

This is the peer reviewed version of the following article:

Hamiltonian/Stroh formalism for reversible poroelasticity (and thermoelasticity) / Nobili, A.. - In: INTERNATIONAL JOURNAL OF SOLIDS AND STRUCTURES. - ISSN 0020-7683. - 301:(2024), pp. 112935-112935. [10.1016/j.ijsolstr.2024.112935]

*Terms of use:*

The terms and conditions for the reuse of this version of the manuscript are specified in the publishing policy. For all terms of use and more information see the publisher's website.

23/04/2025 20:33

(Article begins on next page)

1 Hamiltonian/Stroh formalism for reversible poroelasticity  
2 (and thermoelasticity)

3 Andrea Nobili<sup>1</sup>

4 *Department of Engineering Enzo Ferrari, University of Modena and Reggio Emilia, via*  
5 *Vivarelli 10, 41125 Modena, Italy*

---

6 **Abstract**

Stroh's sextic formalism represents the equilibrium equations of anisotropic elasticity in a particularly attractive form, that is most suitable for studying layered and composite materials and time harmonic problems. Taking advantage of the fact that the Stroh formalism really amounts to the canonical form of the equations in the Hamiltonian sense, the case of Biot's reversible (i.e. no fluid dissipation) poroelasticity is here addressed, in the absence of a fluid pressure gradient. This framework is the same as thermoelasticity of perfect conductors. Two Hamiltonian formulations are developed: the first describes both the solid and the fluid phases and it exhibits, besides energy conservation, momentum conservation, as a result of pressure uniformity. The second is restricted to the solid skeleton and parallels anisotropic elasticity, although with Stroh matrices that account for fluid coupling. The case of weak fluid-solid coupling is also considered and it produces a perturbation from anisotropic elasticity with the same structure as incompressibility, although in an "opposing" manner. This comparison suggests that the incompressibility limit introduced by Biot should be revised. The energy conservation integral and the edge impedance matrix are also illustrated.

7 *Keywords:* Stroh formalism, reversible poroelasticity, thermoelasticity,  
8 Hamiltonian form

---

*Email address:* [andrea.nobili@unimore.it](mailto:andrea.nobili@unimore.it) (Andrea Nobili)

## 9 1. Introduction

10 The foundation of Stroh sextic formalism is laid out in a pair of celebrated pa-  
11 pers concerning dislocations (Stroh, 1958) and harmonic motion (Stroh, 1962) in  
12 generally anisotropic materials (in plane strain). This framework provides a sub-  
13 stantial improvement over the already established Eshelby-Reid-Shockley form  
14 of the equations of elastostatics (Ting, 1996). Indeed, although both methods  
15 share the fact that mechanical features are interpreted and described under the  
16 unifying lens of linear algebra, Stroh’s approach exhibits very distinctive features  
17 for the involved matrices. In fact, solutions are built in terms of eigenvalues and  
18 eigenvectors of a block matrix, the so-called Stroh fundamental elastic matrix,  
19 endowed with many striking properties (Barnett, 2000). Since then, the method  
20 has been extensively applied to composite materials, harmonic wave propaga-  
21 tion, crack and dislocations, instability and many more topics (Ting, 1996).  
22 Given its success, it is little wonder that extensions of the method have been  
23 proposed outside its original domain, to address, for example, constrained ma-  
24 terials (Chadwick and Smith, 1977), anisotropic plates (Fu, 2007) and internally  
25 constrained micro-polar solids (Nobili and Radi, 2022). In general, the success  
26 of the procedure hangs on the careful choice of the unknown variables, which  
27 can be rather tricky unless somehow guided. In fact, many contributions ex-  
28 ist in the literature where a trial-and-error approach was used (see Fu (2007)).  
29 As an illustration, Hwu (2003) analyses coupled stretching-bending modes in  
30 anisotropic laminates through a modification of the Lekhnitskii formalism (for  
31 details on which see Ting (1996)) in an attempt to recover the properties specific  
32 to the Stroh form. Recently, Fu (2003) developed a Stroh-like formulation for  
33 determining the dispersion relation of edge waves in generally anisotropic plates  
34 under the sole restriction that the mid-plane is a plane of material symmetry.  
35 In that work, Fu capitalized on the observation that the Stroh formalism is re-  
36 ally an Hamiltonian formulation where a space variable is treated in time-like  
37 fashion, already available in the literature (Barnett, 2000), to develop a guid-  
38 ing principle for the right choice of the unknown pairs, namely the principle of

39 energy conjugation. Successively, this approach was used in (Fu and Brookes,  
40 2006) to study edge waves in asymmetrically laminated plates, for which in-  
41 plane and out-of-plane deformations are coupled by anisotropy. By the same  
42 method, Fu (2007) studies incompressible anisotropic materials and anisotropic  
43 plates, and results are later extended by Edmondson and Fu (2009) to generally  
44 constrained and pre-stressed anisotropic materials. The procedure paves the  
45 way for the application of the surface-impedance matrix for studying localized  
46 waves (Fu, 2005).

47 Thus far, a classical Stroh formalism could be retrieved, by which a right  
48 eigenvalue problem is finally obtained (as presently explained). Yet, the Hamil-  
49 tonization of any mechanical model may be carried out by the same principles  
50 and the outcome, in general, may not correspond to a classical Stroh-like struc-  
51 ture. As a case in point, Fu and Kaplunov (2012) study waves localized at the  
52 edge of isotropic thin cylindrical shells and find that the fundamental elastic  
53 matrix is in fact wavenumber dependent. This result, which is typical of dis-  
54 persive systems, is also retrieved by Nobili and Radi (2022) in the context of  
55 the indeterminate couple-stress theory of elasticity. The structure of the Stroh  
56 formalism is now supplemented by a right hand side that is proportional to  
57 the unknowns (i.e. the problem is still linear). Therefore, the very form of  
58 the Stroh-like canonical system already reveals important informations on the  
59 problem under scrutiny.

60 Biot's poroelasticity is a very successful phenomenological theory with enor-  
61 mous practical implications in the fields of seismology and seismic exploration,  
62 geology and geotechnical structures, soil testing and characterization, to name  
63 only a few Dullien (2012). The literature on this topic is very extensive and  
64 moves in many directions, for example, concerning wave propagation in porous  
65 media, see the review paper by Corapcioglu and Tuncay (1996). Efforts in the  
66 direction of connecting this theory to the theory of mixtures or to microme-  
67 chanical theories have been long going, with mixed success, see, among many,  
68 Lopatnikov and Cheng (2004). Extensions of the theory have been proposed  
69 in the many directions, for example introducing double (Berryman and Wang,

70 2000) or multi- porosities (Pramanik et al., 2024), finite elastic deformations  
 71 (Norris and Grinfeld, 1995) or even piezoelectric effects (Sharma, 2010). Still,  
 72 no Stroh-like formalism may be traced in the literature, possibly on the grounds  
 73 that multi-field theories may prove impervious to this framework.

74 In this paper, we Hamiltonize the equations of Biot’s poroelasticity in the  
 75 absence of dissipation, i.e. in the context of reversible processes (thermostatistics)  
 76 and in the absence of a fluid pressure gradient (Biot, 1955, 1956a). This same  
 77 framework may be applied to thermoelastostatics of perfect conductors, where in  
 78 fact temperature plays the role of the fluid pressure (Biot, 1956b). Inertia effects  
 79 are only considered inasmuch as they may be incorporated into the material  
 80 properties in the form of time-harmonic contributions. Focus is set on the  
 81 determination of the canonical formalism and on the properties it reveals.

## 82 2. Reversible poroelasticity

83 Let  $\mathbf{u}$  and  $\mathbf{U}$  denote the displacement in the solid and in the fluid phase,  
 84 respectively. Besides, the fluid-to-solid displacement per unit volume of the  
 85 poroelastic medium reads

$$\mathbf{w} = f(\mathbf{U} - \mathbf{u}), \quad (1)$$

86 where  $f$  is the *effective porosity*, generally not uniform, that represents the  
 87 interconnected pore space. In particular, the porosity is defined as the ratio  
 88 between the volume of interconnected pores,  $V_p$ , over the bulk volume  $V_b$ , the  
 89 latter being obtained by  $V_b = V_p + V_s$ , i.e. summing the pore volume to the  
 90 volume occupied by the solid skeleton, see (Biot, 1955). Following Biot (1962),  
 91 in this theory closed porosity is assumed to be part of the solid skeleton. Also,  
 92 we let

$$e = \operatorname{div} \mathbf{u}, \quad \zeta = -\operatorname{div} \mathbf{w}, \quad (2)$$

93 that provide the volume *increment* for the solid and the fluid phase, respectively  
 94 (indeed the fluid increment is obtained by the inflow of  $\mathbf{w}$ ). In particular, we  
 95 have the connection

$$-\zeta = (\mathbf{U} - \mathbf{u}) \cdot \operatorname{grad} f + f(\epsilon - e), \quad (3)$$

96 where we have let  $\epsilon = \text{div } \mathbf{U}$ . In the case of uniform porosity, we retrieve the  
 97 result given in Biot (1962)

$$\epsilon = e - f^{-1}\zeta. \quad (4)$$

Let the rank-2 tensor  $\mathbf{T}$  denote the *total stress*, that is obtained summing the stress in the solid phase  $\boldsymbol{\sigma}$  with the stress in the fluid phase  $\boldsymbol{\sigma}_f = -fp_f\mathbf{1}$ , where, here and after,  $\mathbf{1}$  is the rank-3 identity tensor and  $p_f$  is the fluid pressure (positive when compressive) per unit area of the fluid phase. Sometimes, to refer pressure to the unit bulk area, the shorthand  $\sigma_f = -fp_f$  is introduced. Let  $(O, x_1, x_2, x_3)$  denote an orthogonal reference frame and  $\mathbf{n}$  be the unit vector normal to any relevant directed surface  $S$ . Alongside the axis  $(x_1, x_2, x_3)$ , we introduce an orthonormal set of basis vectors,  $\mathbf{e}_1$ ,  $\mathbf{e}_2$  and  $\mathbf{e}_3$ , such that  $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$ , with the usual understanding that twice repeated subscripts are summed over in the set  $\{1, 2, 3\}$ . Here,  $\delta_{ij}$  is zero for  $i \neq j$  and 1 for  $i = j$ . We define the fundamental force vectors in a generally anisotropic medium with elastic constants  $c_{ijkl}$

$$\mathbf{t}_1 = \mathbf{T}\mathbf{e}_1 = \mathbf{Q}\mathbf{u}_{,1} + \mathbf{R}\mathbf{u}_{,2} - r\zeta\mathbf{e}_1, \quad (5a)$$

$$\mathbf{t}_2 = \mathbf{T}\mathbf{e}_2 = \mathbf{R}^T\mathbf{u}_{,1} + \mathbf{T}\mathbf{u}_{,2} - r\zeta\mathbf{e}_2, \quad (5b)$$

98 where  $Q_{ij} = c_{i1j1}$ ,  $R_{ij} = c_{i1j2}$ ,  $T_{ij} = c_{i2j2}$  are the usual Stroh matrices. In  
 99 particular,  $\mathbf{Q}$  and  $\mathbf{T}$  are symmetric, i.e.  $\mathbf{Q} = \mathbf{Q}^T$  and  $\mathbf{T} = \mathbf{T}^T$ , and positive def-  
 100 inite, provided the strain energy is a positive function (Ting, 1996, §6.1). Here,  
 101  $r$  denotes the cross coupling term between volume changes in the solid and in  
 102 the fluid (denoted by  $C$  in (Biot, 1962, Eq.(3.5)), and by  $Q/f$  in (Corapcioglu  
 103 and Tuncay, 1996, Eq.(2.16))). In this paper, we assume that cross coupling oc-  
 104 curs in isotropic fashion, for transverse anisotropy see, for example, Biot (1955,  
 105 Eq.(3.2)). Besides, it is assumed that dependent variables are independent from  
 106  $x_3$ , i.e.  $\partial/\partial x_3(\ ) = 0$ . In a steady-state motion with velocity  $v$  in the  $x_1$ -  
 107 direction, the matrix  $\mathbf{Q}$  is simply replaced by  $\mathbf{Q} - \rho v^2\mathbf{I}$ , where  $\rho$  is the density  
 108 of the solid skeleton and  $\mathbf{I}$  is the identity matrix. Besides, for an isotropic solid,  
 109 it is  $c_{ijkl} = 2\mu\delta_{ik}\delta_{jl} + \lambda_c\delta_{kl}\delta_{ij}$ , where  $\mu$  and  $\lambda_c$  are the Lamé moduli (Nobili  
 110 and Radi, 2022).

111 For a compressible fluid, it is

$$p_f = -re + m\zeta, \quad (6)$$

112 where  $m$  is the compressibility modulus for the fluid, defined as the fluid pressure  
 113 required to force a unit volume of fluid into the pore structure while keeping the  
 114 solid volume unchanged, i.e.  $e = 0$ . As pointed out by Biot and Willis (1957),  
 115 the reversibility assumption, by which a stored elastic potential is admitted,  
 116 identifies the coupling coefficient in the last term in (5) with that in (6). Also,  
 117 following Biot and Willis (1957), it is

$$f \leq \alpha = r/m < 1. \quad (7)$$

118 Biot (1956b) showed that this framework parallels that of thermo-elasticity, with  
 119 the pressure  $p_f$  playing the same role as temperature.

120 We are now in the position to write the potential elastic energy minus the  
 121 work done by the applied external forces over the body  $B$  (i.e. the total energy  
 122 in the sense of Eshelby)

$$\mathcal{L} = \int_B W dV - \int_{\partial B} (\mathbf{t}_0 \cdot \mathbf{u} - p_{f0} \mathbf{n} \cdot \mathbf{w}) dS, \quad (8)$$

123 where we have let the stored potential energy density

$$W = \frac{1}{2} (\mathbf{T} \cdot \text{grad } \mathbf{u} + p_f \zeta). \quad (9)$$

124 Here,  $\mathbf{t}_0$  and  $p_{f0}$  are the prescribed surface force and fluid pressure over the  
 125 body boundary  $\partial B$  with unit normal  $\mathbf{n}$ . For the sake of simplicity, no body  
 126 force is considered. With a little abuse of notation yet in favour of tidiness, an  
 127 interposed dot denotes the scalar product between both tensor and vector pairs,  
 128 i.e. in components  $\mathbf{A} \cdot \mathbf{B} = A_{ij} B_{ij}$  and  $\mathbf{a} \cdot \mathbf{b} = a_i b_i$ , respectively. By strong  
 129 ellipticity, Q and T are positive definite and  $m > 0$ .

130 In a reversible process (i.e. thermostatics), the imposed boundary pressure  
 131 should not trigger movement of the fluid phase and, therefore,  $p_{f0}$  is constant  
 132 on the surface  $\partial B$  and  $p_f$  is equally constant *throughout the body* (Biot, 1962).  
 133 This pressure distribution holds also in steady state motion of the solid provided

134 that we take no account of dissipation. The minus sign associated with the fluid  
 135 pressure  $p_{f0}$  is a consequence of it being positive in compression. In Eq.(8), we  
 136 have let the shorthand

$$\mathbf{T} \cdot \text{grad } \mathbf{u} = \mathbf{t}_1 \cdot \mathbf{u}_{,1} + \mathbf{t}_2 \cdot \mathbf{u}_{,2},$$

137 where it is understood that  $\text{grad } \mathbf{u} = \mathbf{u}_{,1} \otimes \mathbf{e}_1 + \mathbf{u}_{,2} \otimes \mathbf{e}_2$  and a subscript comma  
 138 denotes differentiation, e.g.  $\mathbf{u}_{,1} = \partial \mathbf{u} / \partial x_1$ . Here, we have used the vector  
 139 dyadic that, for any pair of vectors  $\mathbf{a}$  and  $\mathbf{b}$ , yields the rank-2 tensor  $\mathbf{a} \otimes \mathbf{b}$  such  
 140 that, for any vector  $\mathbf{c}$ ,  $(\mathbf{a} \otimes \mathbf{b})\mathbf{c} = (\mathbf{b} \cdot \mathbf{c})\mathbf{a}$ .

The equilibrium equations read

$$\mathbf{t}_{1,1} + \mathbf{t}_{2,2} = \mathbf{o}, \quad (10a)$$

$$\text{grad } p_f = \mathbf{o}, \quad (10b)$$

141 where the last is the equilibrium version of Darcy's law<sup>1</sup>, see Biot (1962, Eq.(7.2)).  
 142 Without loss of generality, we assume that the boundary conditions are only ex-  
 143 pressed in terms of forces

$$\mathbf{T}\mathbf{n} = \mathbf{t}_0 \quad \text{and} \quad p_f = p_{f0}, \quad \text{for } \mathbf{x} \in \partial B, \quad (11)$$

144 where  $\partial B$  is the frontier of the body  $B$ . By the divergence theorem, one can  
 145 rewrite the total energy in terms of a single volume integral

$$\mathcal{L} = - \int_B L dV,$$

146 where we have introduced the Lagrangian density  $L$

$$L(\mathbf{u}_{,1}, \mathbf{u}_{,2}, \text{div } \mathbf{w}) = \frac{1}{2} \mathbf{T} \cdot \text{grad } \mathbf{u} + \frac{1}{2} p_f \zeta. \quad (12)$$

In light of Eqs.(5) and (6), this may be rewritten as

$$\begin{aligned} L(\mathbf{u}_{,1}, \mathbf{u}_{,2}, \mathbf{w}_{,1}, \mathbf{w}_{,2}) &= \frac{1}{2} \mathbf{u}_{,1} \cdot \mathbf{Q}\mathbf{u}_{,1} + \mathbf{u}_{,1} \cdot \mathbf{R}\mathbf{u}_{,2} + \frac{1}{2} \mathbf{u}_{,2} \cdot \mathbf{T}\mathbf{u}_{,2} \\ &\quad - \frac{1}{2} r \zeta (\mathbf{e}_1 \cdot \mathbf{u}_{,1} + \mathbf{e}_2 \cdot \mathbf{u}_{,2}) + \frac{1}{2} (-re + m\zeta) \zeta, \end{aligned} \quad (13)$$

---

<sup>1</sup>Darcy's law emerges from considering an irreversible process and the attached *dissipation function*, that is a quadratic form in  $\partial \mathbf{w} / \partial t$



where  $\mathbf{u}_{,1} \cdot \mathbf{e}_1 + \mathbf{u}_{,2} \cdot \mathbf{e}_2 = \text{div } \mathbf{u} = e$ . Clearly, this formulation admits the Stroh formalism because, unlike internally constrained solids (Nobili and Radi, 2022), both displacement vectors,  $\mathbf{u}$  and  $\mathbf{w}$ , appear only in differentiated form. The Lagrangian density becomes

$$L(\mathbf{u}_{,1}, \mathbf{u}_{,2}, \mathbf{w}_{,1}, \mathbf{w}_{,2}) = \frac{1}{2} \mathbf{u}_{,1} \cdot \mathbf{Q} \mathbf{u}_{,1} + \mathbf{u}_{,1} \cdot \mathbf{R} \mathbf{u}_{,2} + \frac{1}{2} \mathbf{u}_{,2} \cdot \mathbf{T} \mathbf{u}_{,2} + r(\mathbf{e}_1 \cdot \mathbf{w}_{,1} + \mathbf{e}_2 \cdot \mathbf{w}_{,2})(\mathbf{e}_1 \cdot \mathbf{u}_{,1} + \mathbf{e}_2 \cdot \mathbf{u}_{,2}) + \frac{1}{2} m(\mathbf{e}_1 \cdot \mathbf{w}_{,1} + \mathbf{e}_2 \cdot \mathbf{w}_{,2})^2, \quad (14)$$

whence the Euler-Lagrange equations read

$$\frac{d}{dx_1} \frac{\partial L}{\partial \mathbf{u}_{,1}} + \frac{d}{dx_2} \frac{\partial L}{\partial \mathbf{u}_{,2}} = 0, \quad (15a)$$

$$\frac{d}{dx_1} \frac{\partial L}{\partial \mathbf{w}_{,1}} + \frac{d}{dx_2} \frac{\partial L}{\partial \mathbf{w}_{,2}} = 0, \quad (15b)$$

147 which is clearly in the Stroh form once we settle for either coordinate to act as  
148 a time-like variable, say  $x_2$  as in Fu (2007). Eq.(15a) gives

$$(\mathbf{Q} \mathbf{u}_{,1} + \mathbf{R} \mathbf{u}_{,2} - r\zeta \mathbf{e}_1)_{,1} + (\mathbf{R}^T \mathbf{u}_{,1} + \mathbf{T} \mathbf{u}_{,2} - r\zeta \mathbf{e}_2)_{,2} = \mathbf{o}, \quad (16)$$

149 that corresponds to the equilibrium equation (10a), provided that we account  
150 for (5). Similarly, Eq.(15b) lends

$$(re - m\zeta)_{,1} \mathbf{e}_1 + (re - m\zeta)_{,2} \mathbf{e}_2 = \mathbf{o}, \quad (17)$$

151 that indeed amounts to Eq.(10b), once acknowledging for (6).

### 152 3. Hamiltonian formalism

153 We now introduce the Hamiltonian formalism by treating  $x_2$  as a time-like  
154 variable (Fu, 2007). Consequently, differentiation with respect to  $x_2$  will be  
155 denoted by a superscript dot. For reasons that shall be presently apparent, we  
156 let

$$\bar{\mathbf{Q}} = \mathbf{Q} - \frac{r^2}{m} \mathbf{e}_1 \otimes \mathbf{e}_1, \quad \bar{\mathbf{R}} = \mathbf{R} - \frac{r^2}{m} \mathbf{e}_1 \otimes \mathbf{e}_2, \quad \bar{\mathbf{T}} = \mathbf{T} - \frac{r^2}{m} \mathbf{e}_2 \otimes \mathbf{e}_2, \quad (18)$$

whence we may rewrite (5) as

$$\mathbf{t}_1 = \bar{\mathbf{Q}} \mathbf{u}_{,1} + \bar{\mathbf{R}} \dot{\mathbf{u}} - r \frac{p_f}{m} \mathbf{e}_1, \quad (19a)$$

$$\mathbf{t}_2 = \bar{\mathbf{R}}^T \mathbf{u}_{,1} + \bar{\mathbf{T}} \dot{\mathbf{u}} - r \frac{p_f}{m} \mathbf{e}_2. \quad (19b)$$

Eq.(14) becomes

$$L(\mathbf{u}_{,1}, \dot{\mathbf{u}}, \mathbf{w}_{,1}, \dot{\mathbf{w}}) = \frac{1}{2} \mathbf{u}_{,1} \cdot \bar{\mathbf{Q}} \mathbf{u}_{,1} + \mathbf{u}_{,1} \cdot \bar{\mathbf{R}} \dot{\mathbf{u}} + \frac{1}{2} \dot{\mathbf{u}} \cdot \bar{\mathbf{T}} \dot{\mathbf{u}} \\ + \frac{1}{2} m^{-1} [r(\mathbf{e}_1 \cdot \mathbf{u}_{,1} + \mathbf{e}_2 \cdot \dot{\mathbf{u}}) + m(\mathbf{e}_1 \cdot \mathbf{w}_{,1} + \mathbf{e}_2 \cdot \dot{\mathbf{w}})]^2. \quad (20)$$

from which conjugate momenta are immediately obtained

$$\mathbf{p}_1 = \frac{\partial L}{\partial \dot{\mathbf{u}}} = \mathbf{t}_2, \quad (21a)$$

$$\mathbf{p}_2 = \frac{\partial L}{\partial \dot{\mathbf{w}}} = (re - m\zeta) \mathbf{e}_2 = -p_f \mathbf{e}_2. \quad (21b)$$

157 Solving Eq.(21b) for  $\zeta$  gives

$$\zeta = m^{-1} (p_f + re), \quad (22)$$

158 while solving Eq.(19b) for  $\dot{\mathbf{u}}$  gives

$$\dot{\mathbf{u}} = \bar{\mathbf{T}}^{-1} \left( \mathbf{t}_2 - \bar{\mathbf{R}}^T \mathbf{u}_{,1} + \frac{r}{m} p_f \mathbf{e}_2 \right). \quad (23)$$

159 Scalar multiplication of (23) throughout by  $\mathbf{e}_2$  lends

$$\dot{\mathbf{u}} \cdot \mathbf{e}_2 = \zeta_1 \bar{\mathbf{T}}^{-1} \bar{\mathbf{t}}_2 \cdot \mathbf{e}_2, \quad (24)$$

160 where we have let the shorthand

$$\bar{\mathbf{t}}_2 = \mathbf{t}_2 - \bar{\mathbf{R}}^T \mathbf{u}_{,1} + r \left( \zeta - \frac{r}{m} \mathbf{u}_{,1} \cdot \mathbf{e}_1 \right) \mathbf{e}_2,$$

161 and, as in Fu (2007, Eq.(3.12)), it is

$$\zeta_1^{-1} = 1 + \frac{r^2}{m} \mathbf{e}_2 \cdot \bar{\mathbf{T}}^{-1} \mathbf{e}_2 > 1, \quad (25)$$

162 whose last term is always positive by virtue of strong ellipticity (see the Ap-

163 pendix). Hence, plugging (24) into (23), it is finally

$$\dot{\mathbf{u}} = \bar{\mathbf{T}}^{-1} \mathbf{P} \bar{\mathbf{t}}_2, \quad (26)$$

164 having let the projector (we have used the symmetry of  $\bar{\mathbf{T}}$ )

$$\mathbf{P} = \mathbf{1} - \frac{r^2}{m} \zeta_1 \mathbf{e}_2 \otimes \bar{\mathbf{T}}^{-1} \mathbf{e}_2. \quad (27)$$

165 We note that

$$\mathbf{P}\mathbf{e}_2 = \zeta_1\mathbf{e}_2, \quad \text{and} \quad \bar{\mathbf{T}}^{-1}\mathbf{P} \in \text{Sym}, \quad (28)$$

166 whence Eq.(26) may be rewritten as

$$\dot{\mathbf{u}} = \bar{\mathbf{T}}^{-1}\mathbf{P}(\mathbf{t}_2 - \bar{\mathbf{R}}^T\mathbf{u}_{,1}) + r\zeta_1\left(\zeta - \frac{r}{m}\mathbf{u}_{,1} \cdot \mathbf{e}_1\right)\bar{\mathbf{T}}^{-1}\mathbf{e}_2. \quad (29)$$

167 Indeed, scalar multiplication by  $\mathbf{e}_2$ , in view of the properties (28), immediately  
168 lends (24).

169 In similar fashion, in light of Eqs.(2,24), Eq.(22) yields

$$-\dot{\mathbf{w}} \cdot \mathbf{e}_2 = \mathbf{w}_{,1} \cdot \mathbf{e}_1 + \frac{p_f}{m} + \frac{r}{m}\mathbf{u}_{,1} \cdot \mathbf{e}_1 + \frac{r}{m}\bar{\mathbf{T}}^{-1}\left(\mathbf{t}_2 - \bar{\mathbf{R}}^T\mathbf{u}_{,1} + \frac{r}{m}p_f\mathbf{e}_2\right) \cdot \mathbf{e}_2. \quad (30)$$

We introduce the Hamiltonian density

$$\begin{aligned} H &= \mathbf{t}_2 \cdot \dot{\mathbf{u}} + \mathbf{p}_2 \cdot \dot{\mathbf{w}} - L \\ &= \mathbf{t}_2 \cdot \dot{\mathbf{u}} - p_f\mathbf{e}_2 \cdot \dot{\mathbf{w}} - \frac{1}{2}\mathbf{u}_{,1} \cdot \bar{\mathbf{Q}}\mathbf{u}_{,1} - \mathbf{u}_{,1} \cdot \bar{\mathbf{R}}\dot{\mathbf{u}} - \frac{1}{2}\dot{\mathbf{u}} \cdot \bar{\mathbf{T}}\dot{\mathbf{u}} - \frac{1}{2}m^{-1}p_f^2, \end{aligned}$$

whence

$$\begin{aligned} H &= \frac{1}{2}\left(\mathbf{t}_2 - \bar{\mathbf{R}}^T\mathbf{u}_{,1} + \frac{r}{m}p_f\mathbf{e}_2\right) \cdot \bar{\mathbf{T}}^{-1}\left(\mathbf{t}_2 - \bar{\mathbf{R}}^T\mathbf{u}_{,1} + \frac{r}{m}p_f\mathbf{e}_2\right) \\ &\quad + p_f\left(\mathbf{w}_{,1} \cdot \mathbf{e}_1 + \frac{p_f}{m} + \frac{r}{m}\mathbf{u}_{,1} \cdot \mathbf{e}_1\right) - \frac{1}{2}\mathbf{u}_{,1} \cdot \bar{\mathbf{Q}}\mathbf{u}_{,1} - \frac{1}{2}m^{-1}p_f^2, \end{aligned}$$

and finally

$$\begin{aligned} H &= \frac{1}{2}\left(\mathbf{t}_2 - \bar{\mathbf{R}}^T\mathbf{u}_{,1} + \frac{r}{m}p_f\mathbf{e}_2\right) \cdot \bar{\mathbf{T}}^{-1}\left(\mathbf{t}_2 - \bar{\mathbf{R}}^T\mathbf{u}_{,1} + \frac{r}{m}p_f\mathbf{e}_2\right) \\ &\quad + p_f\left(\mathbf{w}_{,1} + \frac{r}{m}\mathbf{u}_{,1}\right) \cdot \mathbf{e}_1 - \frac{1}{2}\mathbf{u}_{,1} \cdot \bar{\mathbf{Q}}\mathbf{u}_{,1} + \frac{1}{2}\frac{p_f^2}{m}. \quad (31) \end{aligned}$$

170 As well known, the canonical equations may be grouped in two sets, described  
171 by the vector canonical equations

$$\dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}}, \quad \text{and} \quad \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}}. \quad (32)$$

172 In the first group we have

$$\dot{\mathbf{u}} = \frac{\partial H}{\partial \mathbf{t}_2} = \bar{\mathbf{T}}^{-1}\left(\mathbf{t}_2 - \bar{\mathbf{R}}^T\mathbf{u}_{,1} + \frac{r}{m}p_f\mathbf{e}_2\right), \quad (33)$$

173 and

$$\dot{\mathbf{w}} \cdot \mathbf{e}_2 = -\frac{\partial H}{\partial p_f} = -\mathbf{w}_{,1} \cdot \mathbf{e}_1 - \frac{p_f}{m} - \frac{r}{m} \mathbf{u}_{,1} \cdot \mathbf{e}_1 - \frac{r}{m} \mathbf{e}_2 \cdot \bar{\mathbf{T}}^{-1} \left( \mathbf{t}_2 - \bar{\mathbf{R}}^T \mathbf{u}_{,1} + \frac{r}{m} p_f \mathbf{e}_2 \right), \quad (34)$$

174 that correspond to Eq.(23) and to (30), respectively. The second group provides  
175 the equilibrium equations. Indeed, one gets

$$\dot{\mathbf{t}}_2 = -\frac{\partial H}{\partial \mathbf{u}} = -\left[ \bar{\mathbf{R}} \bar{\mathbf{T}}^{-1} \left( \mathbf{t}_2 - \bar{\mathbf{R}}^T \mathbf{u}_{,1} + \frac{r}{m} p_f \mathbf{e}_2 \right) + \bar{\mathbf{Q}} \mathbf{u}_{,1} - \frac{r}{m} p_f \mathbf{e}_1 \right]_{,1} \quad (35)$$

176 that, accounting for (23), whereby  $\bar{\mathbf{T}}^{-1}$  times the term in round brackets gives  
177  $\dot{\mathbf{u}}$ , and in light of the first of (5), amounts to (10a). By the same token,

$$-p_f \mathbf{e}_2 = -\frac{\partial H}{\partial \mathbf{w}} = (p_f \mathbf{e}_1)_{,1} \quad (36)$$

178 that is immediately (10b). Incorporating the dissipation function into this for-  
179 mulation, may provide the starting point for addressing the general case of  
180 irreversible poroelasticity.

### 181 3.1. Reduced Hamiltonian

182 Looking at Eq.(21b) and recalling that  $p_f$  is constant throughout the body,  
183 as a result of the equilibrium equation (10b), one realises that, besides energy  
184 conservation, another motion invariant is available. Indeed, this formulation  
185 possesses translational invariance with respect to  $\dot{\mathbf{w}}$ . This is an outcome of the  
186 fact that, unlike  $\mathbf{u}$ ,  $\mathbf{w}$  appears in the Lagrangian only through its divergence  
187  $\zeta$ , and therefore one may assume  $\mathbf{w} = \text{grad } \varphi$  without loss of generality, the  
188 solenoidal contribution to  $\mathbf{w}$  being irrelevant to the present purposes, see (Biot,  
189 1962, Eq.(7.13)). This feature is specific to reversible poroelasticity and it is  
190 lost when encompassing for dissipation. Consequently,  $\mathbf{w}_{,1} \cdot \mathbf{e}_1$  and  $\dot{\mathbf{w}} \cdot \mathbf{e}_2$   
191 are not (globally) independent from one another. To avoid dealing with this  
192 constraint, a more convenient approach consists of replacing  $\zeta$  in (20) through  
193 the connection (22) to get

$$\hat{L}(\mathbf{u}_{,1}, \dot{\mathbf{u}}) = \frac{1}{2} \mathbf{u}_{,1} \cdot \bar{\mathbf{Q}} \mathbf{u}_{,1} + \mathbf{u}_{,1} \cdot \bar{\mathbf{R}} \dot{\mathbf{u}} + \frac{1}{2} \dot{\mathbf{u}} \cdot \bar{\mathbf{T}} \dot{\mathbf{u}}, \quad (37)$$

194 having dispensed with the irrelevant constant term  $\frac{1}{2} p_f^2 / m$ . In this form, the  
195 system matches anisotropic elasticity, provided that the Stroh matrices (18) are

196 used. It is also emphasized that, in this reduced formulation (37), only the solid  
 197 skeleton is represented. The Euler-Lagrange equation reads

$$\hat{\mathbf{t}}_{1,1} + \hat{\mathbf{t}}_2 = 0, \quad (38)$$

198 having let the force vectors

$$\hat{\mathbf{t}}_1 = \bar{\mathbf{Q}}\mathbf{u}_{,1} + \bar{\mathbf{R}}\dot{\mathbf{u}}, \quad \hat{\mathbf{t}}_2 = \bar{\mathbf{R}}^T\mathbf{u}_{,1} + \bar{\mathbf{T}}\dot{\mathbf{u}}. \quad (39)$$

199 This amounts to defining the new stress tensor  $\hat{\mathbf{T}}$ , which differs from the total  
 200 stress  $\mathbf{T}$  by the constant hydrostatic pressure  $\frac{r}{m}p_f\mathbf{1}$ , and corresponds to Biot's  
 201 *effective stress*  $\sigma_{ij}$ , that is the force in excess to pressure applied to the solid per  
 202 unit surface of the bulk material, see (Biot, 1956a, Eq.(3.2)) and (Biot, 1962,  
 203 Eq.(3.9)). The corresponding momentum immediately follows

$$\hat{\mathbf{p}} = \frac{\partial L}{\partial \dot{\mathbf{u}}} = \hat{\mathbf{t}}_2, \quad (40)$$

204 and it can be solved for the conjugate coordinate  $\dot{\mathbf{u}}$  giving again (23), yet as-  
 205 suming that  $p_f = 0$ , i.e.

$$\dot{\mathbf{u}} = \bar{\mathbf{T}}^{-1} (\hat{\mathbf{t}}_2 - \bar{\mathbf{R}}^T\mathbf{u}_{,1}). \quad (41)$$

206 The possibility to invert  $\bar{\mathbf{T}}$  is granted by strong ellipticity, as discussed in the Ap-  
 207 pendix. The corresponding Hamiltonian is similarly obtained from (31) letting  
 208  $p_f = 0$ ,

$$\hat{H} = \hat{\mathbf{p}} \cdot \dot{\mathbf{u}} - \hat{L} = \frac{1}{2} (\hat{\mathbf{t}}_2 - \bar{\mathbf{R}}^T\mathbf{u}_{,1}) \cdot \bar{\mathbf{T}}^{-1} (\hat{\mathbf{t}}_2 - \bar{\mathbf{R}}^T\mathbf{u}_{,1}) - \frac{1}{2}\mathbf{u}_{,1} \cdot \bar{\mathbf{Q}}\mathbf{u}_{,1}. \quad (42)$$

209 The canonical equations are

$$\dot{\mathbf{u}} = \frac{\partial \hat{H}}{\partial \hat{\mathbf{t}}_2}, \quad (43)$$

210 that indeed gives (41), and

$$\dot{\hat{\mathbf{t}}}_2 = -\frac{\partial \hat{H}}{\partial \mathbf{u}} = -[\bar{\mathbf{R}}\bar{\mathbf{T}}^{-1} (\hat{\mathbf{t}}_2 - \bar{\mathbf{R}}^T\mathbf{u}_{,1}) - \bar{\mathbf{Q}}\mathbf{u}_{,1}]_{,1}, \quad (44)$$

211 that corresponds to (38).

212 For a homogeneous material, letting the stress potential  $\hat{\phi} = \int \hat{\mathbf{t}}_2 dx_1$  and  
 213 the vector of unknowns

$$\boldsymbol{\xi} = \begin{bmatrix} \mathbf{u} \\ \hat{\phi} \end{bmatrix}, \quad (45)$$

214 we can write the Stroh formalism

$$\frac{\partial}{\partial x_2} \boldsymbol{\xi} = \mathbf{N} \frac{\partial}{\partial x_1} \boldsymbol{\xi}, \quad (46)$$

215 where  $\mathbf{N}$  is the *fundamental elasticity block-matrix* (Ting, 1996, §6)

$$\mathbf{N} = \begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 \\ \mathbf{N}_3 & \mathbf{N}_1^T \end{bmatrix}, \quad (47)$$

216 and we have let the 3 by 3 block-matrices

$$\mathbf{N}_1 = -\bar{\mathbf{T}}^{-1} \bar{\mathbf{R}}^T, \quad \mathbf{N}_2 = \bar{\mathbf{T}}^{-1}, \quad \mathbf{N}_3 = \bar{\mathbf{R}} \bar{\mathbf{T}}^{-1} \bar{\mathbf{R}}^T - \bar{\mathbf{Q}}. \quad (48)$$

217 We observe that  $\boldsymbol{\xi}$  has mixed dimensions, namely length and force over length  
 218 for the first and for the second vector component, respectively. Consequently,  
 219  $\mathbf{N}_1$  is dimensionless, while  $\mathbf{N}_3$  and  $\mathbf{N}_2$  have dimension of stress and inverse of  
 220 stress (compliance), respectively.

221 Letting the 6 by 6 constant matrix (Ting, 1996, Eq.(5.5-7))

$$\hat{\mathbf{I}} = \begin{bmatrix} \mathbf{O} & \mathbf{I} \\ \mathbf{I} & \mathbf{O} \end{bmatrix}, \quad (49)$$

222 and in view of the symmetry of  $\mathbf{N}_2$  and  $\mathbf{N}_3$ , one retrieves the fundamental  
 223 symmetric matrix

$$\hat{\mathbf{I}} \mathbf{N} = \begin{bmatrix} \mathbf{N}_3 & \mathbf{N}_1^T \\ \mathbf{N}_1 & \mathbf{N}_2 \end{bmatrix} = (\hat{\mathbf{I}} \mathbf{N})^T. \quad (50)$$

224 Following Ting (1996, §5.5),  $\mathbf{N}_2$  is positive definite and  $-\mathbf{N}_3$  is positive semidef-  
 225 inite. When looking for travelling solutions of the form  $\boldsymbol{\xi} = \boldsymbol{\Xi} f(x_1 + px_2)$ , a  
 226 right eigenvalue problem is retrieved

$$\mathbf{N} \boldsymbol{\Xi} = p \boldsymbol{\Xi}, \quad (51)$$

227 The Hamiltonian density (42) may be rewritten as the quadratic form asso-  
 228 ciated with the fundamental matrix

$$\hat{H} = \frac{1}{2} \boldsymbol{\xi} \cdot \hat{\mathbf{N}} \boldsymbol{\xi}, \quad (52)$$

229 whence the first integral associated with energy conservation (given that the  
 230 Lagrangian is  $x_2$ -independent) reads

$$\int_{\Sigma} \boldsymbol{\xi} \cdot \hat{\mathbf{N}} \boldsymbol{\xi} dx_1 dx_3 = \text{const}, \quad (53)$$

231 in the assumption that we may decompose the domain as  $B = \Sigma \times I$ , where  $I$   
 232 is an interval in the  $x_2$  coordinate. Finally, we may define the *edge impedance*  
 233 *matrix*  $\mathbf{M}$  as

$$\hat{\boldsymbol{\phi}} = \imath \mathbf{M} \mathbf{u}, \quad (54)$$

234 and in light of (45,46), we may write

$$\dot{\hat{\boldsymbol{\phi}}} = \imath \mathbf{M} \dot{\mathbf{u}} = (\imath \mathbf{M} \mathbf{N}_1 - \mathbf{M} \mathbf{N}_1 \mathbf{M}) \mathbf{u}_{,1} = (\mathbf{N}_3 + \imath \mathbf{N}_1^T \mathbf{M}) \mathbf{u}_{,1},$$

235 whence  $\mathbf{M}$  satisfies the matrix equation (Fu, 2007, Eq.(4.40))

$$\mathbf{N}_3 + \imath \mathbf{N}_1^T \mathbf{M} - \imath \mathbf{M} \mathbf{N}_1 + \mathbf{M} \mathbf{N}_1 \mathbf{M} = \mathbf{O}. \quad (55)$$

236 This matrix provides a very simple procedure to determine localized waves, for  
 237 which  $\mathbf{t}_2 \equiv \mathbf{o}$  on the body surface  $x_2 = 0$ . Indeed, the dispersion relation is  
 238 simply obtained by admitting non-trivial solutions to the system

$$\mathbf{M} \mathbf{u} = \mathbf{o},$$

239 hence the major obstacle lying in the way is the determination of the impedance  
 240 matrix through the connection (55). This result is most simply achieved through  
 241 the integral representation originally introduced by Barnett and Lothe (1974).

#### 242 4. Weak reversible poroelasticity and the incompressible limit

243 We shall now consider the limit where the coupling effect is weaker than the  
 244 elastic response. For this, we let  $\tau_0 = \|\mathbf{T}\|$  be the norm of the matrix  $\mathbf{T}$ , and we

245 assume that  $\tau_0^{-1}r^2/m = \varepsilon \ll 1$  is small. We name this condition *weak reversible*  
 246 *poroelasticity*. In this case,

$$\bar{\mathbf{T}}^{-1} = (\mathbf{1} + \varepsilon\tau_0\mathbf{T}^{-1}\mathbf{e}_2 \otimes \mathbf{e}_2) \mathbf{T}^{-1} + O(\varepsilon^2), \quad (56)$$

247 and

$$\zeta_1 = 1 - \varepsilon\tau_0\zeta_0^{-1} + O(\varepsilon)^2, \quad \zeta_0 = 1/(\mathbf{e}_2 \cdot \mathbf{T}^{-1}\mathbf{e}_2),$$

whence  $\zeta_1 \approx 1$ . Then, expanding to first order terms in  $\varepsilon$ , one gets (collecting  
 dimensionality terms)

$$\mathbf{N}_1 = -\mathbf{T}^{-1}\mathbf{R}^T + \varepsilon\tau_0\mathbf{T}^{-1}\mathbf{e}_2 \otimes \mathbf{e}_T \quad (57a)$$

$$\mathbf{N}_2 = \mathbf{T}^{-1} (\mathbf{I} + \varepsilon\tau_0\mathbf{e}_2 \otimes \mathbf{T}^{-1}\mathbf{e}_2), \quad (57b)$$

$$\mathbf{N}_3 = \mathbf{R} (\mathbf{T}^{-1} + \varepsilon\tau_0\mathbf{R}^{-1}\mathbf{e}_T \otimes \mathbf{R}^{-1}\mathbf{e}_T) \mathbf{R}^T - \mathbf{Q}, \quad (57c)$$

248 with  $\mathbf{e}_T = \mathbf{e}_1 - \mathbf{R}\mathbf{T}^{-1}\mathbf{e}_2$ . It is pointed out that Eqs.(57) are indeed valid asymp-  
 249 totic expansions inasmuch as  $\tau_0^{-1}\|\mathbf{R}\| = O(1)$  and  $\tau_0^{-1}\|\mathbf{Q}\| = O(1)$  or bigger.  
 250 Physically, this amounts to requiring that all elastic constants are of the same  
 251 order, i.e. contrast is excluded. Formally, Eqs.(57) match the corresponding ma-  
 252 trices in incompressible anisotropic elasticity (Fu, 2007, Eqs.(3.14-16)), provided  
 253 that  $\varepsilon\tau_0$  is replaced by  $\zeta_0$  and the opposite sign is taken in the incompressibility  
 254 contributions, that are given by the correction term in each of Eqs.(57). Indeed,  
 255 a similar expansion of Eq.(41) yields

$$\dot{\mathbf{u}} = \mathbf{T}^{-1} (\hat{\mathbf{t}}_2 - \mathbf{R}^T\mathbf{u}_{,1} - p_0\mathbf{e}_2) \quad (58)$$

256 with

$$p_0 = \varepsilon\tau_0 \{ -\mathbf{T}^{-1} (\hat{\mathbf{t}}_2 - \mathbf{R}^T\mathbf{u}_{,1}) \cdot \mathbf{e}_2 - \mathbf{u}_{,1} \cdot \mathbf{e}_1 \}. \quad (59)$$

257 Providing again that  $\varepsilon\tau_0 = \zeta_0$  and  $p_0$  is sign reversed, such equations are for-  
 258 mally equivalent to (3.7) and (3.11) of Fu (2007), respectively giving  $\dot{\mathbf{u}}$  and the  
 259 Lagrange multiplier enforcing incompressibility for incompressible anisotropic  
 260 solids. This analysis reveals that the weak poroelastic limit is similar to incom-  
 261 pressible anisotropic elasticity, with yet two important differences. First, given  
 262 that  $\tau_0 \sim \zeta_0$ , the condition  $\varepsilon\tau_0 = \zeta_0$  can only be achieved in a correction sense



263 and therefore incompressibility is to be intended as a perturbation from the  
 264 unconstrained leading solution. Second, the sign reversal of  $p_0$  reveals that this  
 265 perturbation is taken in the opposite direction, i.e. the role of the fluid phase  
 266 in the weak limit is opposite to that of the incompressibility constraint. At any  
 267 rate, incompressibility cannot be achieved for the solid skeleton in the general  
 268 sense.

269 Biot, on heuristic grounds, claims that the incompressible limit is obtained  
 270 letting  $m \rightarrow +\infty$  and  $\alpha = r/m = 1$ , see for example Biot (1962). Although,  
 271 just looking at (6), it is manifest that the former condition is sufficient for fluid  
 272 incompressibility, the latter needs some revision. Indeed, the condition  $\alpha = 1$   
 273 merely demands that the fluid response is the same under fluid and solid vol-  
 274 umetric changes, and therefore one may deduce that, for a given pressure  $p_f$ ,  
 275 it must be  $\zeta - e = -\operatorname{div}(\mathbf{u} + \mathbf{w}) = p_f/m$ . When the fluid phase becomes in-  
 276 compressible, i.e.  $m \rightarrow +\infty$ , one needs to specify how the pressure  $p_f$  behaves  
 277 compared to  $m$ . If  $p_f/m \rightarrow 0$ , then zero net flow of both fluid and solid out of  
 278 the control volume is approached and this limit amounts to an isochoric trans-  
 279 formation. This line of reasoning led Biot to the concept of incompressible limit,  
 280 as in Biot (1955). However, while the fluid may behave as incompressible, the  
 281 foregoing analysis shows that the solid does not. In fact, the solid behaves just  
 282 like an anisotropic elastic solid whose Stroh matrices (46) become unbounded  
 283 as  $r \sim m \rightarrow +\infty$ . Besides, to support strong ellipticity (A.2), the elastic con-  
 284 stants must also become unbounded, hence it is concluded that this limit is  
 285 questionable. In fact, the actual physical regime is determined by the ratio  $\tau_0\varepsilon$   
 286 of the poroelastic effect to the elastic effect. In general, when  $\tau_0\varepsilon = O(1)$ , the  
 287 solid behaves like an ordinary anisotropic solid whose material properties are  
 288 affected by the fluid phase. Instead, in the weak limit  $\tau_0\varepsilon \ll 1$ , the fluid acts as  
 289 a perturbation to the anisotropic solid and this perturbation operates similarly  
 290 to incompressibility, yet in opposing fashion, i.e. a positive pressure accompa-  
 291 nies positive volumetric changes. Finally, when  $\tau_0\varepsilon \gg 1$ , the solid behaves like a  
 292 perturbation of an ideal liquid with small viscosity  $O(\tau_0\varepsilon)^{-1}$  given by the elastic  
 293 phase.

## 294 5. Conclusions

295 When deriving the Stroh-like formulation of a mechanical system, one is  
296 confronted with the crucial step of designating the right variable pairs, which  
297 unlock the full potential of the formalism. Recently, Fu (2007) pointed out that  
298 energy conjugation is really the guiding tool which drives such designation, thus  
299 getting away from guess-working and problem intuition, which may not suffice in  
300 complex situations. Indeed, the Stroh formalism is really a canonical formalism  
301 in the Hamiltonian sense, where a coordinate is treated in time-like fashion. In  
302 this paper, we adopt this viewpoint to deal with Biot's reversible poroelasticity,  
303 that dispenses with dissipation and occurs in the absence of a fluid pressure  
304 gradient. This is the same framework as thermoelasticity of perfect conductors,  
305 the pressure playing the role of temperature. Although this framework is insuf-  
306 ficient to deal with any poroelastic problem, it may well provide the starting  
307 point for the general formulation. Also, it investigates the most useful setting for  
308 specimen testing. Spotlight is here set on emphasizing the canonical approach  
309 and the features it brings out. Two formulations are derived: the first accounts  
310 for both the solid and the fluid and it possesses, besides energy conservation,  
311 translational invariance with respect to the fluid velocity. This feature, that is  
312 a result of the absence of a pressure gradient, reveals constraints on the con-  
313 jugate variables. To avoid dealing with such constraints, a second approach is  
314 developed that is restricted to the solid skeleton only. The corresponding Stroh  
315 formulation matches anisotropic elasticity where, however, the Stroh matrices  
316 incorporate fluid coupling. Besides, strong ellipticity warrants their positive def-  
317 inite character. Energy conservation and the impedance matrix follow naturally.  
318 The special case of weak poroelasticity, whereby fluid-solid coupling is weaker  
319 than the elastic response, is also investigated and shows remarkable similarities  
320 with incompressible anisotropic elasticity with yet two important differences,  
321 namely incompressibility acts as a small perturbation with opposite sign. This  
322 analysis leads to reconsider the incompressible limit originally introduced by  
323 Biot, that seems to show some inconsistencies.

## 324 6. Acknowledgments

325 This research was supported by FARD-2023 under the project "3D printing  
326 of skin bandages for healing wounds (3DSkinHealing)". The author acknowl-  
327 edges support from the project "Mathematics for Industry 4.0 (Math4I4)",  
328 PRIN 2020F3NCPX. He is also grateful for support by the European Union  
329 Horizon 2020 MSCA-RISE-2020, Research and Innovation Staff Exchange pro-  
330 gramme, grant agreement No 101008140, EffectFact, within the H2020-EU.1.3.  
331 - EXCELLENT SCIENCE - Marie Skłodowska-Curie Actions Main Programme.  
332 Finally, he recognizes support from the National Group of Mathematical Physics  
333 (GNFM) inside INdAM.

## 334 Conflict of Interest statement

335 The author has no conflict of interest to declare.

## 336 Data Availability

337 This paper makes use of no data.

## 338 Appendix: Strong ellipticity in reversible poroelasticity

339 We now discuss the role of strong ellipticity in poroelasticity in the absence  
340 of dissipation. For a reversible process, we have a uniform pressure distribution  
341  $p_f$  and the motion equation for the solid skeleton is given by Eq.(10a) where we  
342 write inertia explicitly (i.e. it is not hidden inside the Stroh matrices)

$$c_{ijkl}u_{k,lj} - r\zeta_{,j}\delta_{ij} = \rho\ddot{u}_i. \quad (\text{A.1})$$

As well known (Edmondson and Fu, 2009), strong ellipticity may be equally  
retrieved demanding that the speed  $v$  of *any amplitude* propagating body wave  
in *any direction* is real (and positive, without loss of generality). To this aim,  
let's assume  $\mathbf{u} = \boldsymbol{\alpha}e^{i(\boldsymbol{\beta}\cdot\mathbf{x}-vt)}$ , whence

$$e = \alpha_k\beta_k e^{i(\boldsymbol{\beta}\cdot\mathbf{x}-vt)}, \quad \zeta = m^{-1} \left( p_f + r\alpha_k\beta_k e^{i(\boldsymbol{\beta}\cdot\mathbf{x}-vt)} \right).$$

Then, Eq.(A.1) becomes

$$c_{ijkl}\alpha_k\beta_l\beta_j - \frac{r^2}{m}\delta_{ij}\alpha_k\beta_k\beta_j = \rho v^2\alpha_i,$$

which, multiplied through by  $\alpha_i$  and summed over  $i$ , gives

$$\left(c_{ijkl} - \frac{r^2}{m}\delta_{ij}\delta_{kl}\right)\alpha_i\alpha_k\beta_l\beta_j = \rho v^2\alpha_i\alpha_i > 0,$$

343 for any  $\alpha, \beta$  different from zero. This is a variant of the incompressibility  
 344 constraint. In particular, letting  $\beta = \mathbf{e}_2$ , one gets that

$$\bar{\mathbf{T}} = \mathbf{T} - \frac{r^2}{m}\mathbf{e}_2 \otimes \mathbf{e}_2 \text{ is positive definite,} \quad (\text{A.2})$$

345 and therefore Eq.(40) may be solved.

## 346 References

- 347 Barnett, D., Lothe, J., 1974. Consideration of the existence of surface wave  
 348 (Rayleigh wave) solutions in anisotropic elastic crystals. Journal of physics  
 349 F: Metal physics 4, 671.
- 350 Barnett, D.M., 2000. Bulk, surface, and interfacial waves in anisotropic linear  
 351 elastic solids. International Journal of Solids and Structures 37, 45–54.
- 352 Berryman, J.G., Wang, H.F., 2000. Elastic wave propagation and attenuation in  
 353 a double-porosity dual-permeability medium. International Journal of Rock  
 354 Mechanics and Mining Sciences 37, 63–78.
- 355 Biot, M.A., 1955. Theory of elasticity and consolidation for a porous anisotropic  
 356 solid. Journal of applied physics 26, 182–185.
- 357 Biot, M.A., 1956a. Theory of deformation of a porous viscoelastic anisotropic  
 358 solid. Journal of Applied physics 27, 459–467.
- 359 Biot, M.A., 1956b. Thermoelasticity and irreversible thermodynamics. Journal  
 360 of applied physics 27, 240–253.
- 361 Biot, M.A., 1962. Mechanics of deformation and acoustic propagation in porous  
 362 media. Journal of applied physics 33, 1482–1498.

363 Biot, M.A., Willis, D.G., 1957. The elastic coefficients of the theory of consoli-  
364 dation. *Journal of Applied Mechanics* 24, 594–601.

365 Chadwick, P., Smith, G., 1977. Foundations of the theory of surface waves in  
366 anisotropic elastic materials. *Advances in applied mechanics* 17, 303–376.

367 Corapcioglu, M.Y., Tuncay, K., 1996. Propagation of waves in porous media,  
368 in: *Advances in porous media*. Elsevier. volume 3, pp. 361–440.

369 Dullien, F.A., 2012. *Porous media: fluid transport and pore structure*. Academic  
370 press.

371 Edmondson, R., Fu, Y., 2009. Stroh formulation for a generally constrained and  
372 pre-stressed elastic material. *International Journal of Non-Linear Mechanics*  
373 44, 530–537.

374 Fu, Y., 2003. Existence and uniqueness of edge waves in a generally anisotropic  
375 elastic plate. *Quarterly Journal of Mechanics and Applied Mathematics* 56,  
376 605–616.

377 Fu, Y., 2005. An explicit expression for the surface-impedance matrix of a  
378 generally anisotropic incompressible elastic material in a state of plane strain.  
379 *International Journal of Non-Linear Mechanics* 40, 229–239.

380 Fu, Y., 2007. Hamiltonian interpretation of the Stroh formalism in anisotropic  
381 elasticity. *Proceedings of the Royal Society A: Mathematical, Physical and*  
382 *Engineering Sciences* 463, 3073–3087.

383 Fu, Y., Brookes, D., 2006. Edge waves in asymmetrically laminated plates.  
384 *Journal of the Mechanics and Physics of Solids* 54, 1–21.

385 Fu, Y., Kaplunov, J., 2012. Analysis of localized edge vibrations of cylindrical  
386 shells using the stroh formalism. *Mathematics and mechanics of solids* 17,  
387 59–66.

388 Hwu, C., 2003. Stroh-like formalism for the coupled stretching–bending analysis  
389 of composite laminates. *International Journal of Solids and Structures* 40,  
390 3681–3705.

391 Lopatnikov, S.L., Cheng, A.D., 2004. Macroscopic lagrangian formulation of  
392 poroelasticity with porosity dynamics. *Journal of the Mechanics and Physics*  
393 *of Solids* 52, 2801–2839.

394 Nobili, A., Radi, E., 2022. Hamiltonian/stroh formalism for anisotropic media  
395 with microstructure. *Philosophical Transactions of the Royal Society A* 380,  
396 20210374.

397 Norris, A.N., Grinfeld, M.A., 1995. Nonlinear poroelasticity for a layered  
398 medium. *The Journal of the Acoustical Society of America* 98, 1138–1146.

399 Pramanik, D., Manna, S., Nobili, A., 2024. Theory of elastic wave propagation  
400 in a fluid saturated multiporous medium with multi-permeability. *Proceedings*  
401 *of the Royal Society of London Under review.*

402 Sharma, M., 2010. Piezoelectric effect on the velocities of waves in an anisotropic  
403 piezo-poroelastic medium. *Proceedings of the Royal Society A: Mathematical,*  
404 *Physical and Engineering Sciences* 466, 1977–1992.

405 Stroh, A., 1958. Dislocations and cracks in anisotropic elasticity. *Philosophical*  
406 *magazine* 3, 625–646.

407 Stroh, A., 1962. Steady state problems in anisotropic elasticity. *Journal of*  
408 *Mathematics and Physics* 41, 77–103.

409 Ting, T., 1996. *Anisotropic elasticity: theory and applications.* 45, Oxford  
410 *University Press on Demand.*