TEACHING CHALLENGES IN FOSTERING STUDENTS' DISCURSIVE SHIFTS TO EXPLORATIVE PARTICIPATION IN MATHEMATICS THROUGH REMEDIAL INTERVENTIONS: THE CASE OF LINEAR EQUATIONS

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Remedial interventions, which are typically provided outside regular school hours to students who are in need of additional support, can play a significant role in improving low-performing students' mathematical attainment. In this paper, we present excerpts from a technology-based remedial intervention of two students guided by a tutor. The intervention concerned equivalence of algebraic expressions using the 'balancing' approach with applets developed in Desmos and GeoGebra. We analyse the pattern of their interactions through the lens of commognition and 'attractor states'. We conclude by arguing that students' prior school experiences can challenge the tutor's attempts to foster shifts to explorative participation to mathematical discourse.

Keywords: Digital technology, DynaMat project, mathematics learning difficulties, remedial interventions, secondary school mathematics

INTRODUCTION, RATIONALE AND CONTEXTUALIZATION OF THE RESEARCH

The PISA 2022 results (OECD, 2023) reveal a significant decline in secondary school students' mathematics performance, with an average drop of 15 points, equivalent to three-quarters of a school year, and approximately 30% of students failing to reach proficiency Level 2. In Italy, this figure stands at 29.6%. Supporting this trend, the 2022 Italian national assessment indicates that 82.7% of students in vocational upper secondary schools are classified as *low-performing* in mathematics. This alarming scenario underscores the urgent need for educational actions, such as *remedial interventions*, at the secondary school level, along with the reinforcement of effective mathematics instruction from the early grades.

In the Italian educational context, students with low achievement or mathematical learning difficulties (MLD), identified by their teachers or through clinical certification, are included in mainstream classrooms and constitute up to 20% of the class. *Remedial interventions*, when provided, typically take place outside regular school hours and are often led by teachers who are not the students' regular classroom teachers. Traditional interventions mainly focus on reteaching content and correcting surface-level errors, and re-teaching procedures or definitions of mathematics that have already been presented in class. This is consistent with international literature that suggests using explicit instruction, graphic organizers and mnemonics; mathematics contents from a lower-class level than the students' class level (Nelson et al., 2022). However, it is unclear how these practices overcome the limitations of the unengaging standard instruction.

Technology can play a key role in enhancing mathematics education by offering interactive tools that provide immediate feedback, personalized support, and multiple representations of mathematical concepts. Although digital technology is widely used in mathematics interventions, its application in remedial interventions is often limited to reinforcing procedural skills or providing explicit instruction through video modeling (Herfort et al., 2023). Yet, research shows that well-designed digital resources can be especially

supportive for struggling students and have the potential to go well beyond procedural reinforcement (Baccaglini-Frank, 2021).

The mathematics content that we focus on is that of linear equations, which is a topic well-known for causing difficulties to students. Research has shown that the arithmetic ways of thinking that students relied on in primary school do not equip them to handle the alphanumeric symbols representing unknowns and variables in secondary school algebra (e.g., Kieran, 2022). Frequently students misinterpret variables, symbols, and the equals sign, which is viewed by many students as an instruction to compute, rather than as a statement of equivalence (Knuth et al., 2006).

DESIGN OF THE REMEDIAL ACTIVITIES IN A COMPLEX DYNAMIC SYSTEM'S PERSPECTIVE AND THROUGH THE LENS OF COMMOGNITION

To overcome such difficulties, the DynaMat project designed a sequence of activities, some of which make use of a virtual scale[6], a dynamic explorative object[7] (DEO) available on the DESMOS platform. We call it the Two-Pans-Scale (TPS) DEO[8]. Indeed, this model effectively conveys the principle of equivalence through the idea of balance, reflecting the relational meaning of the equal sign. With appropriate tasks and prompts, it can help promote equation-solving strategies based on the principle of "doing the same thing on both sides," rather than relying on the mechanical "change side, change sign" approach, used in traditional Italian classrooms. Figure 1 shows a screenshot of the TPS DEO as it is set in the activity we will be considering in this paper.

We take a commognitive perspective, and look at mathematics as a particular kind of discourse, with specific words, visual mediators, routines and narratives (Sfard, 2008). The TPS DEO can be considered a special kind of visual mediator[9]. We will now introduce routines, using the constructs of task situation and precedent, which we will be using in this paper. Following Lavie and colleagues (2019) we distinguish between a *task situation* (TS), a context in which an individual feels obliged to act, for example the one described in Figure 1a, and a *task*, that is the student's personal interpretation of a task situation, which may differ from that of the teacher or of another student. Such interpretation is shaped by *precedents*, previous experiences that the student perceives as sufficiently similar to the current context to justify applying the same procedure. A *routine* consists of a task (interpreted) paired with a *procedure*, which is the set of actions the person feels compelled to do. For a same task, different people might carry out different procedures (and hence routines).

Especially in the context of MLD, another important distinction is between ritualistic and explorative participation, two forms that lie at opposite ends of a continuum. Students' participation in mathematical discourse is *ritualistic* when it focuses on carrying out memorised procedural steps, with attention placed on the execution itself rather than its outcome. For example, this can be witnessed when a student manipulates the symbols in an equation, following steps of a memorized procedure, for the sake of finishing it. Participation can be *explorative* when routines are carried out with a focus on the mathematical objects they produce and meaningful narratives about it (Sfard, 2008; Lavie et al., 2019). For example, (an expert) student, who has conquered the mathematical object "equation" might transition from the balanced scale to a

[6] Benefits and limitations of the balance model are widely discussed in the literature (see Otten et al., 2019).

[7] "An interactive and dynamic computer-based model or tool that capitalizes on visualization and is developed to explore a mathematical concept or conjecture, or real-world situation", (Muller et al., 2009, p. 64)

[8] The activities we designed can be accessed here https://www.carme.center/risorse-dynamat

[9] Previously the construct of Dynamic Interactive Mediator (DIM) was coined to describe this kind of visual mediator that gives feedback (Baccaglini-Frank, 2021).

different *realization*, writing an equality between two algebraic expressions like "$4 + x + 5 + 1 + 8 = 3 + x + x + 4 + 2 + 2$". The process of recognizing these as realizations of the same thing, known as *saming*, is a key step in developing explorative participation. Then the student might determine the weight of a red ball by solving the equation symbolically (reaching $x = 7$), and reinterpret the solution (7) as such a weight.

Persistent difficulties in mathematics, including many forms of MLD that lead to failure to learn mathematics, can be seen discursively as patterns that keep the student's discourse stuck in a ritualistic form of participation. Learning can be seen as "deritualizing" and shifting to explorative discursive participation. Two of the key processes involved in such a shift are: *substantiability* (a shift from the process to the outcome, for example by justifying the performed routine) and *objectification* (talking about mathematical objects as if they exist independently from actions). Commognition and the analytic tools it offers allow for fine-grained detail analyses of students' discourse. However, to also synergistically capture a more holistic perspective, recently Heyd-Metzuyanim (2025) has discussed how such cycles of failure can be captured combining such theoretical tools with a Complex Dynamic Systems (CDS) perspective. She introduces the notions of attractor state, feedback loops, and task cycles, among others and explains how CDSs tend to stabilize into *attractor states,* "patterns of interaction between the elements of the CDS that reflect an equilibrium of influences between internal and external dynamics" (Heyd-Metzuyanim, 2025, p. 3). *Feedback events* and *loops* that impact on the behaviors of the participants involved in the system, stabilizing the system and making it resistant to change until significant *perturbations* occur, that can lead to the shift into a new attractor state.

Using this perspective to design and analyse remedial interventions, we can consider a CDS constituted by agents (in our case a pair of students and a tutor) and technology-enhanced activities. Heyd-Metzuyanim (2025) has introduced the notion of *task-cycle* to model these agents' participation in mathematics. A task-cycle starts with the agents engaging in a task situation attempting to achieve the task (through a procedure or set of sub-routines), and it ends when the agents performing the task determine (collectively or each on their own) that the task is completed. We can design activities with DEOs and tutor prompts with the aim of promoting significant perturbations of task-cycles involving undesirable attractor states, that foster the system's shift to a more desirable attractor state. In the methods section we will elaborate on how we designed the potential perturbations.

RESEARCH QUESTIONS

Taking a CDS perspective and Commognition as a lens for analyzing students' participation discursively, we focus on perturbations of attractor states in the CDS formed by interactions between the students, a task situation (in which a DEO is involved) and the tutor during our remedial intervention; and we ask (1) what are (some) identifiable effects of such perturbations? (2) what are (some) key aspects that seem to be influencing such effects?

METHODS

Within the Italian research project *Dynamic Math for Inclusive Education* (DynaMat, PRIN 2020BKWEXR), we recruited 10th-grade students in Italian high schools with a history of persistent difficulties in mathematics, as identified by themselves or their teachers, and offered them a 10-hour remedial intervention. Data collection occurred at the Center for Advanced Research in Mathematics Education (CARME) in Pistoia, in a specially equipped room featuring advanced technology for the in-depth study of teaching and learning processes. We focus here on Aldo and Giulio (pseudonyms), who were 2 of the 23 recruited participants. They attended weekly afternoon sessions lasting 2 hours, between October and November 2023. All sessions were fully recorded using audio-video and screen recording technology. The first session for each of them was an individual diagnostic interview to assess their forms of participation to mathematics discourse about

equations. Equations seemed to be familiar to Aldo and Giulio, yet their approach to the TSs featuring equations was largely guided by memorized procedures learned at school, resulting in mostly ritualistic participation, and to non-canonical (incorrect or meaningless) narratives.

The sequence of activities that was proposed to Aldo and Giulio by a tutor, one of the DynaMat researchers, started with the TPS DEO introduced earlier. As part of the design, some of the tutor's prompts aimed at bringing symbolic equations (as learned in school) into the students' meaningful discourse about scales. For example, the tutor's prompts included requests like: *"Before you act on the balance, write in your notes what the balance looks like at the beginning, so we don't forget it"* or "*write out the balance*". In response, for example, Aldo and Giulio transcribed the numbers from the two pans separated by a semicolon and used an "and" between the two groups of numbers. They also samed the balanced scale to the equal sign, stating: "*because if it is balanced [...] we're okay with the equal sign in the middle*"

In this paper we will analyze Task Situation 5 (Figure 1a), as explored by Aldo and Giulio, the first instance in which unknown weights (red balls) were introduced.

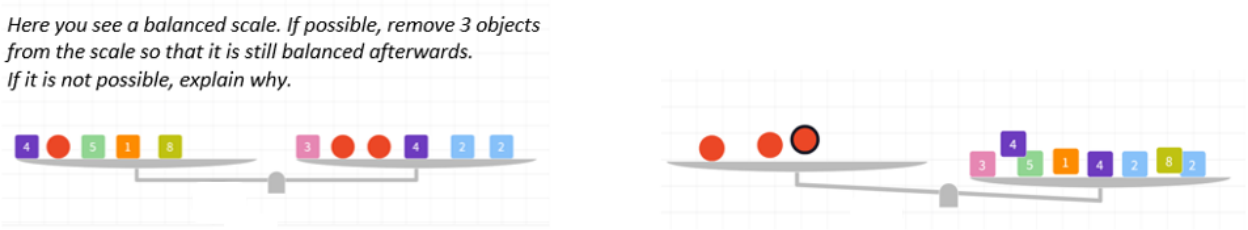

Figure 1: a) TPS DEO of TS 5 in the initial state; b) The scale resulting from the direct application of the "change sides, change sign" rule in the second excerpts.

We present two excerpts from this session. The first excerpt responds to the initial task situation: *"If possible, remove three objects from the scale so that it is still balanced afterwards"* and illustrates a task-cycle in which the tutor promotes explorative participation, and the system appears to stabilize in a desirable attractor state, in line with the design of the remedial intervention. The second excerpt, in response to the prompt *"Can we find out how much the little ball weighs?"*, shows a ritual task-cycle that resists perturbations.

DATA ANALYSIS AND RESULTS

The commognitive analysis of the first excerpt highlights Aldo and Giulio's predominantly explorative participation. The students manage multiple realizations of the mathematical object "unknown": the word *unknown*, the red little balls, and x. In their discourse these different realizations are samed and used as subjects in meaningful narratives consistent with canonical mathematical discourse. The term *unknown* is explicitly used as a synonym for the red balls with an unknown value: "two little balls, I mean, two unknowns" [Figure 2, line 36]. A bit before this excerpt happened, the students had introduced x as a new visual mediator and written the equation: $4 + x + 5 + 1 + 8 = 3 + x + x + 4 + 2 + 2$. Their use of x and the term unknown to describe the red ball is consistent with mathematical discourse: "we don't know the value of the little red ball" [13], "x is an unknown, and in this case, the little ball…" [20]. Moreover, when the tutor asks, "Why are we sure that two unknowns would stay balanced?" [38], the students respond that, even though the value is unknown, the two are equal to each other, showing substantiability.

From a complexity perspective, the discursive task-cycle illustrated in Figure 2 exemplifies momentary interactions reflecting Aldo and Giulio's explorative participation. The tutor presents an exploration-requiring TS, to which the students respond by engaging with the DEO and maintaining active control over the balancing scale. As intended by the design, their participation appears to stabilize in the desired

explorative state of “A&G as explorer with DEO”. Similarly, all prior activities up to this point suggest a willingness on their part to engage more exploratively, apparently stabilising this attractor state.

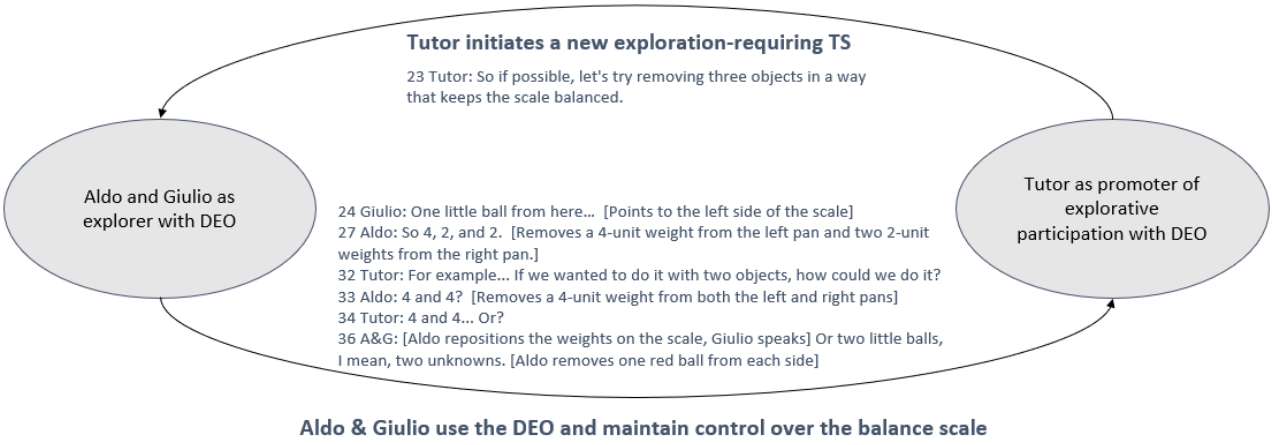

Figure 2. Task-cycles depicting explorative participation. Legend: States (positions) signified by ovals; processes signalled by black continuous arrows.

In the second excerpt, in response to the question “Can we find out how much the little ball weighs?” [Figure 3, line 72], Aldo and Giulio recall a school-taught procedure for solving linear equations, which leads them into a strongly ritualistic form of participation. Giulio suggests applying the ‘change sides, change sign’ rule directly to the scale, “I mean, you move all of these to one side” [Figure 3, line 78], but then shifts to working with written algebraic equations. The task (interpreted by the students) is that of solving the algebraic equation following a memorized procedure. The steps are described impersonally in third person, using expressions like “you do,” “you move,” and “it becomes”, with the focus entirely on the execution of the process. The procedure followed is non-canonical, suggesting that both students have reverted to a ritualistic form of participation.

The discursive task-cycle illustrated in Figure 3 captures momentary interactions that reinforce Aldo and Giulio’s ritualistic participation. The tutor, as a promoter of explorative participation with the DEO, introduces the new exploration-requiring TS. However, Giulio responds by recalling the standard school-taught procedure for solving linear equations. In an effort to perturb the ritual cycle by trying to shift back to the students’ meaningful discourse about the DEO, the tutor asks: “Does this make sense on the scale? What happens if you move all these to one side?” [Figure 3, line 81]. The tutor hopes that such a perturbation will help steer the system back toward the attractor state with Aldo and Giulio as explorers with the DEO, which had emerged in earlier task situations with the balance scale. However, the attempt is unsuccessful. Giulio ignores the tutor and, supported by Aldo, remains focused solely on completing the memorized procedure, verbally describing each step (Figure 3, line 82-88) without checking that the scale stays balanced, an aspect they had previously always attended to. The strong pull of the procedural routine acts as a stable, self-reinforcing pattern, that stabilizes the attractor state of Aldo and Giulio as students who rely on memorized procedures.

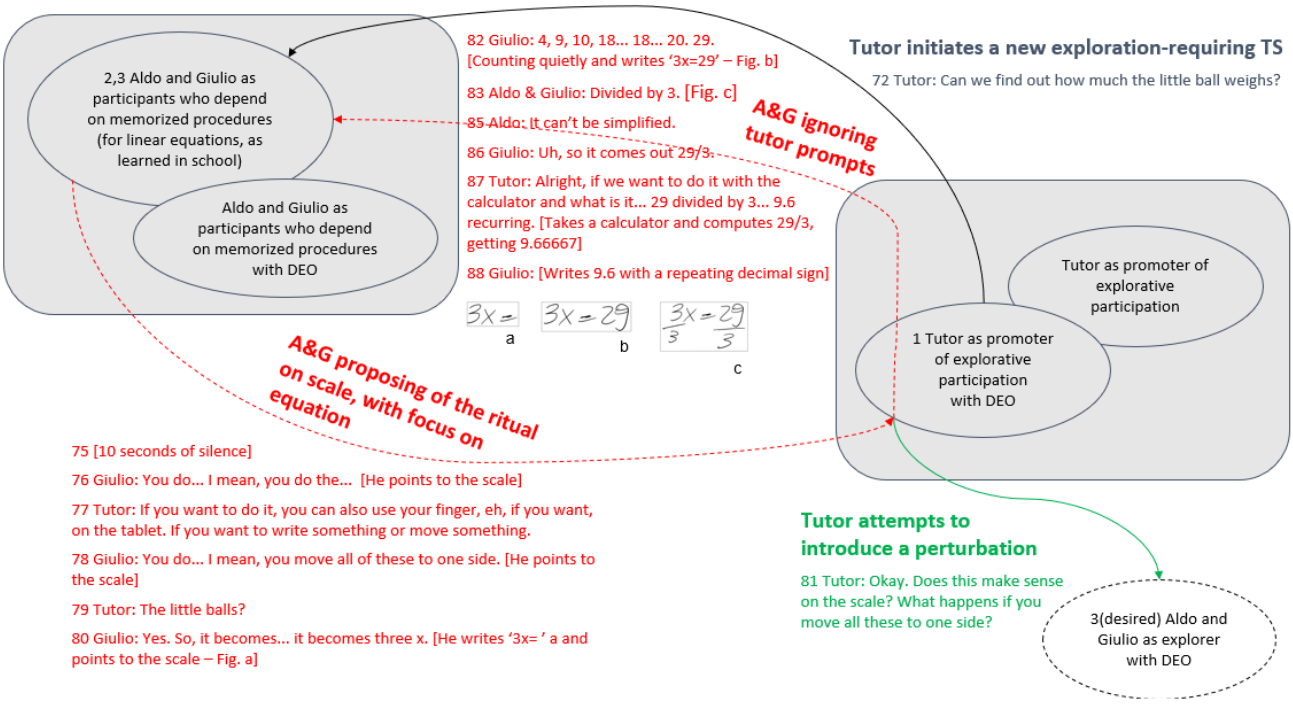

Figure 3. First ritual task-cycle that resists perturbations. Legend: feedback events signalled by red dashed arrow; perturbations signified by green arrows.

At this point [89–104], the tutor asks the students to explain what they had done, and their discourse is once again about the algebraic equation, not the balanced scale. In response to the question, "Why did you do it like that...?" [99], Aldo replied "it's an equation" [101], which is probably, together with "x", an important part of the precedent he is recalling and following as he enacts his "equation" ritual. This supports our interpretation of the students being sucked back into an undesirable (for the tutor) attractor state of participants who depend on memorized procedures.

Since their procedure is non-canonical (they do not change the sign when changing sides) and lacking substantiability, the tutor encourages revisiting the task and restarting the cycle (Figure 4) by introducing a new exploration-requiring TS: "But what if we wanted to do it on the scale?" [Figure 4, line 105]. However, Giulio immediately responds, "I mean, still like this" [points to the equation], I mean, you would move… [points to the scale]" "all the numbers to one side and the balls to the other" [Figure 4, line 110, 112], once again recalling the memorized procedure and suggesting its application to the balance scale, a suggestion that Aldo promptly implements on the DEO. The tutor makes another attempt to introduce a perturbation, "But the scale is not balanced" [Figure 4, line 116], aimed at shifting the students toward the pattern of "explorers with the DEO." However, this attempt also fails, as it is again disregarded by both students, who remain strongly influenced by the pull of the memorized procedure and the perceived need to complete it. This acts as an additional feedback loop, once again stabilizing, and further reinforcing, Aldo and Giulio's being "participants who depend on memorized procedures."

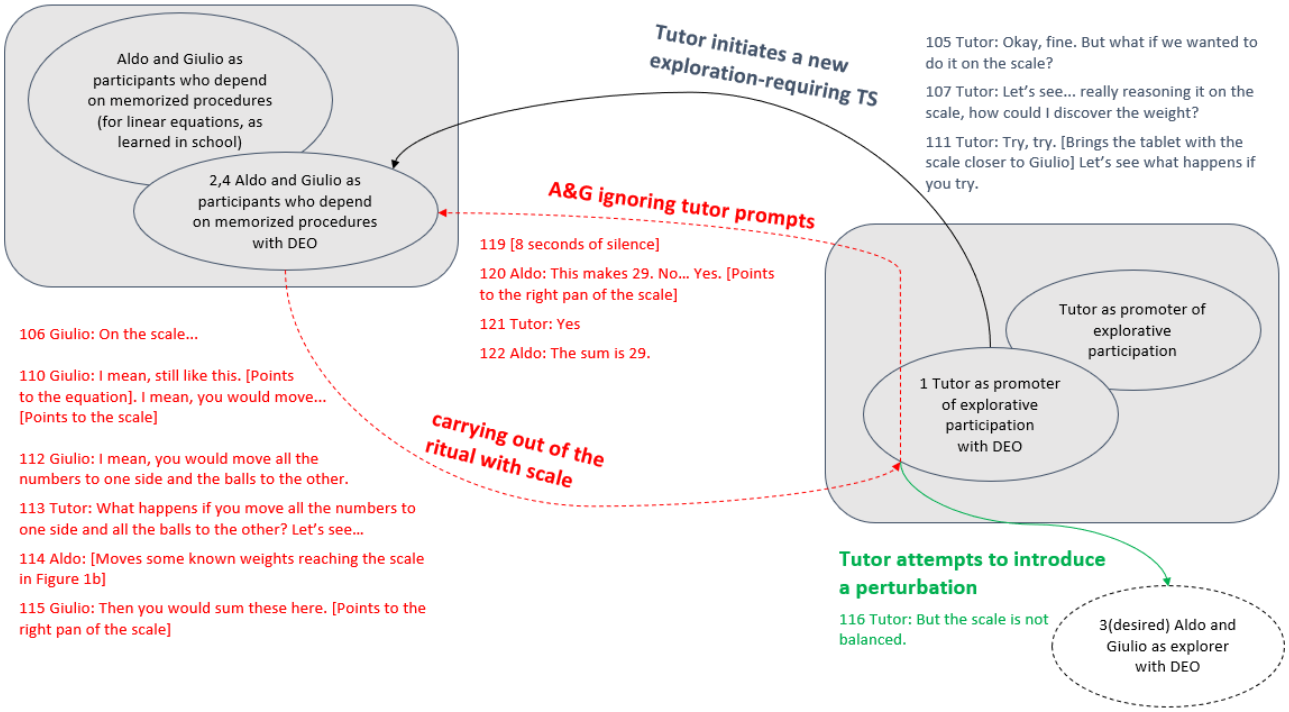

Figure 4. Second ritual task-cycle that resists perturbations.

DISCUSSION AND CONCLUSIONS

Our analysis shows that designed perturbations, such as the use of DEOs and carefully crafted tutor prompts, can lead to momentary shifts toward explorative participation. Aldo and Giulio initially engaged meaningfully with the concept of balance, using multiple representations and showing signs of objectification and substantiability in their discourse. Despite these promising shifts, strongly internalized procedural routines, particularly the "change side, change sign" rule, acted as powerful precedents, re-stabilizing the system into a ritualistic attractor state. This occurred even in the face of persistent and well-designed perturbations. Although in the long term the intervention was successful, in this activity the tutor's interventions, while aligned with the intended exploratory goals, were ultimately unable to overcome the pull of these ingrained school practices.

While our simplified CDS focused on interactions among a tutor, two students, and a DEO, we acknowledge that a fuller system would also include their regular classroom, teachers, and textbooks. Moreover, we acknowledge other limitations in trying to look at this episode through the lens of CDS, such as the need to consider more long term processes in order to capture attractor states more convincingly. However, we see potential in this new (in mathematics education) perspective on MLD and designing and implementing remedial interventions, that combines the theory of CDS with Commognition: within this bounded system, our findings point to the resilience of ritualistic attractor states, which is consistent with literature on MLD and mathematical failure (Lavie et al., 2019; Heyd-Metzuyanim, 2025). Insights from this study reinforce the importance of designing interventions that not only promote explorative participation but also anticipate and counteract the powerful pull into ritualistic attractor states of certain precedents.

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