PROCEEDINGS A

royalsocietypublishing.org/journal/rspa

Research

Cite this article: Mucci D, Saracco A, Sopio C. 2026 Weak elastic energy of rectifiable curves in the sphere. *Proc. R. Soc. A* **482**: 20250782. https://doi.org/10.1098/rspa.2025.0782

Received: 10 September 2025
Accepted: 13 February 2026

Subject Areas:
analysis, geometry

Keywords:
irregular spherical curves, curvature, elastic energy, relaxation

Author for correspondence:
Domenico Mucci
e-mail: domenico.mucci@unipr.it

Weak elastic energy of rectifiable curves in the sphere

Domenico Mucci[1], Alberto Saracco[1] and Cristian Sopio[2]

[1]Dipartimento di Scienze Matematiche, Fisiche e Informatiche, Università di Parma, Campus—Parco Area delle Scienze 53/A, 43124 Parma, Italy
[2]Dipartimento di Matematica e Informatica, Università degli Studi di Ferrara, Via Machiavelli 35, 44121 Ferrara, Italy

DM, 0000-0001-7459-3740; AS, 0000-0001-5776-4747; CS, 0009-0002-4996-4209

We introduce for any exponent $p>1$ the p-curvature functional for rectifiable curves in the two-dimensional sphere. We prove that this functional is finite and agrees with the integral of the geodesic curvature raised to the power p on curves whose arc-length parameterization is in the Sobolev class $W^{2,p}$.

1. Introduction

This work is a first approach to the project of studying the *p-curvature* of *irregular curves* in Riemannian surfaces M. For a comprehensive introduction to irregular curves, refer to [1–4].

In [2], the definition of *total curvature* $\mathrm{TC}(c)$ for a rectifiable curve c as the supremum of the *rotation* of inscribed polygonals was introduced.

An *intrinsic* theory of rectifiable curves with finite total curvature in a Riemannian surface M was recently developed by the first two authors in [5] and [6] for the plastic case ($p=1$). Here we want to address, following [7], the elastic case ($p>1$) for $M=\mathbb{S}^2$, the unit sphere in $\mathbb{R}^3$. When the sectional curvature is positive, the expected monotonicity formula fails to hold, see remark 2.3. To obtain a good intrinsic notion of total curvature $\mathrm{TC}_M(c)$, we need to use the *modulus* $\mu_c(P)$ of an inscribed polygonal P introduced in [1] that is equal to the maximum of the geodesic diameter of the arcs c determined by the consecutive vertices in P, see

THE ROYAL SOCIETY
PUBLISHING

definition 2.1. By [8], it turns out that a representation formula holds for the total curvature of piecewise smooth curves c:

$$\mathrm{TC}_M(c) = \int_c |k_M(c)|\,\mathrm{d}s + \sum_i |\beta_i|, \tag{1.1}$$

where β_i is the *signed turning angle* at the ith corner point of the curve c and $k_M = k_M(c)$ is the tangential part of the curvature vector at smooth points of c. We shall always consider the turning angles $\vartheta_i = |\beta_i| \in (0, \pi)$.

The plastic case is completely different from the elastic one, even in the Euclidean setting. Indeed, as we can see from equation (1.1), a curve with finite total curvature may have corner points. In the elastic case, as it is pointed out in [7, §4b], if the *total p-curvature* is finite, the curve turns out to be in the Sobolev space $W^{2,p}(I, \mathbb{R}^n)$. Also in the spherical setting, a rectifiable curve in $\mathbb{S}^2$ of finite total curvature may have corner points, but as observed in remark 5.2, the curve in the elastic case cannot have any corner points.

We recall that for general rectifiable curves with finite total curvature in a Riemannian surface M, a representation formula for the total curvature holds true using the language of functions of bounded variation, see [5]. Here, for $p > 1$ we obtain the expected Sobolev regularity.

(a) Results

We concentrate on studying the integral of the geodesic curvature raised to the power p for rectifiable curves in the sphere $\mathbb{S}^2$. We extend the results in [7] obtained by the first two authors in the Euclidean setting.

Our main result is to obtain a geometric functional $\mathcal{F}_p(c)$ on a rectifiable curve c parametrized by arc length that relies on the Sobolev regularity of the curve. It turns out that if $\mathcal{F}_p(c) < \infty$ the curve is in $W^{2,p}(I, \mathbb{S}^2)$ and $\mathcal{F}_p(c)$ coincides with the integral of the p-power of the absolute value of the geodesic curvature.

In the same spirit as Lebesgue–Serrin's relaxed functional, we introduce the *p-curvature* functional $\mathcal{F}_p(c)$ of rectifiable curves c in $\mathbb{S}^2$ as

$$\mathcal{F}_p(c) := \inf\left\{ \liminf_{h\to\infty} \mathbf{k}_p(P_h) \;\middle|\; \{P_h\} \ll c, \mu_c(P_h) \to 0 \right\} \quad \text{for } p > 1,$$

where the *p-rotation* $\mathbf{k}_p(P)$ of an inscribed polygonal P is obtained by distributing the turning angles at corner points of P via an optimal curve $\gamma(P)$ of locally constant geodesic curvature and integrating its geodesic curvature raised to power p, see §3, definition 3.3 and appendix A.

Theorem 1.1. *Let $c : [0, L] \to \mathbb{S}^2$ be a rectifiable and open curve in $\mathbb{S}^2$ parametrized by arc length, and let $p > 1$. Then*

$$\mathcal{F}_p(c) < \infty \Longleftrightarrow c \in W^{2,p}([0, L], \mathbb{S}^2)$$

and in this case, there holds

$$\mathcal{F}_p(c) = \int_c |k_{\mathbb{S}^2}|^p\,\mathrm{d}s = \int_0^L \|\ddot{c}^\top\|^p\,\mathrm{d}s.$$

Here $\ddot{c}^\top$ stands for the projection of the Euclidean second derivative vector of the curve $c : [0, L] \to \mathbb{S}^2 \subseteq \mathbb{R}^3$ onto the tangent plane $T_{c(s)}\mathbb{S}^2$, see §2b(ii).

(b) Comments and further directions

The case of closed spherical curves can be treated in a similar way, taking account of straightforward modifications. Comparing this work with [7], one can see that for an equilateral approximating sequence $\{P_h\}$ of a curve $c : I \to \mathbb{S}^2$ with side length ℓ and turning angles ϑ_i, the p-rotation behaves like in the Euclidean case, more precisely $\mathbf{k}_p(P_h) \sim \sum_i \ell^{1-p} \vartheta_i^p$, see remark 3.4. We expect that the same holds for curves in the hyperbolic plane and, more general, for curves in CAT(k) two-dimensional Riemannian manifolds.

In the proofs, we take advantage of local comparisons with planar curves obtained by means of conformal images of pieces of the spherical curve. This argument can be extended to the case of a curve in a Riemannian surface, but difficulties arise from how to define the p-curvature functional on rectifiable curves.

2. Preliminaries

(a) Length and inscribed polygonals

Let M denote an immersed surface in $\mathbb{R}^3$. We assume M smooth (at least of class $\mathcal{C}^3$) and closed, our model case being $M=\mathbb{S}^2$, the standard unit sphere in $\mathbb{R}^3$. Also, we denote by d_M the geodesic distance on M, and by $d_{\mathbb{R}^3}$ the Euclidean distance in $\mathbb{R}^3$.

Let us consider an open curve c contained in M and parameterized by the continuous map $c: I \to M$, where $I=[0,L]$. We say that P is a *polygonal* curve *inscribed* in the curve c, say $P \ll c$, if it is obtained by choosing a finite partition $\mathcal{P} := \{0=t_0 < \ldots < t_k = L\}$ of I, say $P=P(\mathcal{P})$, such that $P(t_i)=c(t_i)$ for $i=0,\ldots,k$, and $P_{|[t_i,t_{i+1}]}$ is a minimal geodesic arc of M that joins $c(t_i)$ to $c(t_{i+1})$. The length of the polygonal is

$$\mathcal{L}(P)=\sum_{i=0}^{k-1} d_M(c(t_{i+1}),c(t_i)).$$

The *length* $\mathcal{L}(c)$ of the curve c is defined by

$$\mathcal{L}(c) := \sup\{\mathcal{L}(P) \mid P \ll c\},$$

and the curve c is said to be *rectifiable* if $\mathcal{L}(c)<\infty$. In that case, its arc-length parametrization $c(s)$ is Lipschitz continuous and hence, by Rademacher's theorem, the derivative $\dot{c}$ exists almost everywhere (a.e.).

Assume from now on that $c(s)$ is the arc-length parametrization of a rectifiable and open curve in M. For any inscribed polygonal P as above, we denote

$$\operatorname{mesh}\mathcal{P} := \sup_{0\le i\le k-1} |t_{i+1}-t_i| \quad \text{and} \quad \operatorname{mesh} P := \sup_{0\le i\le k-1} d_M(c(t_{i+1}),c(t_i)),$$

and the following definition goes back to Alexandrov–Reshetnyak [1].

Definition 2.1. The *modulus* $\mu_c(P)$ of P is the maximum of the geodesic diameter of the arcs of c determined by two consecutive vertices in P, namely

$$\mu_c(P) := \max_{i=1,\ldots,k} \max_{s_1,s_2\in[t_{i-1},t_i]} d_M(c(s_1),c(s_2)).$$

We observe that there holds

$$\operatorname{mesh} P \le \operatorname{mesh}\mathcal{P} \quad \text{and} \quad \operatorname{mesh} P \le \mu_c(P).$$

In general, we have $\operatorname{mesh}\mathcal{P} \not\le \mu_c(P)$, but supposing $\mathrm{TC}_M(c)<\infty$, see definition 2.6, we have that $\operatorname{mesh}\mathcal{P}$ goes to zero as $\mu_c(P)$ goes to zero. Moreover, taking $P_h=P(\mathcal{P}_h)$, where $\{\mathcal{P}_h\}$ is any sequence of partitions of I such that $\operatorname{mesh}\mathcal{P}_h \to 0$, by uniform continuity we get $\operatorname{mesh} P_h \to 0$ and the convergence $\mathcal{L}(P_h)\to\mathcal{L}(c)$ of the length functional.

Definition 2.2. The *Fréchet distance* $d_F(c_1,c_2)$ between two rectifiable curves in M is the infimum, over all strictly monotonic reparameterizations, of the maximum pointwise Euclidean distance between them, see [9]. More precisely, if β_1,β_2 are reparametrizations of the interval I, we have

$$d_F(c_1,c_2) := \inf_{\beta_1,\beta_2} \max_{t\in I} d_{\mathbb{R}^3}(c_1(\beta_1(t)),c_2(\beta_2(t))).$$

Since the surface M is smooth and compact, the convergence induced by d_F is equivalent to the one obtained by replacing $d_{\mathbb{R}^3}$ with d_M in the definition of Fréchet distance. Moreover, if $\{c_h\}$

is a sequence of rectifiable curves in M such that $d_F(c_h, c) \to 0$ as $h \to \infty$ for some rectifiable curve c, then by lower semicontinuity

$$\mathcal{L}(c) \leq \liminf_{h\to\infty} \mathcal{L}(c_h). \tag{2.1}$$

(b) Total intrinsic curvature

In this section we focus on the plastic case and point out the different features between curves in the Euclidean space and in an immersed surface in $\mathbb{R}^3$. Moreover, we observe that for surfaces with positive sectional curvature, the expected monotonicity formula on the rotation of polygonals does not hold.

(i) Euclidean total curvature

In the Euclidean setting, given a rectifiable curve c in $\mathbb{R}^n$ and an inscribed polygonal P, we recall that its rotation $\mathbf{k}^*_{\mathbb{R}^n}(P)$ is the sum of the turning angles in $(0, \pi)$ between consecutive segments.

Remark 2.3. The following facts hold:

— if P and P' are inscribed polygonals and P' is obtained by adding a vertex in c to the vertices of P, then $\mathbf{k}^*_{\mathbb{R}^n}(P) \leq \mathbf{k}^*_{\mathbb{R}^n}(P')$;
— if c has finite total curvature, for each point p in c, small open arcs of c with an end point equal to p have small total curvature.

For a proof of the first statement see [9, corollary 2.2]. The second statement follows from the fact that the total curvature is the total variation of the *tantrix* (that is given by the a.e. derivative of the arc-length parametrization of the curve c), a function of bounded variation.

By the above remark, starting with any polygonal P inscribed in the curve c and performing a sequence $\{P_h\}$ by adding vertices to P, we get that the rotation of P_h increases and the sequence converges in length to the curve, if the mesh of the polygonals goes to zero. Then, the total curvature can be defined as the supremum of the rotation of any polygonal sequence $\{P_h\}$ with mesh $P_h \to 0$.

(ii) Geodesic curvature

Let M be an immersed surface in $\mathbb{R}^3$ as above, and let c be a smooth and regular curve in M parametrized by arc length. The Darboux frame along the curve c is the triad $(\mathbf{t}, \mathbf{n}, \mathbf{u})$, where $\mathbf{t}(s) := \dot{c}(s)$ is the unit tangent vector, $\mathbf{n}(s) := \boldsymbol{\nu}(c(s))$, $\boldsymbol{\nu}(p)$ being the oriented unit normal to the tangent 2-space T_pM and $\mathbf{u}(s) := \mathbf{n}(s) \times \mathbf{t}(s)$, where $\times$ denotes the vector product in $\mathbb{R}^3$, is the unit conormal. Therefore, the tangent space $T_{c(s)}M$ is spanned by $(\mathbf{t}(s), \mathbf{u}(s))$.

The curvature vector $\ddot{c} = \dot{\mathbf{t}}(s)$ is orthogonal to $\mathbf{t}(s)$, and thus decomposes as

$$\ddot{c}(s) = k_M(s)\,\mathbf{u}(s) + k_n(s)\,\mathbf{n}(s),$$

where $k_M := \ddot{c} \cdot \mathbf{u}$ and $k_n := \ddot{c} \cdot \mathbf{n}$ denote the *geodesic* and the *normal curvature* of c, respectively, and '$\cdot$' is the scalar product in $\mathbb{R}^3$. The projection $k_M\mathbf{u}$ of $\ddot{c}$ onto the tangent bundle of M is an intrinsic object, see [5], and for the sake of readability is denoted by $\ddot{c}^\top$. If the curve c is a geodesic on M, we have $k_M \equiv 0$, whereas in general

$$|k_M| = |\ddot{c} \cdot \mathbf{u}| = ||\ddot{c}^\top||. \tag{2.2}$$

Only if there is any ambiguity regarding the curve under consideration, shall we explicitly write $k_M(c(s))$.

(iii) Total (geodesic) intrinsic curvature

The *(intrinsic) rotation* $\mathbf{k}^*_M(P)$ of a polygonal P in M, where M is a Riemannian manifold, is the sum of the turning angles in $(0,\pi)$ between the consecutive geodesic arcs of P. The following property has been proved in [8] for curves in a generic Riemannian manifold.

Theorem 2.4. *[8, theorem 3.4]. Let c be a regular curve in M of class $\mathcal{C}^2$, parameterized by arc length. Then, for any sequence $\{P_h\}$ inscribed in c such that mesh$P_h \to 0$, one has*

$$\lim_{h\to\infty} \mathbf{k}^*_M(P_h) = \int_0^L |k_M(c(s))|\, \mathrm{d}s.$$

As a consequence, for a rectifiable curve c supported in an immersed surface M in $\mathbb{R}^3$ as above, one is tempted to define its total intrinsic curvature as in the Euclidean case, i.e. as the supremum of the intrinsic rotation $\mathbf{k}^*_M(P)$ computed among all the polygonals P inscribed in it. However, as observed in [8], if M has positive sectional curvature, as e.g. $M=\mathbb{S}^2$, the latter definition does not work. In fact, if $P, P' \ll c$, and P' is obtained by adding a vertex in c to the vertices of P, then the monotonicity inequality $\mathbf{k}^*_M(P) \le \mathbf{k}^*_M(P')$ holds true in general provided that M has non-positive sectional curvature. Indeed, it relies on the fact that in this case the sum of the interior angles of a geodesic triangle of M is not greater than π, see [8, lemma 4.1].

Example 2.5. In the case of the sphere $M=\mathbb{S}^2$, if we take as c a parallel that is not a great circle and P, P' are inscribed polygonals such that P' is obtained by adding a vertex to P, the other inequality holds:

$$\mathbf{k}^*_{\mathbb{S}^2}(P) > \mathbf{k}^*_{\mathbb{S}^2}(P') \quad \text{and} \quad \mathbf{k}^*_{\mathbb{S}^2}(P) > \int_c |k_{\mathbb{S}^2}|\, \mathrm{d}s.$$

Actually, the good definition turns out to be the one introduced by Alexandrov–Reshetnyak in [1] using the modulus $\mu_c(P)$ of a polygonal P inscribed in c.

According to definition 2.1, for $\varepsilon > 0$, we let

$$\Sigma_\varepsilon(c) := \{P \ll c \mid \mu_c(P) < \varepsilon\}.$$

Definition 2.6. The *total intrinsic curvature* of a curve c in M is

$$\mathrm{TC}_M(c) := \lim_{\varepsilon\to 0^+} \sup\{\mathbf{k}^*_M(P) \mid P \in \Sigma_\varepsilon(c)\}.$$

Clearly, the above limit is equal to the infimum of $\sup\{\mathbf{k}^*_M(P) \mid P \in \Sigma_\varepsilon(c)\}$ as $\varepsilon > 0$. Moreover, arguing as in [10, proposition 2.1], for a polygonal P in M we always have

$$\mathrm{TC}_M(P) = \mathbf{k}^*_M(P).$$

Most importantly, making use of a result by Dekster [11], as a consequence of [10, proposition 2.4] one obtains:

Proposition 2.7. *The total curvature $TC_M(c)$ of any curve c in M is equal to the limit of the rotation $\mathbf{k}^*_M(P_h)$ of any sequence of polygonals $\{P_h\}$ inscribed in c such that $\mu_c(P_h) \to 0$.*

Proposition 2.7 fills in the gap given by the lack of monotonicity, yielding to the conclusion that definition 2.6 involves a control on the modulus and not on the mesh, at least when the sectional curvature of M fails to be non-negative.

As a consequence, by theorem 2.4 one infers that for smooth curves c in M one has

$$\mathrm{TC}_M(c) = \int_c |k_M(c)|\, \mathrm{d}s.$$

By [8, corollary 3.6], for piecewise smooth curves c in M one similarly obtains

$$\mathrm{TC}_M(c) = \int_c |k_M(c)|\, \mathrm{d}s + \sum_i |\beta_i|. \tag{2.3}$$

In this formula, the integral is computed separately outside the corner points of c, where the geodesic curvature k_M is well defined, and the second addendum denotes the finite sum of the absolute value of the signed turning angles β_i between the incoming and outcoming unit tangent vectors at each corner point of c. Therefore, for piecewise smooth curves in $M \subseteq \mathbb{R}^3$ parametrized by arc length, we can rewrite equation (2.3) as

$$\mathrm{TC}_M(c) = \int_0^L ||\ddot{c}^\top(s)||\,\mathrm{d}s + \sum_{s\in J_{\mathbf{t}}} d_{\mathcal{S}^2}(\mathbf{t}(s+), \mathbf{t}(s-)), \tag{2.4}$$

where $\ddot{c}^\top$ is the tangential part of the second derivative of $c(s)$, $d_{\mathcal{S}^2}$ is the distance in the Gauss sphere, $J_{\mathbf{t}}$ is the jump set of the bounded variation function $\mathbf{t}$ and $\mathbf{t}(s\pm)$ are the right and left limits of $\mathbf{t}$ at the jump points.

Equality (2.3) shows that a rectifiable curve with finite total curvature may have corners in the plastic case. In the elastic case, this is not true.

(c) Functional setting

We deal with Sobolev maps defined in an open interval $I \subset \mathbb{R}$ and taking values in the two-sphere $\mathbb{S}^2$. For any $p > 1$ and for $k = 1, 2$, we define the Sobolev class:

$$W^{k,p}(I, \mathbb{S}^2) := \{u \in W^{k,p}(I, \mathbb{R}^3) \mid ||u|| = 1 \text{ a.e. on } I\}.$$

We shall work with curves c parametrized by arc length. If $c : I \to \mathbb{S}^2 \subseteq \mathbb{R}^3$ is of class $\mathcal{C}^2$ and $||\dot{c}|| \equiv 1$, the curvature $k_{\mathbb{S}^2}(c(t))$ coincides with the projection onto the tangent space to $\mathbb{S}^2$ at $c(t)$ of the second derivative $\ddot{c}$, namely

$$|k_{\mathbb{S}^2}(c(s))| = ||\ddot{c}^\top(s)||, \tag{2.5}$$

where $\ddot{c}^\top = \ddot{c} + c$.

Therefore, asking for the covariant derivative $\nabla_{\dot{c}}\dot{c}$ to be in L^p is equivalent to the curve being in $W^{2,p}(I, \mathbb{S}^2)$.

We observe that, by standard Sobolev embeddings $W^{2,p}(I, \mathbb{S}^2) \hookrightarrow \mathcal{C}^1(I, \mathbb{R}^3)$, i.e. a curve $u \in W^{2,p}(I, \mathbb{S}^2)$ is at least $\mathcal{C}^1$ smooth.

3. The *p*-curvature functional

(a) The *p*-rotation of polygonals

In this section, we deal with the definition of p-rotation for polygonals and of p-curvature functional for rectifiable curves in the sphere $\mathbb{S}^2$. We first recall how they can be defined in the Euclidean case and then, we perform an analogous construction in the spherical setting.

(i) The Euclidean case

From a given polygonal P inscribed in an open curve $c : I \to \mathbb{R}^n$, the first two authors in [7] construct a curve $\gamma(P)$ that is piecewise smooth and it is related to P as follows: for each vertex of P, we consider the lengths of the two adjacent edges, take half of the minimum of these lengths, and construct a circular arc which is $\mathcal{C}^1$-joined to the polygonal curve at the points on the two edges corresponding to this length. The idea is to redistribute the *curvature measure* of P concentrated in the points $\{P(t_1), \ldots, P(t_{k-1})\}$ along pieces of smooth curves with constant curvature, i.e. pieces of circles.

Example 3.1. For a given $\alpha \in (0, \pi)$, let P be the polygonal with a vertex in $(0, 0) \in \mathbb{R}^2$ and given by the two segments parametrized by $(t, 0)$ and $(t\cos\alpha, t\sin\alpha)$, where $t \in [0, \varepsilon]$ for some $\varepsilon > 0$. The

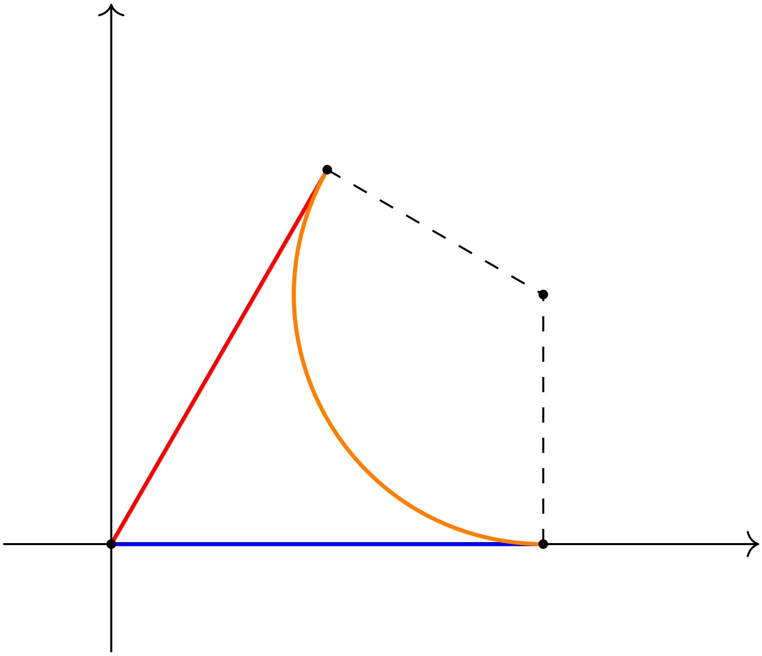

Figure 1. The curvature $k_{\mathbb{R}^2}(\gamma)$ is the inverse of the radius $R=\varepsilon/\tan(\vartheta/2)$ of the orange curve γ.

curve

$$\gamma(s)=\left(\varepsilon,\varepsilon\tan\left(\frac{\alpha}{2}\right)\right)+\varepsilon\tan\left(\frac{\alpha}{2}\right)(\cos s,\sin s),$$

where $s\in[\pi/2+\alpha,3\pi/2]$, coincides at order 1 with the end points of the two segments, see figure 1. If $\vartheta:=\pi-\alpha$ is the turning angle, then

$$\int_\gamma |k_{\mathbb{R}^2}|\,\mathrm{d}s=\vartheta \quad\text{and}\quad \int_\gamma |k_{\mathbb{R}^2}|^p\,\mathrm{d}s=\varepsilon^{1-p}\vartheta\tan^{p-1}\left(\frac{\vartheta}{2}\right).$$

Then, for a given polygonal P inscribed in an open curve $c:I\to\mathbb{R}^n$, with edge length ℓ, with $k+1$ vertices and with turning angles $\vartheta_i\in(0,\pi)$, for $i=1,\dots,k-1$, they define the p-rotation of P as

$$\mathbf{k}_p(P):=\sum_{i=1}^{k-1}\left(\frac{\ell}{2}\right)^{1-p}\vartheta_i\tan^{p-1}\left(\frac{\vartheta_i}{2}\right).$$

It turns out that $\mathbf{k}_1(P)=\mathbf{k}^*_{\mathbb{R}^n}(P)=\mathrm{TC}_{\mathbb{R}^n}(P)$, so this definition of p-rotation includes the definition of *total curvature* for $p=1$.

(ii) The spherical case

From now on, we consider the case of $M=\mathbb{S}^2$.

We define a p-curvature functional on rectifiable spherical curves that gives a notion of *total p-curvature*.

Given a rectifiable curve $c:I:\to\mathbb{S}^2$ parametrized by arc length and an inscribed polygonal $P\ll c$, using appendix A, we define the curve $\gamma(P)$ as follows (see figure 2).

— Around every vertex $P(t_i)$, we look at the length of the edges of P that meet in the corner. We start by looking at $P(t_1)$, let $\overline{\ell}_0,\overline{\ell}_1$ be the length of $P_{|[t_0,t_1]},P_{|[t_1,t_2]}$, respectively. Let $\ell_1=\min\{\overline{\ell}_0,\overline{\ell}_1\}$ and perform the construction given in appendix A on $\ell_1/2$;

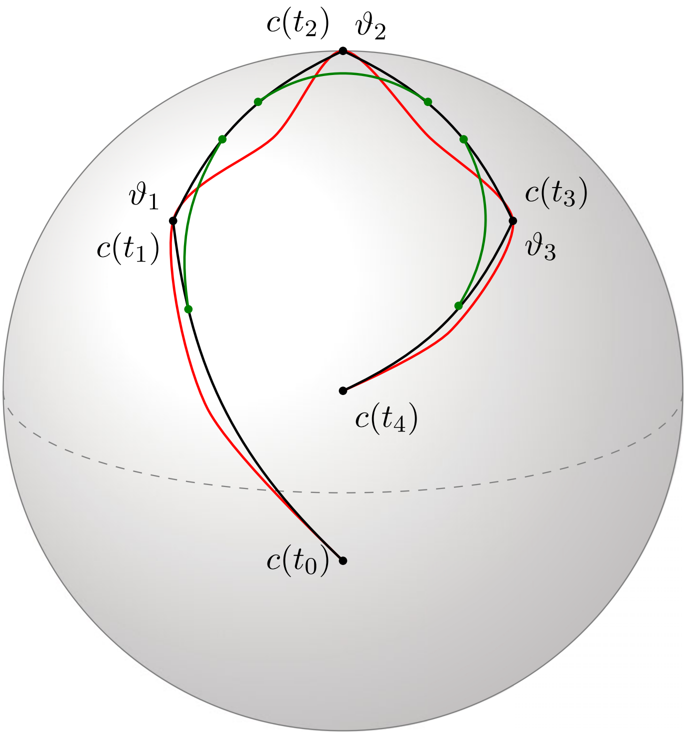

Figure 2. Construction applied to a geodesic polygonal with $k=4$. In red the curve $c(t)$, in black the polygonal with vertices $\{c(t_1), c(t_2), c(t_3)\}$, respectively turning angles ϑ_1, ϑ_2, ϑ_3 and in green the curve $\gamma(P)$.

- repeat the construction on the next vertex $P(t_2)$, and so on, using as $\ell_i = \min\{\overline{\ell}_{i-1}, \overline{\ell}_i\}$, where $\overline{\ell}_i$ is the length of the polygonal restricted to $[t_i, t_{i+1}]$;
- take as $\gamma(P) : [0, L] \to \mathbb{S}^2$ the obtained curve where near to the corners we glue the previous construction with the polygonal.

Notice that the curve $\gamma(P)$ is automatically in the Sobolev class $W^{2,1}(I, \mathbb{S}^2)$.

Definition 3.2. Given a spherical polygonal P, we define the p-rotation of P as

$$\mathbf{k}_p(P) := \int_{\gamma(P)} |k_{\mathbb{S}^2}|^p \, \mathrm{d}t.$$

If $k-1$ is the number of interior vertices of P, the curve $\gamma(P)$ that we have obtained is smooth except for $2k-2$ points where it is $\mathcal{C}^1$. Moreover by [12], it turns out that

$$\mathbf{k}_1(P) := \int_{\gamma(P)} |k_{\mathbb{S}^2}| \, \mathrm{d}t = \int_0^{\mathcal{L}(\gamma(P))} ||\ddot{c_P}^{\top}(s)|| \, \mathrm{d}s,$$

where c_P is the arc-length parametrization of the curve $\gamma(P)$.

Here, it is no longer true, as in the Euclidean case, that $\mathbf{k}_1(P) = \mathrm{TC}_{\mathbb{S}^2}(P)$, but for sequences $\{P_h\} \ll c$ with $\mu_c(P_h) \to 0$, we have $\mathbf{k}_1(P_h) \sim \mathrm{TC}_{\mathbb{S}^2}(c)$, see remark 3.4.

We define the p-curvature of the curve c as the Lebesgue–Serrin relaxed functional of p-rotations of polygonals inscribed in the curve.

Definition 3.3. The *p-curvature* functional of a rectifiable curve c is defined as

$$\mathcal{F}_p(c) := \inf_{\varepsilon > 0} \left\{ \liminf_{h \to \infty} \mathbf{k}_p(P_h) : \{P_h\} \subseteq \Sigma_\varepsilon(c) \right\}.$$

It turns out that for an equilateral and open polygonal P with k edges of length ℓ and turning angles $\vartheta_1, \ldots, \vartheta_{k-1}$, the p-rotation $\mathbf{k}_p(P)$ is given by

$$\mathbf{k}_p(P) := \sum_{i=1}^{k-1} 2\arctan\left(\frac{\Psi(\vartheta_i,\ell)}{\cos(\vartheta_i/2)\cos(\ell/2)}\right) \cdot \frac{1}{\Psi(\vartheta_i,\ell)} \cdot \frac{\sin^p(\vartheta_i/2)}{(\cos(\vartheta_i/2)\sin(\ell/2))^{p-1}}, \tag{3.1}$$

where $\Psi(\vartheta,\ell) := \sqrt{\sin^2(\ell/2) + \sin^2(\vartheta/2)\cos^2(\ell/2)}$. For an explicit computation, see appendix A.

Remark 3.4. We observe that for small turning angles ϑ_i and side length ℓ, the expression of the spherical p-rotation of an equilateral polygonal P is almost identical to the expression of the Euclidean p-rotation of a Euclidean polygonal with the same turning angles and side length, namely

$$\mathbf{k}_p(P) \sim 2\sum_{i=1}^{k-1} \tan^p\left(\frac{\vartheta_i}{2}\right)\sin^{1-p}\left(\frac{\ell}{2}\right) \sim \sum_{i=1}^{k-1} \vartheta_i^p \ell^{1-p}.$$

This will be quantitatively used to prove theorem 5.1 through the local conformality of the sphere to the plane explored in proposition 4.2.

4. Technical lemmas

The results of the following lemma are well known. However, to the best of our knowledge, we are not able to give a specific reference for the formulas that we need, even for smooth curves. The proof is an application of the rules of differential calculus.

Lemma 4.1 (Geodesic curvature via conformal maps). *Let $f:(M,g) \to (N,g')$ be a conformal map between two-dimensional oriented Riemannian manifolds such that $g' = \mathrm{e}^{2\lambda} g$ and let the curve $\gamma:[0,L] \to M$ be $\mathcal{C}^2$ and $\boldsymbol{u} \in T_\gamma M$ the conormal.*

Suppose that the image curve $c := f \circ \gamma : [0,L] \to N$ is parametrized by arc length. Then if k_M and k_N are the geodesic curvature of γ and c, respectively, for every $t \in (0,L)$ there holds

$$k_N(t) = \mathrm{e}^{-\lambda(\gamma(t))}(k_M(t) - \partial_{\boldsymbol{u}}\lambda(\gamma(t))). \tag{4.1}$$

Moreover, the arc-length element $s(t)$ of the curve γ is given by

$$s(t) = \int_0^t \mathrm{e}^{-\lambda(\gamma(\tau))}\,\mathrm{d}\tau. \tag{4.2}$$

Proof. By definition, the geodesic curvature of the curve c in N is defined as

$$k_N = g'(\nabla'_{\dot c}\dot c, \mathbf{u}'),$$

where $\mathbf{u}'$ is the conormal in N such that $\{\dot c(t), \mathbf{u}'(t)\}$ is an oriented basis of $T_{c(t)}N$ and ∇' is the Levi-Civita connection on N. Now, via conformal changes, we have

$$\mathrm{e}^{\lambda}||\dot\gamma||_g = ||\dot c||_{g'} = 1 \quad \text{and} \quad ||\dot\gamma||_g = \mathrm{e}^{-\lambda},$$

and if $\mathbf{u}(t)$ is the conormal vector in M, then $(df)_\gamma \mathbf{u}$ is orthogonal to $\dot c$ and $||(df)_\gamma \mathbf{u}||_{g'} = \mathrm{e}^{\lambda}$. We use $\mathbf{u}' = \mathrm{e}^{-\lambda}(df)_\gamma \mathbf{u}$ and by [13] there holds

$$\nabla'_{(df)_\gamma\dot\gamma}(df)_\gamma\dot\gamma = (df)_\gamma(\nabla_{\dot\gamma}\dot\gamma) + 2\dot\gamma(\lambda)(df)_\gamma\dot\gamma - g(\dot\gamma,\dot\gamma)(df)_\gamma(\nabla\lambda), \tag{4.3}$$

and

$$\nabla_{\mathrm{e}^{\lambda}\dot\gamma}(\mathrm{e}^{\lambda}\dot\gamma) = \mathrm{e}^{\lambda}(\nabla_{\dot\gamma}(\mathrm{e}^{\lambda}\dot\gamma)) = \mathrm{e}^{\lambda}(\mathrm{e}^{\lambda}\nabla_{\dot\gamma}\dot\gamma + \mathrm{e}^{\lambda}\dot\gamma(\lambda)\dot\gamma),$$

so

$$\nabla_{\dot\gamma}\dot\gamma = \mathrm{e}^{-2\lambda}\nabla_{\mathrm{e}^{\lambda}\dot\gamma}(\mathrm{e}^{\lambda}\dot\gamma) - \dot\gamma(\lambda)\dot\gamma.$$

Using equation (4.3) we get

$$\nabla'_{\dot c}\dot c = (df)_\gamma(\mathrm{e}^{-2\lambda}\nabla_{\mathrm{e}^{\lambda}\dot\gamma}(\mathrm{e}^{\lambda}\dot\gamma)) + \dot\gamma(\lambda)(df)_\gamma\dot\gamma - g(\dot\gamma,\dot\gamma)(df)_\gamma(\nabla\lambda).$$

Finally, we compute the geodesic curvature k_N on N as

$$
\begin{aligned}
k_N &= g'(\nabla'_{\dot{c}}\dot{c}, \mathbf{u}') = g'(\nabla'_{\dot{c}}\dot{c}, \mathrm{e}^{-\lambda}(df)_\gamma \mathbf{u}) \\
&= g'((df)_\gamma(\mathrm{e}^{-2\lambda}\nabla_{\mathrm{e}^{\lambda}\dot{\gamma}}(\mathrm{e}^{\lambda}\dot{\gamma})) + \dot{\gamma}(\lambda)(df)_\gamma\dot{\gamma} - g(\dot{\gamma},\dot{\gamma})(df)_\gamma(\nabla\lambda), \mathrm{e}^{-\lambda}(df)_\gamma\mathbf{u}) \\
&= \mathrm{e}^{-3\lambda} g'((df)_\gamma \nabla_{\mathrm{e}^{\lambda}\dot{\gamma}}(\mathrm{e}^{\lambda}\dot{\gamma}), (df)_\gamma\mathbf{u}) + \mathrm{e}^{-\lambda}\dot{\gamma}(\lambda) g'((df)_\gamma\dot{\gamma}, (df)_\gamma\mathbf{u}) \\
&\quad - \mathrm{e}^{-3\lambda} g'((df)_\gamma(\nabla\lambda), (df)_\gamma\mathbf{u}) \\
&= \mathrm{e}^{-\lambda} g(\nabla_{\mathrm{e}^{\lambda}\dot{\gamma}}(\mathrm{e}^{\lambda}\dot{\gamma}), \mathbf{u}) + \mathrm{e}^{\lambda}\dot{\gamma}(\lambda) g(\dot{\gamma}, \mathbf{u}) - \mathrm{e}^{-\lambda} g(\nabla\lambda, \mathbf{u}) \\
&= \mathrm{e}^{-\lambda}(k_M - \partial_{\mathbf{u}}\lambda),
\end{aligned}
$$

where we have used that $g(\dot{\gamma}, \mathbf{u}) = 0$ and $g(\dot{\gamma}, \dot{\gamma}) = \mathrm{e}^{-2\lambda}$. ■

Our purpose is to use lemma 4.1 locally for a curve $c : I_\delta := [-\delta, \delta] \to \mathbb{S}^2_*$, parameterized by arc length, pushed forward from a curve $\gamma : I_\delta \to \mathbb{R}^2$ by a conformal map f, where $\mathbb{S}^2_* := \mathbb{S}^2 \setminus \{(0, 0, -1)\}$. More precisely, suppose that $c(0) = (0, 0, 1)$ and define $\gamma := f^{-1} \circ c$, where the map f is a conformal change of the inverse of the stereographic projection in order to have $\lambda(0, 0) = 0$, given by $f : \mathbb{R}^2 \longrightarrow \mathbb{S}^2_*$,

$$(x, y) \mapsto \left(\frac{4x}{4 + x^2 + y^2}, \frac{4y}{4 + x^2 + y^2}, \frac{8}{4 + x^2 + y^2} - 1 \right).$$

The map f is conformal with conformal factor

$$\exp\lambda(x, y) = \frac{4}{4 + x^2 + y^2}.$$

We work with the stereographic projection since it is conformal and therefore it preserves angles. In particular, a polygonal curve on $\mathbb{S}^2_*$, inscribed in the curve c and consisting of two geodesic segments meeting at $c(0)$ with turning angle ϑ is mapped by f^{-1} to a polygonal curve in $\mathbb{R}^2$ whose edges meet at $\gamma(0)$ with the same turning angle ϑ. Moreover, the image polygonal is inscribed in the planar curve γ, which allows us to relate the turning angle at $c(0)$ to the geodesic curvature of γ by [7, theorem 5.3]. Finally, by means of lemma 4.1 we relate the geodesic curvature of the curve c to the geodesic curvature of the curve γ. In [7, theorem 5.3], the curvature vector of a curve in $\mathbb{R}^n$, parametrized by arc length, is identified with the second derivative of the curve itself. Concerning spherical curves, since this property no longer holds, we are not aware of a direct way to relate the geodesic curvature to the turning angles of polygonal curves inscribed in it.

We need to apply lemma 4.1 to curves c in the Sobolev class $W^{2,p}(I_\delta, \mathbb{S}^2)$ and $f : \mathbb{R}^2 \to \mathbb{S}^2_*$. In order to do this, if φ is a smooth curve, there holds

$$\int_{I_\delta} k^p_{\mathbb{S}^2}(\varphi(t))\,\mathrm{d}t = \int_{-\delta}^{\delta} \left(\frac{4 + x(t)^2 + y(t)^2}{4} (k_{\mathbb{R}^2}(\gamma(t)) - \gamma(t) \cdot \mathbf{u}(t)) \right)^p \mathrm{d}t,$$

where $\gamma = f^{-1} \circ \varphi = (x(t), y(t))$ and $\mathbf{u}$ is the conormal. By the dominated convergence theorem, the same formula holds for weak derivatives in $W^{2,p}(I, \mathbb{S}^2)$.

Proposition 4.2 (Arc-length parametrization). *Fix* $c : I_\delta = [-\delta, \delta] \to \mathbb{S}^2_*$ *a curve parametrized by arc length such that* $c(0) = (0, 0, 1)$, *and assume that* $c \in W^{2,p}(I_\delta, \mathbb{S}^2)$. *Then, the arc-length parametrization* $\varGamma(s) = (\tilde{x}(s), \tilde{y}(s))$ *of the curve* $\gamma(t) := f^{-1} \circ c(t) = (x(t), y(t))$ *has the same regularity of c and satisfies*

$$\int_{I_\delta} k_{\mathbb{S}^2}(c(t))\,\mathrm{d}t = \int_0^{\mathcal{L}(\gamma)} \left(k_{\mathbb{R}^2}(\varGamma(s)) + \frac{C}{4 + \tilde{x}(s)^2 + \tilde{y}(s)^2} \varGamma(s) \cdot \tilde{\boldsymbol{u}}(s) \right) \mathrm{d}s, \tag{4.4}$$

where C is a constant independent from the curve and $\boldsymbol{u}(t)$ *is the conormal vector in* $T_{\gamma(t)}\mathbb{S}^2$ *orthogonal to* $\dot{\gamma}(t)$, *and*

$$\varGamma(s) := \gamma(\varphi(s)) \quad \text{and} \quad \tilde{\boldsymbol{u}}(s) := \boldsymbol{u}(\varphi(s)),$$

where $s = s(t) \Longleftrightarrow t = \varphi(s)$.

Moreover, there exists a non-negative real constant $C := C(\delta, f, p)$ *and a positive constant* $\tilde{C}_\delta$ *close to 1, independent from the curve, such that*

$$\int_{I_\delta} |k_{\mathbb{S}^2}(c(t))|^p \,\mathrm{d}t \geq \tilde{C}_\delta \left(\int_0^{\mathcal{L}(\gamma)} |k_{\mathbb{R}^2}(\Gamma(s))|^p \,\mathrm{d}s - C\delta \int_0^{\mathcal{L}(\gamma)} |k_{\mathbb{R}^2}(\Gamma(s))|^{p-1} \,\mathrm{d}s \right). \tag{4.5}$$

Proof. Observe that for $t \in I_\delta = [-\delta, \delta]$, the arc element is

$$s(t) := \int_{-\delta}^{t} \mathrm{e}^{-\lambda(\gamma(t))} \,\mathrm{d}t \leq \int_{-\delta}^{\delta} \frac{4 + x(t)^2 + y(t)^2}{4} \,\mathrm{d}t = \mathcal{L}(\gamma),$$

and if $\delta \leq \sqrt{\varepsilon}$ for some $\varepsilon > 0$, then $\delta \leq \mathcal{L}(\gamma)/2 \leq \delta(1+\varepsilon)$. By definition, we have $||\dot{\Gamma}|| = 1$ and $\varphi'(s) = \mathrm{e}^{\lambda(\Gamma(s))}$. So, by changing variables we obtain

$$\begin{aligned}\int_{I_\delta} \mathrm{e}^{-\lambda(\gamma(t))}(k_{\mathbb{R}^2}(\gamma(t)) - \partial_{\mathbf{u}}\lambda(\gamma(t)))\,\mathrm{d}t &= \int_0^{\mathcal{L}(\gamma)} \mathrm{e}^{-\lambda(\Gamma(s))}(k_{\mathbb{R}^2}(\Gamma(s)) - \partial_{\tilde{\mathbf{u}}}\lambda(\Gamma(s)))\varphi'(s)\,\mathrm{d}s \\ &= \int_0^{\mathcal{L}(\gamma)} (k_{\mathbb{R}^2}(\Gamma(s)) - \partial_{\tilde{\mathbf{u}}}\lambda(\Gamma(s)))\,\mathrm{d}s,\end{aligned}$$

where in the second equality we have used that $\varphi'(s) = \mathrm{e}^{\lambda(\Gamma(s))}$.

Moreover,

$$\partial_{\tilde{\mathbf{u}}}\lambda(\Gamma(s)) = -\frac{1}{2}\,\mathrm{e}^{\lambda(\Gamma(s))}\,\Gamma(s)\cdot\tilde{\mathbf{u}}(s),$$

and equation (4.4) holds.

Now, from equation (4.1), passing through the arc-length parametrization of the curve γ we have

$$\begin{aligned}\int_{I_\delta} |k_{\mathbb{S}^2}(c(t))|^p \,\mathrm{d}t &= \int_0^{\mathcal{L}(\gamma)} \mathrm{e}^{(1-p)\lambda(\Gamma(s))} |k_{\mathbb{R}^2}(\Gamma(s)) - \partial_{\tilde{\mathbf{u}}}\lambda(\Gamma(s))|^p \,\mathrm{d}s \\ &\geq \int_0^{\mathcal{L}(\gamma)} \mathrm{e}^{(1-p)\lambda(\Gamma(s))} |\,|k_{\mathbb{R}^2}(\Gamma(s))| - |\partial_{\tilde{\mathbf{u}}}\lambda(\Gamma(s))|\,|^p \,\mathrm{d}s,\end{aligned}$$

where we used the second triangle inequality.

We now observe that there exists a constant $\tilde{C}_\delta > 1$, that can change line by line, such that $\tilde{C}_\delta \to 1$ as $\delta \to 0$, for which we can estimate

$$\frac{1}{\tilde{C}_\delta} \leq \mathrm{e}^{\lambda(\Gamma(s))} \leq 1 \quad \text{and} \quad \frac{\delta}{2} \leq \partial_{\tilde{\mathbf{u}}}\lambda(\Gamma(s)) \leq \frac{\tilde{C}_\delta \delta}{2}.$$

Moreover, we postpone the proof of the following:

Lemma 4.3. *For* $a, b > 0$, *there exists a real constant* $C := C(p) > 0$ *such that*

$$|a - b|^p \geq a^p - Cba^{p-1}. \tag{4.6}$$

We thus have for a.e. $s \in (0, \mathcal{L}(\gamma))$

$$\begin{aligned}|\,|k_{\mathbb{R}^2}(\Gamma)| - |\partial_{\tilde{\mathbf{u}}}\lambda(\Gamma)|\,|^p &\geq \min\left\{ \left| |k_{\mathbb{R}^2}(\Gamma)| - \frac{\tilde{C}_\delta \delta}{2} \right|^p, \quad \left| |k_{\mathbb{R}^2}(\Gamma)| - \frac{\delta}{2} \right|^p \right\} \\ &\geq |k_{\mathbb{R}^2}(\Gamma)|^p - C(p)\frac{\tilde{C}_\delta \delta}{2} |k_{\mathbb{R}^2}(\Gamma)|^{p-1},\end{aligned}$$

and hence we readily obtain the inequality (4.5). ■

We finally give:

Proof of lemma 4.3. Dividing by b^p both terms in inequality (4.6), we need to prove that there exists a constant C such that

$$|t-1|^p \geq t^p - Ct^{p-1} \quad \forall t > 0.$$

Now, the maps $t \mapsto |t-1|^p \ - t^p$ and $t \mapsto t^{p-1}$ are continuous, on every compact set $[0,m]$ with $m \in \mathbb{R}^+$, so there exists a constant C_m for which the inequality holds in $[0,m]$. To have that the constant C_m does not blow up as $m \to +\infty$, we observe that the following limit exists:

$$\lim_{t\to+\infty} \frac{(t-1)^p - t^p}{t^{p-1}} = -p.$$

Then, there exists a real constant $C = \sup_{m\in[1,\infty]} C_m$ such that inequality (4.6) holds. ■

5. Results

The aim of this section is to prove the following theorem.

Theorem 5.1. *Let $c : [0,L] \to \mathbb{S}^2$ be a rectifiable and open curve in $\mathbb{S}^2$ parametrized by arc length. Then*

$$\mathcal{F}_p(c) < \infty \quad \text{for some } p > 1 \Longleftrightarrow c \in W^{2,p}([0,L], \mathbb{S}^2),$$

and in this case, there holds

$$\mathcal{F}_p(c) = \int_c |k_{\mathbb{S}^2}|^p \, \mathrm{d}s = \int_0^L ||\ddot{c}^\top(s)||^p \, \mathrm{d}s.$$

To prove this theorem, we split it into two theorems in which we prove an upper and a lower bound on the integral of the geodesic curvature of the curve raised to the power p.

Remark 5.2. We first observe that the finiteness of the functional $\mathcal{F}_p(c)$ implies that the curve has no corners. Indeed, suppose that the curve has a corner in $c(\bar{t})$ with a turning angle ϑ, then we construct a polygonal sequence $\{P_h\}$ using partitions that contain $\bar{t} - 1/h, \bar{t}, \bar{t} + 1/h$ as consecutive points for each h, for which

$$\mathbf{k}_p(P_h) \geq h^{p-1} (\vartheta_{\bar{t}}^{(h)})^p,$$

where $\vartheta_{\bar{t}}^{(h)}$ is the turning angle in $P_h(\bar{t})$. As $h \to +\infty$, we have $\vartheta_{\bar{t}}^{(h)} \to \vartheta$ and

$$h^{p-1} (\vartheta_{\bar{t}}^{(h)})^p \to +\infty.$$

Then, the curve cannot have any corners if its p-curvature is finite.

(i) Upper bound

We start by proving the upper bound on the integral of the geodesic curvature raised to the power p of a spherical curve with finite p-curvature.

Theorem 5.3. *Let c be a rectifiable and open curve in $\mathbb{S}^2$ parametrized by arc length such that $\mathcal{F}_p(c) < \infty$ for some $p > 1$. Then $c \in W^{2,p}([0,L], \mathbb{S}^2)$ and*

$$\int_0^L ||\ddot{c}^\top(s)||^p \, \mathrm{d}s \leq \mathcal{F}_p(c) < \infty.$$

Proof. Let $I_L := [0,L]$, where $L := \mathcal{L}(c)$. By finiteness of $\mathcal{F}_p(c)$, we can take a sequence $\{P_h\}$ of polygonal curves inscribed in the curve c satisfying $\mu_c(P_h) \to 0$ and $\mathbf{k}_p(P_h) \to \mathcal{F}_p(c)$. For each h, let $c_{P_h} : [0, L_h] \to \mathbb{S}^2$ be the arc-length parametrization of the piecewise smooth and $\mathcal{C}^1$ curve $\gamma(P_h)$

obtained by construction A.1, where $L_h := \mathcal{L}(\gamma(P_h))$, and let $\gamma_h : [0, L] \to \mathbb{S}^2$ be given by

$$\gamma_h(s) := c_{P_h}\left(\frac{sL_h}{L}\right).$$

By piecewise smoothness, apart from a finite set of points, one has

$$|k_{\mathbb{S}^2}(c_{P_h}(\lambda))| = ||\ddot{c}_{P_h}{}^{\top}(\lambda)||$$

for $\lambda \in [0, L_h]$ and

$$\ddot{\gamma}_h^{\top}(s) = \left(\frac{L_h}{L}\right)^2 \ddot{c}_{P_h}(\lambda) + c_{P_h}(\lambda)$$

for $s \in I_L$, with $\lambda = s(L_h/L)$. Therefore,

$$\mathbf{k}_p(P_h) := \int_{\gamma(P_h)} |k_{\mathbb{S}^2}|^p \, \mathrm{d}s = \int_0^L \left\| \left(\frac{L_h}{L}\right)^2 \ddot{c}_{P_h}(\lambda) + c_{P_h}(\lambda) \right\|^p \mathrm{d}\lambda. \tag{5.1}$$

Now, we have $d_F(\gamma(P_h), P_h) \le \mu_c(P_h)$ for every h, whereas $d_F(P_h, c) \to 0$. Since $\mu_c(P_h) \to 0$, we obtain $d_F(\gamma(P_h), c) \to 0$, and hence by lower semicontinuity we infer that

$$\mathcal{L}(c) \le \liminf_{h \to +\infty} \mathcal{L}(\gamma(P_h)).$$

Using the fact that $\mathcal{L}(\gamma(P_h)) \le \mathcal{L}(P_h) \le \mathcal{L}(c)$ for every h, we deduce that $L_h \to L$. As a consequence, recalling that $\mathbf{k}_p(P_h) \to \mathcal{F}_p(c)$, by equation (5.1) we obtain

$$\lim_{h \to \infty} \int_0^L \left\| \left(\frac{L_h}{L}\right)^2 \ddot{c}_{P_h}(\lambda) + c_{P_h}(\lambda) \right\|^p \mathrm{d}\lambda = \mathcal{F}_p(c). \tag{5.2}$$

We recall that each $\dot{\gamma}_h$ is in the Sobolev space $W^{1,1}(I_L, \mathbb{R}^3)$ by the construction made in §3a(ii).

Since $p > 1$, the sequence $\{\dot{\gamma}_h\}$ converges strongly in $W^{1,1}$ to some function $v \in W^{1,1}(I_L, \mathbb{R}^3)$. By using that $\{\gamma_h\}$ converges to the Lipschitz function c strongly in $L^1(I_L, \mathbb{S}^2)$, we obtain $v = \dot{c}$ a.e.; hence, possibly passing to a (not relabelled) subsequence, $\{\dot{\gamma}_h\}$ converges to $\dot{c}$ weakly in $W^{1,p}(I_L, \mathbb{R}^3)$. In particular, $\dot{c} \in W^{1,p}(I_L, \mathbb{S}^2)$ and $\{\ddot{\gamma}_h\}$ converges to $\ddot{c}$ weakly in L^p and, using that the normal component is the curve c itself, we have that $\{\ddot{\gamma}_h^{\top}\}$ converges to $\ddot{c}^{\top}$ weakly in L^p, as $h \to +\infty$.

Hence the curve c is in $W^{2,p}(I_L, \mathbb{S}^2)$ and, by lower semicontinuity,

$$\int_0^L ||\ddot{c}^{\top}(s)||^p \, \mathrm{d}s \le \mathcal{F}_p(c) < \infty.$$

■

Remark 5.4. Suppose that the spherical curve parametrized by arc length $c : I_L \to \mathbb{S}^2$ is in the Sobolev class $W^{2,p}(I_L, \mathbb{S}^2)$, so that the functional

$$J \mapsto \int_J |k_{\mathbb{S}^2}(c(t))|^p \, \mathrm{d}t$$

is absolutely continuous. Then, for every $\varepsilon > 0$, there exists $\delta_{AC} := \delta_{AC}(\varepsilon)$ such that if an interval $J \subseteq I$ is small, i.e. $|J| < \delta_{AC}$, then

$$\int_J |k_{\mathbb{S}^2}(c(t))|^p \, \mathrm{d}t \le \varepsilon.$$

Moreover, for every $q \in [1, p]$, we have

$$\left(\fint_J |k_{\mathbb{S}^2}(c(t))|^q \, \mathrm{d}t \right)^{p/q} \le \fint_J |k_{\mathbb{S}^2}(c(t))|^p \, \mathrm{d}t,$$

and

$$\int_J |k_{\mathbb{S}^2}(c(t))|^q \, \mathrm{d}t \le \delta_{AC}^{1-q/p} \left(\int_J |k_{\mathbb{S}^2}(c(t))|^p \, \mathrm{d}t \right)^{q/p} \le \varepsilon^{q/p} \delta_{AC}^{(p-q)/p}. \tag{5.3}$$

(ii) Lower bound

Now, we are able to prove the following lower bound. The idea is to localize the Sobolev spherical curve, use the comparison result given by proposition 4.2, and [7, theorem 5.3] for the Euclidean comparison curve.

Theorem 5.5. *Let c be a rectifiable and open curve in $\mathbb{S}^2$ parametrized by arc length of class $W^{2,p}([0,L],\mathbb{S}^2)$ for some $p>1$. Then, for every $\varepsilon>0$ small, there exists a polygonal P_ε inscribed in the curve c and a positive constant C_ε such that $\mu_c(P_\varepsilon)\to 0$, $C_\varepsilon\to 1$ as $\varepsilon\to 0$ and*

$$C_\varepsilon \boldsymbol{k}_p(P_\varepsilon)-g(\varepsilon)\leq \int_0^L |k_{\mathbb{S}^2}(c(s))|^p\,\mathrm{d}s<\infty, \tag{5.4}$$

where $g(\varepsilon)\to 0$ as $\varepsilon\to 0$.

Therefore

$$\mathcal{F}_p(c)\leq \int_0^L |k_{\mathbb{S}^2}(c(s))|^p\,\mathrm{d}s. \tag{5.5}$$

Proof. Fix $\varepsilon>0$.

Step 1. We find upper and lower bounds on the length of the edges of an equilateral polygonal inscribed in the curve c and we analyse the p-curvature functional of this polygonal.

We start producing an equilateral polygonal curve inscribed in the curve c. We observe that there exists $\delta_1:=\delta_1(\varepsilon)$ such that if the edges of any equilateral polygonal, inscribed in the curve, have length ℓ less than δ_1, then

$$\ell(1+\varepsilon)\geq |t_{i+1}-t_i|\geq \ell. \tag{5.6}$$

This is implied by the regularity of the curve c and using that it is parametrized by arc length. Indeed, the curve c is 1-Lipschitz and

$$\ell=d_{\mathbb{S}^2}(c(t_i),c(t_{i+1}))\leq |t_{i+1}-t_i|.$$

For the other inequality, we use that if $(t_{i+1}-t_i)\to 0$, then

$$\frac{d_{\mathbb{S}^2}(c(t_{i+1}),c(t_i))}{|t_{i+1}-t_i|}\to 1,$$

and by the uniform continuity of the metric derivative of the curve c, we have the existence of such δ_1.

Fix $\tilde{\ell}=\delta/2$ where $\delta:=\min\{\delta_1,\delta_{AC},\sqrt{\varepsilon}\}$ and let $k=k(\varepsilon)$ be the unique integer for which there is a sequence of times $0=t_0<t_1<\cdots<t_{k-1}<t_k=L$ such that $d_{\mathbb{S}^2}(c(t_{i-1}),c(t_i))=\tilde{\ell}$ for $i=1,\dots,k-1$ and $d_{\mathbb{S}^2}(c(t_{k-1}),c(t_k))=\hat{\ell}\leq\tilde{\ell}$. Then, we have a partition $\mathcal{P}_\varepsilon$ of $[0,L]$ of the form

$$\mathcal{P}_\varepsilon=\{t_0,\dots,t_k\},\quad I_i=[t_{i-1},t_i]\quad\text{and}\quad |I_i|=\tilde{\ell}_i,$$

where the constraint

$$\sum_{i=1}^k \tilde{\ell}_i=L, \tag{5.7}$$

holds. If $\hat{\ell}=\tilde{\ell}$, we end up with an equilateral polygonal on $\mathcal{P}_\varepsilon$ choosing $\ell=\tilde{\ell}$. If this is not the case, consider

$$\begin{aligned}\ell:=\max\{l\leq\delta/2:&\text{ there exist }0=t_0<t_1<\cdots<t_{k-1}<t_k=L\\ &\text{such that }d_{\mathbb{S}^2}(c(t_{i-1}),c(t_i))=l\ \forall\, i=1,\dots,k\},\end{aligned}$$

where the maximum exists because we have fixed the number of edges k and the length ℓ can be obtained enlarging the length $\hat{\ell}$ of the last edge and shrinking uniformly the length of the first $k-1$ edges. Notice that the length ℓ must be greater than $\delta/3$. Indeed, supposing by contradiction that $\ell < \delta/3$, from inequality (5.6) and equation (5.7) we have the inequalities

$$k\frac{\delta}{3}(1+\varepsilon) \geq \sum_{i=1}^{k} \tilde{\ell}_i = L \geq (k-1)\frac{\delta}{2} + \tilde{\ell}_k$$

that makes $\tilde{\ell}_k$ negative, which cannot hold.

We observe that from $\ell \geq \delta/3$ and $k\ell(1+\varepsilon) \leq L$, we obtain

$$k \leq \frac{3\tilde{C}_\varepsilon L}{\delta(1+\varepsilon)}. \tag{5.8}$$

Moreover, the obtained equilateral polygonal $P_\varepsilon := P(\mathcal{P}_\varepsilon)$ with turning angles ϑ_i in $P(t_i) = c(t_i)$ for $i = 1, \ldots, k-1$ has small angles ϑ_i if δ is small, indeed localizing around $P(t_i)$ and using proposition 4.2 one has $\vartheta_i \leq \delta(1+\varepsilon)$ and

$$\tan\left(\frac{\vartheta_i}{2}\right) \leq (1+\varepsilon)\frac{\vartheta_i}{2}.$$

We define as $\Psi(\ell, \vartheta)$ the continuous function

$$\Psi(\ell, \vartheta) := \sqrt{\sin^2\left(\frac{\ell}{2}\right) + \sin^2\left(\frac{\vartheta}{2}\right)\cos^2\left(\frac{\ell}{2}\right)}.$$

From $\ell < \sqrt{2\varepsilon}$, we have $\sin(\ell) \geq \ell(1 - \ell^2/6) \geq \ell(1-\varepsilon)$ and $\cos(\ell) \geq (1-\varepsilon)$.

From now on, we denote by $\tilde{C}_\varepsilon$ a constant that can change line by line such that $\tilde{C}_\varepsilon \to 1$ as $\varepsilon \to 0$.

We first bound the p-rotation $\mathbf{k}_p(P)$ of such polygonal from above, namely

$$\begin{aligned}\mathbf{k}_p(P_\varepsilon) &= \sum_{i=1}^{k-1} 2\arctan\left(\frac{\Psi(\vartheta_i, \ell)}{\cos(\vartheta_i/2)\cos(\ell/2)}\right) \cdot \frac{1}{\Psi(\vartheta_i, \ell)} \cdot \frac{\sin^p(\vartheta_i/2)}{(\cos(\vartheta_i/2)\sin(\ell/2))^{p-1}} \\ &\leq \sum_{i=1}^{k-1} 2\frac{\tan^p(\vartheta_i/2)}{\cos(\ell/2)\sin^{p-1}(\ell/2)} \leq \sum_{i=1}^{k-1} 2^p \tilde{C}_\varepsilon \tan^p\left(\frac{\vartheta_i}{2}\right)\ell^{1-p},\end{aligned}$$

whence we estimate

$$\mathbf{k}_p(P_\varepsilon) \leq \tilde{C}_\varepsilon \sum_{i=1}^{k-1} \ell^{1-p}\vartheta_i^p. \tag{5.9}$$

Step 2. We bound from below the integral of the p-power of the geodesic curvature.

Let $\psi : [0, k\ell] \to [0, L]$ be a reparametrization such that $\psi(s_i) = t_i$ and ψ is affine in $[s_i, s_{i+1}]$, with

$$\psi'(s)_{|(s_i, s_{i+1})} = \frac{t_{i+1} - t_i}{s_{i+1} - s_i}.$$

Following [14], we have

$$\int_0^L |k_{\mathbb{S}^2}(c(t))|^p \,\mathrm{d}t \geq \sum_{i=1}^{k-1} \frac{1}{\ell} \int_0^\ell \left(\int_{\psi(s_{i-1}+a)}^{\psi(s_i+a)} |k_{\mathbb{S}^2}(c(t))|^p \,\mathrm{d}t\right) \mathrm{d}a.$$

Using proposition 4.2, we compare the geodesic curvature of the localizing curves:

$$c_i := c_{|[\psi(s_{i-1}+a), \psi(s_i+a)]},$$

with the curvature in $\mathbb{R}^2$ of the comparison curves $\Gamma_i(s)$, namely

$$\int_{\psi(s_{i-1}+a)}^{\psi(s_i+a)} |k_{\mathbb{S}^2}(c_i(t))|^p \,\mathrm{d}t = \int_0^{\mathcal{L}(a,i)} \mathrm{e}^{(1-p)\lambda(\Gamma_i(s))} |k_{\mathbb{R}^2}(\Gamma_i(s)) - \partial_{\tilde{\mathbf{u}}_i}\lambda(\Gamma_i(s))|^p \,\mathrm{d}s,$$

where $\mathcal{L}(a,i) := \mathcal{L}(c_{i|_{[\psi(s_{i-1}+a),\psi(s_i+a)]}}) \le \tilde{C}_\varepsilon \delta$. By using inequality (4.5), we obtain for every $i = 1, \dots, k-1$ and $a \in (0, \ell)$

$$\int_0^{\mathcal{L}(a,i)} \mathrm{e}^{(1-p)\lambda(\Gamma_i(s))} |k_{\mathbb{R}^2}(\Gamma_i(s)) - \partial_{\tilde{\mathbf{u}}_i} \lambda(\Gamma_i(s))|^p \, \mathrm{d}s$$
$$\ge \tilde{C}_\varepsilon \int_0^{\mathcal{L}(a,i)} |k_{\mathbb{R}^2}(\Gamma_i)|^p \, \mathrm{d}s - C\delta \int_0^{\mathcal{L}(a,i)} |k_{\mathbb{R}^2}(\Gamma_i)|^{p-1} \, \mathrm{d}s,$$

where $C := C(\varepsilon, p)$ is a positive constant that is bounded as $\varepsilon \to 0$, that can change line by line, and $\tilde{C}_\varepsilon \to 1$ as $\varepsilon \to 0$.

We denote by I_1, I_2

$$I_1 := \sum_{i=1}^{k-1} \tilde{C}_\varepsilon \fint_0^\ell \left(\int_0^{\mathcal{L}(a,i)} |k_{\mathbb{R}^2}(\Gamma_i(s))|^p \right) \mathrm{d}a$$

and

$$I_2 := \sum_{i=1}^{k-1} C\delta \fint_0^\ell \left(\int_0^{\mathcal{L}(a,i)} |k_{\mathbb{R}^2}(\Gamma_i(s))|^{p-1} \, \mathrm{d}s \right) \mathrm{d}a.$$

We first deal with I_1.

In the Euclidean setting, if the curve Γ_i is parametrized by arc length, then the curvature is given by the norm of the second derivative $\Gamma_i''(s)$ of the curve, i.e.

$$\int_0^{\mathcal{L}(a,i)} |k_{\mathbb{R}^2}(\Gamma_i(s))|^p \, \mathrm{d}s = \int_0^{\mathcal{L}(a,i)} ||\Gamma_i''(s)||^p \, \mathrm{d}s.$$

Applying twice the Jensen inequality and supposing $t_{i+1} - t_i \ge t_i - t_{i-1}$, we have

$$\tilde{C}_\varepsilon \fint_0^\ell \left(\int_0^{\mathcal{L}(a,i)} ||\Gamma_i''(s)||^p \, \mathrm{d}s \right) \mathrm{d}a \ge \tilde{C}_\varepsilon \fint_0^\ell ||\Gamma_i'(\psi(s_i + a)) - \Gamma_i'(\psi(s_{i-1} + a))||^p \, \mathrm{d}a$$
$$\ge \tilde{C}_\varepsilon \ell^{1-p} \left\| \int_0^\ell (\Gamma_i'(\psi(s_i + a)) - \Gamma_i'(\psi(s_{i-1} + a))) \, \mathrm{d}a \right\|^p$$
$$\ge \tilde{C}_\varepsilon \ell^{1-p} \left\| \frac{\Gamma_i(t_{i+1}) - \Gamma_i(t_i)}{t_{i+1} - t_i} - \frac{\Gamma_i(t_i) - \Gamma_i(t_{i-1})}{t_i - t_{i-1}} \right\|^p.$$

Finally, using [7, theorem 5.3], we have

$$\tilde{C}_\varepsilon \fint_0^\ell \left(\int_0^{\mathcal{L}(a,i)} ||\Gamma_i''(s)||^p \, \mathrm{d}s \right) \mathrm{d}a \ge \tilde{C}_\varepsilon \ell^{1-p} \vartheta_i^p.$$

Thanks to the conformality of the projection, the turning angles ϑ_i for $i = 1, \dots, k-1$ are exactly the turning angles of P_ε and by inequality (5.9) there holds

$$I_1 \ge \tilde{C}_\varepsilon \sum_{i=1}^{k-1} \ell^{1-p} \vartheta_i^p \ge \tilde{C}_\varepsilon \mathbf{k}_p(P_\varepsilon).$$

Now, we prove that $I_2 \le g(\varepsilon) \to 0$ as $\varepsilon \to 0$.

By remark 5.4, we have

$$\int_0^{\mathcal{L}(a,i)} |k_{\mathbb{R}^2}(\Gamma_i(s))|^q \, \mathrm{d}s \le \varepsilon^{q/p} \delta^{(p-q)/p}.$$

Then, using $q = p - 1$ we obtain

$$I_2 = \sum_{i=1}^{k-1} C\delta \fint_0^\ell \left(\int_0^{\mathcal{L}(a,i)} |k_{\mathbb{R}^2}(\Gamma_i(s))|^{p-1} \, \mathrm{d}s \right) \mathrm{d}a \le \sum_{i=1}^{k-1} C\delta \varepsilon^{(p-1)/p} \delta^{1/p}.$$

Summing over the index i and using inequality (5.8), we get

$$\sum_{i=1}^{k-1} C\varepsilon^{(p-1)/p}\delta^{1+(1/p)} \leq C\varepsilon^{(p-1)/p}\delta^{1/p} =: g(\varepsilon).$$

Then, combining we obtain

$$\int_0^L |k_{\mathbb{S}^2}(c(s))|^p = I_1 - I_2 \geq \tilde{C}_\varepsilon \mathbf{k}_p(P_\varepsilon) - g(\varepsilon).$$

Now, inequality (5.5) follows taking the sequence $\{P_{\varepsilon_h}\}$, where ε_h is any sequence such that $\varepsilon_h \to 0$ and for each ε_h the polygonal P_{ε_h} is the one constructed before. ■

Data accessibility. This article has no additional data.

Declaration of AI use. We have not used AI-assisted technologies in creating this article.

Authors' contributions. D.M.: conceptualization, investigation; A.S.: conceptualization, investigation; C.S.: investigation, writing—original draft, writing—review and editing.

All authors gave final approval for publication and agreed to be held accountable for the work performed therein.

Conflict of interest declaration. We declare we have no competing interests.

Funding. The authors were partially supported by the GNAMPA of INDAM.

Acknowledgements. We thank the reviewers for their contributions which led to an improvement of the article.

Appendix A. Construction of spherical bends

In this appendix, we explain how to construct from a given polygonal P with vertices $P(t_i)$ and turning angles ϑ_i for $i = 1, \dots, k-1$ a local bending curve around $P(t_i)$ that glues $\mathcal{C}^1$ with P near to t_i with constant geodesic curvature. Working locally, up to isometry, we can suppose that the polygonal near $P(t_i)$ is given by

$$P(t) = \begin{cases} (-K\sin t, S\sin t, \cos t) & t \in [-\delta, 0], \\ (K\sin t, S\sin t, \cos t) & t \in [0, \delta], \end{cases}$$

where $K := \cos\alpha_i$ and $S := \sin\alpha_i$, with $\alpha_i = (\pi - \vartheta_i)/2$. We want to perform the construction only on half of each geodesic segment with $\delta = \ell/2$.

The idea of the construction is as follows: first we change coordinates on the sphere, via a rotation, in order to have a simpler expression for the tangent vectors at the ending points of the polygonal (the tangent vectors have zero vertical component). Then, we construct the parallel joining the two endpoints of the polygonal: in these coordinates it is easier to compute the integral of the p-power of the geodesic curvature of such constructed curve. Since such an integral is invariant under isometries, we get the desired quantity.

Construction A.1. For any fixed length $\ell \in (0, \pi)$, we start with the localized polygonal curve:

$$P(t) = \begin{cases} (-K\sin t, S\sin t, \cos t) & t \in \left[-\frac{\ell}{2}, 0\right], \\ (K\sin t, S\sin t, \cos t) & t \in \left[0, \frac{\ell}{2}\right], \end{cases}$$

where $K := \cos\alpha$ and $S := \sin\alpha$, with $\alpha := (\pi - \vartheta)/2 \in (0, \pi/2)$, since $\vartheta \in (0, \pi)$. Then,

$$P'(t) = \begin{cases} (-K\cos t, S\cos t, -\sin t) & t \in \left(-\frac{\ell}{2}, 0\right), \\ (K\cos t, S\cos t, -\sin t) & t \in \left(0, \frac{\ell}{2}\right). \end{cases}$$

Denote $P_\pm := P(\pm\ell/2)$ and $v_\pm := P'(\pm\ell/2)$, so that with $s := \sin(\ell/2)$ and $c := \cos(\ell/2)$ we have

$$P_- = (Ks, -Ss, c), \quad P_+ = (Ks, Ss, c), \quad v_- = (-Kc, Sc, s) \quad \text{and} \quad v_+ = (Kc, Sc, -s).$$

We choose a rotation matrix $R \in SO(3)$ with rotational axis e_2 and angle β. With $\tau = \cos\beta$ and $\sigma = \sin\beta$, we have

$$v_- R^T = (-(\tau Kc + \sigma s), Sc, -(\sigma Kc - \tau s)) \quad \text{and} \quad v_+ R^T = ((\tau Kc + \sigma s), Sc, (\sigma Kc - \tau s)),$$

and choose β so that $\sigma Kc - \tau s = 0$, i.e.

$$\tau = \frac{Kc}{\sqrt{s^2 + K^2c^2}} \quad \text{and} \quad \sigma = \frac{s}{\sqrt{s^2 + K^2c^2}}.$$

This way, the rotated velocity vectors $v_\pm R^T$ at the end points have zero third components. We correspondingly have

$$P_- R^T = (x, -y, z) \quad \text{and} \quad P_+ R^T = (x, y, z), \quad x := \tau Ks - \sigma c,\ y := Ss,\ z := \sigma Ks + \tau c,$$

so that explicitly

$$x = -\frac{S^2 sc}{\sqrt{s^2 + K^2c^2}} \quad \text{and} \quad z = \frac{K}{\sqrt{s^2 + K^2c^2}}.$$

We now choose the angle $\varPhi \in (0, \pi/2)$ so that

$$\cos\varPhi = z \quad \text{and} \quad \sin\varPhi = \sqrt{x^2 + y^2} = \frac{Ss}{\sqrt{s^2 + K^2c^2}}.$$

This way, we reduce to compute the integral of the pth power of the curvature of the parallel $\gamma(P)$ connecting the rotated points $P_\pm R^T$. For $s_0 \in (0, \pi/2)$ to be chosen, we set

$$\gamma(P)(s) = (-\sin\varPhi\cos s, -\sin\varPhi\sin s, \cos\varPhi), \quad s \in [-s_0, s_0].$$

By imposing that $\gamma(P)(s_0) = P_- R^T = (x, -y, z)$, we infer that $\sin s_0 = \sqrt{s^2 + K^2c^2}$ and $\cos s_0 = Sc$. Therefore, the curve $\gamma(P)$ has length

$$\mathcal{L}(\gamma(P)) = 2\arctan\left(\frac{\sqrt{s^2 + K^2c^2}}{Sc}\right) \cdot R, \quad R = \sin\varPhi.$$

Moreover, the geodesic curvature density of $\gamma(P)$ is equal to the constant

$$k_{\mathbb{S}^2} = \cot\varPhi = \frac{K}{Ss},$$

and we obtain

$$\int_{\gamma(P)} |k_{\mathbb{S}^2}|^p \, \mathrm{d}s = 2\arctan\left(\frac{\sqrt{s^2 + K^2c^2}}{Sc}\right) \cdot \frac{Ss}{\sqrt{s^2 + K^2c^2}} \cdot \left(\frac{K}{Ss}\right)^p.$$

Replacing $K = \cos\alpha = \sin(\vartheta/2)$, $S = \sin\alpha = \cos(\vartheta/2)$, $c = \cos(\ell/2)$, and $s := \sin(\ell/2)$, we finally get for every $p \geq 1$ the quantity

$$\mathcal{F}_p(\ell, \vartheta) := 2\arctan\left(\frac{\varPsi(\ell, \vartheta)}{\cos(\vartheta/2)\cos(\ell/2)}\right) \cdot \frac{1}{\varPsi(\ell, \vartheta)} \cdot \frac{\sin^p(\vartheta/2)}{(\cos(\vartheta/2)\sin(\ell/2))^{p-1}},$$

where

$$\varPsi(\ell, \vartheta) := \sqrt{\sin^2\left(\frac{\ell}{2}\right) + \sin^2\left(\frac{\vartheta}{2}\right)\cos^2\left(\frac{\ell}{2}\right)}.$$

References

1. Alexandrov VV, Reshetnyak YG. 2012 *General theory of irregular curves*. Mathematics and its Applications. Netherlands: Springer.
2. Milnor JW. 1950 On the total curvature of knots. *Ann. Math.* **52**, 248–257. (doi:10.2307/1969467)
3. Milnor J. 1953 On total curvatures of closed space curves. *Math. Scand.* **1**, 289–296. (doi:10.7146/math.scand.a-10387)

4. Reshetnyak YG. 2005 The theory of curves in differential geometry from the viewpoint of the theory of functions of a real variable. *Russ. Math. Surveys* **60**, 1165–1181. (doi:10.1070/RM2005v060n06ABEH004286)
5. Mucci D, Saracco A. 2021 The total intrinsic curvature of curves in Riemannian surfaces. *Rend. Circ. Mat. Palermo (2)* **70**, 521–557. (doi:10.1007/s12215-020-00516-3)
6. Mucci D, Saracco A. 2021 Correction to: the total intrinsic curvature of curves in Riemannian surfaces. *Rend. Circ. Mat. Palermo (2)* **70**, 1137–1138. (doi:10.1007/s12215-021-00612-y)
7. Mucci D, Saracco A. 2023 Weak elastic energy of irregular curves. *Phil. Trans. R. Soc. A* **381**, 20220370. (doi:10.1098/rsta.2022.0370)
8. López MC, Mateos VF, Masqué JM. 2010 Total curvature of curves in Riemannian manifolds. *Differ. Geom. Appl.* **28**, 140–147. (doi:10.1016/j.difgeo.2009.10.008)
9. Sullivan J. 2006 Curves of finite total curvature. *Discrete Differ. Geom.* **38**, 137–161. (doi:10.1007/978-3-7643-8621-4_7)
10. Maneesawarng C, Lenbury Y. 2003 Total curvature and length estimate for curves in CAT(K) spaces. *Differ. Geom. Appl.* **19**, 211–222. (doi:10.1016/S0926-2245(03)00031-7)
11. Dekster BV. 1979 Upper estimates of the length of a curve in a Riemannian manifold with boundary. *J. Differ. Geom.* **14**, 149–166. (doi:10.4310/jdg/1214434965)
12. Acerbi E, Mucci D. 2017 Curvature-dependent energies: the elastic case. *Nonlinear Anal.* **153**, 7–34. (doi:10.1016/j.na.2016.05.012)
13. do Carmo MP. 1976 Differential geometry of curves and surfaces. Englewood Cliffs, NJ: Prentice-Hall, Inc.
14. Bruckstein AM, Netravali AN, Richardson TJ. 2001 Epi-convergence of discrete elastica. *Appl. Anal.* **79**, 137–171. (doi:10.1080/00036810108840955)