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A matrix unified framework for deriving various impulse responses in Markov switching VAR: Evidence from oil and gas markets

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ABSTRACT

We propose a new method to compute various impulse response functions (IRF) for a Markov switching VAR model in terms of neat matrix expressions in closed form. The key is to derive a suitable closed form representation for Markov switching VAR models using a state-space representation. By this representation, the IRF analysis can be processed with respect to either an asymmetric discrete or a symmetric continuous shocks. A simulation study demonstrates the actual advantages of the proposed matrix methodology. To illustrate the feasibility and the usefulness of our approach, we present empirical applications to oil and natural gas markets showing the relevance of accommodating asymmetries in the relationship between their price shocks and economic activities.

1. Introduction

Since the seminal papers by Hamilton (1989, 1990), VAR models subject to Markov switching have been used actively in econometrics and statistics to model various time series. Stationarity, existence of moments, geometric ergodicity, statistical inference and asymptotic theory for Markov switching vector autoregressive (MS VAR) models have been studied by several authors (see, e.g., Krolzig, 1997; Yang, 2000; Francq & Zakoïan, 2001; Alvarez et al., 2018; Kasahara & Shimotsu, 2019; Stelzer, 2009). Results on estimation, consistency, hypothesis testing and model selection of MS VAR models can be found in Zhang and Stine (2001), Breunig et al. (2003), Cavicchioli (2014a, 2014b, 2015, 2021), Fu and Wu (2022), Li and Kwok (2021), Qu and Zhuo (2021). Methods to derive the spectral density of such models were proposed by Pataracchia (2011), Cavicchioli (2013) and Cheng (2016). Matrix expressions for higher order moments and asymptotic Fisher information matrix of MS VAR models have been provided by Cavicchioli (2017a, 2017b), respectively.

For a survey on the literature on regime changes together with empirical applications in a number of areas of macroeconomics see, for example, Hamilton (1994, §22). See also Hamilton (2016). A nice investigation whether the US economy responds negatively to oil price uncertainty using a MS GARCH-in-MeanVAR can be found in Serletis and Xu (2019). These authors find evidence of an asymmetric relationship between economic activity and oil price. Otranto (2016) proposes a MS model which provides oscillations of the level of the time series within each state. This allows for consideration of extreme jumps in a parsimonious way, without the adoption of a large number of regimes. Pohle et al. (2021) provide a comprehensive overview of the techniques related to coupling in MS models for modeling multiple observation sequences whose underlying state variables interact. Dufrénot and Keddad (2014) employ a time-varying transition probability MS framework to analyze the relationships between the business cycles in East Asia.

Some related interesting papers concern with the classes of MS bilinear models and doubly MS autoregressive models, introduced by Bibi and Ghezal (2016) and Ghezal (2023b), respectively, in order to model economic series that exhibit structural breaks.

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One advantage of such models is their capability to take into account certain phenomena commonly observed in practice, such as asymmetric distribution, leptokurtosis and leverage effect. Methods to derive the spectral and bispectral density functions of MS bilinear models can be found in Ghezal (2023a) and Ghezal and Zemmouri (2023a). Some probabilistic and statistical properties of MS asymmetric logGARCH processes and periodic Threshold GARCH models have been investigated by Ghezal (2021), Ghezal and Zemmouri (2023b, 2023d), where conditions for strong consistency and asymptotic normality of the quasi-maximum likelihood estimators (QMLE) are given. Finally, Ghezal and Zemmouri (2023c) propose a broad class of MS autoregressive stochastic volatility models to capture the habitual changing behavior of volatility due to economic forces and abrupt abnormal events.

A popular tool for dynamics investigation in applied macroeconomics is based on the *Impulse Response Functions* (IRF hereafter). It provides a global picture of what happens in a dynamic system hit by an exogenous shock within a given horizon. See, for example, Koop et al. (1996). With nonlinear specifications, and particularly in the MS VAR literature, the IRF analysis deals with the important question to capture state, sign and/or size *asymmetries* in the economic and financial mechanisms. Lin (1997) proposes IRF for conditional volatility in GARCH models, and Guerron-Quintana et al. (2017) describe impulse response matching estimators for DSGE models. Camacho (2005) postulates a MS common stochastic trends model to study both the short-run responses to permanent shocks and the effects of recessions in the long-run growth. The obtained findings are used to explore the short- and long-run asymmetric relationships among output, consumption and investment. Functional approximations of IRFs have been provided by Barnichon and Matthes (2018).

Ehrmann et al. (2003) derive sets of *regime-dependent impulse response functions* (RD IRF hereafter) in a MS VAR that characterize the different patterns of the economy in various regimes, and provide insights on the dynamics at stake within the regime in which the shock occurs. Such functions are used by Qiao et al. (2011) to investigate the dynamic relationships among the stock markets of three industrialized economies. Other impulse response functions in MS structural VARs have been proposed by Karamé (2010, 2012). More precisely, Karamé (2010) introduces a more general IRF, called *exact* IRF (EIRF hereafter) that captures the global response of the system in the wake of an identified shock, whatever the states visited in the wake of the shock. Successively, Karamé (2015) proposes a *generalized* IRF (GIRF hereafter) for MS VARs which serves to implement a test for sign and size asymmetries on aggregate gross job flows. Structural VARs with Markov switching have been used in Lanne et al. (2010) to identify shocks when the reduced form error covariance matrix varies across states. Such models serve to test restrictions which are just-identifying in a standard structural VAR analysis.

A MS VAR approach and IRF analysis have been used in several applied papers to study the asymmetric shocks of macrovariables on the stock market. Here we mention some of them which are related to our empirical applications. Shahrestani and Rafei (2020) and Gong et al. (2021) analyze the impact relationship between different factors and global oil price fluctuations under different regimes. The findings based on RD IRFs show that the effects of the oil shocks on the stock market are positive and negative in various regimes. The impact of oil prices on gas prices has been discussed in Hou and Nguyen (2018), showing that it is relatively small and regime dependent. Asymmetric evidence of gasoline price responses in France by using a Markov switching approach can be found in Boroumand et al. (2016). Simo-Kengne et al. (2013), Chowdhury and MacLennan (2014), Matsuki et al. (2015), and Papadamou and Markopoulos (2018) study the asymmetric effects of traditional monetary policy on the house prices, the bank interest rates and the inflation rates, employing MS VARs and associated RD IRFs. The results suggest that the effect of traditional monetary policy is not neutral and the impact varies significantly during boom and bust periods. The transmission mechanism of monetary policy to macroeconomic variables in some European and American economies has been investigated by Darvas (2013), Camacho and Perez-Quiros (2014), and Allen and Robinson (2015). Lange (2018) evaluates asymmetries in the systematic responses of the Canadian economy to movements in the business cycle in the US economy. The findings of these authors lead to believe that information asymmetry plays an important role in the economy and in monetary transmission mechanism. A comprehensive picture of market behavior has been given by Oelschläger and Adam (2023) by using hierarchical hidden Markov models. Chevallier (2011) shows that two-regime MS VAR models provide a sound statistical framework for a comprehensive analysis of the nonlinear adjustment between industrial production and carbon prices. Liu, et al. (2022) provide evidence that the impacts of economic policy uncertainty on oil-stock correlations are regime-dependent both at the aggregate and industry levels.

The goals of the present paper can be summarized as follows: (1) A unified framework for various concepts of IRFs and suitable state space representations of MS VARs; (2) Explicit neat expressions in closed form for the RD IRFs and the EIRFs in terms of the matrices involved in the state space representation of the model specification. The derived matrix formulas of impulse responses are completely new, easily tractable, and cannot be found, to our best knowledge, in the literature, filling a gap on the considered topic. Moreover, they are readily programmable in addition of greatly reducing the computational cost; (3) The RD IRF and EIRF analysis can be processed with respect to either an asymmetric discrete shock or to a symmetric continuous shock; (4) Useful tools for practitioners, illustrating them via empirical applications. In particular, we provide new methods to easily evaluate the responses to shocks in different states of the economy, allowing asymmetric features and differentiated impacts. The proposed derivations of IRFs based on very simple and tractable matrix formulas are the actual advantages of computing the impulse responses of measurements to a shock following our approach over using Ehrmann et al. (2003) and Karamé (2010, 2012) directly.

The paper is organized as follows. Section 2 presents the econometric context (the model specification, assumptions, state space and Markovian representations) concerning a MS VAR approach. Matrix formulas in closed form to easily calculate RD IRF and EIRF for a MS VAR are derived in Section 3. Empirical applications are proposed in Section 4 to illustrate the usefulness and the actual advantages of the proposed matrix methodology over the existing methods. Particularly, empirical applications to oil and natural gas markets show the relevance of accommodating asymmetries in the relationship between their price shocks and economic activities. Section 5 contains some concluding remarks. Proofs are given in Appendix.

2. A Markov switching VAR approach

2.1. The model specification

Let $\mathbf{y} = (\mathbf{y}_t)$ be a K-dimensional random vector with values in $\mathbb{R}^K$ which satisfies a M-state Markov switching VAR(p) model, in short MS(M) VAR(p):

$$\mathbf{A}_{s_t}(L)\,\mathbf{y}_t = \boldsymbol{\nu}_{s_t} + \mathbf{u}_t \tag{1}$$

where $\boldsymbol{\nu}_{s_t} \in \mathbb{R}^K$, $\mathbf{A}_{s_t}(L) = \mathbf{I}_K - \sum_{i=1}^{p} \mathbf{A}_{i,s_t} L^i$ is a $(K \times K)$ matrix polynomial in the lag operator L, $\mathbf{u}_t = \boldsymbol{\Sigma}_{s_t}\boldsymbol{\eta}_t$, $\boldsymbol{\Sigma}_{s_t}$ non-singular $(K \times K)$ matrix, and $\boldsymbol{\eta}_t \sim \mathrm{IID}(\mathbf{0}, \mathbf{I}_K)$. Here $\mathbf{I}_K$ denotes the $(K \times K)$ identity matrix, as usual. The model parameters $\boldsymbol{\nu}_{s_t}$, $\mathbf{A}_{i,s_t}$, for $i = 1, \ldots, p$, and $\boldsymbol{\Sigma}_{s_t}$ are regime-dependent, that is, they are driven by a Markov chain (s_t) with values in the set $\Xi = \{1, \ldots, M\}$.

Assumption 1. The process (s_t) follows an M-state irreducible and aperiodic Markov chain with transition probability matrix $\mathbf{P} = (p_{ij})$, where $p_{ij} = Pr(s_t = j | s_{t-1} = i)$, for all $i, j = 1, \ldots, M$, and unconditional (or steady state) probabilities $\pi_i = Pr(s_t = i)$, for all $i = 1, \ldots, M$.

Since the chain (s_t) is finite, assuming it irreducible and aperiodic is enough for it to be ergodic (i.e., regular). Ergodicity implies the existence of a stationary vector of probabilities $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_M)'$ satisfying $\boldsymbol{\pi} = \mathbf{P}'\boldsymbol{\pi}$ and $\mathbf{i}'_M\boldsymbol{\pi} = 1$, where $\mathbf{i}_M$ denotes the usual $(M \times 1)$ vector of ones. Then the vector $\boldsymbol{\pi}$ is defined as the eigenvector of $\mathbf{P}'$ associated with the unit eigenvalue. Irreducibility implies that $\pi_i > 0$, for $i \in \Xi$, meaning that all unobservable states are possible. It is also assumed that $\pi_1 \geq \pi_2 \geq \cdots \geq \pi_M$ for identifiability of regimes, and $\boldsymbol{\nu}_i \neq \boldsymbol{\nu}_j$, for $i, j \in \Xi$, $i \neq j$. In order to clarify the definition of aperiodic Markov chain, we need some preliminaries. The period d_i of a state $i \in \Xi$ is given by $d_i := \mathrm{gcd}\,\{m \geq 1 \ : \ p_{ii}^{(m)} > 0\}$, where gcd denotes the greatest common divisor and $p_{ii}^{(m)} = Pr(s_{t+m} = i \,|\, s_t = i)$. Set $d_i = \infty$ if $p_{ii}^{(m)} = 0$ for all $m \geq 1$. If $d_i = 1$, then the state $i \in \Xi$ is called aperiodic. A Markov chain is said to be *aperiodic* if all its states are aperiodic.

Assumption 2. The pair $(s_t, \boldsymbol{\eta}_t)$ is a strictly stationary process defined in some probability space, and the regime variable s_t is independent of $\boldsymbol{\eta}_t$ for every t.

2.2. A state space representation

An useful representation for (s_t) is obtained by letting ξ_t denote a random $(M \times 1)$ vector whose ith element is equal to unity if $s_t = i$ and zero otherwise. Then the Markov chain follows a VAR(1) process

$$\xi_t = \mathbf{P}'\,\xi_{t-1} + \mathbf{v}_t \tag{2}$$

where $\mathbf{v}_t = \xi_t - E[\xi_t|\xi_{t-1}]$ is a zero mean martingale difference sequence.

By direct computations, we have the following standard properties:

$$E[\xi_t] = \boldsymbol{\pi} \qquad E[\xi_t\,\xi_t'] = \mathbf{D}$$
$$E[\xi_t\,\xi_{t+h}'] = \mathbf{D}\mathbf{P}^h \qquad \mathbf{v}_t \sim \mathrm{IID}(\mathbf{0}, \boldsymbol{\Gamma}_{\mathbf{v}})$$

where $\mathbf{D} = \mathrm{diag}(\pi_1 \ldots \pi_M)$, $\boldsymbol{\Gamma}_{\mathbf{v}} = \mathbf{D} - \mathbf{P}'\mathbf{D}\mathbf{P}$, and $h > 0$.

Define $\boldsymbol{\Lambda} = (\boldsymbol{\nu}_1 \ \ldots \ \boldsymbol{\nu}_M) \in \mathbb{R}^{K\times M}$, $\mathbf{A}_i = (\mathbf{A}_{i,1} \ \ldots \ \mathbf{A}_{i,M}) \in \mathbb{R}^{K\times(KM)}$ for every $i = 1, \ldots, p$, and $\boldsymbol{\Sigma} = (\boldsymbol{\Sigma}_1 \ \ldots \ \boldsymbol{\Sigma}_M) \in \mathbb{R}^{K\times(KM)}$.

Then the process $\mathbf{y} = (\mathbf{y}_t)$ in (1) admits the following state-space representation

$$\begin{cases} \mathbf{y}_t = \boldsymbol{\Lambda}\,\xi_t + \sum_{i=1}^{p} \mathbf{A}_i\,(\xi_t \otimes \mathbf{I}_K)\,\mathbf{y}_{t-i} + \mathbf{u}_t \\ \xi_t = \mathbf{P}'\,\xi_{t-1} + \mathbf{v}_t \end{cases} \tag{3}$$

where

$$\mathbf{u}_t = \boldsymbol{\Sigma}_{s_t}\,\boldsymbol{\eta}_t = \boldsymbol{\Sigma}(\xi_t \otimes \mathbf{I}_K)\,\boldsymbol{\eta}_t.$$

In Eq. (3), $\mathbf{u}_t$ is the symmetric continuous shock of the time-series process, while $\mathbf{v}_t$ is the asymmetric discrete shock of the latent state variable $\xi_t \in \mathbb{R}^M$.

2.3. The Markovian representation

Using (3), we get a Markovian vectorial form of the initial model (through the section, set $n = pK + M$):

$$\mathbf{z}_t = \boldsymbol{\Phi}_t\,\mathbf{z}_{t-1} + \mathbf{e}_t \tag{4}$$

where

$$\mathbf{z}_t = (\mathbf{y}_t' \quad \mathbf{y}_{t-1}' \quad \cdots \quad \mathbf{y}_{t-p+1}' \quad \xi_{t+1}')' \in \mathbb{R}^n$$

$$\mathbf{e}_t = (\mathbf{u}_t' \quad \mathbf{0}' \quad \cdots \quad \mathbf{0}' \quad \mathbf{v}_{t+1}')' \in \mathbb{R}^n$$

and

$$\boldsymbol{\Phi}_t = \begin{pmatrix} \mathbf{A}_{1,s_t} & \mathbf{A}_{2,s_t} & \cdots & \mathbf{A}_{p-1,s_t} & \mathbf{A}_{p,s_t} & \boldsymbol{\Lambda} \\ \mathbf{I}_K & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_K & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I}_K & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{P}' \end{pmatrix} \in \mathbb{R}^{n\times n}$$

where $\mathbf{A}_{i,s_t} = \mathbf{A}_i(\xi_t \otimes \mathbf{I}_K)$ for $i = 1, \ldots, p$. In this vectorial representation, it is implicitly assumed that $p \geq 1$ without loss of generality because the matrix coefficient $\mathbf{A}_{p,s_t}$ can be equal to $\mathbf{0}$ in $\mathbf{A}_{s_t}(L)$. In the sequel, let $\boldsymbol{\Phi}_i$ denote the matrix obtained by replacing s_t by i in $\boldsymbol{\Phi}_t$.

3. Impulse response functions

This section offers a derivation of impulse responses (IRs) of measurements to a shock in the MS VAR model based on its state-space representation. There are two types of IRs to be computed in such a model. The first is state-specific, that is, conditional on the states (presuming that the regime will not change over the period for which the IRs are computed). In this case, the formulas for the IRs are the same as in the standard vector autoregressions. The other type is one takes into account the possibility of future regime changes. The potential difficulty in computing these IRs is that they should account for the forecasted state probabilities that are computed using the transition probability matrix. The given Markovian representation of the initial model is used to derive the matrix formulas for the IRFs.

In (4) there are two independent (under Assumption 2) shocks: a symmetric continuous shock $\mathbf{u}_t$ and an asymmetric discrete shock $\mathbf{v}_t$. See Appendix.

The results of Subsection (**3.1**) were first proved in Cavicchioli (2023), but we report them to make the reading self-contained and because the proposed matrix formulas for the RD IRF's will be used in the empirical applications treated in the present paper.

New results on the Exact IRF's are derived in Subsection (**3.2**), where the proposed matrix formulas are very neat and easy to apply. They cannot be found in the existing theoretical literature and constitute a novel and different method with respect to Karamé (2012, 2015), giving a substantial academic contribution on the topic.

3.1. Regime dependent impulse response analysis

Given (4) we derive the closed-form matrix expressions of the Regime Dependent Impulse Response Functions (RD IRFs) with respect to these shocks. Below we always assume that $p \geq h$ since otherwise zeros of $\mathbf{A}_{p,s_t}$ can be filled in.

Theorem 1. *Let $(\mathbf{y}_t)$ be the process driven by the MS(M) VAR(p) model in (1). Under Assumptions 1–2, the RD IRFs for $(\mathbf{y}_t)$ with respect to the symmetric continuous shock $\mathbf{u}_t$ or $\boldsymbol{\eta}_t$ are given by*

$$\frac{\partial\, E_t[\mathbf{y}_{t+h}]}{\partial\, \mathbf{u}_t'}\Big|_{s_t=i_0,\ldots,s_{t+h}=i_h} = \boldsymbol{L}\boldsymbol{\Phi}_{i_h} \cdots \boldsymbol{\Phi}_{i_1}\, \boldsymbol{L}'$$

and

$$\frac{\partial\, E_t[\mathbf{y}_{t+h}]}{\partial\, \boldsymbol{\eta}_t'}\Big|_{s_t=i_0,\ldots,s_{t+h}=i_h} = \boldsymbol{L}\boldsymbol{\Phi}_{i_h} \cdots \boldsymbol{\Phi}_{i_1}\, \boldsymbol{L}'\, \boldsymbol{\Sigma}_{i_0}$$

for all $h > 0$, where $E_t[\mathbf{y}_{t+h}] := E[\mathbf{y}_{t+h}|\mathbf{Y}_t]$, being $\mathbf{Y}_t$ the information set up to time t, that is, $\mathbf{Y}_t = \{\mathbf{y}_t, \mathbf{y}_{t-1}, \ldots\}$, and $\boldsymbol{L} = (\boldsymbol{I}_K \quad \mathbf{0} \quad \cdots \quad \mathbf{0}) \in \mathbb{R}^{K\times n}$. If $h = 0$, the first resp. second matrix function equals $\boldsymbol{I}_K$ resp. $\boldsymbol{\Sigma}_{i_0}$. The same formulas also work replacing $E_t[\mathbf{y}_{t+h}]$ by $\mathbf{y}_{t+h}$.

Corollary 1. *Under the assumptions of Theorem 1, it follows that*

$$\frac{\partial\, E_t[\mathbf{y}_{t+h}]}{\partial\, \mathbf{u}_t'}\Big|_{s_t=\cdots=s_{t+h}=i} = \boldsymbol{L}\boldsymbol{\Phi}_i^h\, \boldsymbol{L}'$$

and

$$\frac{\partial\, E_t[\mathbf{y}_{t+h}]}{\partial\, \boldsymbol{\eta}_t'}\Big|_{s_t=\cdots=s_{t+h}=i} = \boldsymbol{L}\boldsymbol{\Phi}_i^h \boldsymbol{L}'\, \boldsymbol{\Sigma}_i := \boldsymbol{\Theta}_{i,h}$$

where

$$\boldsymbol{\Theta}_{i,0} = \boldsymbol{\Sigma}_i \qquad \boldsymbol{\Theta}_{i,1} = \boldsymbol{A}_{1,i}\, \boldsymbol{\Theta}_{i,0} = \boldsymbol{A}_{1,i}\, \boldsymbol{\Sigma}_i$$

and in general

$$\boldsymbol{\Theta}_{i,h} = \boldsymbol{A}_{1,i}\, \boldsymbol{\Theta}_{i,h-1} + \boldsymbol{A}_{2,i}\, \boldsymbol{\Theta}_{i,h-2} + \ldots + \boldsymbol{A}_{p,i}\, \boldsymbol{\Theta}_{i,h-p}$$

for every $i = 1, \ldots, M$ and $h = 1, 2, \ldots$, with $\boldsymbol{\Theta}_{i,h} = \mathbf{0}$ if $h < 0$.

Theorem 2. *Let $(\mathbf{y}_t)$ be the process driven by the MS(M) VAR(p) model in (1). Under Assumptions 1–2, the RD IRF for $(\mathbf{y}_t)$ with respect to the discrete asymmetric shock $\mathbf{v}_t$ is given by*

$$\frac{\partial \mathbf{y}_{t+h}}{\partial \mathbf{v}'_{t+1}}|_{s_t=i_0,\dots,s_{t+h}=i_h} = \boldsymbol{L}\boldsymbol{\Phi}_{i_h}\cdots\boldsymbol{\Phi}_{i_1}\boldsymbol{R}'$$

where $\boldsymbol{L}$ is as in Theorem 1, and $\boldsymbol{R} = (\mathbf{0} \quad \cdots \quad \mathbf{0} \quad \boldsymbol{I}_M) \in \mathbb{R}^{M\times n}$.

Corollary 2. *Under the assumptions of Theorem 1, we have*

$$\frac{\partial \mathbf{y}_{t+h}}{\partial \mathbf{v}'_{t+1}}|_{s_t=\dots=s_{t+h}=i} = \boldsymbol{L}\boldsymbol{\Phi}_i^h\boldsymbol{R}' := \boldsymbol{\Psi}_{i,h}$$

where

$$\boldsymbol{\Psi}_{i,h} = \sum_{j=1}^{h-1}\boldsymbol{\Theta}_{i,j}\,\boldsymbol{\Sigma}_i^{-1}\,\boldsymbol{\Lambda}\,(\boldsymbol{P}')^{h-j-1} + \boldsymbol{\Lambda}\,(\boldsymbol{P}')^{h-1}$$

for $i = 1,\dots,M$ and $h = 2,\dots$, $\boldsymbol{\Psi}_{i,1} = \boldsymbol{\Lambda}$, and $\boldsymbol{\Psi}_{i,h} = \mathbf{0}$ if $h \le 0$. Here for a $(M\times M)$ matrix $\boldsymbol{X}$, set $\boldsymbol{X}^k = \boldsymbol{I}_M$ if $k = 0$.

3.2. Exact impulse response analysis

To get matrix expressions in closed form for the Exact Impulse Response Functions (EIRFs) of the process with respect to the shocks $\mathbf{u}_t$ and $\mathbf{v}_t$, we need the following $(Mn)\times(Mn)$ matrix, with $n = pK + M$:

$$\boldsymbol{\Phi} = \begin{pmatrix} p_{11}\boldsymbol{\Phi}_1 & p_{21}\boldsymbol{\Phi}_1 & \cdots & p_{M1}\boldsymbol{\Phi}_1 \\ p_{12}\boldsymbol{\Phi}_2 & p_{22}\boldsymbol{\Phi}_2 & \cdots & p_{M2}\boldsymbol{\Phi}_2 \\ \vdots & \vdots & & \vdots \\ p_{1M}\boldsymbol{\Phi}_M & p_{2M}\boldsymbol{\Phi}_M & \cdots & p_{MM}\boldsymbol{\Phi}_M \end{pmatrix}$$

where $\boldsymbol{\Phi}_i$ is obtained from the above-defined matrix $\boldsymbol{\Phi}_t$ replacing s_t by i, for $i = 1,\dots,M$.

Theorem 3. *Let $(\mathbf{y}_t)$ be the process driven by the initial* MS(M) VAR(p) *model. Under Assumptions 1–2, the EIRFs for $(\mathbf{y}_t)$ with respect to the symmetric continuous shock $\mathbf{u}_t$ or $\boldsymbol{\eta}_t$ are given by*

$$\frac{\partial E_t[\mathbf{y}_{t+h}]}{\partial \mathbf{u}'_t} = (\boldsymbol{L}\otimes\boldsymbol{i}'_M)\boldsymbol{\Phi}^h(\boldsymbol{L}'\otimes\boldsymbol{\pi})$$

and

$$\frac{\partial E_t[\mathbf{y}_{t+h}]}{\partial \boldsymbol{\eta}'_t} = (\boldsymbol{L}\otimes\boldsymbol{i}'_M)\boldsymbol{\Phi}^h(\boldsymbol{L}'\otimes\boldsymbol{D})\boldsymbol{\Sigma}'$$

for all $h > 0$, where $\boldsymbol{i}_M$ is the $(M\times1)$ vector of ones, $\boldsymbol{\pi} = (\pi_1,\dots,\pi_M)'$ is the stationary vector of the Markov chain, and $\boldsymbol{L}$ is as in Theorem 1.

Theorem 4. *Let $(\mathbf{y}_t)$ be the process driven by the initial* MS(M) VAR(p) *model. Under Assumptions 1–2, the EIRFs for $(\mathbf{y}_t)$ with respect to the asymmetric discrete shock $\mathbf{v}_t$ is given by*

$$\frac{\partial E_t[\mathbf{y}_{t+h}]}{\partial \mathbf{v}'_{t+1}} = (\boldsymbol{L}\otimes\boldsymbol{i}'_M)\boldsymbol{\Phi}^h(\boldsymbol{R}'\otimes\boldsymbol{i}_M)$$

for all $h > 0$, where $\boldsymbol{i}_M$ is the $(M\times1)$ vector of ones, $\boldsymbol{\pi} = (\pi_1,\dots,\pi_M)'$ is the stationary vector of the Markov chain, and $\boldsymbol{L}$ and $\boldsymbol{R}$ are as in Theorems 1 and 2.

For practical inference purposes, the matrices involved in the statements of the above theorems are replaced by their maximum likelihood (ML) estimates which can be obtained by using the ML estimates of the model parameters. See, for example, Cavicchioli (2014b, 2021). This gives a convenient plug-in approach to approximate the regime-dependent impulse response functions theoretically derived in this paper. The obtained matrix formulas display exponentially increasing complexity as regards the prediction horizon. This agrees with the usual findings in the existing literature.

4. Applications

4.1. Oil price shocks from Shahrestani and Rafei (2020)

We first depict the regime dependent and the exact impulse response functions for the two-state bivariate MS VAR model estimated in Table 6, p.6, from Shahrestani and Rafei (2020), which we report in Table 1 to make the reading self-contained. Such authors study the impulse responses of the Tehran stock market to world oil price shocks. The used data consists of monthly prices of the world price of oil (West Texas Intermediate, WTI) and Tehran Stock Exchange Index (Tehran Price Index, TPI) covering the period from 2002/04/01 to 2017/02/31. Particularly, these authors consider a bivariate MS(2) VAR(1) model as in (1), namely

Table 1
Parameter estimations (approximated up to the fourth decimal number) for the bivariate MS(2) VAR(1) model from Shahrestani and Rafei (2020, Table 6, p.6) and sufficient conditions for the second-order stationarity (bottom line).

	ν_{s_t}	A_{s_t}	Ω_{s_t}	P
Regime 1	$\begin{pmatrix} 0.0242 \\ -0.0157 \end{pmatrix}$	$\begin{pmatrix} 0.4040 & 0.1905 \\ 0.0773 & 0.5304 \end{pmatrix}$	$\begin{pmatrix} 0.0028 & 0 \\ 0 & 0.0065 \end{pmatrix}$	$\begin{pmatrix} 0.8940 & 0.1060 \\ 0.0939 & 0.9061 \end{pmatrix}$
Regime 2	$\begin{pmatrix} 0.0008 \\ 0.0229 \end{pmatrix}$	$\begin{pmatrix} 0.3201 & -0.0758 \\ 0.5270 & 0.0671 \end{pmatrix}$	$\begin{pmatrix} 0.0008 & 0 \\ 0 & 0.0039 \end{pmatrix}$	
Stationarity		Regime 1:$\rho(\mathbf{A}_1) = 0.604$	Regime 2:$\rho(\mathbf{A}_2) = 0.248$	Global:$\rho(\mathbf{A}) = 0.548$

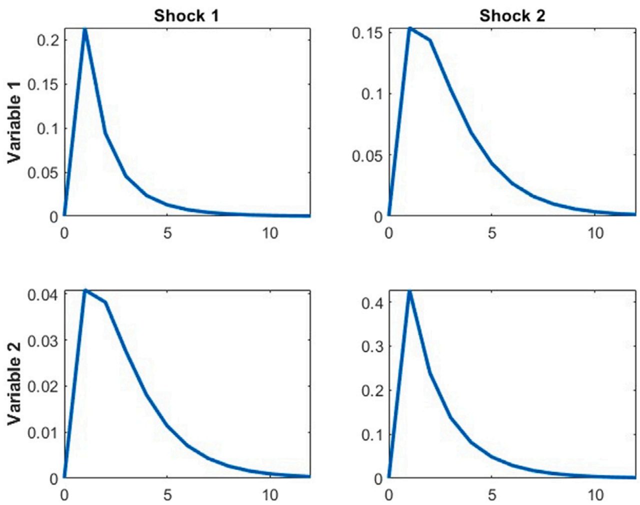

Fig. 1. Regime-dependent impulse response functions for the bivariate MS(2) VAR(1) model in Table 1: continuous shock in regime 1.

$\mathbf{y}_t = \boldsymbol{\nu}_{s_t} + \mathbf{A}_{s_t}\mathbf{y}_{t-1} + \mathbf{u}_t$, $\mathbf{u}_t = \boldsymbol{\Sigma}_{s_t}\boldsymbol{\eta}_t$, $\boldsymbol{\eta}_t \sim \text{IID}(\mathbf{0}, \mathbf{I}_2)$ and $s_t \in \{1, 2\}$. To verify the global stationarity we apply the sufficient condition given in Theorem 2 of Francq and Zakoïan (2001), that is, $\rho(\mathbf{A}) < 1$, where the matrix $\mathbf{A}$ is defined by

$$\mathbf{A} = \begin{pmatrix} p_{11}\,\mathbf{A}_1 & (1-p_{22})\,\mathbf{A}_1 \\ (1-p_{11})\,\mathbf{A}_2 & p_{22}\,\mathbf{A}_2 \end{pmatrix}$$

and $\rho(\cdot)$ denotes the spectral radius. As shown in the last line of Table 1, the two regimes are second-order stationary and the model is globally stationary as well.

Using the proposed matrix expressions for the IRFs, the findings in Shahrestani and Rafei (2020) are confirmed with respect to the regime-dependent continuous shocks in both regimes (see Figs. 1 and 2). Particularly, with respect to the effect of oil price on stock market, we see that in regime 1 the effect is positive and declines from the second month; in regime 2 the immediate effect is opposite and becomes stable from the fourth month. We complete this example including the impulse responses with respect to the discrete shocks (Fig. 3). Moreover, we report the exact impulse response functions that show the global response of the system whatever the states visited by the shock (see Fig. 4). Here we note that shock 1 only affects the first variable, which is reasonable given that a shock in Tehran market has no impact on the world oil price. On the contrary, a drop in the world oil price has a global effect, which is positive for the considered stock market.

4.2. US gas market

Motivated by the last relevant changes in gas prices, we model the US natural gas market using a two-state MS VAR(1) to allow for possible recurrent structural shifts and capture asymmetric responses to shocks. We focus on a trivariate system including gas production, a variable describing demand for natural gas and its price. Concerning the production side, we use monthly gas withdrawals from the US Department of Energy (EIA), which is seasonally adjusted and transformed using the first difference of the logs. With regards to the demand side, we proxy the US economic activity with the monthly US industrial production index,

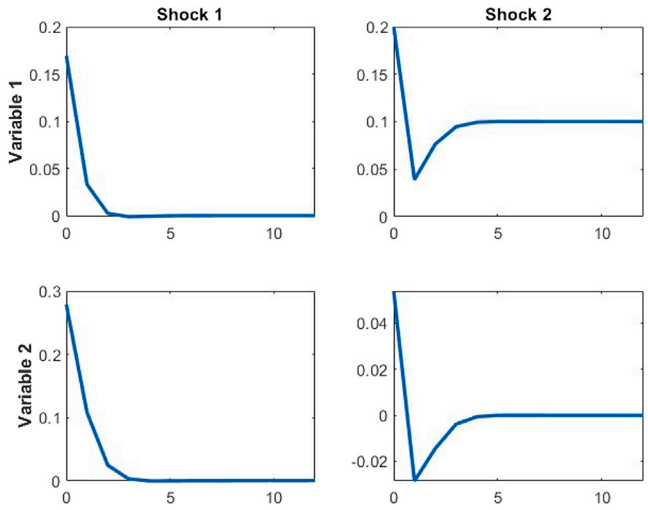

Fig. 2. Regime-dependent impulse response functions for the bivariate MS(2) VAR(1) model in Table 1: continuous shock in regime 2.

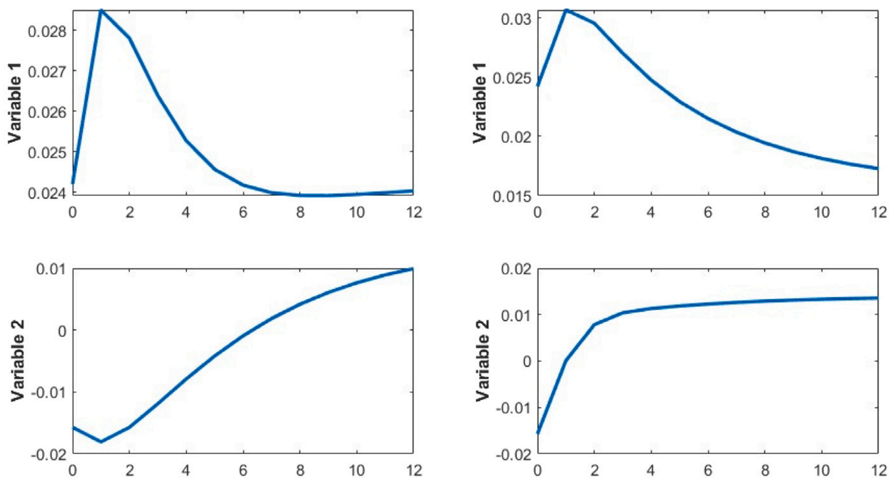

Fig. 3. Regime-dependent impulse response functions for the bivariate MS(2) VAR(1) model in Table 1: discrete shock in regime 1 (left plot) and discrete shock in regime 2 (right plot) .

seasonally adjusted and taken from FRED database (the same transformation applies). Finally, we consider the real gas price obtained as the ratio between the nominal price series (wellhead price) taken from EIA and the US CPI from FRED. This time series is also transformed by using the first difference of the logs. The time span runs from January 1980 to September 2022 and the considered series are reported in Fig. 5.

Model selection of the regime number and VAR lag order is performed by using the methods proposed in Cavicchioli (2014a, 2015) based on the associated stable VARMA representation of the considered model. Then the system is modeled as a trivariate MS(2) VAR(1), and it is estimated by ML (resp. OLS) approach as described in Cavicchioli (2014b, 2021). The two obtained regimes can also be supported by some empirical arguments. Since including the periods of crisis change the results significantly, it follows that natural gas price uncertainty may give rise to asymmetric effects on output with respect to expansions and contractions in real economic activity. Therefore, two regimes should be able to describe the dynamic interactions between the real price of natural gas and the real output growth rate across contractionary and expansionary phases of the business cycle.

ML point estimates and relative standard errors in parentheses are given in Table 2. The second regime shows periods of lower gas production and higher price together with economic contraction and high volatility. Whereas the first regime is much persistent than the second one, and it is related to normal times. Moreover, expected durations of the first and second regimes are 41 months (about 3 years) and 2.5 months, respectively, so that the economy visits most in regime 1. To check local and global stationarity of

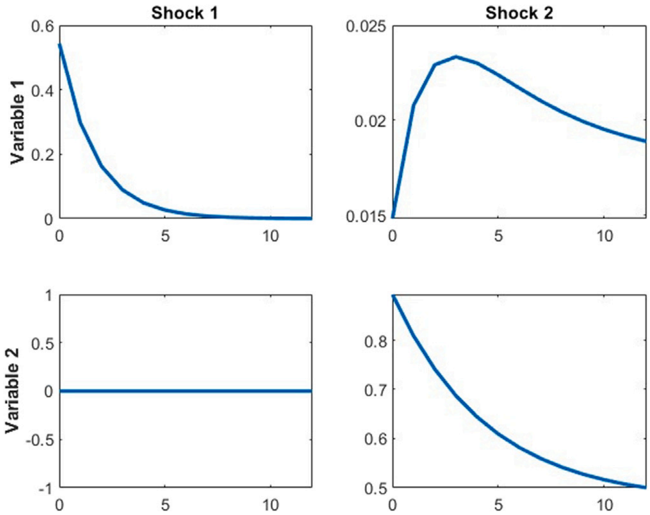

Fig. 4. Exact impulse response functions for the bivariate MS(2) VAR(1) model in Table 1.

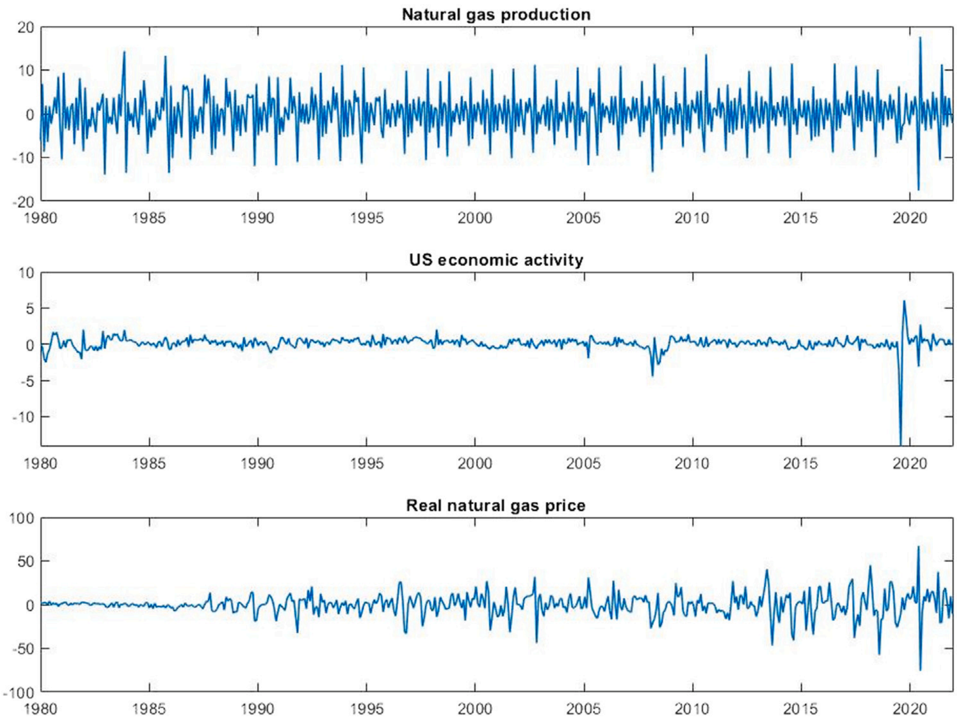

Fig. 5. US gas production, economic activity and gas price from January 1980 to September 2022; details are described in Section 4.2.

the system, we compute the spectral radius of the matrices $\boldsymbol{\Phi}_1$, $\boldsymbol{\Phi}_2$ and $\boldsymbol{\Phi}$, which correspond to the first regime, the second regime and the global level, respectively. The obtained values are $\rho(\boldsymbol{\Phi}_1) = 0.9995$, $\rho(\boldsymbol{\Phi}_2) = 0.5887$ and $\rho(\boldsymbol{\Phi}) = 0.9275$. In particular, it is satisfied the sufficient condition for the second-order stationarity of the process. See Theorem 2 from Francq and Zakoïan (2001). The smoothed probabilities of regime 2 are depicted in Fig. 6. They generally correspond to turbulent periods with higher gas prices and contractions in the business cycle. Furthermore, to understand the dynamic responses of the variables in the system, we depict exact impulse response functions, together with one-standard error bands, in Fig. 7. The relevant facts are the following: first, gas demand and price shocks have marginal impact on the gas supply; second, price shocks shrink the gas demand; third, the gas price

Table 2
Parameter ML estimates for the trivariate MS(2) VAR(1) in the log difference of US gas production, economic activity and gas price from January 1980 to September 2022. Standard errors are in parentheses.

Parameter	Regime 1	Regime 2
$\boldsymbol{\nu}_{s_t}$	$\begin{pmatrix} 0.349\,(.069) \\ 0.116\,(.021) \\ 1.616\,(.487) \end{pmatrix}$	$\begin{pmatrix} -0.247\,(.082) \\ 0.087\,(.013) \\ 2.404\,(.741) \end{pmatrix}$
$\mathbf{A}_{s_t}$	$\begin{pmatrix} 0.417(.088) & -0.153(.056) & 0.015(.003) \\ 0.037(.014) & 0.504(.024) & -0.017(.013) \\ 1.271(.168) & -0.939(.341) & 0.924(.415) \end{pmatrix}$	$\begin{pmatrix} 0.586(.262) & -0.533(.098) & -0.012(.052) \\ 0.014(.077) & -0.209(.054) & -0.007(.002) \\ -0.190(.008) & -0.266(.0.086) & 0.468(.153) \end{pmatrix}$
$\boldsymbol{\Sigma}_{s_t}$	$\begin{pmatrix} 4.651(.887) & 0.363(.074) & 0.037(.006) \\ - & 0.793(.043) & -0.896(.153) \\ - & - & 6.999(.954) \end{pmatrix}$	$\begin{pmatrix} 6.494(1.035) & 0.483(.781) & 1.010(.633) \\ - & 0.932(.549) & 0.103(.099) \\ - & - & 8.854(1.486) \end{pmatrix}$
$\mathbf{P}$	$\begin{pmatrix} 0.976\,(.011) & 0.398\,(.015) \\ 0.024\,(.004) & 0.602\,(.013) \end{pmatrix}$	

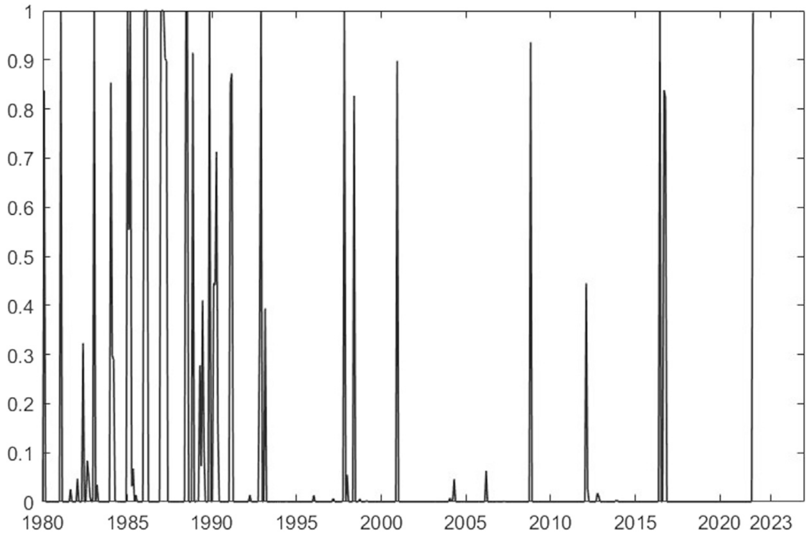

Fig. 6. Smoothed probabilities of regime 2 for the empirical application described in Section 4.2.

is mainly guided by demand shocks rather than production side. The investigation of the global responses (taking into consideration asymmetric effects of the economic activity) guides us to a comprehensive understanding of the price and demand dynamics.

Our findings agree with those obtained in Rubaszek et al. (2021) and Wiggins and Etienne (2017). The first paper investigates the role of structural shocks for the dynamics of the US natural gas market within the Bayesian structural VAR framework. There, the short-term price elasticity of natural gas supply and demand can be estimated. The results indicate that the former is slow, whereas the latter is higher than the average estimate in the literature. Moreover, Wiggins and Etienne (2017) investigate supply and demand shocks in the US natural gas market, focusing on how the effects of these shocks have changed over time. Such authors find that supply and demand shocks are the main drivers of natural gas price fluctuations during 1993–2015, with speculative activities playing a minor role during a portion of the sample period.

5. Conclusion

In this paper we investigate various concepts of impulse response functions (IRFs) in Markov switching vector autoregressions (MS VAR). Representing the considered model in a VAR(1) form and casting it into a state-space representation is useful to derive impulse responses of measurements to a shock in the system. Then we derive neat matrix formulas in closed form to compute the regime-dependent and the exact IRFs in MS VARs. Using such formulas, the impulse response function analysis can be directly processed with respect to symmetric continuous shocks and asymmetric discrete shocks. Empirical applications are proposed to illustrate the usefulness and the actual advantages of the proposed matrix methodology over the existing methods. In particular, we apply our matrix methods to identify the link between the pass-through of world oil price to Tehran stock exchange using a two-state MS VAR(1) model. We confirm the results in Shahrestani and Rafei (2020) and deepen the understanding of such mechanisms based on a different impulse response approach. A further investigation examines whether regime switching exists in the US gas market

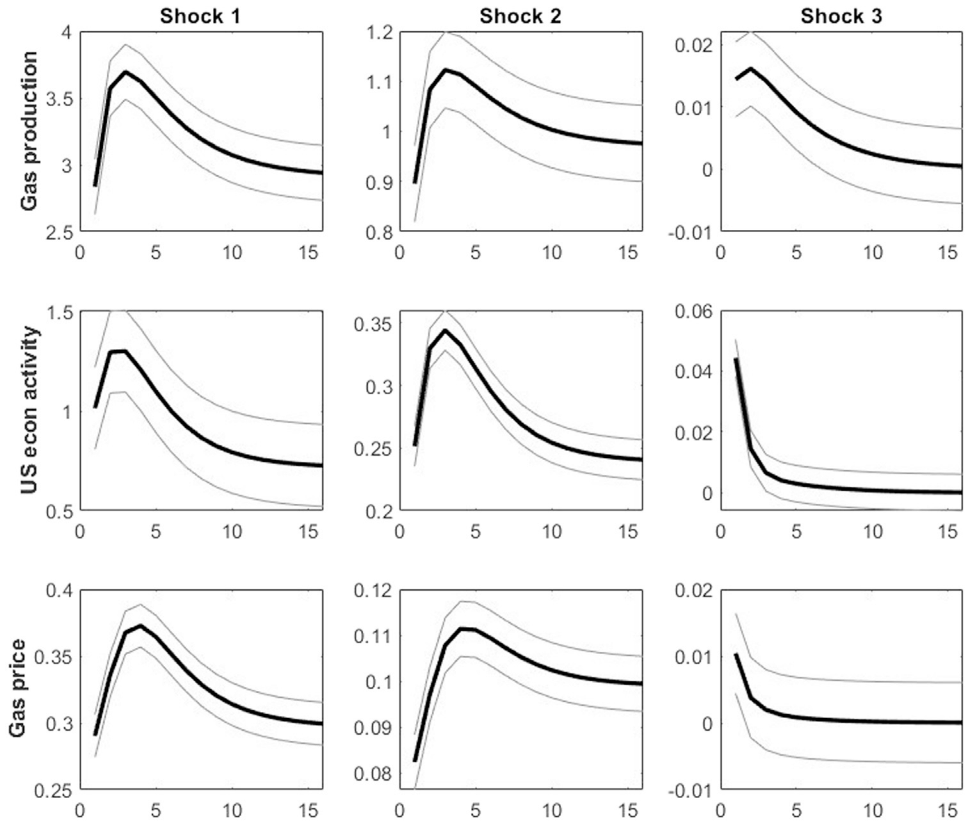

Fig. 7. Exact impulse response functions along with one-standard error bands for the empirical application described in Section 4.2.

and analyzes reactions to shocks across various regimes. Our findings show the importance of regime switching modeling to evaluate asymmetric features of responses.

CRediT authorship contribution statement

Maddalena Cavicchioli: Conceptualization, Data curation, Formal analysis, Methodology, Resources, Software, Visualization, Writing – original draft, Writing – review & editing.

Declaration of competing interest

None.

Data availability

Data are publicly available.

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Appendix

I. *The shock* $\mathbf{u}_t$ *is independent of* $\mathbf{v}_\tau$ *for all* t *and* τ. In fact, we have

$$E[\mathbf{u}_t\,\mathbf{v}_\tau'] = E[\boldsymbol{\Sigma}(\xi_t \otimes \mathbf{I}_K)\,\boldsymbol{\eta}_t\,(\xi_\tau - \mathbf{P}'\,\xi_{\tau-1})']$$
$$= \boldsymbol{\Sigma}\left\{E[\xi_t\,\xi_\tau'] \otimes \mathbf{I}_K \;-\; E[\xi_t\,\xi_{\tau-1}'\,\mathbf{P}] \otimes \mathbf{I}_K\right\}\,E[\boldsymbol{\eta}_t] \otimes \mathbf{I}_M = \mathbf{0}$$

as $E[\boldsymbol{\eta}_t] = \mathbf{0}$ and $\boldsymbol{\eta}_t$ is independent of ξ_τ for all t and τ by Assumption 2.

(II) *Proof of Theorem* 1. The law of iterated expectation implies that

$$E_t[\mathbf{z}_{t+h}]|_{s_t=i_0,\ldots,s_{t+h}=i_h} \;=\; E[E[\mathbf{z}_{t+h}|\mathbf{Y}_t,\, s_{t+h}=i_h]|s_t=i_0,\ldots,s_{t+h-1}=i_{h-1}]$$
$$= \boldsymbol{\Phi}_{i_h}\,E[E[\mathbf{z}_{t+h-1}|\mathbf{Y}_t,\, s_{t+h-1}=i_{h-1})|s_t=i_0,\ldots,s_{t+h-2}=i_{h-2}]$$
$$= \boldsymbol{\Phi}_{i_h}\,\boldsymbol{\Phi}_{i_{h-1}}\,\cdots\,\boldsymbol{\Phi}_{i_1}\,E_t[\mathbf{z}_t]|_{s_t=i_0}$$

where

$$E_t[\mathbf{z}_t] \;=\; (\mathbf{y}_t' \quad \mathbf{y}_{t-1}' \quad \cdots\, \mathbf{y}_{t-p+1}' \quad \xi_{t+1|t}')'$$

and $\xi_{t+1|t} = \mathbf{P}'\,\xi_{t|t}$. Here $\xi_{t|t} = E[\xi_t|\mathbf{Y}_t]$ denotes the conditional mean of ξ_t given $\mathbf{Y}_t$. The components of $\xi_{t|t}$ are called *filtered regime probabilities*. A fast iterative algorithm to compute $\xi_{t|t}$ can be found in Hamilton (1994), §22, or Krolzig (1997), §5, given the initial value $\xi_{0|0} = \boldsymbol{\pi}$.

Since $\mathbf{y}_{t+h}$ is the first component of $\mathbf{z}_{t+h}$, we have

$$E_t[\mathbf{y}_{t+h}]|_{s_t=i_0,\ldots,s_{t+h}=i_h} \;=\; \mathbf{L}\,\boldsymbol{\Phi}_{i_h}\,\boldsymbol{\Phi}_{i_{h-1}}\,\cdots\,\boldsymbol{\Phi}_{i_1}\,E_t[\mathbf{z}_t]|_{s_t=i_0}.$$

Taking the first derivative of this $(K \times 1)$ vector with respect to $\mathbf{u}_t$ gives the $(K \times K)$ matrix function

$$\frac{\partial\,E_t[\mathbf{y}_{t+h}]}{\partial\,\mathbf{u}_t'}|_{s_t=i_0,\ldots,s_{t+h}=i_h} \;=\; \mathbf{L}\,\boldsymbol{\Phi}_{i_h}\,\boldsymbol{\Phi}_{i_{h-1}}\,\cdots\,\boldsymbol{\Phi}_{i_1}\,\frac{\partial\,E_t[\mathbf{z}_t]}{\partial\,\mathbf{u}_t'}|_{s_t=i_0}$$
$$= \mathbf{L}\,\boldsymbol{\Phi}_{i_h}\,\boldsymbol{\Phi}_{i_{h-1}}\,\cdots\,\boldsymbol{\Phi}_{i_1}\,\mathbf{L}'.$$

This proves the first formula in Theorem 1. The second one easily follows as $\frac{\partial\,\mathbf{u}_t}{\partial\,\boldsymbol{\eta}_t'}|_{s_t=i_0} = \boldsymbol{\Sigma}_{i_0}$.

(III) *Proof of Theorem* 2. Taking the first derivative of the $(K \times 1)$ vector $\mathbf{y}_{t+h}$ with respect to $\mathbf{v}_{t+1}$ given the regimes yields the $(K \times M)$ matrix function

$$\frac{\partial\,\mathbf{y}_{t+h}}{\partial\,\mathbf{v}_{t+1}'}|_{s_t=i_0,\ldots,s_{t+h}=i_h} \;=\; \mathbf{L}\,\boldsymbol{\Phi}_{i_h}\,\boldsymbol{\Phi}_{i_{h-1}}\,\cdots\,\boldsymbol{\Phi}_{i_1}\,\frac{\partial\,\mathbf{z}_t}{\partial\,\mathbf{v}_{t+1}'}|_{s_t=i_0}$$
$$= \mathbf{L}\,\boldsymbol{\Phi}_{i_h}\,\boldsymbol{\Phi}_{i_{h-1}}\,\cdots\,\boldsymbol{\Phi}_{i_1}\,\frac{\partial\,\mathbf{e}_t}{\partial\,\mathbf{v}_{t+1}'}|_{s_t=i_0}$$
$$= \mathbf{L}\,\boldsymbol{\Phi}_{i_h}\,\boldsymbol{\Phi}_{i_{h-1}}\,\cdots\,\boldsymbol{\Phi}_{i_1}\,\mathbf{R}'.$$

The formulas in the statement of Corollary 2 follow from standard matrix calculus.

(IV) *Proof of Theorem* 3 . From (4) with $h > 0$, we get

$$E_t[\mathbf{z}_{t+h}] = E[\mathbf{z}_{t+h}|\mathbf{Y}_t] = \sum_{i_h=1}^{M} E_t[\mathbf{z}_{t+h},\, s_{t+h}=i_h] = \sum_{i_h=1}^{M} E_t[\mathbf{z}_{t+h}|\, s_{t+h}=i_h]\,\pi_{i_h}$$
$$= \sum_{i_h=1}^{M} \boldsymbol{\Phi}_{i_h}\,E_t[\mathbf{z}_{t+h-1}|s_{t+h}=i_h]\,\pi_{i_h}$$
$$= \sum_{i_h=1}^{M}\sum_{i_{h-1}=1}^{M} \boldsymbol{\Phi}_{i_h}\,E_t[\mathbf{z}_{t+h-1},\, s_{t+h-1}=i_{h-1}|s_{t+h}=i_h]\,\pi_{i_h}$$
$$= \sum_{i_h=1}^{M}\sum_{i_{h-1}=1}^{M} \boldsymbol{\Phi}_{i_h}\,E_t[\mathbf{z}_{t+h-1}|s_{t+h-1}=i_{h-1}]\,Pr(s_{t+h}=i_h|s_{t+h-1}=i_{h-1})\,\pi_{i_h}$$
$$= \sum_{i_h=1}^{M}\sum_{i_{h-1}=1}^{M} \boldsymbol{\Phi}_{i_h}\,\boldsymbol{\Phi}_{i_{h-1}}\,E_t[\mathbf{z}_{t+h-2}|s_{t+h-1}=i_{h-1}]\,p_{i_{h-1},i_h}\,\pi_{i_h}$$
$$= \sum_{i_0=1}^{M}\sum_{i_1=1}^{M}\cdots\sum_{i_{h-1}=1}^{M}\sum_{i_h=1}^{M} \boldsymbol{\Phi}_{i_1}\,\cdots\,\boldsymbol{\Phi}_{i_{h-1}}\,\boldsymbol{\Phi}_{i_h}\,p_{i_0,i_1}\,\cdots\,p_{i_{h-2},i_{h-1}}\,p_{i_{h-1},i_h}$$
$$\times\,\pi_{i_h}\,E_t[\mathbf{z}_t|s_t=i_0]$$

hence

$$\frac{\partial\,E_t[\mathbf{z}_{t+h}]}{\partial\,\mathbf{u}_t'} = \sum_{i_0=1}^{M}\sum_{i_1=1}^{M}\cdots\sum_{i_{h-1}=1}^{M}\sum_{i_h=1}^{M} \boldsymbol{\Phi}_{i_1}\,\cdots\,\boldsymbol{\Phi}_{i_{h-1}}\,\boldsymbol{\Phi}_{i_h}\,p_{i_0,i_1}\,\cdots\,p_{i_{h-2},i_{h-1}}\,p_{i_{h-1},i_h}\,\pi_{i_h}\,\mathbf{L}'$$

as

$$\frac{\partial E_t[\mathbf{z}_t|s_t = i_0]}{\partial \mathbf{u}_t'} = \mathbf{L}',$$

being $\mathbf{L} = (\mathbf{I}_K \quad \mathbf{0} \quad \cdots \quad \mathbf{0}) \in \mathbb{R}^{K\times n}$. This implies

$$\frac{\partial E_t[\mathbf{y}_{t+h}]}{\partial \mathbf{u}_t'} = \mathbf{L} \sum_{i_0=1}^{M} \sum_{i_1=1}^{M} \cdots \sum_{i_{h-1}=1}^{M} \sum_{i_h=1}^{M} \boldsymbol{\Phi}_{i_1} \dots \boldsymbol{\Phi}_{i_{h-1}} \boldsymbol{\Phi}_{i_h} \times p_{i_0,i_1} \dots p_{i_{h-2},i_{h-1}} p_{i_{h-1},i_h} \pi_{i_h} \mathbf{L}'.$$

In matrix form the first formula in Theorem 3 holds. In fact, we see that a typical element of the power matrix $\boldsymbol{\Phi}^h$ is a linear combination of products $\boldsymbol{\Phi}_{i_1} \dots \boldsymbol{\Phi}_{i_{h-1}} \boldsymbol{\Phi}_{i_h}$ with coefficients $p_{i_0,i_1} \dots p_{i_{h-2},i_{h-1}} p_{i_{h-1},i_h}$. Finally, the second formula in the statement of Theorem 3 follows from

$$\frac{\partial E_t[\mathbf{y}_{t+h}]}{\partial \boldsymbol{\eta}_t'} = \mathbf{L} \sum_{i_0=1}^{M} \sum_{i_1=1}^{M} \cdots \sum_{i_{h-1}=1}^{M} \sum_{i_h=1}^{M} \boldsymbol{\Phi}_{i_1} \dots \boldsymbol{\Phi}_{i_{h-1}} \boldsymbol{\Phi}_{i_h} \times p_{i_0,i_1} \dots p_{i_{h-2},i_{h-1}} p_{i_{h-1},i_h} \pi_{i_h} \mathbf{L}' \boldsymbol{\Sigma}_{i_0}$$

as

$$\frac{\partial \mathbf{u}_t}{\partial \boldsymbol{\eta}_t'}|_{s_t=i_0} = \boldsymbol{\Sigma}_{i_0}.$$

(V) *Proof of Theorem 4* . Reasoning in the same manner, we derive the matrix expression in the statement of Theorem 4. In fact, we have

$$\frac{\partial E_t[\mathbf{z}_{t+h}]}{\partial \mathbf{v}_{t+1}'} = \sum_{i_0=1}^{M} \sum_{i_1=1}^{M} \cdots \sum_{i_{h-1}=1}^{M} \sum_{i_h=1}^{M} \boldsymbol{\Phi}_{i_1} \dots \boldsymbol{\Phi}_{i_{h-1}} \boldsymbol{\Phi}_{i_h} p_{i_0,i_1} \dots p_{i_{h-2},i_{h-1}} p_{i_{h-1},i_h} \pi_{i_h} \mathbf{R}'$$

as

$$\frac{\partial E_t[\mathbf{z}_t|s_t = i_0]}{\partial \mathbf{v}_{t+1}'} = \mathbf{R}',$$

being $\mathbf{R} = (\mathbf{0} \quad \cdots \quad \mathbf{0} \quad \mathbf{I}_M) \in \mathbb{R}^{M\times n}$. This implies

$$\frac{\partial E_t[\mathbf{y}_{t+h}]}{\partial \mathbf{v}_{t+1}'} = \mathbf{L} \sum_{i_0=1}^{M} \sum_{i_1=1}^{M} \cdots \sum_{i_{h-1}=1}^{M} \sum_{i_h=1}^{M} \boldsymbol{\Phi}_{i_1} \dots \boldsymbol{\Phi}_{i_{h-1}} \boldsymbol{\Phi}_{i_h} \times p_{i_0,i_1} \dots p_{i_{h-2},i_{h-1}} p_{i_{h-1},i_h} \pi_{i_h} \mathbf{R}'.$$

In matrix form the last expression yields the formula in the statement of Theorem 4.

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