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A variational property of critical speed to travelling waves in presence of nonlinear diffusion

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Abstract

Let f be a continuous function in $[0,1]$ with $f(0) = 0 = f(1)$ and $f > 0$ on $]0,1[$. We show that, under additional mild conditions on f, the minimal speed for travelling waves of

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}\right] + f(u), \tag{0.1}$$

may be computed via a constrained minimum problem which in turn is related to the solution of a singular boundary value problem in the half line.

Keywords: travelling wave; nonlinear diffusion; critical speed; constrained minimum

Mathematics subject classification: 34C37, 35C07, 35K57.

1 Introduction

Throughout this note, let $f : [0,1] \to \mathbb{R}$ be a continuous function such that $f(0) = f(1) = 0$ and $f(u) > 0$ if $u \in (0,1)$. In the theory of Fisher-Kolmogorov-Petrovski-Piskounov (FKPP) equations, such a function is sometimes referred to as a function of *type A* (see e.g. [3]).

Also, let $p > 1$.

In [4] the notions of admissible speed and critical (i. e. minimal) speed have been introduced for travelling waves to reaction-diffusion equations driven by the one-dimensional p-Laplacian operator, namely

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}\right] + f(u), \tag{1.2}$$

The relevant front wave profiles $u(x+ct)$ with speed c are given by the (monotone) solutions of the second order problem

$$(|u'|^{p-2}u')' - cu' + f(u) = 0, \quad u(-\infty) = 0,\ u(+\infty) = 1. \tag{1.3}$$

Let q be the conjugate of p, that is $\frac{1}{p} + \frac{1}{q} = 1$. The solutions of the parametric first order boundary value problem (where we write $y_+ = \max(y,0)$)

$$y' = q(c\,{y_+}^{\frac{1}{p}} - f(u)), \quad 0 \le u \le 1, \quad y(0) = 0 = y(1), \quad y > 0 \text{ in }]0.1[\tag{1.4}$$

yield the trajectories of solutions of (1.3) via the relationship

$$u' = y(u(t))^{1/p}.$$

We recall the following assumptions, used in [4].

(H_p) $\quad M = M_p := \sup_{0<u<1} \frac{f(u)}{u^{q-1}} < +\infty;$ $\qquad (H_p')$ $\quad \mu := \lim_{u\to 0^+} \frac{f(u)}{u^{q-1}}$ exists, $0 \le \mu < +\infty$.

It follows from results in [4] that there is a 1-1 correspondence between solutions of (1.3) (up to translation) taking values in $]0,1]$ and solutions of (1.4) that are strictly positive in $]0,1[$. These sets of solutions are nonempty provided (H_p) holds. Also, basic properties of the profiles and their speeds, now classical in the FKPP theory ($p=2$), were extended in [4] to the p-Laplacian model. In particular, if (H_p) holds, the set of admissible speeds – that is, values of the parameter c such that (1.4) has a solution – is an interval $[c^*,+\infty[$ where

$$\mu^{\frac{1}{q}}p^{\frac{1}{p}}q^{\frac{1}{q}} \leq c^* \leq M^{\frac{1}{q}}p^{\frac{1}{p}}q^{\frac{1}{q}} \tag{1.5}$$

(the first inequality being valid if the stronger $(H'p)$ holds). The minimum admissible value c^* of the parameter c is called *critical speed*.

Remark 1.1. An elementary calculation on the basis of (1.4) shows that, given a number $a>0$, c is an admissible speed with respect to f if and only if $ca^{\frac{1}{p}}$ is admissible with respect to af.

For the case of linear diffusion ($p=2$), variational caracterizations of the critical speed c^* are known: in [1] a variational formulation is presented, based on the second order ordinary differential equation satisfied by the wave profiles; in [2] the authors use the first order model that represents the wave trajectories in a phase plane to establish another defining property of variational type for c^*.

The purpose of this note is to obtain a variational property of c^* in the framework of (1.3). We shall use some ideas from [1].

Remark 1.2. It will be useful for our purpose to recall the role played by functions of *type B*. A function $f:[0,1]\to\mathbb{R}$ is said to be of type B if it is continuous and there exists $\delta\in]0,1[$ such that $f(s)=0$ if $0\leq s\leq\delta$ or $s=1$, and $f(s)>0$ if $\delta<s<1$.

It is known that if f is of type B there exists exactly one admissible speed c^* of (1.3), that is, (1.4) has a positive solution for exactly this value of the parameter c. Moreover, if f_n is a nondecreasing sequence of functions of type B and $\lim_{n\to\infty} f_n(x) = f(x)$, then with obvious notation $\lim_{n\to\infty} c^*(f_n) = c^*(f)$. See [4], section 4.

2 Some equivalent boundary value problems

For convenience, we start by considering a different model, with homogeneity of degree $p-1$ in the derivatives. Consider the problem

$$(u'^{p-1})' - c^{p-1}u'^{p-1} + f(u) = 0, \quad u(-\infty)=0,\ u(+\infty)=1 \tag{2.6}$$

which, by the way, may be seen as the search for travelling waves of the form $u(x+ct)$ for the quasilinear parabolic equation in one spacial dimension

$$\frac{\partial(u^{p-1})}{\partial t} = \frac{\partial}{\partial x}\left[\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}\right] + f(u), \tag{2.7}$$

Related quasilinear PDEs have been considered in the literature, for example from the point of view of subtle analytic properties of solutions: see e.g. [5].

The homogeneity appearing in the quasilinear term of (2.6) is used in the following way. If we perform the change of variable $s=e^{kt}$ with $k>0$, and define $v(s)=u(t)$, this problem is seen to be equivalent to the following boundary value problem in $[0,+\infty[$

$$(v'^{p-1})' + \frac{1}{k^p}\frac{f(v(s))}{s^p} = 0, \quad v(0)=0,\ v(+\infty)=1,\ \ v'>0 \tag{2.8}$$

provided

$$c^{p-1} = k(p-1).$$

Another convenient interpretation of the problem (2.6) is given by the first order model that describes a phase portrait of the second order equation. Letting φ denote the function such that $u'=\varphi(u)$ we easily see that φ satisfies

$$(p-1)\varphi^{p-2}\varphi\varphi' = c^{p-1}\varphi^{p-1} - f(u)$$

so that $\psi=\varphi^p$ solves

$$\psi' = q\left(c^{p-1}\psi^{\frac{1}{q}} - f(u)\right), \quad \psi(0)=0,\ \psi(1)=0,\ \ \psi>0 \ \text{ in } \ ]0,1[. \tag{2.9}$$

Acording to what has been recalled in the Introduction, (2.9) has solutions provided that

(H_q) $\qquad M_q := \sup_{0<u<1}\frac{f(u)}{u^{p-1}} < +\infty.$

Moreover, writing (2.9) as

$$\psi' = p\left(c^{p-1}\frac{q}{p}\psi^{\frac{1}{q}} - \frac{q}{p}f(u)\right) \tag{2.10}$$

we assert that the set of admissible speeds c is an interval $[c^*, +\infty[$ where $c^{*\,p-1} \leq M_q^{\frac{1}{p}} p$. If, in addition, we assume the stronger assumption

(H_q') $\quad \nu := \lim_{u\to 0^+} \frac{f(u)}{u^{p-1}}$ exists, $0 \leq \nu < +\infty$

then we also have the lower estimate

$$c^{*\,p-1} \geq \nu^{\frac{1}{p}} p. \tag{2.11}$$

The preceeding considerations may be summarized in the following statement.

Proposition 2.1. *Let f be of type A and (H_q) hold, or let f be of type B. Then the following are equivalent:*

- *(2.6) has a monotone solution with $u' > 0$ in some interval $]-\infty, b[$, and $u(b^-) = 1$*
- *(2.9) has a solution which is positive in $]0,1[$*
- *(2.8) with $k = \frac{c^{p-1}}{p-1}$ has a (concave) solution with $v' > 0$ in some interval $]-\infty, \beta[$, and $u(\beta^-) = 1$.*

Remark 2.2. $b = +\infty$ (and therefore also $\beta = +\infty$) if $q \leq 2$ and $\sup_{0<u<1} \frac{f(u)}{(1-u)^{q-1}} < +\infty$. See [4], sction 6. In this case the heteroclinics that solve (2.6) are *nondegenerate*, never taking the value 1. The same can be said of the solution of the corresponding problem (2.8).

Remark 2.3. If f is of type B, (2.8) is solvable only for $k = k^* := \frac{(c^*)^{p-1}}{p-1}$.

Proposition 2.4. *Suppose that ψ solves (2.9) with $c > c^*$. Then*

$$\lim_{u\to 0} \frac{\psi(u)}{u^p} < \left(\frac{c^{p-1}}{p}\right)^p.$$

Proof. See [4], Theorem 3.3 and page 175, in view of (2.10).

3 A constrained minimum problem

The purpose of this section is to relate (2.6) with the nonlinear singular boundary value problem

$$(v'^{p-1})' + \lambda\frac{f(v(s))}{s^p} = 0, \quad v(0) = 0,\ v(+\infty) = 1,\ \ v' > 0 \tag{3.12}$$

where λ is a positive parameter.

Let us fix some notation. We still denote by f the extension of f with zero value outside $[0,1]$ and set

$$F(u) = \int_0^u f(z)\,dz.$$

In addition we consider the space of functions

$$E = \{v \in AC([0,+\infty[,\mathbb{R}) \,|\, v' \in L^p(0,+\infty) \ \ , \ \ v(0) = 0.\}$$

and the following real functionals on E

$$J(v) = \frac{1}{p}\int_0^{+\infty} |v'(s)|^p\,ds, \quad \Gamma(v) = \int_0^{+\infty} \frac{F(v(s))}{s^p}\,ds.$$

Remark 3.1. 1. If V is a subset of E such that $J(V)$ is bounded, then by Hőlder's inequatity there exists a number $C > 0$ such that

$$|v(s)| \le C s^{\frac{1}{q}} \quad \forall s \ge 0, \quad \forall v \in V.$$

2. The assumption (H_q) is sufficient for Γ to be well defined and C^1 in E. In fact this follows from Hardy's inequality:

$$\int_0^{+\infty} |v'(s)|^p \, ds < q^p \int_0^{+\infty} \frac{|v(s)|^p}{s^p} \, ds \quad \forall v \in E \setminus 0.$$

Set

$$\theta = \inf_{v \in E \setminus 0} \frac{J(v)}{\Gamma(v)}. \tag{3.13}$$

Theorem 3.2. *Let f be of type B, or of type A and such that (H'_q) holds. We have $\nu q^p \theta \le 1$. If $\nu q^p \theta < 1$ then the* inf *in (3.13) is attained. In any case $\theta^{1/p} = \frac{p-1}{{c^*}^{p-1}}$ where c^* is the least admissible value of c so that (2.9)has solutions.*

Proof. *Step 1* $\nu q^p \theta \le 1$. Let $\xi(x) = \inf_{0<z\le x} \frac{F(z)}{z^p}$. Because of (H'_q) $\lim_{x\to 0} \xi(x) = \frac{\nu}{p}$. Let $\alpha > \frac{1}{q}$ and define $v_r(s) = \min(s^\alpha, r^\alpha)$ for $r > 0$ small. Then $J(v_r) = \frac{\alpha^p r^{\alpha p - p + 1}}{p(\alpha p - p + 1)}$ and $\Gamma(v_r) > \xi(r^\alpha) \int_0^r s^{\alpha p - p} \, ds$. It follows that $\frac{J(v_r)}{\Gamma(v_r)} < \frac{\alpha^p}{p \xi(r^\alpha)}$. Taking the limit as $r \to 0$ and then the limit as $\alpha \to \frac{1}{q}$ yields the statement.

Step 2 Let $u_n \to 0$ weakly in E, u_n bounded in $L^\infty(0,\infty)$ and $\Gamma(u_n) = 1$.
Then $\liminf J(u_n) \ge \frac{1}{\nu q^p}$. For each $r > 0$, denote by J_r and Γ_r the functionals obtained by replacing the integration interval with $[0, r]$. Since $\Gamma - \Gamma_r$ is obviously weakly sequentially continuous in E, we have $\lim \Gamma_r(u_n) = 1$ for each $r > 0$. Similarly to step 1, we write $\eta(x) = \sup_{0<z\le x} \frac{F(z)}{z^p}$; then $F(x) \le \eta(x) x^p$ and $\lim_{x \to 0} \eta(x) = \frac{\nu}{p}$. Using Hardy's inequality and noting that there exists a constant C such that $\sup_{s>0} \frac{|u_n(s)|}{s^{1/q}} \le C$ for all n, we obtain

$$J(u_n) \ge J_r(u_n) \ge q^{-p} \frac{1}{p} \int_0^r \frac{u_n(s)^p}{s^p} \, ds \ge \frac{\Gamma_r(u_n)}{p q^p \eta(C r^{1/q})}.$$

Applying liminf as $n \to \infty$ and then the limit as $r \to 0$ we conclude.

Step 3 Consider the functional $I_\lambda = J - \lambda \Gamma$ and let $\lambda \le \frac{1}{q^p \nu}$. Then if v_n converges weakly to v in E and v_n is bounded in $C[0, +\infty[$, we have $I_\lambda(v) \le \liminf I_\lambda(v_n)$.
Let us decompose

$$I_\lambda = A + B, \quad A(w) = J(w) - \lambda \nu \int_0^\infty \frac{|w(s)|^p}{p s^p} \, ds, \;\; B(w) = \lambda \left(\nu \int_0^\infty \frac{|w(s)|^p}{p s^p} \, ds - \int_0^\infty \frac{F(w(s))}{s^p} \, ds \right).$$

We prove our claim by showing that

$$\lim B(v_n) = B(v), \quad A(v) \le \liminf A(v_n).$$

We start with the assertion about B. By assumption, taking Remark 3.1 into account, we may fix a constant $C > 0$ such that

$$|v_n| \le C, \; |v| \le C, \; \sup_{s>0} \frac{|v_n(s)|}{s^{1/q}} \le C, \; \sup_{s>0} \frac{|v(s)|}{s^{1/q}} \le C, \; \sup_{n \in \mathbb{N}} \int_0^\infty \frac{|v_n(s)|^p}{s^p} \, ds \le C.$$

Now let $\varepsilon > 0$ be given. There exists δ such that $x \le \delta \Longrightarrow |\frac{F(x)}{x^p} - \frac{\nu}{p}| \le \varepsilon$. Putting $\eta^{1/q} = \delta / C$ we have

$$\int_0^\eta \left| \nu \frac{|v_n(s)|^p}{p s^p} - \frac{F(v_n(s))}{s^p} \right| ds = \int_0^\eta \left(\left| \frac{\nu}{p} - \frac{F(v_n(s))}{|v_n(s)|^p} \right| \right) \frac{|v_n(s)|^p}{s^p} \, ds \le C \varepsilon$$

Also, we may fix $T > 0$ such that

$$\int_T^\infty \left| \nu \frac{|v_n(s)|^p}{p s^p} - \frac{F(v_n(s))}{s^p} \right| ds \le \varepsilon$$

and both estimates above hold with v in the place of v_n. By the compact embedding of E into $C([\eta, T])$ we have $v_n|_{[\eta,T]} \to v|_{[\eta,T]}$ uniformly. It follows that

$$B(v) - 2(C+1)\varepsilon \le \liminf B(v_n) \le \limsup B(v_n) \le B(v) + 2(C+1)\varepsilon.$$

The claim follows by the arbitrariness of ε.

Next let us consider A. Let $\varepsilon > 0$ be given and choose a sufficiently large T as before. For each $r > 0$, we write

$$\int_r^\infty \frac{|v'(s)|^p}{p} - \lambda\nu\frac{|v(s)|^p}{ps^p}\,ds \le \int_r^T \frac{|v'(s)|^p}{p} - \lambda\nu\frac{|v(s)|^p}{ps^p}\,ds + \varepsilon$$

$$\le \liminf \int_r^T \frac{|v_n'(s)|^p}{p} - \lambda\nu\frac{|v_n(s)|^p}{ps^p}\,ds + \varepsilon \le \liminf \left(\int_r^\infty \left(\frac{|v_n'(s)|^p}{p} - \lambda\nu\frac{|v_n(s)|^p}{ps^p}\right) ds + \lambda\nu\int_T^\infty \frac{|v_n(s)|^p}{ps^p}\right) + \varepsilon$$

$$\le \liminf \int_0^\infty \left(\frac{|v_n'(s)|^p}{p} - \lambda\nu\frac{|v_n(s)|^p}{ps^p}\,ds\right) + 2\varepsilon$$

where in the last inequality we use the fact that by the choice of λ and Hardy's inequality

$$\int_0^r \left(\frac{|v_n'(s)|^p}{p} - \lambda\nu\frac{|v_n(s)|^p}{ps^p}\right) ds > 0.$$

Letting $\varepsilon \to 0$ and then $r \to 0$ the claim follows.

Step 4 The case $\nu q^p\theta < 1$. Now assume $\nu q^p\theta < 1$. Take $z_n \in E$, $z_n \neq 0$ with $\frac{J(z_n)}{\Gamma(z_n)} \to \theta$. Since F is constant outside $[0,1]$ we may assume that $0 \le z_n \le 1$. Put $\rho_n = (\Gamma(z_n))^{-1}$, $v_n(s) = z_n(\rho_n s)$, so that

$$\Gamma(v_n) = \rho_n\Gamma(z_n) = 1, \quad J(v_n) = \rho_n J(z_n) = \frac{J(z_n)}{\Gamma(z_n)} \to \theta.$$

Since v_n is bounded in E we may assume $v_n \rightharpoonup v \in E$. Hence

$$0 \le I_\theta(v) \le \liminf I_\theta(v_n) = \lim J(v_n) - \theta = 0.$$

Certainly $v \neq 0$, otherwise by Step 2 we obtain the contradiction

$$\theta \ge \frac{1}{\nu q^p}.$$

We have seen that $I_\theta(v) = 0$, that is, $\frac{J(v)}{\Gamma(v)} = \theta$. Hence I_θ attains a minimum at v and so v is a solution of (3.12) with $\lambda = \theta$. (It is easy to see that v satisfies the boundary conditions.) Therefore (2.9) has a solution ψ with $c^{p-1} = (p-1)\theta^{-1/p}$. Let $k = \theta^{-1/p}$. The function v is related with ψ by

$$v(s) = u(\frac{\ln s}{k}), \quad \text{where} \quad u'(t) = \psi(u(t))^{1/p}\ \forall t \in \mathbb{R}.$$

Assume, in view of a contradiction, that $c > c^*$. Then by Proposition 2.4

$$\lim_{u\to 0}\frac{\psi(u)}{u^p} < (\frac{k}{q})^p.$$

Let $\delta > 0$ be fixed so that

$$\frac{\psi(x)}{x^p} < (\frac{k}{q})^p \quad \forall x \in]0,\delta]$$

and let η be such that

$$0 \le s \le \eta \Rightarrow v(s) \le \delta.$$

Since $v'(s) = \frac{\psi(v(s))^{1/p}}{ks}$ we obtain

$$v'(s) < \frac{v(s)}{qs} \quad 0 < s \le \eta.$$

Integrating in $[s, s_0]$ where $0 < s < s_0 < \eta$ we see that there exists a constant $C > 0$ such that

$$v(s) \ge Cs^{1/q}, \quad 0 < s \le s_0.$$

This is impossible since the fact that $v \in E$ implies $\lim_{s\to 0}\frac{v(s)}{s^{1/q}} = 0$.

Step 5 If $\nu q^p\theta = 1$ then $\theta^{1/p} = \frac{p-1}{c^{*\,p-1}}$. The critical speed for a given f may be approached by the critical speeds c_n of an increasing sequence of functions of type B (see Remark 1.2). Denote by θ_n the corresponding minima, by Step 4 we have $\theta_n^{1/p} = \frac{p-1}{c_n^{*\,p-1}}$. Obviously $\theta_n \ge \theta$ so that

$$\theta^{-1/p} \ge \frac{c_n^{*\,p-1}}{p-1} \to \frac{c^{*p-1}}{p-1} \ge \theta^{-1/p}$$

where the last inequality comes from (2.11) and our assumption.

Remark 3.3. The condition $\nu q^p \theta < 1$ holds for instance if

$$\int_0^1 f(x)\,dx > (p-1)q^p \nu.$$

In fact, with $v(s) = \min(s, 1)$ we obtain

$$\Gamma(v) \geq \int_1^\infty \frac{F(1)}{s^p}\,ds \geq \frac{\int_0^1 f(x)\,dx}{p-1}.$$

Hence $\Gamma(v) > q^p \nu$ and, since $J(v) = 1$, the claim follows.

4 Conclusion

We now come back to the caracterization of the critical speed for (1.2) where f is of type A.

The front wave profiles with speed c are the monotone solutions of the second order boundary value problem

$$(|u'|^{p-2}u')' - cu' + f(u) = 0, \quad u(-\infty) = 0,\; u(+\infty) = 1 \tag{4.14}$$

under assumption H_p'). As recalled in the Introduction, the admissible values of c are those for which (1.4) has solutions.

Consider the space of functions

$$F = \{v \in AC([0, +\infty[, \mathbb{R}) \,|\, v' \in L^q(0, +\infty) \;\;,\;\; v(0) = 0.\}$$

In the previous section we have given a variational characterization of the least value c such that (2.9) is solvable. By interchanging p and q, noting that (2.9) can also be read as (2.10) and taking into account Remark 1.1, we easily obtain the following statement.

Theorem 4.1. *Let f be a function of type A and assume (H_p'). Define*

$$\gamma = \inf_{v \in F \setminus 0} \frac{\frac{1}{q}\int_0^{+\infty} |v'(s)|^q\,ds}{\int_0^{+\infty} \frac{F(v(s))}{s^q}\,ds}.$$

Then the critical speed for (4.14) is the number c^* given by

$$\gamma = \frac{q}{pc^{*q}}.$$

Moreover γ is attained if $\mu p^q \gamma < 1$.

Remark 4.2. In Theorem 3.2, the minimizer, say, $\bar{v}$, yields the heteroclinic that solves (2.6) via the change of variable $\bar{u}(t) = \bar{v}(e^{\frac{c^{*p-1}t}{p-1}})$.

In Theorem 4.1 the relationship between the minimizer and the solution of (1.3) is less direct unless, of course, $p = 2$. In this case, after defining $\bar{u}$ as above, one obtains a solution ψ of

$$\psi' = p\left(c^{q-1}\psi^{\frac{1}{p}} - f(u)\right), \quad \psi(0) = 0,\; \psi(1) = 0$$

by $\psi = \varphi^q$ where $\bar{u}' = \varphi(\bar{u})$. Then the heteroclinic $w(t)$ that solves (1.3) is recovered via $w' = \psi(w)^{\frac{1}{p}}$.

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