearity,

$$(4.1) \quad [x_0, {}_n \bar{h}] = 0$$

in  $\tilde{L}$ . Note, however, that  $\tilde{L}_{(v)}$  does not consist only of  $\mathbb{Z}[\omega]$ -multiples of elements of  $L_{(v)}$ . Nevertheless, (4.1) looks like the  $n$ -Engel identity, which implies its linearization, which in turn survives extension of the ground ring and then implies the required property due to the condition  $p > n$  making  $n!$  an invertible element of the ground ring. However, we cannot simply make a reference to these well-known facts, since this is not exactly an identity, so we reproduce these familiar arguments in our specific situation (jumping over one of the steps).

We substitute  $a_1 + \dots + a_n$  for  $\bar{h}$  in (4.1) with arbitrary homogeneous elements  $a_i \in L_{(v)}$  (the indices of the  $a_i$  are used for numbering). Thus,

$$[x_0, {}_n(a_1 + \dots + a_n)] = 0$$

for any elements  $a_i \in L_{(v)}$ , some of which may also be equal to one another. After expanding all brackets, we obtain the equation

$$(4.2) \quad 0 = [x_0, {}_n(a_1 + \dots + a_n)] = \sum_{\substack{i_1 \geq 0, \dots, i_n \geq 0 \\ i_1 + \dots + i_n = n}} \varkappa_{i_1, \dots, i_n},$$

where  $\varkappa_{i_1, \dots, i_n}$  denotes the sum of all commutators of degree  $i_j$  in  $a_j$ . Replacing  $a_1$  with 0 (only this formal occurrence, keeping intact all other  $a_i$ 's even if some are equal to  $a_1$ ) shows that

$$0 = \sum_{\substack{i_1=0, i_2 \geq 0, \dots, i_n \geq 0 \\ i_1 + \dots + i_n = n}} \varkappa_{i_1, \dots, i_n}.$$

Hence we can remove from the right-hand side of (4.2) all terms not involving  $a_1$  as a formal entry (keeping the other  $a_i$ 's even if some are equal to  $a_1$ ). We obtain

$$0 = [x_0, {}_n(a_1 + \dots + a_n)] = \sum_{\substack{i_1 \geq 1, i_2 \geq 0, \dots, i_n \geq 0 \\ i_1 + \dots + i_n = n}} \varkappa_{i_1, \dots, i_n}.$$

Then we do the same with  $a_2$  for the resulting equation, and so on, consecutively with all the  $a_i$ 's. As a result we obtain

$$0 = \sum_{\substack{i_1 \geq 1, \dots, i_n \geq 1 \\ i_1 + \dots + i_n = n}} \varkappa_{i_1, \dots, i_n} = \varkappa_{1, \dots, 1},$$

that is,

$$(4.3) \quad 0 = \sum_{\pi \in S_n} [x_0, a_{\pi(1)}, \dots, a_{\pi(n)}],$$

where the right-hand side is the desired linearization. Every element  $l \in \tilde{L}_{(v)}$  can be written as a linear combination  $l = m_0 + \omega m_1 + \omega^2 m_2 + \dots + \omega^{q-2} m_{q-2}$ , where

$m_i \in L_{(v)}$ . Then

$$\begin{aligned}
 [x_0, n l] &= [x_0, n(m_0 + \omega m_1 + \omega^2 m_2 + \dots + \omega^{q-2} m_{q-2})] \\
 &= \sum_{i=0}^{n(q-2)} \omega^i \sum_{j_1+2j_2+\dots+(q-2)j_{q-2}=i} \lambda_{j_0, j_1, \dots, j_{q-2}},
 \end{aligned}$$

where  $\lambda_{j_0, j_1, \dots, j_{q-2}}$  denotes the sum of all commutators in the expansion of the left-hand side with weight  $j_s$  in  $m_s$ . But each of these sums is clearly symmetric and therefore is equal to 0 as a consequence of (4.3), where, if an element  $a_i$  is required to be repeated  $n_i$  times, the coefficient  $n_i!$  appears, which is invertible in the ground ring, since  $n_i < p$  and the additive group is a  $p$ -group. The lemma is proved.  $\square$

**Lemma 4.6.** *If  $p > n$ , then for any  $v$  and  $k$  any homogeneous eigenvector  $l_k \in \tilde{L}_{(v)k}$  is ad-nilpotent of index bounded in terms of  $q, n, c$ .*

*Proof.* First consider the case  $k = 0$ . Then  $l_0 = \sum_{i=0}^{q-2} \omega^i \bar{y}_i$ , where  $\bar{y}_i$  is the image of an element  $y_i \in C_T(\varphi) \cap \gamma_v(T)$  in  $\gamma_v(T)/\gamma_{v+1}(T)$  (the indices of the  $y_i$  are used for numbering). For each  $i$ , since  $y_i^{-1}$  is a right  $n$ -Engel element of  $T$  by hypothesis,  $y_i$  is a left  $(n + 1)$ -Engel element by a result of Heineken [6] (see also [14, 12.3.1]). For any homogeneous element  $\bar{h} \in L_{(u)}$  there is an element  $h \in T \cap \gamma_u(T)$  such that  $\bar{h}$  is the image of  $h$  in  $\gamma_u(T)/\gamma_{u+1}(T)$ . Since  $[h, n_{+1}y_i] = 1$  in the group  $T$ , we have  $[\bar{h}, n_{+1}\bar{y}_i] = 0$  in  $L$  for every  $i$ . Hence, by linearity, each  $\bar{y}_i$  is ad-nilpotent in  $L$  of index at most  $n + 1$ . Let  $M$  be the subring of  $L$  generated by  $\bar{y}_0, \bar{y}_1, \dots, \bar{y}_{q-2}$ . Since  $M \leq C_L(\varphi)$ , the subring  $M$  is nilpotent of class at most  $c$  by Lemma 4.4. We can now apply Lemma 2.4, by which

$$[L, \underbrace{M, M, \dots, M}_\varepsilon] = 0$$

for some  $\varepsilon = \varepsilon(q-1, n+1, c)$  bounded in terms of  $q, n, c$ . This equation remains valid after extension of the ground ring. In particular,  $l_0 = \sum_{i=0}^{q-2} \omega^i \bar{y}_i$  is ad-nilpotent in  $\tilde{L}$  of index at most  $\varepsilon = \varepsilon(q-1, n+1, c)$ , as required.

Now suppose that  $k \neq 0$ . For a homogeneous eigenvector  $x_j \in \tilde{L}_{(u)j}$ , the commutator

$$(4.4) \quad [x_j, q+n-1 l_k] = [x_j, \underbrace{l_k, \dots, l_k}_s, l_k, \dots, l_k]$$

has an initial segment of length  $s + 1 \leq q$  that is a homogeneous eigenvector  $x_0 = [x_j, s l_k] \in \tilde{L}_{(w)0}$  (for some weight  $w$ ). Indeed, the congruence  $j + sk \equiv 0 \pmod{q}$  has a solution  $s \in \{0, 1, \dots, q-1\}$  since  $k \not\equiv 0 \pmod{q}$ . There remain at least  $n$  further entries of  $l_k$  in (4.4), so that we have a subcommutator of the form  $[x_0, n l_k]$ , which is equal to 0 by Lemma 4.5. Thus, by linearity,  $l_k$  is ad-nilpotent of index at most  $q + n - 1$ .  $\square$

We now finish the proof of Proposition 4.3. By Lemma 4.6, for  $p > n$  every commutator in the generators  $t_{ij}$  of the Lie ring  $\tilde{L}$  is ad-nilpotent of index bounded in terms of  $q, n, c$ . The same is true for the generators in the  $\varphi$ -invariant set

$$\{\omega^k t_{ij} \mid i = 1, \dots, d; j = 0, \dots, q-1; k = 0, \dots, q-1\},$$

which consists of  $q^2d$  elements. The fixed-point subring  $C_{\tilde{L}}(\varphi)$  is nilpotent of class at most  $c$  by Lemma 4.4. Thus, for  $p > n$  the Lie ring  $\tilde{L}$  and its group of automorphisms  $\langle \varphi \rangle$  satisfy the hypotheses of Proposition 2.3. By this proposition, there exist positive integers  $e$  and  $r$  depending only on  $d, q, n, c$  such that  $e\gamma_r(\tilde{L}) = 0$ . The additive group of  $\tilde{L}$  is a  $p$ -group. Therefore, if  $p > e$ , then  $e$  is invertible in the ground ring, so that we obtain  $\gamma_r(L) = 0$ . It remains to put  $N_3(d, q, n, c) = \max\{n, e\}$  and  $f(d, q, n, c) = r - 1$ .

We thus have proved that for  $p > N_3(d, q, n, c)$  every finite quotient of  $P$  by a  $\varphi$ -invariant normal open subgroup is nilpotent of class at most  $f(d, q, n, c)$ . Therefore  $P$  is nilpotent of class at most  $f(d, q, n, c)$  if  $p > N_3(d, q, n, c)$ .  $\square$

We finally combine all the results in the proof of the main theorem.

*Proof of Theorem 1.1.* Recall that  $G$  is a profinite group admitting a coprime automorphism  $\varphi$  of prime order  $q$  all of whose fixed points are right Engel in  $G$ ; we need to prove that  $G$  is locally nilpotent. Any finite set  $S \subseteq G$  is contained in the  $\varphi$ -invariant finite set  $S^{(\varphi)} = \{s^{\varphi^k} \mid s \in S, k = 0, 1, \dots, q-1\}$ . Therefore we can assume that the group  $G$  is finitely generated, say, by  $d$  elements, and then need to prove that  $G$  is nilpotent. As noted in the Introduction, the group  $G$  is pronilpotent, so we only need to prove that all Sylow  $p$ -subgroups of  $G$  are nilpotent of class bounded by some number independent of  $p$ .

Let  $n$  and  $N_1$  be the numbers given by Lemma 4.1 and let  $c$  and  $N_2$  be the numbers given by Lemma 4.2. Further, let  $N_3(d, q, n, c)$  be the number given by Proposition 4.3. Then for every prime  $p > \max\{N_1, N_2, N_3(d, q, n, c)\}$  the Sylow  $p$ -subgroup of  $G$  is nilpotent of class at most  $f(d, q, n, c)$  for the function given by Proposition 4.3. Since every Sylow  $p$ -subgroup is nilpotent by Theorem 3.1, we obtain a required uniform bound for the nilpotency classes of Sylow  $p$ -subgroups independent of  $p$ .  $\square$

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