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A random time-dependent noncooperative equilibrium problem

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Abstract
The paper deals with the random time-dependent oligopolistic market equilibrium problem. For such a problem the firms' point of view has been analyzed in Barbagallo and Guarino Lo Bianco (Optim. Lett. **14**: 2479–2493, 2020) while here the policymaker's point of view is studied. The random dynamic optimal control equilibrium conditions are expressed by means of an inverse stochastic time-dependent variational inequality which is proved to be equivalent to a stochastic time-dependent variational inequality. Some existence and well-posedness results for optimal regulatory taxes are obtained. Moreover a numerical scheme to compute the solution to the stochastic time-dependent variational inequality is presented. Finally an example is discussed.

Keywords Inverse stochastic time-dependent variational inequality · Random dynamic optimal control equilibrium problem · Existence results · Well-posedness analysis

1 Introduction

In the recent years stochastic dynamic optimization models received a lot of attention and have applications in many different areas. In particular stochastic variational inequalities arise from problems with conditions of randomness where

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the data are affected by a certain degree of uncertainty (see [17]). Such topics involved a lot of authors: see for example [14] or the recent paper [24] where stochastic variational inequalities with anticipativity in a dynamic multistage setting have been studied. Moreover, it is worth to highlight that inverse and control problems associated with stochastic PDEs also lead to stochastic variational inequalities (see [18]).

The purpose of this note is to analyze an effective oligopolistic market equilibrium model using the theory of stochastic and time-dependent variational inequalities combined with the Nash equilibrium theory. Similar analysis has been obtained in [7] where the authors studied a random time-dependent oligopolistic market equilibrium problem in presence of both production and demand excesses from the firms' point of view. Here we focus our attention on the policymaker's point of view. More precisely, control policies are implemented by imposing higher taxes or subsidies in order to restrict or encourage exportations. We prove that the equilibrium conditions can be formulated by means of an inverse stochastic time-dependent variational inequality. We investigate also the well-posedness of the inverse stochastic time-dependent variational inequality and the connection between the well-posedness of a stochastic time-dependent variational inequality and its inverse. Moreover we present a numerical method to compute the random dynamic oligopolitistic market equilibrium distribution. In the literature numerical methods to solve stochastic variational inequalities are available. A first numerical method was proposed in [21] but the convergence was ensured under very strong hypothesis on the function of the variational inequality. Later many other stochastic approximation methods for stochastic variational inequalities have been developed (see for instance [16, 20]). In our case we deal with a stochastic time-dependent variational inequality problem. Then we introduce a first instance of an iterative procedure for such inequalities which is based on the stochastic continuity result obtained in [7]. Thanks to that, a discretization of the time interval can be performed and, then, a projection method to solve the stochastic variational inequalities can be applied. Some reference for the numerical resolution of dynamic variational inequalities can be found, for instance, in [2, 3].

The time-dependent oligopolistic market equilibrium problem was intensively studied in the deterministic setting. It was introduced and deep analyzed starting by [4]. After that several variations of the model have been considered. More precisely in [11, 12] the presence of production and demand excesses, occurring during an economic crisis period or when the physical transportation of commodity between a firm and a demand market is evidently limited, has been introduced. Recently, the oligopolistic market model has been extended by allowing the possibility to a company to produce more than one good: the theoretical tools is based, in this case, on the tensor variational inequality theory (see [8–10] and the reference therein). The attention on the policymaker's point of view (whose aim is to control the commodity exportations by means of the imposition of taxes or incentives) was studied in [13] where the authors formulate the resulting optimization problem as an inverse variational inequality. Inverse variational inequalities can be considered as a special case of general variational inequalities and can be used to model various control problems (see [15] for details). Note that only recently the strict connection between classical variational inequalities and inverse variational inequalities has been studied (see, e.g., [25]).

The introduction of uncertainty in the oligopolistic market equilibrium problem arises because the constraints or the data vary in a non-regular and unpredictable manner (think about unpredictable events). Thus a suitable model has to permit to handle random constraints. The models in presence of uncertainty have been analyzed in [6] while in [5] the authors add production and demand excesses.

The mathematical setting is the one of Hilbert spaces, which allows us to obtain existence and regularity results. In this paper we study inverse stochastic time-dependent variational inequalities: results are obtained concerning existence and well-posedness.

The paper is organized as follows. In Sect. 2, the random oligopolistic market equilibrium problem is presented. We describe the firms' point of view obtaining the equivalence between the random Cournot-Nash equilibrium condition and an appropriate stochastic time-dependent variational inequality. Then we introduce the policy-maker's point of view, proving that the random dynamic optimal regulatory tax is a solution to an inverse stochastic time-dependent variational inequality. In Sect. 3 some existence results are obtained. In Sect. 4 we study the well-posedness of an inverse stochastic time-dependent variational inequality, which is, under suitable conditions, equivalent to the existence and uniqueness of its solution. In Sect. 5 a numerical scheme to compute the random dynamic oligopolistic market equilibrium distribution, based on a combination between a discretization procedure and a projection method, is presented. Finally an example is provided.

2 The random dynamic oligopolistic market equilibrium model

The aim of the section is to present the random dynamic oligopolistic market equilibrium problem. This is the problem of finding a trade equilibrium in a supply-demand market between a finite number of spatially separated firms which produce a homogeneous commodity and ship the commodity to some demand markets. We suppose that the data are affected by a certain degree of uncertainty and depending on the time. We analyze first the firms' point of view of the problem and then we introduce the policymaker's point of view, namely the random dynamic optimal control equilibrium problem.

2.1 The firms' point of view

Let us start to consider the firms' point of view of the random dynamic oligopolistic market equilibrium problem. Let $T > 0$, let Ω be an open subset of $\mathbb{R}$ and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $([0,T] \times \Omega, \mathcal{B}([0,T]) \otimes \mathcal{F}, \mathcal{L}^1 \otimes \mathbb{P})$ be the product measure space, where $\mathcal{B}([0,T])$ is the Borel σ-field of [0, T] and $\mathcal{L}^1$ is the 1-dimensional Lebesgue measure on [0, T]. A point in $[0,T] \times \Omega$ will be denoted by the couple (t, ω). If $Y = Y(t, \omega)$ is a measurable function on $[0,T] \times \Omega$, then the function defined as

$$X(\omega) = \int_0^T Y(\xi, \omega)\, d\xi$$

is a random variable on Ω and its expectation is given by

$$\mathbb{E}(X) = \int_\Omega X(\omega)\, d\mathbb{P}.$$

A *stochastic process* is a family of random variables $X_t(\omega) = X(t, \omega)$ on Ω indexed by the time variable $t \in [0, T]$. Correspondingly, the random function $t \longmapsto X(t, \omega)$, $\omega \in \Omega$ is called *sample path* at ω. Let us denote by $L^2([0, T] \times \Omega, \mathbb{R}^k, \mathbb{P})$ the Hilbert space of stochastic processes X from $[0, T] \times \Omega$ to $\mathbb{R}^k$ such that the expectation $\mathbb{E}(X) < \infty$. Moreover, we define in $L^2([0, T] \times \Omega, \mathbb{R}^k, \mathbb{P})$ the following bilinear form on $(L^2([0, T] \times \Omega, \mathbb{R}^k, \mathbb{P}))^* \times L^2([0, T] \times \Omega, \mathbb{R}^k, \mathbb{P})$, through the expectation, by

$$\ll \Xi, X \gg_{\mathbb{E}} = \int_0^T \int_\Omega \langle \Xi(t, \omega), X(t, \omega) \rangle dt\, d\mathbb{P},$$

where $\Xi \in (L^2([0, T] \times \Omega, \mathbb{R}^k, \mathbb{P}))^* = L^2([0, T] \times \Omega, \mathbb{R}^k, \mathbb{P})$, $X \in L^2([0, T] \times \Omega, \mathbb{R}^k, \mathbb{P})$ and

$$\langle \Xi(t, \omega), X(t, \omega) \rangle = \sum_{l=1}^k \Xi_l(t, \omega) X_l(t, \omega), \quad \text{a.e. in } [0, T],\ \mathbb{P} - \text{a.s.}$$

Let us consider m firms P_i, $i = 1, \ldots, m$, which produce a homogeneous commodity and n demand markets Q_j, $j = 1, \ldots, n$, which are generally spatially separated. Assume that the homogeneous commodity, produced by the m firms and consumed by the n markets, depends on random variables. Let p_i be the random time-dependent variable expressing the nonnegative commodity output produced by firm P_i and suppose that $p_i = p_i(t, \omega)$, $(t, \omega) \in [0, T] \times \Omega$, $i = 1, \ldots, m$. Let q_j be the random time-dependent variable expressing the nonnegative demand for the commodity of demand market Q_j, namely $q_j = q_j(t, \omega)$, $(t, \omega) \in [0, T] \times \Omega$, $j = 1, \ldots, n$. Let x_{ij} be the random time-dependent variable expressing the nonnegative commodity shipment between the supply producer P_i and the demand market Q_j, namely $x_{ij} = x_{ij}(t, \omega)$, $(t, \omega) \in [0, T] \times \Omega$, $i = 1, \ldots, m$, $j = 1, \ldots, n$. Finally let x_i be the strategy vector for the firm P_i, namely $x_i(t, \omega) = (x_{i1}(t, \omega), \ldots, x_{in}(t, \omega))$, $(t, \omega) \in [0, T] \times \Omega$, $i = 1, \ldots, m$. For technical reasons, we analyze the model in the Hilbert space $L^2([0, T] \times \Omega, \mathbb{R}^{mn}_+, \mathbb{P})$.

Let us assume that the following feasibility conditions hold:

$$p_i(t, \omega) = \sum_{j=1}^n x_{ij}(t, \omega), \quad i = 1, \ldots, m,\ \text{a.e. in } [0, T],\ \mathbb{P} - \text{a.s.}, \tag{1}$$

$$q_j(t, \omega) = \sum_{i=1}^m x_{ij}(t, \omega), \quad j = 1, \ldots, n,\ \text{a.e. in } [0, T],\ \mathbb{P} - \text{a.s.} \tag{2}$$

The condition (1) expesses that the random time-dependent quantity produced by each firm P_i has to be equal to the random commodity shipments from that firm to all the demand markets. Instead, condition (2) expresses that the random time-dependent quantity demanded by each demand market Q_j has to be equal to the random commodity shipments from all the firms to that demand market.

Furthermore let us assume that the nonnegative random time-dependent commodity shipment between the producer P_i and the demand market Q_j has to satisfy two capacity constraints, namely there exist two nonnegative random time-dependent variables $\underline{x}, \overline{x} \in L^2([0,T] \times \Omega, \mathbb{R}^{mn}_+, \mathbb{P})$ such that

$$0 \leq \underline{x}_{ij}(t,\omega) \leq x_{ij}(t,\omega) \leq \overline{x}_{ij}(t,\omega), \quad \forall i = 1, \dots, m, \ \forall j = 1, \dots, n, \ \text{a.e. in } [0,T], \ \mathbb{P} - \text{a.s.} \tag{3}$$

Therefore, the set of feasible distributions $x \in L^2([0,T] \times \Omega, \mathbb{R}^{mn}_+, \mathbb{P})$ is

$$\mathbb{K} = \left\{ x \in L^2([0,T]\times\Omega, \mathbb{R}^{mn}_+, \mathbb{P}) : 0 \leq \underline{x}_{ij}(t,\omega) \leq x_{ij}(t,\omega) \leq \overline{x}_{ij}(t,\omega), \quad \forall i = 1, \dots, m, \ \forall j = 1, \dots, n, \ \text{a.e. in } [0,T], \ \mathbb{P} - \text{a.s.} \right\}.$$

It worth to underline that $\mathbb{K}$ is a convex closed bounded subset of $L^2([0,T] \times \Omega, \mathbb{R}^{mn}_+, \mathbb{P})$.

At last we introduce the costs. More precisely, let f_i be a random time-dependent variable denoting the production cost of firm P_i such that $f_i = f_i(t,\omega,x(t,\omega))$, $(t,\omega) \in [0,T] \times \Omega$, $i = 1, \dots, m$. Similarly, let d_j be a random time-dependent variable denoting the demand price for unity of the commodity for the demand market Q_j such that $d_i = d_i(t,\omega,x(t,\omega))$, $(t,\omega) \in [0,T] \times \Omega$, $j = 1, \dots, n$. Finally, let c_{ij} be the random variable expressing the transaction cost, which includes the transportation cost associated with trading the commodity between firm P_i and demand market Q_j such that $c_{ij} = c_{ij}(t,\omega,x(t,\omega))$, $(t,\omega) \in [0,T] \times \Omega$, $i = 1, \dots, m$, $j = 1, \dots, n$. Let η_{ij} be the random variable expressing the supply or resource tax imposed on the supply market P_i for the transaction with the demand market Q_j, namely $\eta_{ij} = \eta_{ij}(t,\omega)$, $(t,\omega) \in [0,T] \times \Omega$, $i = 1, \dots, m$, $j = 1, \dots, n$. Let λ_{ij} be the random variable expressing the incentive pay imposed on the supply market P_i for the transaction with the demand market Q_j namely $\lambda_{ij} = \lambda_{ij}(t,\omega)$, $(t,\omega) \in [0,T] \times \Omega$, $i = 1, \dots, m$, $j = 1, \dots, n$. Moreover, let h_{ij} be the random variable expressing the difference between the supply tax and the incentive pay imposed on the supply market P_i for the transaction with the demand market Q_j, namely $h_{ij}(t,\omega) = \eta_{ij}(t,\omega) - \lambda_{ij}(t,\omega)$, $(t,\omega) \in [0,T] \times \Omega$, $i = 1, \dots, m$, $j = 1, \dots, n$. As a consequence, the profit $v_i(t,\omega,x(t,\omega))$ of the firm P_i is

$$v_i(t,\omega,x(t,\omega)) = \sum_{j=1}^{n} d_j(t,\omega,x(t,\omega))x_{ij}(t,\omega) - f_i(t,\omega,x(t,\omega))$$
$$- \sum_{j=1}^{n} c_{ij}(t,\omega,x(t,\omega))x_{ij}(t,\omega) - \sum_{j=1}^{n} h_{ij}(t,\omega)x_{ij}(t,\omega),$$
$$i = 1,\dots,m, \text{ a.e. in } [0,T], \ \mathbb{P} - \text{a.s.},$$

namely, it is equal to the price which the demand markets are disposed to pay minus the production costs, the transportation costs and the taxes.

Let us assume the following:

(i) $v_i(\cdot)$ is continuously differentiable for each $i = 1,\dots,m$,

(ii) $\nabla_D v(\cdot)$ is a Carathéodory function such that

$$\exists h \in L^2(\Omega,\mathbb{P}) : \left\|\nabla_D v(t,\omega,x(t,\omega))\right\| \le h(t,\omega)\|x(t,\omega)\|, \quad \forall x \in L^2([0,T]\times\Omega,\mathbb{R}^{mn}_+,\mathbb{P}), \text{ a.e. in } [0,T], \ \mathbb{P} - \text{ a.s. }, \tag{4}$$

(iii) $v_i(\cdot)$ is pseudoconcave[1] with respect to the variables $x_i, i = 1,\dots,m$.

Let us set $\nabla_D v = \left(\dfrac{\partial v_i}{\partial x_{ij}}\right)_{\substack{i=1,\dots,m\\ j=1,\dots,n}} \in L^2([0,T]\times\Omega,\mathbb{R}^{mn}_+,\mathbb{P})$.

The model is based on the fact that the m firms supply the commodity in a noncooperative fashion, each one tries to maximize its own profit function considered the optimal distribution pattern for the other firms. The aim is to find a nonnegative commodity distribution for which the m firms and the n demand markets will be in a state of equilibrium as defined below.

Definition 1 A feasible distribution $x^* \in \mathbb{K}$ is a random time-dependent oligopolistic market equilibrium distribution if and only if, for each $i = 1,\dots,m$, a.e. in [0, T], $\mathbb{P}$-a.s., we have

$$v_i(t,\omega,x^*(t,\omega)) \ge v_i(t,\omega,x_i(t,\omega),x^*_{-i}(t,\omega)), \tag{5}$$

where

$$x^*_{-i}(t,\omega) = (x^*_1(t,\omega),\dots,x^*_{i-1}(t,\omega),x^*_{i+1}(t,\omega),\dots,x^*_m(t,\omega)).$$

The previous definition generalizes the random Cournot-Nash principle in the random time-dependent case. Moreover, under assumptions (i), (ii) and (iii), it is characterized by means of the stochastic variational inequality (see [7])

[1] A function $v_i(\cdot)$, continuously differentiable, is called *pseudoconcave* with respect to x_i, $i = 1,\dots,m$, (see [22]) if the following condition holds

$$\left\langle \frac{\partial v_i}{\partial x_i}(x_1,\dots,x_i,\dots,x_m), x_i - y_i \right\rangle \ge 0 \ \Rightarrow \ v_i(x_1,\dots,x_i,\dots,x_m) \ge v_i(x_1,\dots,y_i,\dots,x_m).$$

$$\ll -\nabla_D v(x^*), x - x^* \gg_{\mathbb{E}} \geq 0, \quad \forall x \in \mathbb{K}, \tag{6}$$

namely

$$\int_0^T \int_\Omega -\sum_{i=1}^m \sum_{j=1}^n \frac{\partial v_i(t,\omega,x^*(t,\omega))}{\partial x_{ij}} \Big(x_{ij}(t,\omega) - x^*_{ij}(t,\omega) \Big) dt\, d\mathbb{P} \geq 0, \quad \forall x \in \mathbb{K}.$$

2.2 The policymaker's point of view

Let us left the producers' point of view in the analysis of the problem and let us introduce the random time-dependent optimal control model in which the term h, which is a fixed parameter in the firms' point of view model, will become a variable. The random time-dependent resource exploitations $x(t, \omega, h(t, \omega))$ can be controlled by adjusting taxes $h(t, \omega)$, a.e. in $[0, T]$, $\mathbb{P}$-a.s. In this prospective, the random time-dependent tax adjustment has the role of regulating exportation. More precisely, if the policymaker wants to reduce exportations and, hence, the production of the commodity, then higher taxes will be enforced, otherwise if the policymaker wants to force exportations of the commodity, subventions will be provided.

Let $x(h) = x(t, \omega, h(t, \omega))$ be the random time-dependent function of regulatory taxes, with $h \in L^2([0,T] \times \Omega, \mathbb{R}^{mn}, \mathbb{P})$. Let us assume that $x(t, \omega, h(t, \omega))$ is a Carathéodory function and there exists $\gamma \in L^2([0,T] \times \Omega, \mathbb{P})$ such that

$$\begin{aligned} \|x(t,\omega,h(t,\omega))\| \leq & \gamma(t,\omega) + \|h(t,\omega)\|, \\ & \forall h \in L^2([0,T]\times\Omega, \mathbb{R}^{mn}_+, \mathbb{P}), \text{ a.e. in } [0,T], \ \mathbb{P}-\text{a.s.} \end{aligned} \tag{7}$$

Hence the set of feasible states is given by

$$W = \Big\{ w \in L^2([0,T]\times\Omega, \mathbb{R}^{mn}, \mathbb{P}) : \underline{x}_{ij}(t,\omega) \leq w_{ij}(t,\omega) \leq \overline{x}_{ij}(t,\omega), \\ \forall i = 1,\dots,m, \ \forall j = 1,\dots,n, \text{ a.e. in } [0,T], \ \mathbb{P}-\text{a.s.} \Big\}.$$

We notice immediately that W is a convex closed bounded subset of $L^2([0,T]\times\Omega, \mathbb{R}^{mn}_+, \mathbb{P})$.

Definition 2 A random dynamic regulatory tax $h^* \in L^2([0,T]\times\Omega, \mathbb{R}^{mn}_+, \mathbb{P})$ is a random dynamic optimal regulatory tax if $x(h^*) \in W$ and, for $i = 1,\dots,m$, $j = 1,\dots,n$, a.e. in $[0, T]$ and $\mathbb{P}$-a.s. , the following conditions hold:

$$x_{ij}(t,\omega,h^*(t,\omega)) = \underline{x}_{ij}(t,\omega) \Rightarrow h^*_{ij}(t,\omega) \leq 0, \tag{8}$$

$$\underline{x}_{ij}(t,\omega) < x_{ij}(t,\omega,h^*(t,\omega)) < \overline{x}_{ij}(t,\omega) \Rightarrow h^*_{ij}(t,\omega) = 0, \tag{9}$$

$$x_{ij}(t,\omega,h^*(t,\omega)) = \overline{x}_{ij}(t,\omega) \Rightarrow h^*_{ij}(t,\omega) \geq 0. \tag{10}$$

Definition 2 must be interpreted as follows: the random time-dependent optimal regulatory tax h^* is such that the correspondig state $x(t,\omega,h^*(t,\omega))$ has to satisfy capacity constraints, namely $\underline{x}(t,\omega) \le x(t,\omega,h^*(t,\omega)) \le \overline{x}(t,\omega)$, a.e. in [0, T], $\mathbb{P}$-a.s. Moreover, if $x_{ij}(t,\omega,h^*(t,\omega)) = \underline{x}_{ij}(t,\omega)$, then the random time-dependent exportations have to be encouraged, namely taxes must be less than or equal to the random dynamic incentive pays. If $x_{ij}(t,\omega,h^*(t,\omega)) = \overline{x}_{ij}(t,\omega)$, then the random time-dependent exportations have to be reduced, hence random dynamic taxes must be greater than or equal to the random dynamic incentive pays. Finally, if $\underline{x}_{ij}(t,\omega) < x_{ij}(t,\omega,h^*(t,\omega)) < \overline{x}_{ij}(t,\omega)$ is satisfied, random dynamic taxes must be equal to random dynamic incentive pays.

Let us establish the inverse stochastic time-dependent variational formulation of the random dynamic optimal equilibrium control problem.

Theorem 1 *A random dynamic regulatory tax* $h^* \in L^2([0,T]\times\Omega,\mathbb{R}^{mn}_+,\mathbb{P})$ *is a random dynamic optimal regulatory tax if and only if it solves the inverse stochastic time-dependent variational inequality*

$$\int_0^T \int_\Omega \sum_{i=1}^m \sum_{j=1}^n h^*_{ij}(t,\omega)\big(w_{ij}(t,\omega) - x_{ij}(t,\omega,h^*(t,\omega))\big)dt\,d\mathbb{P} \le 0, \quad \forall w \in W. \tag{11}$$

Proof For the reader's convinience, we present the details of the proof. Let h^* be a random dynamic optimal regulatory tax and let $w \in W$. For $i \in \{1,\dots,m\}$ and $j \in \{1,\dots,n\}$ fixed, it results that $\underline{x}_{ij}(t,\omega) \le w_{ij}(t,\omega) \le \overline{x}_{ij}(t,\omega)$, a.e. in [0, T], $\mathbb{P}$-a.s. One has

1. If $x_{ij}(t,\omega,h^*(t,\omega)) = \underline{x}_{ij}(t,\omega)$, a.e. in [0, T], $\mathbb{P}$-a.s., by (8) it follows that $h^*_{ij}(t,\omega) \le 0$, a.e. in [0, T], $\mathbb{P}$-a.s., and, as a consequence, $h^*_{ij}(t,\omega)\big(w_{ij}(t,\omega) - x_{ij}(t,\omega,h^*(t,\omega))\big) \le 0$, a.e. in [0, T], $\mathbb{P}$-a.s.;
2. If $\underline{x}_{ij}(t,\omega) < x_{ij}(t,\omega,h^*(t,\omega)) < \overline{x}_{ij}(t,\omega)$, a.e. in [0, T], $\mathbb{P}$-a.s., by using (9) we get $h^*_{ij}(t,\omega) = 0$, a.e. in [0, T], $\mathbb{P}$-a.s., and, then, $h^*_{ij}(t,\omega)\big(w_{ij}(t,\omega) - x_{ij}(t,h^*(\omega))\big) = 0$, a.e. in [0, T], $\mathbb{P}$-a.s.;
3. If $x_{ij}(t,\omega,h^*(t,\omega)) = \overline{x}_{ij}(t,\omega)$, a.e. in [0, T], $\mathbb{P}$-a.s., by (10) we deduce that $h^*_{ij}(t,\omega) \ge 0$, a.e. in [0, T], $\mathbb{P}$-a.s., and, hence, $h^*_{ij}(t,\omega)\big(w_{ij}(t,\omega) - x_{ij}(t,\omega,h^*(t,\omega))\big) \le 0$, a.e. in [0, T], $\mathbb{P}$-a.s.

We have shown that for every $i = 1,\dots,m$, $j = 1,\dots,n$ and $w \in W$, we have

$$h^*_{ij}(t,\omega)\big(w_{ij}(t,\omega) - x_{ij}(t,\omega,h^*(t,\omega))\big) \le 0, \quad \text{a.e. in } [0,T],\ \mathbb{P} - \text{a.s.}$$

By summing over $i = 1,\dots,m$, $j = 1,\dots,n$ and integrating both on [0, T] and on Ω, we obtain the inverse stochastic time-dependent variational inequality (11).

Vice versa, let h^* be satisfied the inverse stochastic time-dependent variational inequality (11). We fix $i \in \{1,\dots,m\}$, $j \in \{1,\dots,n\}$ and set $w_{hk}(t,\omega) = x_{ij}(t,\omega,h^*(\omega))$, a.e. in [0, T], $\mathbb{P}$-a.s., for every $h \ne i$, $k \ne j$. Making use of (11), it results

$$\int_0^T \int_\Omega h^*_{ij}(t,\omega)\big(w_{ij}(t,\omega) - x_{ij}(t,\omega,h^*(t,\omega))\big)dt\,d\mathbb{P} \le 0,$$
$$\forall w_{ij}(t,\omega) \in L^2([0,T]\times\Omega,\mathbb{P}) : \; \underline{x}_{ij}(t,\omega) \le w_{ij}(t,\omega) \le \overline{x}_{ij}(t,\omega). \tag{12}$$

We claim that if $x_{ij}(t,\omega,h^*(t,\omega)) = \underline{x}_{ij}(t,\omega)$, a.e. in $[0, T]$, $\mathbb{P}$-a.s., then, $h^*_{ij}(t,\omega) \le 0$, a.e. in $[0, T]$, $\mathbb{P}$-a.s. Indeed, by contradiction, we suppose that there exists or a set $I \subset [0,T]$ either a set $\Xi \subseteq \Omega$, with $\mathbb{P}(\Xi) > 0$, such that $h^*_{ij}(t,\omega) > 0$, or a.e. in I, either $\mathbb{P}$-a.s. in Ξ. Let us suppose that we are in the case in which there exists a set $I \subset [0,T]$ such that $h^*_{ij}(t,\omega) > 0$, a.e. in I, $\mathbb{P}$-a.s. Then we choose

$$w_{ij}(t,\omega) = \begin{cases} \overline{x}_{ij}(t,\omega), & \text{a.e. in } I, \; \mathbb{P}-\text{a.s.}, \\ x_{ij}(t,\omega,h^*(t,\omega)), & \text{a.e. in } [0,T]\setminus I, \; \mathbb{P}-\text{a.s.} \end{cases}$$

Therefore we obtain

$$\begin{aligned} &\int_0^T \int_\Omega h^*_{ij}(t,\omega)\big(w_{ij}(t,\omega) - x_{ij}(t,\omega,h^*(t,\omega))\big)\,dt\,d\mathbb{P} \\ &\quad = \int_I \int_\Omega h^*_{ij}(t,\omega)(\overline{x}_{ij}(t,\omega) - \underline{x}_{ij}(t,\omega))\,dt\,d\mathbb{P} > 0, \end{aligned} \tag{13}$$

which is in contradiction with (12). The case in which there exists a set $\Xi \subseteq \Omega$, with $\mathbb{P}(\Xi) > 0$, such that $h^*_{ij}(t,\omega) > 0$, a.e. in $[0, T]$, $\mathbb{P}$-a.s. in Ξ, is analogous.

Similarly we can proceed in the other cases deducing:

- if $x_{ij}(t,\omega,h^*(t,\omega)) = \overline{x}_{ij}(t,\omega)$, a.e. in $[0, T]$, $\mathbb{P}$-a.s., then $h^*_{ij}(t,\omega) \ge 0$, a.e. in $[0, T]$, $\mathbb{P}$ -a.s.,
- if $\underline{x}_{ij}(t,\omega) < x_{ij}(t,\omega,h^*(t,\omega)) < \overline{x}_{ij}(t,\omega)$, a.e. in $[0, T]$, $\mathbb{P}$-a.s., then $h^*(\omega) = 0$, a.e. in $[0, T]$, $\mathbb{P}$-a.s.

□

We are interested to express the random dynamic optimal equilibrium control problem by means of a suitable stochastic time-dependent variational inequality. For this reason we set

$$Z = L^2([0,T]\times\Omega,\mathbb{R}^{mn},\mathbb{P}) \times W, \qquad F : [0,T]\times\Omega\times Z \to L^2([0,T]\times\Omega,\mathbb{R}^{2mn},\mathbb{P}),$$
$$z(t,\omega) = \begin{pmatrix} h(t,\omega) \\ w(t,\omega) \end{pmatrix} \in Z, \quad \text{a.e. in } [0,T], \; \mathbb{P}-\text{a.s.},$$
$$F(t,\omega,z(t,\omega)) = \begin{pmatrix} w(t,\omega) - x(t,\omega,h(t,\omega)) \\ -h(t,\omega) \end{pmatrix}, \quad \text{a.e. in } [0,T], \; \mathbb{P}-\text{a.s.}.$$

Let us highlight that Z is a closed, convex (as product of convex sets) and not bounded subset of $L^2([0,T]\times\Omega,\mathbb{R}^{2mn}_+,\mathbb{P})$. The following result holds.

Theorem 2 *The inverse stochastic time-dependent variational inequality* (11) *is equivalent to the stochastic time-dependent variational inequality*

$$\int_0^T \int_\Omega \sum_{l=1}^{2m} \sum_{j=1}^{n} F_{lj}(t,\omega,z^*(t,\omega))\Big(z_{lj}(t,\omega) - z^*_{lj}(t,\omega)\Big)\,dt\,d\mathbb{P} \geq 0, \quad \forall z \in Z. \tag{14}$$

Proof We suppose that (14) holds true. Therefore, one has $z^* = (h^*, w^*)^T \in Z$, and

$$\begin{aligned}&\int_0^T \int_\Omega \left(\sum_{i=1}^{m}\sum_{j=1}^{n}(w^*_{ij}(t,\omega) - x_{ij}(t,\omega,h^*(t,\omega)))(h_{ij}(t,\omega) - h^*_{ij}(t,\omega))\right) dt\,d\mathbb{P}\\ &-\int_0^T \int_\Omega \left(\sum_{i=1}^{m}\sum_{j=1}^{n} h^*_{ij}(t,\omega)(w_{ij}(t,\omega) - w^*_{ij}(t,\omega))\right) dt\,d\mathbb{P} \geq 0, \quad \forall z = (h,w)^T \in Z.\end{aligned} \tag{15}$$

Let us take $h(t,\omega) = h^*(t,\omega) - w^*(t,\omega) + x(t,\omega,h^*(t,\omega))$, a.e. in $[0,T]$, $\mathbb{P}$-a.s., and $w(t,\omega) = w^*(t,\omega)$, a.e. in $[0,T]$, $\mathbb{P}$-a.s., in (15). As a consequence, we get

$$-\int_0^T \int_\Omega \sum_{i=1}^{m}\sum_{j=1}^{n}(w^*_{ij}(t,\omega) - x_{ij}(t,\omega,h^*(t,\omega)))^2\,dt\,d\mathbb{P} \geq 0,$$

then $x(t,\omega,h^*(t,\omega)) = w^*(t,\omega)$, a.e. in $[0,T]$, $\mathbb{P}$-a.s. Hence $x(h^*) \in W$ and, by using (15), we obtain (11).

Vice versa if $h^* \in L^2([0,T]\times\Omega, \mathbb{R}^{mn}_+, \mathbb{P})$ is a solution to (11), it results

$$\underbrace{\int_0^T \int_\Omega \sum_{i=1}^{m}\sum_{j=1}^{n}(x_{ij}(t,\omega,h^*(t,\omega)) - x_{ij}(t,\omega,h^*(t,\omega)))(h_{ij}(t,\omega) - h^*_{ij}(t,\omega))\,dt\,d\mathbb{P}}_{=0}$$
$$\underbrace{-\int_0^T \int_\Omega \sum_{i=1}^{m}\sum_{j=1}^{n} h^*_{ij}(t,\omega)(w_{ij}(t,\omega) - x_{ij}(t,\omega,h^*(t,\omega)))\,dt\,d\mathbb{P}}_{\geq 0} \geq 0.$$

Hence, $z^* = (h^*, x(h^*))^T \in Z$ is a solution to (14). □

3 Existence results

Let us establish some existence results for the random time-dependent oligopolistic market equilibrium problem. For what concerns the firms' point of view model, we refer to [7]. Here, we investigate on the existence of the random dynamic optimal regulatory tax. First of all, let us remark that thanks to the equivalent stochastic time-dependent variational formulation of (11), the existence can be obtained applying the results shown in [7]. In the sequel, we concentrate our investigation on the inverse stochastic time-dependent variational inequality (11).

In order to prove an existence result for the inverse stochastic time-dependent variational inequality (11), we assume the following:

(a) $x(t,\omega,h) : [0,T]\times\Omega\times L^2([0,T]\times\Omega,\mathbb{R}^{mn},\mathbb{P}) \to L^2([0,T]\times\Omega,\mathbb{R}^{mn},\mathbb{P})$ is a Carathéodory function such that there exists a function $\gamma \in L^2([0,T]\times\Omega,\mathbb{R},\mathbb{P})$:

$$\|x(t,\omega,h(t,\omega))\| \le \gamma(t,\omega) + \|h(t,\omega)\|,$$
$$\forall h \in L^2([0,T]\times\Omega,\mathbb{R}^{mn},\mathbb{P}), \text{ a.e. in } [0,T], \ \mathbb{P}-\text{a.s.};$$

(b) $x(t,\omega,h) : [0,T]\times\Omega\times L^2([0,T]\times\Omega,\mathbb{R}^{mn},\mathbb{P}) \to L^2([0,T]\times\Omega,\mathbb{R}^{mn},\mathbb{P})$ is anti-monotone with respect to h, namely

$$\langle h_1(t,\omega)-h_2(t,\omega), x(t,\omega,h_1(t,\omega)) - x(t,\omega,h_2(t,\omega))\rangle \le 0,$$
$$\forall h_1,h_2 \in L^2([0,T]\times\Omega,\mathbb{R}^{mn},\mathbb{P}), \text{ a.e. in } [0,T], \ \mathbb{P}-\text{a.s.};$$

(c) there exists a constant $M>0$ such that for any $h \in L^2([0,T]\times\Omega,\mathbb{R}^{mn},\mathbb{P})$ with $\|h\|_{L^2([0,T]\times\Omega,\mathbb{R}^{mn},\mathbb{P})} > M$, it results

$$\int_0^T \int_\Omega \langle h(t,\omega), w_0^{proj}(t,\omega) - x(t,\omega,h(t,\omega))\rangle \, dt\, d\mathbb{P} > 0, \tag{16}$$

where w_0^{proj} is the projection of $w_0(t,\omega)=x(t,\omega,0)$ onto the feasible set W, namely

$$w_0^{proj}(t,\omega) = \begin{cases} \underline{x}(t,\omega), & \text{if } w_0(t,\omega) \le \underline{x}(t,\omega), \text{ a.e. in } [0,T], \ \mathbb{P}-\text{a.s.}, \\ w_0(t,\omega), & \text{if } \underline{x}(t,\omega) \le w_0(t,\omega) \le \overline{x}(t,\omega), \text{ a.e. in } [0,T], \ \mathbb{P}-\text{a.s.}, \\ \overline{x}(t,\omega), & \text{if } w_0(t,\omega) \ge \overline{x}(t,\omega), \text{ a.e. in } [0,T], \ \mathbb{P}-\text{a.s.} \end{cases}$$

Assumption (16) means that if we adjust the price in a suitable way (so if $w_0(t,\omega)$ is enough positive for $w_0(t,\omega) \le \overline{x}(t,\omega)$ and is enough negative for $w_0(t,\omega) \le \underline{x}(t,\omega)$), then the resultant curve $x(t,\omega,h(t,\omega))$ will be strictly controlled in the interior of the feasible set W. Now, we are able to prove an existence result under the above assumptions and making use of Corollary 3.7 in [23].

Theorem 3 *Let us assume that conditions* (*a*), (*b*) *and* (*c*) *are satisfied. Then the inverse stochastic time-dependent variational inequality* (11) *admits a solution.*

Proof Making use of assumptions (b) and (c), we have that any $h \in L^2([0,T]\times\Omega,\mathbb{R}^{mn},\mathbb{P})$ with $\|h\|_{L^2([0,T]\times\Omega,\mathbb{R}^{mn},\mathbb{P})} > M$ is too large to be a solution to (11). Let us set $L^2_M = \{x \in L^2([0,T]\times\Omega,\mathbb{R}^{mn},\mathbb{P}) : \|h\|_{L^2([0,T]\times\Omega,\mathbb{R}^{mn},\mathbb{P})} \le M\}$. Consequently, we consider the stochastic time-dependent variational inequality on the bounded set $Z' = L^2_M \times W$, namely

$$\int_0^T \int_\Omega \sum_{l=1}^{2m} \sum_{j=1}^{n} F_{lj}(t,\omega,z^*(t,\omega))(z_{lj}(t,\omega) - z^*_{lj}(t,\omega))\, dt\, d\mathbb{P} \ge 0, \quad \forall z \in Z'.$$

By assumption (b), it is easy to prove

$$\langle z(t,\omega)-\hat{z}(t,\omega), F(t,\omega,z(t,\omega))-F(t,\omega,\hat{z}(t,\omega))\rangle = \langle x(t,\omega,h(t,\omega))-x(t,\omega,\hat{h}(t,\omega)), h(t,\omega)-\hat{h}(t,\omega)\rangle \geq 0, \tag{17}$$

for every $z=(h,w)^T, \hat{z}=(\hat{h},\hat{w})^T \in Z'$, a.e. in $[0, T]$, $\mathbb{P}$-a.s. Let us set $C : Z' \to (L^2([0,T]\times\Omega,\mathbb{R}^{mn}_+,\mathbb{P})\times L^2([0,T]\times\Omega,\mathbb{R}^{mn}_+,\mathbb{P}))^*$ such that

$$\ll C(z), u \gg_{\mathbb{E}} = \int_0^T \int_\Omega \langle F(t,\omega,z(t,\omega)), u(t,\omega)\rangle \, dt\, d\mathbb{P},$$
$$\forall z \in Z',\ \forall u \in L^2([0,T]\times\Omega,\mathbb{R}^{mn}_+,\mathbb{P})\times L^2([0,T]\times\Omega,\mathbb{R}^{mn}_+,\mathbb{P}).$$

By using (17), it results

$$\int_0^T \int_\Omega \langle F(t,\omega,z(t,\omega))-F(t,\omega,\hat{z}(t,\omega)), z(t,\omega)-\hat{z}(t,\omega)\rangle \, dt\, d\mathbb{P} \geq 0, \quad \forall z,\hat{z}\in Z',$$

and, then, the operator C is monotone[2]. Taking into account assumption (a) and the Lebesgue theorem, we can prove that

$$\lim_n \int_0^T \int_\Omega \|F(t,\omega,\lambda_n z(t,\omega)+(1-\lambda_n)\hat{z}(t,\omega)) - F(t,\omega,\lambda z(t,\omega)+(1-\lambda)\hat{z}(t,\omega))\|^2 \, dt\, d\mathbb{P} = 0$$

holds for every sequence $\{\lambda_n\}\subset[0,1]$, such that $\lambda_n \to \lambda \in [0,1]$ and for every $z,\hat{z}\in W'$. Therefore we deduce

$$\lim_n \int_0^T \int_\Omega \langle F(t,\omega,\lambda_n z(t,\omega)+(1-\lambda_n)\hat{z}(t,\omega)), z(t,\omega)-\hat{z}(t,\omega)\rangle \, dt\, d\mathbb{P}$$
$$= \int_0^T \int_\Omega \langle F(t,\omega,\lambda z(t,\omega)+(1-\lambda)\hat{z}(t,\omega)), z(t,\omega)-\hat{z}(t,\omega)\rangle \, dt\, d\mathbb{P},$$

namely the operator $C(z)$ is hemicontinuous along line segments[3]. Moreover, since W is convex closed bounded and for the definition of $L^2_{M'}$, Z' is also a convex closed bounded set. Hence, taking into account Corollary 3.7 in [23], (14) admits a solution. As a consequence, by Theorem 2, (11) has also a solution. □

4 Well-posedness conditions

Let us investigate on the well-posedness of (11) and, then, we establish its relationship with the well-posedness of (14).

[2] Let X be a reflexive Banach space over the reals, let K be a nonempty closed convex subset of X and let X^* be the dual space of X equipped with the weak* topology. A mapping $C : K \longrightarrow X^*$ is called monotone if and only $\langle C(u)-C(v), u-v\rangle \geq 0$, for all $u, v \in K$.

[3] A mapping $C : K \longrightarrow X^*$ is called hemicontinuous along line segments if and only if the function $\xi \mapsto \langle C(\xi), u-v\rangle$ is continuous on the line segments $[u, v]$, for all $u, v \in K$.

Let us define that a sequence $\{h_n\} \subset L^2([0,T]\times\Omega,\mathbb{R}^{mn},\mathbb{P})$ is called an approximating sequence for (11) if and only if there exists a sequence $\{\varepsilon_n\}$, with $\varepsilon_n > 0$, for every $n \in \mathbb{N}$, and $\varepsilon_n \to 0$, such that

$$\int_0^T \int_\Omega \langle h_n(t,\omega), w(t,\omega) - x(t,\omega,h_n(t,\omega))\rangle \, dt \, d\mathbb{P} \leq \varepsilon_n, \quad \forall w \in W, \ \forall n \in \mathbb{N}.$$

Definition 3 We say that (11) is well-posed if and only if (11) has a unique solution and every approximating sequence converges to the unique solution.

The following well-posedness result for the inverse stochastic time-dependent variational inequality holds.

Theorem 4 *Let* $x : [0,T]\times\Omega\times L^2([0,T]\times\Omega,\mathbb{R}^{mn},\mathbb{P}) \to L^2([0,T]\times\Omega,\mathbb{R}^{mn},\mathbb{P})$ *be an hemicontinuous along line segments and anti-monotone mapping. Then,* (11) *is well-posed if and only if it has a unique solution.*

Proof The necessity holds trivially. Let us prove that the sufficiency holds also true. Hence we assume that (11) has a unique solution h^*. Consequently, we have

$$\int_0^T \int_\Omega \langle h^*(t,\omega), w(t,\omega) - x(t,\omega,h^*(t,\omega))\rangle dt \, d\mathbb{P} \leq 0, \quad \forall w \in W.$$

Since x is anti-monotone, we get

$$\begin{aligned} &\int_0^T \int_\Omega \langle w(t,\omega) - x(t,\omega,h(t,\omega)), h(t,\omega) - h^*(t,\omega)\rangle dt \, d\mathbb{P} \\ &+ \int_0^T \int_\Omega \langle h(t,\omega), w(t,\omega) - x(t,\omega,h^*(t,\omega))\rangle dt \, d\mathbb{P} \\ &\leq \int_0^T \int_\Omega \langle h^*(t,\omega), w(t,\omega) - x(t,\omega,h^*(t,\omega))\rangle dt \, d\mathbb{P} \leq 0, \\ &\forall h \in L^2([0,T]\times\Omega,\mathbb{R}^{mn}_+,\mathbb{P}), \ \forall w \in W. \end{aligned} \tag{18}$$

Let $\{h_n\} \subset L^2([0,T]\times\Omega,\mathbb{R}^{mn}_+,\mathbb{P})$ be an approximating sequence for (11). This means that there exists a sequence $\{\varepsilon_n\}$, with $\varepsilon_n > 0$, for every $n \in \mathbb{N}$, and $\varepsilon_n \to 0$ such that

$$\int_0^T \int_\Omega \langle h_n(t,\omega), w(t,\omega) - x(t,\omega,h_n(t,\omega))\rangle dt \, d\mathbb{P} \leq \varepsilon_n, \quad \forall w \in W, \ \forall n \in \mathbb{N}.$$

For the anti-monotonicity of x, we deduce

$$
\begin{aligned}
&\int_0^T \int_\Omega \langle w(t,\omega) - x(t,\omega,h(t,\omega)), h(t,\omega) - h_n(t,\omega)\rangle dt\, d\mathbb{P} \\
&+ \int_0^T \int_\Omega \langle h(t,\omega), w(t,\omega) - x(t,\omega,h_n(t,\omega))\rangle dt\, d\mathbb{P} \\
&\le \int_0^T \int_\Omega \langle h_n(t,\omega), w(t,\omega) - x(t,\omega,h_n(t,\omega))\rangle dt\, d\mathbb{P} \le \varepsilon_n, \\
&\forall h \in L^2([0,T]\times\Omega, \mathbb{R}^{mn}_+, \mathbb{P}),\ \forall w \in W.
\end{aligned} \tag{19}
$$

Let us consider

$$z^*(t,\omega) = (h^*(t,\omega), x(t,\omega,h^*(t,\omega)))^T, \quad \text{a.e. in } [0,T],\ \mathbb{P} - \text{a.s.}$$

and

$$z_n(t,\omega) = (h_n(t,\omega), x(t,\omega,h_n(t,\omega)))^T, \quad \forall n \in \mathbb{N},\ \text{a.e. in } [0,T],\ \mathbb{P} - \text{a.s.}$$

If $\{z_n\}$ is unbounded, without loss of generality, we can assume that $\|z_n\| \to +\infty$. Let us set

$$\lambda_n = \frac{1}{\|z_n - z^*\|},$$

and

$$
\begin{aligned}
\zeta_n(t,\omega) = (k_n(t,\omega), y_n(t,\omega)) &= z^*(t,\omega) + \lambda_n(z_n(t,\omega) - z^*(t,\omega)) \\
&= (h^*(t,\omega) + \lambda_n(h_n(t,\omega) - h^*(t,\omega)), x(t,\omega,h^*(t,\omega)) \\
&\quad + \lambda_n(x(t,\omega,h_n(t,\omega)) - x(t,\omega,h^*(t,\omega)))).
\end{aligned}
$$

Without loss of generality, we can assume that $\lambda_n \in (0,1]$ and $\zeta_n \to \zeta = (k,y) \neq z^*$. Moreover it results that $y \in W$, being W closed and convex. For any $w \in W$ and any $h \in L^2([0,T]\times\Omega, \mathbb{R}^{mn}_+, \mathbb{P})$, we get

$$
\begin{aligned}
&\int_0^T \int_\Omega \langle w(t,\omega) - x(t,\omega,h(t,\omega)), h(t,\omega) - k(t,\omega)\rangle dt\, d\mathbb{P} \\
&\qquad + \int_0^T \int_\Omega \langle h(t,\omega), w(t,\omega) - y(t,\omega)\rangle dt\, d\mathbb{P} \\
&\quad = \int_0^T \int_\Omega \langle w(t,\omega) - x(t,\omega,h(t,\omega)), k_n(t,\omega) - k(t,\omega)\rangle dt\, d\mathbb{P} \\
&\qquad + \int_0^T \int_\Omega \langle w(t,\omega) - x(t,\omega,h(t,\omega)), h^*(t,\omega) - k_n(t,\omega)\rangle dt\, d\mathbb{P} \\
&\qquad + \int_0^T \int_\Omega \langle w(t,\omega) - x(t,\omega,h(t,\omega)), h(t,\omega) - h^*(t,\omega)\rangle dt\, d\mathbb{P} \\
&\qquad + \int_0^T \int_\Omega \langle h(t,\omega), w(t,\omega) - x(t,\omega,h^*(t,\omega))\rangle dt\, d\mathbb{P} \\
&\qquad + \int_0^T \int_\Omega \langle h(t,\omega), x(t,\omega,h^*(t,\omega)) - y_n(t,\omega)\rangle dt\, d\mathbb{P} \\
&\qquad + \int_0^T \int_\Omega \langle h(t,\omega), y_n(t,\omega) - y(t,\omega)\rangle dt\, d\mathbb{P} \\
&\quad = \left\{ \int_0^T \int_\Omega \langle w(t,\omega) - x(t,\omega,h(t,\omega)), k_n(t,\omega) - k(t,\omega)\rangle dt\, d\mathbb{P} \right. \\
&\qquad \left. + \int_0^T \int_\Omega \langle h(t,\omega), y_n(t,\omega) - y(t,\omega)\rangle dt\, d\mathbb{P} \right\} \\
&\qquad + \left\{ \int_0^T \int_\Omega \langle w(t,\omega) - x(t,\omega,h(t,\omega)), h(t,\omega) - h^*(t,\omega)\rangle dt\, d\mathbb{P} \right. \\
&\qquad + \int_0^T \int_\Omega \langle h(t,\omega), w(t,\omega) - x(t,\omega,h^*(t,\omega))\rangle dt\, d\mathbb{P} \\
&\qquad + \lambda_n \left\{ \int_0^T \int_\Omega \langle w(t,\omega) - x(t,\omega,h(t,\omega)), h^*(t,\omega) - h_n(t,\omega)\rangle dt\, d\mathbb{P} \right. \\
&\qquad \left. + \int_0^T \int_\Omega \langle h(t,\omega), x(t,\omega,h^*(t,\omega)) - x(t,\omega,h_n(t,\omega))\rangle dt\, d\mathbb{P} \right\} \\
&\quad = \left\{ \int_0^T \int_\Omega \langle w(t,\omega) - x(t,\omega,h(t,\omega)), k_n(t,\omega) - k(t,\omega)\rangle dt\, d\mathbb{P} \right. \\
&\qquad \left. + \int_0^T \int_\Omega \langle h(t,\omega), y_n(t,\omega) - y(t,\omega)\rangle dt\, d\mathbb{P} \right\} \\
&\qquad + (1-\lambda_n) \left\{ \int_0^T \int_\Omega \langle w(t,\omega) - x(t,\omega,h(t,\omega)), h(t,\omega) - h^*(t,\omega)\rangle dt\, d\mathbb{P} \right. \\
&\qquad + \int_0^T \int_\Omega \langle h(t,\omega), w(t,\omega) - x(t,\omega,h^*(t,\omega))\rangle dt\, d\mathbb{P} \\
&\qquad + \lambda_n \left\{ \int_0^T \int_\Omega \langle w(t,\omega) - x(t,\omega,h(t,\omega)), h(t,\omega) - h_n(t,\omega)\rangle dt\, d\mathbb{P} \right. \\
&\qquad \left. + \int_0^T \int_\Omega \langle h(t,\omega), w(t,\omega) - x(t,\omega,h_n(t,\omega))\rangle dt\, d\mathbb{P} \right\}.
\end{aligned} \tag{20}
$$

By using (18)–(20), it follows

$$\begin{aligned}&\int_0^T \int_\Omega \langle w(t,\omega) - x(t,\omega,h(t,\omega)), h(t,\omega) - k(t,\omega)\rangle dt\, d\mathbb{P} \\ &+ \int_0^T \int_\Omega \langle h(t,\omega), w(t,\omega) - y(t,\omega)\rangle dt\, d\mathbb{P} \\ &\le \int_0^T \int_\Omega \langle w(t,\omega) - x(t,\omega,h(t,\omega)), k_n(t,\omega) - k(t,\omega)\rangle dt\, d\mathbb{P} \\ &+ \int_0^T \int_\Omega \langle h(t,\omega), y_n(t,\omega) - y(t,\omega)\rangle dt\, d\mathbb{P} + \lambda_n \varepsilon_n, \\ &\forall w \in W,\ \forall h \in L^2([0,T] \times \Omega, \mathbb{R}^{mn}_+, \mathbb{P}),\ \forall n \in \mathbb{N}.\end{aligned}$$

Passing to the limit as $n \to +\infty$ in the above inequality, we obtain

$$\begin{aligned}&\int_0^T \int_\Omega \langle w(t,\omega) - x(t,\omega,h(t,\omega)), h(t,\omega) - k(t,\omega)\rangle dt\, d\mathbb{P} \\ &\qquad + \int_0^T \int_\Omega \langle h(t,\omega), w(t,\omega) - y(t,\omega)\rangle dt\, d\mathbb{P} \le 0, \\ &\qquad\qquad \forall w \in W,\ \forall h \in L^2([0,T] \times \Omega, \mathbb{R}^{mn}_+, \mathbb{P}).\end{aligned} \tag{21}$$

For any $h' \in L^2([0,T] \times \Omega, \mathbb{R}^{mn}_+, \mathbb{P})$ and any $y' \in W$, define $k_\lambda(t,\omega) = k(t,\omega) + \lambda(h'(t,\omega) - k(t,\omega))$ and $y_\lambda(t,\omega) = y(t,\omega) + \lambda(y'(t,\omega) - y(t,\omega))$, for all $\lambda \in [0,1]$, a.e. in $[0,T]$, $\mathbb{P}$-a.s. Making use of (21), it results

$$\begin{aligned}&\int_0^T \int_\Omega \langle y_\lambda(t,\omega) - x(t,\omega,k_\lambda(t,\omega)), k_\lambda(t,\omega) - k(t,\omega)\rangle dt\, d\mathbb{P} \\ &\qquad + \int_0^T \int_\Omega \langle k_\lambda(t,\omega), y_\lambda(t,\omega) - y(t,\omega)\rangle dt\, d\mathbb{P} \le 0,\end{aligned}$$

which implies

$$\begin{aligned}&\int_0^T \int_\Omega \langle y_\lambda(t,\omega) - x(t,\omega,k_\lambda(t,\omega)), h'(t,\omega) - k(t,\omega)\rangle dt\, d\mathbb{P} \\ &+ \int_0^T \int_\Omega \langle k_\lambda(t,\omega), y'(t,\omega) - y(t,\omega)\rangle dt\, d\mathbb{P} \le 0, \quad \forall h' \in L^2([0,T] \times \Omega, \mathbb{R}^{mn}_+, \mathbb{P}),\ \forall y' \in W.\end{aligned}$$

Being x hemicontinuous along line segments, passing to the limit as $\lambda \to 0^+$ in the above inequality, we obtain

$$\begin{aligned}&\int_0^T \int_\Omega \langle y(t,\omega) - x(t,\omega,k(t,\omega)), h'(t,\omega) - k(t,\omega)\rangle dt\, d\mathbb{P} \\ &+ \int_0^T \int_\Omega \langle k(t,\omega), y'(t,\omega) - y(t,\omega)\rangle dt\, d\mathbb{P} \le 0, \quad \forall h' \in L^2([0,T] \times \Omega, \mathbb{R}^{mn}_+, \mathbb{P}),\ \forall y' \in W.\end{aligned} \tag{22}$$

By using (22) and being h' arbitrary, it results

$$s\int_0^T\int_\Omega \langle y(t,\omega)-x(t,\omega,k(t,\omega)),v(t,\omega)\rangle dt\,d\mathbb{P}\le \text{constant}, \tag{23}$$

for every real s and every $v\in L^2([0,T]\times\Omega,\mathbb{R}^{mn},\mathbb{P})$. Taking into account (23) and the arbitrary of s and v, we can deduce that $x(t,\omega,k(t,\omega))=y(t,\omega)$, a.e. in $[0,T]$, $\mathbb{P}$ -a.s. As a consequence, we have

$$\int_0^T\int_\Omega \langle y'(t,\omega)-x(t,\omega,k(t,\omega)),k(t,\omega)\rangle dt\,d\mathbb{P}\le 0,\quad \forall y'\in W. \tag{24}$$

By using (24), we get that k solves (11) and then $k(t,\omega)=h^*(t,\omega)$, a.e. in $[0,T]$, $\mathbb{P}$-a.s., since h^* is the unique solution to (11), which contradicts $(h^*(t,\omega),x(t,\omega,h^*(t,\omega)))^T\neq(k(t,\omega),x(t,\omega,k(t,\omega)))^T$, a.e. in $[0,T]$, $\mathbb{P}$-a.s.

Hence we can suppose that $\{z_n\}$ is bounded. Let $\{z_{n_r}\}$ be any subsequence of $\{z_n\}$ such that $z_{n_r}\to(\overline{h},\overline{y})$, as $r\to+\infty$. By virtue of (19), it follows

$$\int_0^T\int_\Omega \langle w(t,\omega)-x(t,\omega,h(t,\omega)),h(t,\omega)-h_{n_r}(t,\omega)\rangle dt\,d\mathbb{P}$$
$$+\int_0^T\int_\Omega \langle h(t,\omega),w(t,\omega)-x(t,\omega,h_{n_r}(t,\omega))\rangle dt\,d\mathbb{P}\le\varepsilon_{n_r},$$
$$\forall h\in L^2([0,T]\times\Omega,\mathbb{R}^{mn}_+,\mathbb{P}),\ \forall w\in W.$$

Passing to the limit as $r\to+\infty$ in the above inequality, we obtain

$$\int_0^T\int_\Omega \langle w(t,\omega)-x(t,\omega,h(t,\omega)),h(t,\omega)-\overline{h}(t,\omega)\rangle dt\,d\mathbb{P}$$
$$+\int_0^T\int_\Omega \langle h(t,\omega),w(t,\omega)-\overline{y}(t,\omega)\rangle dt\,d\mathbb{P}\le 0,$$
$$\forall h\in L^2([0,T]\times\Omega,\mathbb{R}^{mn}_+,\mathbb{P}),\ \forall w\in W.$$

By using same arguments as in (21)–(24), we deduce

$$x(t,\omega,\overline{h}(t,\omega))=\overline{y}(t,\omega),\quad \text{a.e. in } [0,T],\ \mathbb{P}-\text{a.s.},$$

and

$$\int_0^T\int_\Omega \langle \overline{h}(t,\omega),x'(t,\omega)-x(t,\omega,\overline{h}(t,\omega))\rangle dt\,d\mathbb{P}\le 0,\quad \forall x'\in W.$$

This means that $\overline{h}$ solves (11). For the uniqueness of solution to (11), it results that $\overline{h}(t,\omega)=h^*(t,\omega)$, a.e. in $[0,T]$, $\mathbb{P}$-a.s. As a consequence, $\{h_n\}$ converges to h^* and, hence, (11) is well-posed. □

The well-posedness for the stochastic time-dependent variational inequality (14) can be introduced as done in Definition 3. More precisely, a sequence $\{z_n\}\subset L^2([0,T]\times\Omega,\mathbb{R}^{mn},\mathbb{P})$ is called an approximating sequence for (14) if and only if there exists a sequence $\{\varepsilon_n\}$, with $\varepsilon_n > 0$, for every $n \in \mathbb{N}$, and $\varepsilon_n \to 0$, such that

$$\int_0^T \int_\Omega \langle F(t,\omega,z_n(t,\omega)), z_n(t,\omega) - z(t,\omega)\rangle \, dt \, d\mathbb{P} \leq \varepsilon_n, \quad \forall z \in Z, \ \forall n \in \mathbb{N}.$$

Similarly, we say that (14) is well-posed if and only if (14) has a unique solution and every approximating sequence converges to the unique solution.

Theorem 5 *Let* $x : [0,T] \times \Omega \times L^2([0,T] \times \Omega, \mathbb{R}^{mn}, \mathbb{P}) \to L^2([0,T] \times \Omega, \mathbb{R}^{mn}, \mathbb{P})$ *be a continuous mapping. Then,* (11) *is well-posed if and only if* (14) *is well-posed.*

Proof Let us start supposing that (11) is well-posed. Hence (11) has a unique solution $h^* \in L^2([0,T] \times \Omega, \mathbb{R}^{mn}, \mathbb{P})$. Making use of Theorem 2, it follows that $z^* = (h^*, x(h^*))^T \in Z$ is the unique solution to (14). Let us consider an approximating sequence $\{z_n\} = \{(h_n, w_n)^T\} \subset Z$ for (14). Then there exists a sequence $\{\varepsilon_n\}$, with $\varepsilon_n > 0$, for every $n \in \mathbb{N}$, and $\varepsilon_n \to 0$, such that

$$\int_0^T \int_\Omega \langle F(t,\omega,z_n(t,\omega)), z_n(t,\omega) - z(t,\omega)\rangle \, dt \, d\mathbb{P} \leq \varepsilon_n, \quad \forall z = (h, x(h))^T \in Z, \ \forall n \in \mathbb{N}.$$

As a consequence, we have

$$\begin{aligned} &\int_0^T \int_\Omega \langle w_n(t,\omega) - x(t,\omega,h_n(t,\omega)), h_n(t,\omega) - h(t,\omega)\rangle dt \, d\mathbb{P} \\ &\quad - \int_0^T \int_\Omega \langle h_n(t,\omega), w_n(t,\omega) - w(t,\omega)\rangle dt \, d\mathbb{P} \leq \varepsilon_n, \\ &\quad \forall z = (h, w)^T \in Z, \ \forall n \in \mathbb{N}. \end{aligned} \tag{25}$$

Then, we deduce

$$\begin{aligned} &\int_0^T \int_\Omega \langle w_n(t,\omega) - x(t,\omega,h_n(t,\omega)), h_n(t,\omega) - h(t,\omega)\rangle dt \, d\mathbb{P} \\ &\quad \leq \varepsilon_n + \int_0^T \int_\Omega \langle h_n(t,\omega), w_n(t,\omega) - w(t,\omega)\rangle dt \, d\mathbb{P}, \\ &\quad z = (h, w)^T \in Z, \ \forall n \in \mathbb{N}. \end{aligned}$$

Fix $w \in W$, $v \in L^2([0,T] \times \Omega, \mathbb{R}^{mn}, \mathbb{P})$ and consider $h(t,\omega) = h_n(t,\omega) - sv(t,\omega)$, a.e. in $[0, T]$, $\mathbb{P}$-a.s., then

$$s \int_0^T \int_\Omega \langle w_n(t,\omega) - x(t,\omega,h_n(t,\omega)), v(t,\omega)\rangle dt \, d\mathbb{P} \leq \text{constant},$$

where s is arbitrary. Consequently $x(t,\omega,h_n(t,\omega)) = w_n(t,\omega)$, and then by (25) we get

$$
\begin{aligned}
&-\int_0^T \int_\Omega \langle h_n(t,\omega), w_n(t,\omega) - w(t,\omega)\rangle dt\, d\mathbb{P} \\
&= -\int_0^T \int_\Omega \langle h_n(t,\omega), x(t,\omega,h_n(t,\omega)) - w(t,\omega)\rangle dt\, d\mathbb{P} \le \varepsilon_n, \\
&\forall w \in W,\ \forall n \in \mathbb{N}.
\end{aligned}
$$

We have that $\{h_n\} \subset L^2([0,T]\times\Omega, \mathbb{R}^{mn}, \mathbb{P})$ is an approximating sequence for (11). Since (11) is well-posedness, it results $h_n \to h^*$. Therefore, $z_n = (h_n, x(h_n)) \to (h^*, x(h^*))$ and, hence, (14) is well-posed.

Vice versa, let us suppose that (14) is well-posed. Hence it has a unique solution $z^* = (h^*, x(h^*))^T \in Z$. By Theorem 2, it results that h^* is the unique solution to (11). Let $\{h_n\} \subset L^2([0,T]\times\Omega, \mathbb{R}^{mn}, \mathbb{P})$ be an approximating sequence for (11). Then there exists a sequence $\{\varepsilon_n\}$, with $\varepsilon_n > 0$, for every $n \in \mathbb{N}$, and $\varepsilon_n \to 0$, such that

$$
-\int_0^T \int_\Omega \langle h_n(t,\omega), x(t,\omega,h_n(t,\omega)) - w(t,\omega)\rangle\, dt\, d\mathbb{P} \le \varepsilon_n, \quad \forall w \in W,\ \forall n \in \mathbb{N}. \tag{26}
$$

Let us consider

$$
w_n(t,\omega) = x(t,\omega,h_n(t,\omega)), \quad \text{a.e. in } [0,T],\ \mathbb{P}-\text{a.s.},
$$

and

$$
z_n(t,\omega) = (h_n(t,\omega), w_n(t,\omega))^T, \quad \text{a.e. in } [0,T],\ \mathbb{P}-\text{a.s.}
$$

By using (26), one has

$$
\int_0^T \int_\Omega \langle F(t,\omega,z_n(t,\omega)), z_n(t,\omega) - z(t,\omega)\rangle\, dt\, d\mathbb{P} \le \varepsilon_n, \quad \forall z = (h, x(h))^T \in Z,\ \forall n \in \mathbb{N}.
$$

This means that $\{z_n\}$ is an approximating sequence for (14). Making use of the well-posedness of (14), we obtain that $z_n \to z^*$. As a consequence, the sequence $\{h_n\}$ converges to h^*. Then the claim is completely achieved. □

5 Numerical method

In this section we present an algorithm for solving stochastic time-dependent variational inequalities as (6) based on a combination between a discretization procedure and a projection method. The algorithm we present is a generalization of the one in [19] extended to the evolutionary case.

In the deterministic setting, the classical projection method for the following variational inequality

$$
\langle F(x^*), x - x^*\rangle \ge 0, \quad \forall x \in K,
$$

where K is a nonempty closed convex subset of $\mathbb{R}^m$ and $F : K \to \mathbb{R}^m$ is a mapping, akin to the projection method for convex optimization, is

$$x_{k+1} = P_K(x_k - \alpha F(x_k)),$$

where P_K is the projection operator onto K and α is a suitable real number. The convergence of this method is guaranteed assuming F is strongly monotone, Lipschitz continuous and the real number α satisfies a strong condition depending on the modulus of strong monotonicity and the Lipschitz constant. Nevertheless the requirement of strong monotonicity is heavy and often fails to be satisfied in applications. More refined extragradient methods are available in the literature.

In the stochastic setting, the first method for stochastic variational inequalities, proposed in [21], updates iteratively x_k according to the formula

$$x_{k+1} = P_K(x_k - \alpha_k F(v_k, x_k)),$$

where $\{v_k\}$ is a sample of v and $\{\alpha_k\}$ is a sequence of positive stepsizes. Again the convergence is ensured under strong hypotesis on monotonicity of F, Lipschitz constant and tight conditions on the stepsize. After this first instance, the recent research on stochastic approximation methods for stochastic variational inequalities had remarkable developments.

We introduce here an extragradient method for stochastic time-dependent variational inequalities as (6) based on [19]. In [7, Theorem 4] it is proved that under the assumptions (i), (ii), (iii), and assuming the continuity of the data, the solution to (6) is stochastic continuous on $[0, T]$. The stochastic continuity allows us to carry out a discretization procedure in order to reduce the time-dependent problem to some static problems. That is, firstly we consider a partition of $[0, T]$ such that:

$$0 = t_0 < t_1 < \cdots < t_r < \cdots < t_n = T.$$

For each value t_r, $r = 0, 1, ..., n$, we apply an extragradient method to solve the point-to-point stochastic variational inequality

$$\int_\Omega \langle F(t, \omega, x^*(t, \omega)), x(t, \omega) - x^*(t, \omega)\rangle d\mathbb{P} \geq 0, \quad \forall x(t, \omega) \in \mathbb{K}(t), \text{ in } [0, T], \tag{27}$$

where

$$\mathbb{K}(t) = \Big\{ x(t, \omega) \in \mathbb{R}^{mn} : \underline{x}_{ij}(t, \omega) \leq x_{ij}(t, \omega) \leq \overline{x}_{ij}(t, \omega), \\ \forall i = 1, \ldots, m, \ \forall j = 1, \ldots, n, \ \mathbb{P} - \text{a.s.} \Big\},$$

with $t = t_r$, $r = 0, 1, ..., n$. More precisely we have the following:

```
Algorithm
```

- Discretize the interval $[0, T]$: $0 = t_0 < t_1 < \cdots < t_r < \cdots < t_n = T$.
- For each $r = 1, \ldots, n$, choose an initial point x_r^0, a positive stepsize sequence $\{\alpha_k\}$, the sample rate $\{N_k\}$ and initial samples $\{\omega_j^0\}_{j=1}^{N_0}$ and $\{\eta_j^0\}_{j=1}^{N_0}$ of the random variable ω.
- Given the iterate x_r^k, generate the samples $\{\omega_j^k\}_{j=1}^{N_k}$ and $\{\eta_j^k\}_{j=1}^{N_k}$ of ω and define

$$z_r^k = P_{\mathbb{K}(t_r)}\left(x_r^k - \frac{\alpha_k}{N_k} \sum_{j=1}^{N_k} F(t_r, \omega_j^k, x_r^k) \right),$$

$$x_r^{k+1} = P_{\mathbb{K}(t_r)}\left(x_r^k - \frac{\alpha_k}{N_k} \sum_{j=1}^{N_k} F(t_r, \eta_j^k, z_r^k) \right).$$

- Interpolate the numerical solutions for every $r = 1, \ldots, n$.

For each $r = 1, \ldots, n$, the convergence of the method to the solution to the corresponding stochastic variational inequality (27) is guarantees by the convergence results proved in [19]. In order to obtain the convergence of our scheme in the time interval $[0, T]$, we consider a sequence $\{\pi_s\}$ of (not necessarily equidistant) partitions of the time interval $[0, T]$ such that $\pi_s = (t_s^0, t_s^1, \ldots, t_s^{N_s})$, where $0 = t_s^0 < t_s^1 < \ldots < t_s^{N_s} = T$ and assume that

$$k_s = \max\{t_s^r - t_s^{r-1} \; : \; r = 1, 2, \ldots, N_s\},$$

approaches zero, as $s \to +\infty$. Now we construct the approximate solution to (27) by considering

$$x_s(t, \omega) = \sum_{r=1}^{N_s} x(t_s^r, \omega) \chi_{[t_s^{r-1}, t_s^r[}(t),$$

where $x(t_s^r, \omega)$ is the solution to the stochastic variational inequality (27) at $t = t_s^r$, which is computed by means of the projection method. It is possible to show the L^1 -convergence of such approximate solutions by making use of same arguments in [1].

5.1 Example

An example of the random dynamic oligopolistic market equilibrium problem which emphasizes the central role in the study of the model of the uncertainty is provided. To this aim, let us analyze an economic network made up of two firms and two demand markets, as in Fig. 1.

Let $x_{ij}(t, \omega)$ be the commodity shipment from P_i to Q_j, $i = 1, 2$, $j = 1, 2$, and assume that $\underline{x}_{ij}(t, \omega) \leq x_{ij}(t, \omega) \leq \overline{x}_{ij}(t, \omega)$ holds, where $\underline{x}_{ij}(t, \omega)$ and $\overline{x}_{ij}(t, \omega)$ are function on $[0, 25] \times \Omega$ representing the capacity constraints. Assume that

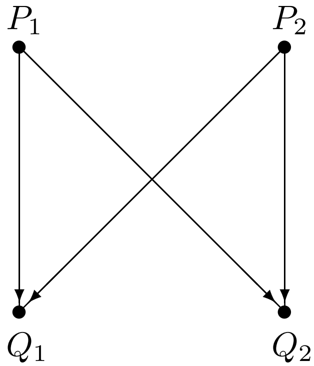


Fig. 1 Network structure of the numerical dynamic spatial oligopoly problem

$\underline{x}_{ij}(t,\omega) = t^2\check{x}_{ij}(\omega)$ and $\overline{x}_{ij}(t,\omega) = t^2\hat{x}_{ij}(\omega)$, where $\hat{x}_{ij}(\omega)$ and $\check{x}_{ij}(\omega)$ are uniformly distributed random variables with probability density functions given by:

$$f_{\hat{x}_{i1}}(z) = \begin{cases} \frac{1}{2}, & \text{if} \quad 0 \le z \le 2, \\ 0, & \text{elsewhere,} \end{cases}$$

$$f_{\hat{x}_{i2}}(z) = \begin{cases} \frac{1}{5}, & \text{if} \quad 0 \le z \le 5, \\ 0, & \text{elsewhere,} \end{cases}$$

$$f_{\check{x}_{i1}}(z) = \begin{cases} \frac{1}{25}, & \text{if} \quad 75 \le z \le 100, \\ 0, & \text{elsewhere,} \end{cases}$$

$$f_{\check{x}_{i2}}(z) = \begin{cases} \frac{1}{20}, & \text{if} \quad 80 \le z \le 100, \\ 0, & \text{elsewhere.} \end{cases}$$

Set now the maximal commodity production of P_i, $i = 1, 2$, and the maximal commodity demand of Q_j, $j = 1, 2$. Let us define $p_i(t,\omega) = t^2\overline{p}_i(\omega)$, $i = 1, 2$, and $q_j(t,\omega) = t^2\overline{q}_j(\omega)$, $j = 1, 2$, where the density function of $\overline{p}_i(\omega)$, $i = 1, 2$, and $\overline{q}_j(\omega)$, $j = 1, 2$, are defined by

$$f_{\overline{p}_1}(z) = \begin{cases} \frac{1}{20}, & \text{if } 80 \le z \le 100, \\ 0, & \text{elsewhere} \end{cases}$$

$$f_{\overline{p}_2}(z) = \begin{cases} \frac{1}{10}, & \text{if } 90 \le z \le 100, \\ 0, & \text{elsewhere} \end{cases}$$

$$f_{\overline{q}_1}(z) = \begin{cases} \frac{1}{30}, & \text{if } 240 \le z \le 270, \\ 0, & \text{elsewhere} \end{cases}$$

$$f_{\overline{q}_2}(z) = \begin{cases} \frac{1}{40}, & \text{if } 150 \le z \le 190, \\ 0, & \text{elsewhere} \end{cases}$$

The feasible set $\mathbb{K}$ is then as in (4) with the above definitions of $\underline{x}_{ij}(t,\omega)$, $\overline{x}_{ij}(t,\omega)$, $\overline{p}_i(t,\omega)$, $\overline{q}_j(t,\omega)$, $i = 1, 2$, $j = 1, 2$.

Then we left to define the profit function $v_i(t,\omega,x(t,\omega))$ for the firms P_i, $i = 1, 2$. We set

$$v_1(t,\omega,x(t,\omega)) = -\,3x_{11}^2(t,\omega) - x_{12}^2(t,\omega) + tx_{12}(t,\omega)x_{21}(t,\omega) + \\ t^2 a_1(t,\omega)x_{11}(t,\omega) + b_1(t,\omega)x_{12}(t,\omega)$$
$$v_2(t,\omega,x(t,\omega)) = -\,2e^t x_{21}^2(t,\omega) - 3tx_{22}^2(t,\omega) + x_{11}(t,\omega)x_{22}(t,\omega) + \\ a_2(t,\omega)x_{21}(t,\omega) + b_2(t,\omega)x_{22}(t,\omega)$$

where $a_i, b_i, i = 1,2$ are uniformly distributed random variables with supports:

$$\operatorname{spt} a_1 = [36,108], \quad \operatorname{spt} b_1 = [10,40]$$
$$\operatorname{spt} a_2 = [40,120], \quad \operatorname{spt} b_2 = [10,40]$$

Let us compute the operator $\nabla_D v$

$$-\nabla_D v(x) = \begin{pmatrix} 6x_{11} - t^2 a_1 & 2x_{12} - tx_{21} - b_1 \\ 4e^t x_{21} - a_2 & 6tx_{22} - x_{11} - b_2 \end{pmatrix},$$

where here, and in the following, we omit the arguments of the variables, simply writing x_{ij} instead of $x_{ij}(t,\omega)$.

The equilibrium conditions is expressed by the following variational inequality problem: find $x^* \in L^2([0,25]\times\Omega,\mathbb{R}^4_+,\mathbb{P})$ such that

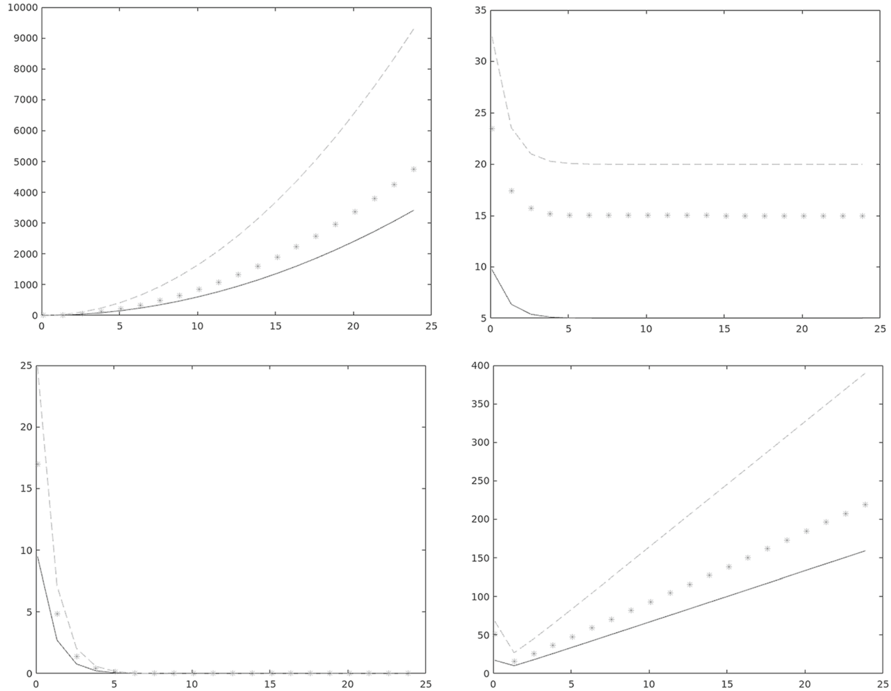
$$\ll -\nabla_D v(x^*), x - x^* \gg_{\mathbb{E}} \geq 0, \quad \forall x \in \mathbb{K}.$$

At first we observe that the existence and uniqueness to the solution to problem above is guaranteed by the theoretical result of [7] and Sect. 3. Moreover all the assumptions to ensure the convergence of the above computational method are satisfied. As a consequence, we can apply the method described before to obtain the numerical solutions x^*_{ij}, $i = 1,2,\ j = 1,2$, for different random evaluation of a_i, b_i, $i = 1,2$, evolving in time. We plot the x^*_{ij}, $i = 1,2,\ j = 1,2$, in the figures of Table 1 for different random variables. Observe that the solution $x^* = (x^*_{ij})$ verifying the following

$$\operatorname{spt} x^*_{11} = [6,18], \qquad \operatorname{spt} x^*_{12} = [20,60]$$
$$\operatorname{spt} x^*_{21} = [10,30], \quad \operatorname{spt} x^*_{22} = [16,58]$$

and, hence, x^* belongs to $\mathbb{K}$ proving that x^* is the solution of the random dynamic oligopolistic market equilibrium problem.

Table 1 Curves of equilibria



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Data availability The authors do not analyze or generate any datasets, because our work proceeds within a theoretical and mathematical approach. One can obtain the relevant materials from the references below.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

References

1. Barbagallo, A.: Existence and regularity of solutions to nonlinear degenerate evolutionary variational inequalities with applications to dynamic network equilibrium problems. Appl. Math. Comput. **208**, 1–13 (2009)
2. Barbagallo, A.: On the regularity of retarded equilibria in time-dependent traffic equilibrium problems. Nonlinear Anal. **71**, e2406–e2417 (2009)
3. Barbagallo, A.: Advances results on variational formulation in oligopolistic market equilibrium problem. Filomat **26**, 935–947 (2012)
4. Barbagallo, A., Cojocaru, M.-G.: Dynamic equilibrium formulation of oligopolistic market problem. Math. Comput. Model. **49**, 966–976 (2009)
5. Barbagallo, A., Di Meglio, G., Ferrara, M., Mauro, P.: Variational approach to a random oligopolistic market equilibrium problem with excesses, submitted
6. Barbagallo, A., Ferrara, M., Mauro, P.: Stochastic variational approach for random Cournot-Nash principle. In: Jadamba, B., Khan, A.A., Migorski, S., Sama, M. (eds.) Deterministic and Stochastic Optimal Control and Inverse Problems, pp. 241–269. CRC Press, Boca Raton (2021)
7. Barbagallo, A., Guarino Lo Bianco, S.: Stochastic variational formulation for a general random time-dependent economic equilibrium problem. Optim. Lett. **14**, 2479–2493 (2020)
8. Barbagallo, A., Guarino Lo Bianco, S.: Variational inequalities on a class of a structured tensors. J. Nonlinear Conv. Anal. **19**, 711–729 (2018)
9. Barbagallo, A., Guarino Lo Bianco, S.: On ill-posedness and stability of tensor variational inequalities: application to an economic equilibrium. J. Glob. Optim. **77**, 125–141 (2020)
10. Barbagallo, A., Guarino Lo Bianco, S., Toraldo, G.: Tensor variational inequalities: thoeretical results, numerical methods and applications to an economic equilibrium model. J. Nonlinear Var. Anal. **4**, 87–105 (2020)
11. Barbagallo, A., Mauro, P.: Evolutionary variational formulation for oligopolistic market equilibrium problems with production excesses. J. Optim. Theory Appl. **155**, 1–27 (2012)
12. Barbagallo, A., Mauro, P.: Time-dependent variational inequality for an oligopolistic market equilibrium problem with production and demand excesses. Abstr. Appl. Anal. 2012, art. no. 651975 (2012)
13. Barbagallo, A., Mauro, P.: Inverse variational inequality approach and applications. Numer. Funct. Anal. Optim. **35**, 851–867 (2014)
14. Barbagallo, A., Scilla, G.: Stochastic weighted variational inequalities in non-pivot Hilbert spaces with applications to a transportation model. J. Math. Anal. Appl. **457**, 1118–1134 (2018)
15. He, B., He, X.Z., Liu, H.X.: Solving a class of "black-box'' inverse variational inequalities. Eur. J. Oper. Res. **234**, 283–295 (2010)
16. Kannan, A., Shanbhag, U.V.: Optimal stochastic extragradient schemes for pseudomonotone stochastic variational inequality problems and their variants. Comput. Optim. Appl. **74**, 779–820 (2019)
17. Gwinner, J., Jadamba, B., Khan, A.A., Raciti, F.: Uncertainty Quantification in Variational Inequalities: Theory, Numerics, and Applications. Chapman and Hall/CRC Press, Boca Raton (2021)
18. Hawks, R., Jadamba, B., Khan, A.A., Sama, M., Yang, Y.: A Variational Inequality Based Stochastic Approximation for Inverse Problems in Stochastic Partial Differential Equations. In: Rassias, T.M., Pardalos, P.M. (eds.) Nonlinear Analysis and Global Optimization, pp. 207–226. Springer Optimization and Its Applications 167, Springer, (2021)
19. Iusem, A., Jofré, A., Oliveira, R.I., Thompson, P.: Extragradient method with variance reduction for stochastic variational inequalities. SIAM J. Optim. **27**, 686–724 (2017)
20. Iusem, A., Jofré, A., Thompson, P.: Incremental constraint projection methods for monotone stochastic variational inequalities. Math. Oper. Res. **44**, 236–263 (2019)
21. Jiang, H., Xu, H.: Stochastic approximation approaches to the stochastic variational inequality problem. IEEE Trans Autom Control **53**, 1462–1475 (2008)
22. Mangasarian, O.: Pseudoconvex functions. J. Soc. Ind. Appl. Math. Ser. A Control **3**, 281–290 (1965)
23. Maugeri, A., Raciti, F.: On existence theorems for monotone and nonmonotone variational inequalities. J. Convex Anal. **16**, 899–911 (2009)
24. Rockafellar, R.T., Wets, R.J.-B.: Stochastic variational inequalities: single-stage to multistage. Math. Program. **165**, 331–360 (2017)

25. Yang, J.: Dynamic power price problem: an inverse variational inequality approach. J. Ind. Manag. Optim. **4**, 673–684 (2008)