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Left-Invariance for Smooth Vector Fields and Applications

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Abstract
Let $X = \{X_0, \dots, X_m\}$ be a family of smooth vector fields on an open set $\Omega \subseteq \mathbb{R}^N$. Motivated by applications to the PDE theory of Hörmander operators, for a suitable class of open sets Ω, we find necessary and sufficient conditions on X for the existence of a Lie group $(\Omega, *)$ such that the operator $L = \sum_{i=1}^m X_i^2 + X_0$ is left-invariant with respect to the operation $*$. Our approach is constructive, as the group law is constructed by means of the solution of a suitable ODE naturally associated to vector fields in X. We provide an application to a partial differential operator appearing in the Finance.

Keywords Hörmander operators · Prolongation of the BCH operation · Lie algebras of vector fields · Baker–Campbell–Hausdorff Theorem · Left-invariance

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1 Introduction and Main Results

Let $X = \{X_0, \dots, X_m\}$ be a family of smooth vector fields in an open set Ω of $\mathbb{R}^N$, satisfying Hörmander's rank condition on Ω (the precise meaning of this hypothesis will be given in a moment). We consider the following operator, that is a Hörmander sum of squares plus a drift:

$$L = \sum_{i=1}^{m} X_i^2 + X_0. \tag{1.1}$$

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In the papers [4, 7, 9], when $\Omega = \mathbb{R}^N$, it was considered the problem of equipping Euclidean space $\mathbb{R}^N$ with the structure of a Lie group $\mathbb{G} = (\mathbb{R}^N, *)$ in such a way that L be left-invariant on that group. For the sake of clarity, we remind the associated notion of left-invariance of L: this means that, setting $\tau_\alpha(x) := \alpha * x$, one has

$$L(u \circ \tau_\alpha) = (Lu) \circ \tau_\alpha \quad \text{on } \mathbb{R}^N,$$

for every function $u \in C^\infty(\mathbb{R}^N, \mathbb{R})$ and for every $\alpha \in \mathbb{R}^N$. This problem was motivated by the great advantage of left-invariance in establishing an appropriate harmonic or potential analysis for L.

For example, left-invariance and homogeneity of L with respect to a family of (possibly non-isotropic) dilations lead to global (i.e., on the whole of $\mathbb{R}^N$) maximal L^p-regularity results (see Folland [17]); more generally, left-invariance with respect to the group operation $*$ allows to write a fundamental solution for L (when it exists) in the convolution form (see one of us and Lanconelli [9])

$$\Gamma(x, y) = \gamma(y^{-1} * x), \quad x \neq y \text{ in } \mathbb{R}^N.$$

Again, left-invariance and suitable decays at infinity of γ and of its second derivatives allow to develop a global L^p-regularity for L via Calderón-Zygmund singular integrals in non-homogeneous quasi-metric spaces (see Bramanti, Cupini, Lanconelli, Priola [12]). The presence of left-invariance is also of paramount help when dealing with a Harnack inequality for L, as it reduces the Harnack inequality near a general point to the study of a fixed one (see for instance Sect. 3).

With these applications in mind, in the papers [4, 7, 9] it was studied the problem of finding necessary and sufficient assumptions on $X_0, \ldots, X_m$ to be left-invariant on a Lie group whose manifold is the entire $\mathbb{R}^N$ space. In the cited papers the vector fields $X_0, \ldots, X_m$ were always required to be real-analytic, and the vector space structure of the underlying manifold $\mathbb{R}^N$ was also used. In [7] it was shown that the above problem is intimately related to the possibility of prolonging the local Baker-Campbell-Hausdorff (BCH, in the sequel) multiplication that $X_0, \ldots, X_m$ determine on $\mathbb{R}^N$ by means of their exponentiation, i.e., by considering the family of their integral curves. In [4] it was proved that this prolongation is always achievable, under a minimal set of assumptions on the Lie algebra $\mathfrak{g}$ generated by the X_i's, when the latter are C^ω. Unique Continuation played a central role in prolongation issues in the cited papers.

Here we improve the results in [4, 7, 9] in two directions: we consider more general open sets Ω instead of $\mathbb{R}^N$, and we remove the C^ω assumption, in favor of the less restrictive and more natural C^∞ requirement. We show that this is possible by using a simple ODE argument, which -most importantly- gives a *constructive ODE procedure* in obtaining the Lie group. Roughly put, Unique Continuation of the C^ω setting will be replaced by uniqueness results for ODEs with smooth coefficients. Once these two goals are achieved, we finally give an application of our results to an operator L as in (1.1) appearing in Mathematical Finance, for which Ω is a half-space in $\mathbb{R}^3$.

A few remarks on the literature related to the intertwining of the BCH multiplication and ODEs are in order. First of all, if $\mathfrak{a}$ is any (real or complex) *finite-dimensional* Lie algebra, it is known (see e.g., [8, Chap. 5]) that the BCH multiplication

$$(x,y)\mapsto Z(x,y) := x+y+\tfrac{1}{2}[x,y]+\tfrac{1}{12}([[x,y],y]+[[y,x],x])-\tfrac{1}{24}[x,[y,[x,y]]]+\cdots \tag{1.2}$$

is well-defined near the origin of $\mathfrak{a}$. More precisely, if $\|\cdot\|$ is any norm[1] on $\mathfrak{a}$ satisfying $\|[x,y]\|\leq\|x\|\cdot\|y\|$ for every $x,y\in\mathfrak{a}$, it can be proved that the series in (1.2) converges whenever $\|x\|+\|y\|<\ln 2$ (see e.g., [5, Theorem 5.3]).

In [21, eq. (7), p. 248], Poincaré first discovered the special ODE solved by the map $t\mapsto Z(x,ty)$, in the framework of Lie groups of transformations: he proved

$$\frac{\mathrm{d}}{\mathrm{d}t}Z(x,ty)=\frac{\mathrm{ad}\,Z(x,ty)}{1-e^{-\mathrm{ad}\,Z(x,ty)}}(y), \tag{1.3}$$

where, as usual, $\mathrm{ad}\,x(y)=[x,y]$. When $\mathfrak{a}$ is a finite dimensional real Lie algebra, and x,y are close to the origin, (1.3) is a genuine ODE, not only a formal one, since the series $Z(x,ty)$ is convergent for $|t|\leq 1$.

Other variants of (1.3) can be given for the maps $Z(tx,y)$ or $Z(tx,ty)$ (see e.g., [5, Sect. 5.3]). For example, the idea of using the (formal) ODE solved by the curve $t\mapsto Z(tx,ty)$ plays a crucial role in [3, 6, 18] too, where the problem of the convergence domain of the BCH series is also addressed. The use of the ODEs solved by Z returns in many contexts related to BCH type theorems: see e.g., the Zassenhaus formula in [2] by Arnal, Casas, Chiralt, or the prolongation problem for the BCH multiplication in [16] by Eggert. We point out that our technique has some common ground with the algebraic approach in the latter paper. Indeed, we shall consider a local multiplication $m(x,y)$ (strongly related to $Z(x,y)$) and we shall study the prolongability of $t\mapsto m(x,ty)$; in doing this, we rely on the prolongation properties of the solutions to ODEs, as we now describe.

We illustrate more closely the prolongation problem we are concerned with in this paper, and our technique in solving it. We suppose $\Omega\subseteq\mathbb{R}^N$ is an open set satisfying the following assumption:

(S): Ω is C^∞-diffeomorphic to $\mathbb{R}^N$.

This is true, for example, in the meaningful case when Ω *is a star-domain*.

Let $\mathcal{X}(\Omega)$ denote the vector space of the smooth vector fields on Ω. We think of any $X\in\mathcal{X}(\Omega)$ as a first order differential operator acting on $C^\infty(\Omega)$, say

$$X=a_1(x)\,\frac{\partial}{\partial x_1}+\cdots+a_N(x)\,\frac{\partial}{\partial x_N},\quad \text{where the } a_i' s \text{ are smooth on } \quad \Omega.$$

With no ambiguity, we interchangeably denote by X_x both the map

$$\Omega\ni x\mapsto (a_1(x),\ldots,a_N(x)),$$

[1] Such a norm always exists: indeed, since $\mathfrak{a}$ is finite-dimensional, the bilinearity of the bracket grants the existence of $M>0$ such that $\|[x,y]\|\leq M\,\|x\|\cdot\|y\|$; thus the new norm $M\|\cdot\|$ satisfies $\|[x,y]\|\leq\|x\|\cdot\|y\|$ for every $x,y\in\mathfrak{a}$.

or the derivation (at x) associated with X as a PDO acting on $C^\infty(\Omega)$. Given $x \in \Omega$, we use the notation

$$t \mapsto \Psi_t^X(x), \tag{1.4}$$

to denote the maximal integral curve of X starting at x, i.e., the maximal solution $\gamma(t)$ (valued in Ω) of the Cauchy problem

$$\dot{\gamma}(t) = X_{\gamma(t)}, \quad \gamma(0) = x.$$

We know that $\gamma \in C^\infty(I, \Omega)$, where I is an open interval containing 0.

We henceforth fix a family $X_0, \dots, X_m$ of vector fields in $\mathcal{X}(\Omega)$, and we write

$$\mathfrak{g} := \mathrm{Lie}\{X_0, \dots, X_m\}$$

to denote the Lie sub-algebra of $\mathcal{X}(\Omega)$ Lie-generated by this family through iterated Lie-bracketing. More explicitly, $\mathfrak{g}$ is the smallest Lie sub-algebra of $\mathcal{X}(\Omega)$ containing $X_0, \dots, X_m$. Following the hypotheses in [4], we make the next assumptions:

(H): $X_0, \dots, X_m$ satisfy Hörmander's bracket generating condition on Ω, i.e.,

$$\dim\{Y_x \in \mathbb{R}^N : Y \in \mathfrak{g}\} = N \quad \text{for every } x \in \Omega;$$

(C): any $X \in \mathfrak{g}$ is complete, i.e., $\Psi_t^X(x)$ is defined for every $t \in \mathbb{R}$ and $x \in \Omega$;
(D): $\dim(\mathfrak{g})$, as a vector subspace of $\mathcal{X}(\Omega)$, is N (where $\Omega \subseteq \mathbb{R}^N$).

We aim to prove via a purely ODE argument that, under conditions (S, H, C, D), Ω *can be equipped with the structure of a Lie group* $\mathbb{G} = (\Omega, *)$ *such that* $\mathfrak{g}$ *coincides with the Lie algebra* $\mathrm{Lie}(\mathbb{G})$ *of* $\mathbb{G}$; moreover, we give an explicit construction of $\mathbb{G}$ (see Theorem 1.5 for the precise statement). In particular, L in (1.1) is a left-invariant operator on Ω equipped with this group structure.

Before providing some remarks related to our assumptions, we give an example of a set of vector fields of relevance in Mathematical Finance fulfilling them.

Example 1 Let us consider, on $\mathbb{R}^3$ (whose points we denote by (x, y, t)),

$$X_0 := x\,\frac{\partial}{\partial y} - \frac{\partial}{\partial t}, \quad X_1 := x\,\frac{\partial}{\partial x}.$$

The (ultra-parabolic) second order PDO $L = X_1^2 + X_0$ is relevant in the study of Asian options (see Sect. 3). The Lie algebra generated by X_0, X_1 is

$$\mathfrak{g} = \mathrm{span}\left\{X_0, X_1, [X_1, X_0] = x\,\frac{\partial}{\partial y}\right\},$$

so that $\mathfrak{g}$ is 3-dimensional. Note that $\mathfrak{g}$ is not nilpotent, as

$$[\underbrace{X_1, \cdots [X_1}_{n\text{times}}, X_0]] = x\,\frac{\partial}{\partial y}, \quad \text{for every } n \in \mathbb{N}.$$

Moreover, X_0, X_1 satisfy Hörmander's condition on the open set $\{x \neq 0\}$. We consider one of the two connected components of the latter: to make a choice,

$$\Omega := \{(x, y, t) \in \mathbb{R}^3 : x > 0\}.$$

Since Ω is convex, it is a star-domain, and is clearly diffeomorphic to $\mathbb{R}^3$. It is not difficult to show that, for any $X \in \mathfrak{g}$, when the starting point P lies in Ω, then the integral curve $\Psi_t^X(P)$ remains in Ω, and is defined for any $t \in \mathbb{R}$ (we carry out this computation in Example 2): thus, hypothesis (C) is also satisfied. Summing up, our assumptions (S, H, C, D) are fulfilled on Ω. In Example 2, we show a simple procedure to equip Ω with the structure of a Lie group with Lie algebra $\mathfrak{g}$.

A few remarks on our assumptions are in order.

Remark 1.1 Hypothesis (D) should not be confused with (H): for example, if

$$X_0 = \frac{\partial}{\partial x_1}, \quad X_1 = x_1 \frac{\partial}{\partial x_2} \quad \text{in } \mathbb{R}^2,$$

then (H) is satisfied, but (D) is not, as $\mathfrak{g} = \text{Lie}\{X_0, X_1\}$ is 3-dimensional. We also stress that X_0 and X_1 are linearly independent *as vector fields* in the vector space $\mathcal{X}(\mathbb{R}^2)$, even if X_1 is null as a derivation at the origin (or on the $\{x_1 = 0\}$ axis).

Remark 1.2 We explicitly point out that, in view of assumptions (H, D), the completeness assumption (C) is actually equivalent to requiring that the generators $X_0, \dots, X_m$ of $\mathfrak{g}$ are complete (see, e.g., [10]), which may be considerably shorter to check than (C). Moreover, (H, C, D) are necessary for the solution of our problem, and they are mutually independent: see [4].

Remark 1.3 When we ask of $\mathfrak{g} = \text{Lie}\{X_0, \dots, X_m\}$ to *coincide* with $\text{Lie}(\mathbb{G})$, we are thinking of $\text{Lie}(\mathbb{G})$ as a subset of $\mathcal{X}(\Omega)$, the elements of the latter being thought of as first order PDOs. We cannot be content with obtaining an *isomorphism* between $\text{Lie}\{X_0, \dots, X_m\}$ and $\text{Lie}(\mathbb{G})$; the latter could be easily deduced by an abstract Theorem of Lie, stating that, given any finite-dimensional (real) Lie algebra $\mathfrak{a}$, there exists a Lie group $\mathbb{G}$ with Lie algebra $\text{Lie}(\mathbb{G})$ isomorphic to $\mathfrak{a}$.

When applying this abstract theorem to the Lie algebra $\mathfrak{g}$, the existence of some Lie group $\mathbb{G}$ with $\text{Lie}(\mathbb{G})$ only isomorphic to $\mathfrak{g}$ is of little use for the left-invariance of $\sum_{i=1}^m X_i^2 + X_0$ on Ω. Thus, we shall avoid the aforementioned Theorem of Lie, which works up to an isomorphism and is all but constructive. To the contrary, the main asset of our technique is that it gives an explicit[2] construction of the group law via ODEs.

Remark 1.4 There is an interesting operation that one may consider in our framework of *complete* vector fields: if we take a linear basis $\{W_1, \dots, W_N\}$ of $\mathfrak{g} = \text{Lie}\{X_0, \dots, X_m\}$, the following map is well-posed:

$$\Omega \times \mathbb{R}^N \ni (x, \lambda) \mapsto \Psi_1^{\lambda_1 W_1 + \dots + \lambda_N W_N}(x).$$

[2] In explicit cases, this construction can also be computer-implemented via *Mathematica*™.

Unfortunately, this operation between x and λ does not directly work in providing a Lie group on Ω turning L into a left-invariant operator. Instead, the right choice is to replace $\lambda_1 W_1 + \cdots + \lambda_N W_N$ with "Log(y)" where Log is a *local* inverse function for the map $\mathfrak{g} \ni W \mapsto \mathrm{Exp}(W) := \Psi_1^W(\omega_0)$. Now the *local* map

$$(x, y) \mapsto f(x, y) := \Psi_1^{\mathrm{Log}\, y}(x)$$

has the disadvantage of not being defined for every y in Ω. Our task in what follows (see Theorem 1.5 below) is to prove that this map can always be prolonged on $\Omega \times \Omega$ in such a way that the position $x * y := f(x, y)$ gives a Lie group $(\Omega, *)$ solving our problem. One can produce simple examples of Lie algebras $\mathfrak{g}$ of vector fields fulfilling our assumptions where Log is not globally defined whereas f is globally extendable; see Example 3 in Sect. 2.

The following is the main result of the paper, already partially announced in Remark 1.4.

Theorem 1.5 *Let $X_0, \ldots, X_m$ be smooth vector fields on the open $\Omega \subseteq \mathbb{R}^N$, where Ω satisfies assumption* (S). *Let $\mathfrak{g}$ be the Lie algebra Lie-generated by $X_0, \ldots, X_m$. Suppose that conditions* (H, C, D) *are fulfilled by $\mathfrak{g}$ on Ω. Let $\omega_0 \in \Omega$ be fixed.*

Then, the map valued in Ω defined by

$$\mathrm{Exp} : \mathfrak{g} \longrightarrow \Omega, \quad X \mapsto \mathrm{Exp}(X) := \Psi_1^X(\omega_0)$$

(*where Ψ_1^X has been defined in* (1.4)) *has an injective restriction to a suitable small neighborhood, $\mathfrak{U}$ say, of the vanishing vector field in $\mathfrak{g}$. We set $V := \mathrm{Exp}(\mathfrak{U})$, the latter providing a neighborhood of ω_0 in Ω. Moreover, let us denote by* Log *the inverse map of*

$$\mathrm{Exp}|_{\mathfrak{U}} : \mathfrak{U} \longrightarrow V.$$

Finally, we consider the function

$$M : \Omega \times V \longrightarrow \Omega, \quad M(a, b) := \Psi_1^{\mathrm{Log}(b)}(a). \tag{1.5}$$

Then, M can be smoothly prolonged to a multiplication $(x, y) \mapsto x * y$ defined on the whole of $\Omega \times \Omega$ equipping Ω with a Lie group structure $\mathbb{G} = (\Omega, *)$ with identity ω_0, and such that $\mathrm{Lie}(\mathbb{G})$ *(thought of as a set of vector fields in Ω) is equal to $\mathfrak{g}$.*

In particular, the following Hörmander operator is left-invariant on Ω

$$L = \sum_{i=1}^{m} X_i^2 + X_0.$$

In order to describe our approach to the proof of Theorem 1.5, we stress the role of (S), which is sufficient to reduce Theorem 1.5 to the case when $\Omega = \mathbb{R}^N$. Indeed, if

(S) holds true, then there exists a C^∞-diffeomorphism $\Phi : \Omega \to \mathbb{R}^N$. For any $X \in \mathfrak{g}$, we denote by $\widetilde{X}$ the smooth vector field on $\mathbb{R}^N$ that is Φ-related to X, i.e.,

$$\widetilde{X}_x = \mathrm{d}_{\Phi^{-1}(x)}\Phi(X_{\Phi^{-1}(x)}), \quad \forall\, x \in \mathbb{R}^N. \tag{1.6}$$

Roughly put, $\widetilde{X}$ is nothing but the vector field corresponding to X in the change of variable associated to Φ. As is well known, by the natural behavior of the Lie-bracket under Φ-relatedness, this also defines a linear map, denoted by $\mathrm{d}\Phi$, from $\mathcal{X}(\Omega)$ to $\mathcal{X}(\mathbb{R}^N)$ sending X to $\widetilde{X}$. Let us denote by $\widetilde{\mathfrak{g}}$ the Lie-algebra generated by $\widetilde{X}_0, \dots, \widetilde{X}_m$.

Then, it is not difficult to show that $\widetilde{X}_0, \dots, \widetilde{X}_m$ satisfy (H, C, D) on $\mathbb{R}^N$. This is a consequence of the following facts (see e.g., [5, Cor. 4.11]):

$$\Phi(\Psi_t^X(\omega)) = \Psi_t^{\mathrm{d}\Phi(X)}(\Phi(\omega)), \quad \forall\, \omega \in \Omega, \quad X \in \mathfrak{g}; \tag{1.7}$$

$$\widetilde{\mathfrak{g}} = \mathrm{d}\Phi(\mathfrak{g}). \tag{1.8}$$

Now, suppose that we can find a Lie group $\widetilde{\mathbb{G}} = (\mathbb{R}^N, \widetilde{*})$ with Lie algebra $\widetilde{\mathfrak{g}}$ (and identity $\Phi(\omega_0)$). Then, it is not difficult to show that the pulled-back operation

$$a * b := \Phi^{-1}(\Phi(a) \,\widetilde{*}\, \Phi(b)), \quad a, b \in \Omega$$

turns Ω into a Lie group $\mathbb{G} = (\Omega, *)$ that is isomorphic to $\widetilde{\mathbb{G}}$ via Φ (and identity ω_0); moreover, thanks to (1.8), the Lie algebra of $\mathbb{G}$ is exactly $\mathfrak{g}$, as desired.

All this being said, it is non-restrictive from now on to assume that $\Omega = \mathbb{R}^N$, where (with no need to change notation) we assume that smooth vector fields $X_0, \dots, X_m$ are given, fulfilling assumptions (H, C, D). Thus, we replicate on $\mathbb{R}^N$ the construction of a local multiplication analogous to M in (1.5), and we aim to prove that this multiplication can be globally prolonged to a group law with assigned Lie algebra $\mathfrak{g} = \mathrm{Lie}\{X_0, \dots, X_m\}$.

As a matter of fact, in order to be sure that Theorem 1.5 can be obtained from the particular case of $\mathbb{R}^N$, we will need to compare the operation M and its replica on $\mathbb{R}^N$, and the integral curves of the original vector fields on Ω with their copies on $\mathbb{R}^N$. This will be rigorously carried out in Remark 1.8.

Given $X \in \mathfrak{g}$ and $x \in \mathbb{R}^N$, thanks to assumption (C), the integral curve $\Psi_t^X(x)$ is defined on $\mathbb{R}$; in view of the exponential BCH-formalism, it is convenient to denote $\Psi_t^X(x)$ by $\exp(tX)(x)$. Thus, the following map is well posed

$$\exp(X)(x) := \exp(tX)(x)\big|_{t=1}.$$

Next we choose any $x_0 \in \mathbb{R}^N$ which will serve as the identity element of the group: the choice is totally immaterial, thus we take once and for all $x_0 = 0$ (a choice that can be achieved at any time through a Euclidean translation). As we did in Theorem 1.5, we consistently denote by Exp the map

$$\mathfrak{g} \ni X \mapsto \exp(X)(0), \quad \text{or equivalently,} \quad \mathrm{Exp}(X) := \Psi_1^X(0). \tag{1.9}$$

Since $\mathfrak{g}$ is finite dimensional by (D), we can fix a norm $\|\cdot\|$ on $\mathfrak{g}$ (all norms being equivalent). Assumptions (H, D) imply that there exists an open neighborhood $\mathfrak{U}$ of $0 \in \mathfrak{g}$ such that $\mathrm{Exp}|_{\mathfrak{U}} : \mathfrak{U} \longrightarrow V := \mathrm{Exp}(\mathfrak{U})$ is a C^∞ diffeomorphism; we denote its inverse by Log. Next, we define a local multiplication by setting

$$m : \mathbb{R}^N \times V \longrightarrow \mathbb{R}^N, \qquad m(x, y) := \exp(\mathrm{Log}(y))(x). \tag{1.10}$$

We may say that m is a "half-local" operation, in that x can vary throughout the space, whilst y lies in the small V. Note that, by the very definition of m (and of Exp and Log) one has, for every $X \in \mathfrak{g}$,

$$m(x, \mathrm{Exp}(tX)) = \Psi_t^X(x), \quad \text{for every } x \in \mathbb{R}^N \quad \text{and every} \quad t \textit{ near } 0. \tag{1.11}$$

This is a key identity, because it expresses the multiplication m in terms of the integral curves of the vector fields in $\mathfrak{g}$, at least when the second factor is of the exponential form $\mathrm{Exp}(tX)$. It is a crucial link between group theory (the left-hand side) and ODE theory (the right-hand one). Furthermore, (1.11) easily implies that

$$m(x, y) = \Psi_1^{\mathrm{Log}(y)}(\Psi_1^{\mathrm{Log}(x)}(0)), \quad \text{for every} \quad x, y \in V,$$

so that the operation m is expressed, in the small, by the composition of two integral curves. Now, it is the BCH series in (1.2) that naturally intervenes if one makes the composition of two flow-maps of the form $\exp(X)$ and $\exp(Y)$: indeed one has the following non-trivial result.

Theorem 1.6 [BCH Theorem for ODEs] *Let $\mathfrak{g}$ be a Lie subalgebra of $\mathcal{X}(\mathbb{R}^N)$ satisfying assumptions (C) and (D). Then there exists $\varepsilon > 0$ such that the BCH series*[3] $Z(X, Y) = \sum_n Z_n(X, Y)$ *is convergent for any $X, Y \in \mathfrak{g}$ with $\|X\|, \|Y\| < \varepsilon$. For any such X and Y, one has*

$$\exp(Y)\big(\exp(X)(x)\big) = \exp(Z(X, Y))(x), \quad \forall\, x \in \mathbb{R}^N. \tag{1.12}$$

In the C^ω case, this result is contained in [7]; for the more general case of C^∞, see the Appendix 5 of the present paper.

By using (1.12) and the associativity-in-the-small of the map $(X, Y) \mapsto Z(X, Y)$, one can prove without difficulties that $(x, y) \mapsto m(x, y)$ defines a local Lie group. By shrinking V if necessary, the associativity of m reads as follows (note that x can be as large as we please):

$$m(m(x, y), z) = m(x, m(y, z)), \quad \text{for every } x \in \mathbb{R}^N \text{ and every } y, z \in V. \tag{1.13}$$

[3] We agree that $Z_n(x, y)$ is the Lie polynomial obtained from $Z(x, y)$ in (1.2) by grouping together the homogeneous Lie-polynomials of degree n in x and y jointly.

Most importantly for us, any $Z \in \mathfrak{g}$ enjoys a suitable left-invariance "in the small", this being the identity

$$Z_{m(x,y)} = \frac{\partial m}{\partial y}(x, y)\, Z_y, \quad \text{for every } x \in \mathbb{R}^N \quad \text{and } y \in V. \tag{1.14}$$

Here we denoted by $\partial m/\partial y$ the differential with respect to y only. Actually, (1.14) can be obtained by differentiating (1.13) with respect to z at 0, and using the fact that (1.14) is valid when $y = 0$ thanks to this calculation:

$$Z_x = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \Psi_t^Z(x) \overset{(1.11)}{=} \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} m(x, \mathrm{Exp}(tZ)) = \frac{\partial m}{\partial y}(x, 0)\, Z_0.$$

For the details, one can argue as in [4, Theorem 3.8].

Our main task is then to show that *the local-Lie-group structure defined by m can be prolonged throughout* $\mathbb{R}^N \times \mathbb{R}^N$, as the next lemma states:

Lemma 1.7 *Let* $X_0, \dots, X_m$ *be smooth vector fields on* $\mathbb{R}^N$. *Let* $\mathfrak{g}$ *be the Lie algebra Lie-generated by* $X_0, \dots, X_m$. *Suppose that conditions* (H, C, D) *are fulfilled by* $X_0, \dots, X_m$ *and* $\mathfrak{g}$. *Finally, let m be the map constructed in* (1.10) *via ODEs.*

Then m can be prolonged to a multiplication $(x, y) \mapsto x * y$ *defined on* $\mathbb{R}^N \times \mathbb{R}^N$ *equipping* $\mathbb{R}^N$ *with a Lie group structure* $\mathbb{G} = (\mathbb{R}^N, *)$ *such that* $\mathrm{Lie}(\mathbb{G})$ *is equal to* $\mathfrak{g}$. *Conditions* (H, C, D) *are also necessary for the latter fact to hold.*

Lemma 1.7 is both an improvement of [9, Th. 1.1] (the latter assuming the existence of a C^ω prolongation of m, along with (H, C, D)), and an improvement of [4, Th. 1.4] (the latter assuming (H, C, D) under the C^ω regularity of $X_0, \dots, X_m$).

Our proof of Lemma 1.7, which we now briefly describe, is consistent with its purely ODE nature (for the details, see Sect. 4). Fixing $x, y \in \mathbb{R}^N$, we consider the curve

$$\gamma_{x,y}(t) := m(x, ty),$$

defined at least for $|t|$ small (depending upon y). Notice that we are implicitly using the vector-space structure of $\mathbb{R}^N$ when defining $\gamma_{x,y}(t)$.

Thanks to Hörmander's condition (H) and to assumption (D), we can fix a basis $J_1, \dots, J_N$ of $\mathfrak{g}$ such that the matrix $J(x)$ whose columns are $J_1(x), \dots, J_N(x)$ is non-singular for every $x \in \mathbb{R}^N$. It is not difficult to show that the curve $t \mapsto \gamma_{x,y}(t)$, say $z(t)$ shortly, satisfies a (non-autonomous) Cauchy problem

$$\dot{z}(t) = \sum_{k=1}^{N} a_k(t, y)\, J_k(z(t)), \quad z(0) = x, \tag{1.15}$$

where $J_1, \dots, J_N$ are as above, and the a_k's are smooth functions. Notice that the reason why the coefficient functions a_k's *do not depend* upon x is to attribute to (1.14). Indeed, (1.15) can be obtained by the calculation

$$\dot{z}(t) = \frac{\partial m}{\partial y}(x, ty)\, y = \frac{\partial m}{\partial y}(x, ty)\, J(ty)(J(ty))^{-1} y \overset{(1.14)}{=} J(m(x, ty))(J(ty))^{-1} y,$$

and then we denote by $a_k(t, y)$ the k-th component of $(J(ty))^{-1}y$. Now, a prolongability result for ODEs proved in [4, Th. 1.1] ensures that $z(t)$ exists *for every* $t \in \mathbb{R}$. The trigger for the prolongation is once again ascribed to (1.14).

We are then entitled to define

$$x * y := \gamma_{x,y}(1) \quad \text{for every} \quad x, y \in \mathbb{R}^N.$$

This is clearly a prolongation of $m(x, y)$; in Sect. 4 we show in details that $(\mathbb{R}^N, *)$ is as required by Lemma 1.7.

Remark 1.8 The proof of Theorem 1.5 follows from Lemma 1.7, once one recognizes that, whatever the diffeomorphism $\Phi : \Omega \to \mathbb{R}^N$, the map m constructed in (1.10) is consistent (through Φ) with the map M of Theorem 1.5: more precisely,

$$\Phi(M(a, b)) = m(\Phi(a), \Phi(b)).$$

Indeed, the latter is equivalent to

$$\Phi(\Psi_1^{\mathrm{Log}(b)}(a)) = \Psi_1^{\widetilde{\mathrm{Log}}(\Phi(b))}(\Phi(a)), \tag{1.16}$$

where Log and $\widetilde{\mathrm{Log}}$ are, respectively, the inverse maps of the Exp-maps naturally associated with the vector fields $X_0, \dots, X_m$ in Ω and the vector fields

$$\widetilde{X}_0 = \mathrm{d}\Phi(X_0), \dots, \widetilde{X}_m = \mathrm{d}\Phi(X_m) \quad \text{in} \quad \mathbb{R}^N,$$

and when the integral curves start, respectively, from ω_0 and $\Phi(\omega_0)$. Now, by using (1.7), identity (1.16) follows from

$$\mathrm{d}\Phi(\mathrm{Log}(b)) = \widetilde{\mathrm{Log}}(\Phi(b)),$$

which is another consequence of (1.7) (with $t = 1$ and $\omega = \omega_0$).

2 A Few Illustrative Examples

In order to illustrate our construction, we consider a few explicit examples.

Example 2 Let X_0, X_1 be the vector fields on the set Ω of Example 1. We choose some point of Ω that will play the role of the identity, say $\omega_0 = (1, 0, 0)$. We compute the map

$$\mathrm{Exp} : \mathfrak{g} \longrightarrow \Omega, \quad X \mapsto \exp(tX)(\omega_0)\big|_{t=1}.$$

Setting $X_2 := [X_1, X_0]$, we take

$$X = -\tau\, X_0 + \xi\, X_1 + \eta\, X_2, \quad (\xi, \eta, \tau) \in \mathbb{R}^3,$$

and we solve the ODE (system) $\dot\gamma = X_\gamma$, $\gamma(0) = (x_0, y_0, t_0)$, i.e.,

$$\begin{cases} \dot\gamma_1(t) = \xi\,\gamma_1(t) & \gamma_1(0) = x_0 \\ \dot\gamma_2(t) = -\tau\,\gamma_1(t) + \eta\,\gamma_1(t) & \gamma_2(0) = y_0 \\ \dot\gamma_3(t) = \tau & \gamma_3(0) = t_0. \end{cases}$$

We get

$$\exp(X)(x_0, y_0, t_0) = \gamma(1) = \left(x_0\,e^{\xi},\, y_0 + (\eta - \tau)x_0\,\frac{e^{\xi} - 1}{\xi},\, t_0 + \tau \right),$$

where $(e^{\xi} - 1)/\xi := 1$ if $\xi = 0$. Choosing $(x_0, y_0, t_0) = \omega_0 = (1, 0, 0)$, we get

$$\mathrm{Exp}(-\tau\,X_0 + \xi\,X_1 + \eta\,X_2) = \left(e^{\xi},\, (\eta - \tau)\frac{e^{\xi} - 1}{\xi},\, \tau \right).$$

Notice that $\mathrm{Exp} : \mathfrak{g} \to \Omega$ is globally invertible with inverse

$$\mathrm{Log}(x, y, t) = -t\,X_0 + \ln x\,X_1 + \left(t + y\,\frac{\ln x}{x - 1} \right) X_2,$$

where $\ln x/(x - 1) := 1$ when $x = 1$. The map M in (1.5) is everywhere well-posed and, after simple computations, we discover that it is equal to

$$M(a, b) = \exp(\mathrm{Log}(b))(a) = (a_1 b_1,\;\; a_2 + a_1 b_2,\;\; a_3 + b_3)\,.$$

Changing notation, the group law on Ω is therefore

$$(x, y, t) * (x', y', t') = \big(x\,x', y + x\,y', t + t'\big)\,. \tag{2.1}$$

Since the Jacobian matrix at $(1, 0, 0)$ of the left translation

$$(x', y', t') \mapsto (x, y, t) * (x', y', t')$$

is equal to

$$\begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

the vector fields associated with the columns of the latter furnish a basis for the Lie algebra of $(\Omega, *)$; as the latter vector fields are

$$x\,\frac{\partial}{\partial x},\quad x\,\frac{\partial}{\partial y},\quad \frac{\partial}{\partial t},$$

we recognize that the original vector fields X_0 and X_1 are indeed left-invariant on $(\Omega, *)$, as predicted by Theorem 1.5.

The typical issue that can arise when dealing with non-invertible Log-maps is not visible in the previous example, where Log is globally invertible. For an example presenting this issue, see the next case:

Example 3 Let us consider on $\mathbb{R}^3$ the vector fields

$$X_1 = \frac{\partial}{\partial x_1}, \quad X_0 = \cos x_1 \frac{\partial}{\partial x_2} + \sin x_1 \frac{\partial}{\partial x_3}.$$

It is easy to see that $\mathfrak{g} := \mathrm{Lie}\{X_0, X_1\}$ is 3-dimensional (whence condition (D) is fulfilled), and it also satisfies hypotheses (H) (as is simple to check) and (C) (for any vector field in $\mathfrak{g}$ has bounded coefficients). The associated Exp map is given by

$$\begin{aligned}&\mathrm{Exp}\big(\xi_1 X_1 + \xi_2 X_0 + \xi_3 [X_1, X_0]\big)\\ &= \left(\xi_1, \xi_2 \frac{\sin \xi_1}{\xi_1} + \xi_3 \frac{\cos \xi_1 - 1}{\xi_1}, -\xi_2 \frac{\cos \xi_1 - 1}{\xi_1} + \xi_3 \frac{\sin \xi_1}{\xi_1}\right).\end{aligned}$$

We remark that Exp is neither injective, nor surjective. After tedious computations, it can be verified that the associated map $M(a, b)$ is equal to

$$\Big(a_1 + b_1, \quad a_2 + b_2 \cos a_1 - b_3 \sin a_1, \quad a_3 + b_2 \sin a_1 + b_3 \cos a_1\Big).$$

By means of our Theorem 1.5, it follows that $L = X_1^2 + X_0$ is left invariant on $\mathbb{G} = (\mathbb{R}^3, *)$, with $a * b = M(a, b)$ as above.

3 An Application to Asian Options

Before embarking in the proof of the results presented in the Introduction, we sketch an application of the left-invariance in Theorem 1.5. We consider the vector fields in Example 2.

The applicative interest of the differential operator

$$L = \left(x \frac{\partial}{\partial x}\right)^2 + x \frac{\partial}{\partial y} - \frac{\partial}{\partial t} \qquad (x, y, t) \in \Omega := (0, \infty) \times \mathbb{R}^2 \tag{3.1}$$

is motivated by the financial problem of pricing arithmetic average Asian options. Indeed, consider the stochastic process

$$\begin{cases} X_t = x_0 \, e^{\sigma W_t - \frac{\sigma^2}{2} t} \\ Y_t = y_0 + \int_0^t X_\tau d\tau, \end{cases} \tag{3.2}$$

where $\sigma > 0$ and $(W_t)_{t\geq 0}$ is a Wiener process. The first component $(X_t)_{t\geq 0}$ is a Geometric Brownian Motion starting from x_0 at $t = 0$, and it describes the evolution of the price of some financial asset in the Black & Scholes setting, while $(Y_t)_{t\geq 0}$ is its time integral and appears whenever the payoff of the option depends on the past average of the price of the asset.

The stochastic differential equation of the process $(X_t, Y_t)_{t\geq 0}$ is

$$\begin{cases} \mathrm{d}X_t = \sigma X_t \mathrm{d}W_t, & X_0 = x_0, \\ \mathrm{d}Y_t = X_t \mathrm{d}t, & Y_0 = y_0, \end{cases}$$

and the density $p = p(x, y, t)$ of $(X_t, Y_t)_{t\geq 0}$ is a smooth function which is a classical solution to the forward Kolmogorov equation

$$\frac{\sigma^2}{2}\left(x\frac{\partial}{\partial x}\right)^2 p(x, y, t) - x\frac{\partial}{\partial y} p(x, y, t) - \frac{\partial}{\partial t} p(x, y, t) = 0.$$

Note that, up to a plain change of variable, it is not restrictive to assume $\frac{\sigma^2}{2} = 1$; moreover the further change of variable $(x, y, t) \mapsto (x, -y, t)$ transforms the above operator into L appearing in (3.1). Theorem 1.5 asserts that this operator is left invariant on $(\Omega, *)$, where $*$ is as in (2.1).

The composition law $*$ in Example 2 was first considered by Monti and Pascucci in [19], and has been used in the articles [1, 14, 20] for the proof of asymptotic bounds of the density p, these bounds relying on a Harnack inequality and on the translation-invariance of the equation $Lp = 0$.

In order to outline the proof of the bounds, we introduce some notation and we recall a statement of the Harnack inequality. For every $r \in (0, 1]$, we set

$$\begin{aligned} H_r &= \left\{(x, y, t) \in \mathbb{R}^3 : \ |x - 1| < r, \quad -r^2 < t \leq 0, \quad |y + t| < r^3\right\}, \\ S_r &= \left\{(x, y, t) \in H_r : \ t < -r^2/2\right\}. \end{aligned}$$

Then the following estimate holds: there exist two universal constants $M > 1$, and $\theta \in (0, 1)$, such that

$$\sup_{S_{\theta r}} u \leq M\, u(1, 0, 0),$$

for every non-negative solution u to $Lu = 0$ defined in an open neighborhood of the set H_r. The translation-invariance property of the differential operator L provides us with the following more general statement of the Harnack inequality:

$$\sup_{S_{\theta r}(x_0, y_0, t_0)} u \leq M\, u(x_0, y_0, t_0), \tag{3.3}$$

where, for every $(x_0, y_0, t_0) \in (0, \infty) \times \mathbb{R}^2$, we have set

$$H_r(x_0, y_0, t_0) = \Big\{(x, y, t) \in \mathbb{R}^3 : \ |x - x_0| < r x_0, \quad -r^2 < t - t_0 \le 0,$$
$$|y - y_0 + x_0(t - t_0)| < r^3\Big\},$$
$$S_r(x_0, y_0, t_0) = \Big\{(x, y, t) \in H_r(x_0, y_0, t_0) : \ t < t_0 - r^2/2\Big\},$$

and the function u is defined in an open neighborhood of the set $H_r(x_0, y_0, t_0)$. We remark that the constant in the Harnack inequality (3.3) does not depend on the point (x_0, y_0, t_0).

The proof of the lower bound of the density p is based on the construction of the so-called *Harnack chains*. By this, we mean a finite sequence of points

$$(x_0, y_0, t_0), \dots, (x_k, y_k, t_k)$$

of the domain A of a non-negative function u, such that

$$u(x_j, y_j, t_j) \le M u(x_{j-1}, y_{j-1}, t_{j-1}), \quad \text{for} \quad j = 1, \dots, k.$$

Clearly, the *invariant* Harnack inequality (3.3) is a useful tool in the construction of Harnack chains. Once a Harnack chain is given, we immediately find that

$$u(x_k, y_k, t_k) \le M^k u(x_0, y_0, t_0).$$

As the Harnack constant is M^k, once a starting-point (x_0, y_0, t_0) and an end-point (x, y, t) are given, it is important to find a Harnack chain of minimal length k whose last point (x_k, y_k, t_k) agrees with (x, y, t).

Optimal Control Theory is used in the article [14] in order to find an optimal Harnack chain, and this optimal choice provides us with accurate asymptotic bounds of the density p. This method extends the one introduced by Aronson and Serrin in the study of uniformly parabolic operators. We refer to the papers [11] for a detailed description of this method, and to [13] for its application to stochastc processes.

4 The Proof of Lemma 1.7

We tacitly inherit all the notations in Sect. 1. We resume the proof of Lemma 1.7 starting from (1.11).

Given $x, y \in V$, we know that $\mathrm{Log}(x), \mathrm{Log}(y) \in \mathfrak{U}$ are well-defined in $\mathfrak{g}$, and we have the following computation, linking our local operation m with the BCH multiplication: by using (1.12), one can easily shrink V in such a way that, for any $x, y \in V$, one has

$$m(x, y) = m(x, \mathrm{Exp}(\mathrm{Log}(y))) \overset{(1.11)}{=} \Psi_1^{\mathrm{Log}(y)}(x) = \Psi_1^{\mathrm{Log}(y)}(\mathrm{Exp}(\mathrm{Log}(x)))$$

$$\stackrel{(1.9)}{=} \Psi_1^{\mathrm{Log}(y)}(\Psi_1^{\mathrm{Log}(x)}(0)) \stackrel{(1.12)}{=} \Psi_1^{Z(\mathrm{Log}(x),\mathrm{Log}(y))}(0),$$

which means, again by (1.9), that

$$m(x, y) = \mathrm{Exp}\Big(Z\big(\mathrm{Log}(x), \mathrm{Log}(y)\big)\Big), \quad \text{for every } x, y \in V. \tag{4.1}$$

This result and the associativity of the (local) operation $(X, Y) \mapsto Z(X, Y)$ (see e.g. [8, Sec. 5.3]) show that, by shrinking V, m is "locally associative", i.e.,

$$m(x, m(y, z)) = m(m(x, y), z), \quad \text{for every } x \in \mathbb{R}^N \quad \text{and every } y, z \in V. \tag{4.2}$$

Thus, $(x, y) \mapsto m(x, y)$ defines a local Lie group, with identity 0 and local inversion

$$x^{-1} = \mathrm{Exp}(-\mathrm{Log}(x)), \quad \forall\, x \in V,$$

such that any $X \in \mathfrak{g}$ is "locally left-invariant". By the last statement we mean, precisely, the following identity

$$X_{m(x,y)} = \frac{\partial m}{\partial y}(x, y) X_y, \qquad \forall\; X \in \mathfrak{g},\; x \in \mathbb{R}^N,\; y \in V. \tag{4.3}$$

The proof of the latter has already been sketched in the Introduction.

Our task here is to show that the local-Lie-group structure defined by m can be prolonged throughout $\mathbb{R}^N$. This is accomplished via a prolongation argument for ODEs which we now describe.

Let us fix a linear basis $\{J_1, \dots, J_N\}$ of $\mathfrak{g}$ as follows (in this choice, we use assumptions (H), (D)):

$$J(0) = \text{identity matrix}, \quad \text{and} \quad \det(J(x)) \neq 0 \quad \text{for every } x \in \mathbb{R}^N; \tag{4.4}$$

here we have denoted by $J(x)$ the matrix whose j-th column is the $N \times 1$ vector whose entries are the coefficients of the vector field J_j with respect to the coordinate partial derivatives.

Fixing $x, y \in \mathbb{R}^N$, we consider the curve

$$\gamma_{x,y}(t) := m(x, ty), \tag{4.5}$$

defined at least for $|t|$ small. As described in the Introduction, one can prove that $z(t) := \gamma_{x,y}(t)$ satisfies the following non-autonomous Cauchy problem

$$\dot{z}(t) = \sum_{k=1}^{N} a_k(t, y)\, J_k(z(t)), \quad z(0) = x, \tag{4.6}$$

where the vector fields J_k are as in (4.4) and the functions a_k are given by

$$\big(a_1(t,y),\dots,a_N(t,y)\big)^T = \big(J(ty)\big)^{-1}\cdot y. \tag{4.7}$$

Since the J_k's satisfy the invariant-type condition (4.3), we are entitled to apply the prolongability result for ODEs in [4, Th. 1.1], ensuring that $\gamma_{x,y}(t)$ exists for every $t\in\mathbb{R}$. We are then allowed to set

$$\Lambda:\mathbb{R}^N\times\mathbb{R}^N\longrightarrow\mathbb{R}^N,\qquad \Lambda(x,y):=\gamma_{x,y}(1). \tag{4.8}$$

It is clear that Λ is smooth and it prolongs m. All that remains to prove is that Λ defines on $\mathbb{R}^N$ a Lie group whose Lie algebra is $\mathfrak{g}$. We claim that both the associativity of $(x,y)\mapsto x*y:=\Lambda(x,y)$ and the left-invariance of any $X\in\mathfrak{g}$ derive from the following identity:

$$X_{\Lambda(x,y)} = \frac{\partial\Lambda}{\partial y}(x,y)X_y,\qquad \forall\ X\in\mathfrak{g},\ \ x,y\in\mathbb{R}^N, \tag{4.9}$$

which is a global version of (4.3). While the left-invariance of $X\in\mathfrak{g}$ is clearly a restatement of (4.9), the associativity of $*$ follows from the fact that (4.9) also implies that the curve $t\mapsto x*(y*(tz))$ satisfies the same Cauchy problem solved by $\gamma_{x*y,z}(t)$; thus, when $t=1$ one gets $x*(y*z))=(x*y)*z$.

Hence (4.9) is the core of the argument and we now prove it:

Proof of identity 4.9 Let the J_k's be as in (4.4). It is not difficult to check that $z(t):=\gamma_{x,y}(t)$ in (4.5) solves the Cauchy problem (4.6), compactly written (with the notation in (4.4)) as

$$\dot z(t)=J(z(t))J(ty)^{-1}y,\quad z(0)=x.$$

By the prolongability result for ODEs in [4, Th. 1.1], we know that $z(t)$ exists for any $t\in\mathbb{R}$; hence the map Λ in (4.8) is well posed and smooth. As the two curves $s\mapsto\gamma_{x,ty}(s),\gamma_{x,y}(ts)$ solve the same Cauchy problem, we have $\gamma_{x,y}(t)=\Lambda(x,ty)$ for any t. From the latter, we easily get

$$\frac{\mathrm{d}}{\mathrm{d}t}\{\Lambda(x,ty)\} = J(\Lambda(x,ty))J(ty)^{-1}y\qquad (\text{for any } x,y\in\mathbb{R}^N). \tag{4.10}$$

After differentiation the sides of (4.10) with respect to y, we get the matrix ODE

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big\{t\,\frac{\partial\Lambda}{\partial y}(x,ty)\Big\} = \frac{\partial}{\partial y}\Big\{J(\Lambda(x,ty))\,J(ty)^{-1}\,y\Big\}. \tag{4.11}$$

Next, we have the following very technical fact: let us consider the structure constants of $\mathfrak{g}$ with respect to the basis $\{J_1,\dots,J_N\}$, that is $\{C^k_{i,j}\}_{i,j,k\le N}$ satisfy

$$[J_i,J_j]=\textstyle\sum_{k=1}^N C^k_{i,j}\,J_k,\quad \forall\ i,j\in\{1,\dots,N\}. \tag{4.12}$$

For any fixed $j \in \{1, \dots, N\}$, we consider the matrix $C(j) := (C^k_{i,j})_{k,i \le N}$; then one has (for every $z \in \mathbb{R}^N$)

$$C(j)\, J(z)^{-1} z = \\ J(z)^{-1} \frac{\partial J_j}{\partial z}(z)\, z - J(z)^{-1} \left(\frac{\partial J_1}{\partial z}(z)\, J_j(z) \cdots \frac{\partial J_N}{\partial z}(z)\, J_j(z) \right) J(z)^{-1} z. \tag{4.13}$$

Indeed (4.13) follows by re-writing (4.12) under its obvious matrix form (then by multiplication times $J(z)^{-1}$ on the left, and $J(z)^{-1}z$ on the right). We claim that

$$\frac{\partial \Lambda}{\partial y}(x, y) = J(\Lambda(x, y))\, J(y)^{-1}, \qquad \forall\, x, y \in \mathbb{R}^N. \tag{4.14}$$

By right multiplication of (4.14) times $J(y)$, we get the desired (4.9) with X replaced by $J_1, \dots, J_N$; then (4.9) will follow by linearity. Thus, all that we have to prove is (4.14). To this end, we fix $x, y \in \mathbb{R}^N$, and we prove that

$$t \frac{\partial \Lambda}{\partial y}(x, ty) = t J(\Lambda(x, ty))\, J(ty)^{-1}, \qquad \forall\, t \in \mathbb{R}. \tag{4.15}$$

We denote by $A(t)$ and $B(t)$, respectively, the left-hand and the right-hand sides of (4.15). If we show that $A(t) = B(t)$ for any t, then (4.14) will follow by taking $t = 1$. Considering that $A(0) = 0 = B(0)$, we show that A and B solve the same (matrix linear) ODE. By (4.11), we see that A solves

$$A'(t) = J(\Lambda(x, ty))\, \frac{\partial}{\partial y}\Big(J(ty)^{-1}\, y \Big) + \sum_{k=1}^{N} (J(ty)^{-1}\, y)_k \frac{\partial J_k}{\partial y}(\Lambda(x, ty))\, A(t). \tag{4.16}$$

We finally claim that $B(t)$ solves the same ODE (4.16): indeed, if one inserts $B(t)$ in place of $A(t)$ in (4.16), after a tedious computation, one discovers that the claimed needed identity is equivalent to the technical identity in (4.13). This ends the proof of (4.9). □

Finally, we sketch a short argument for the existence of the neutral element and of the group inversion.

As regards the former issue we observe that, since $\mathrm{Log}(0) = 0 \in \mathfrak{g}$, we have

$$\Lambda(x, 0) = x \quad \text{for every } x \in \mathbb{R}^N;$$

on the other hand, if $y \in \mathbb{R}^N$ is arbitrarily fixed, from the very definition of Λ (see (4.8)) we have $\Lambda(0, y) = \gamma_{0,y}(1)$, where $t \mapsto \gamma_{0,y}(t)$ is the unique solution (defined on $\mathbb{R}$) of the Cauchy problem

$$\dot z(t) = \sum_{k=1}^{N} a_k(t, y)\, J_k(z(t)), \quad z(0) = 0.$$

Since the curve $z(t) = t\,y$ solves this problem (as it can be directly checked by using (4.7)), we conclude that $\gamma_{0,y} \equiv z$ on $\mathbb{R}$, whence

$$\Lambda(0, y) = z(1) = y.$$

Now that we have proved that 0 is the neutral element of Λ, we turn to demonstrate that every fixed $x \in \mathbb{R}^N$ possesses an inverse with respect to Λ.

First of all, since the map $\iota : V \to \mathbb{R}^N$ defined by

$$\iota(y) := \mathrm{Exp}(-\mathrm{Log}(y))$$

provides a local inversion with respect to m, there exists an open and *convex* neighborhood $W \subseteq V$ of 0 such that $\iota(W) \subseteq V$ and (see also [4, Lemma 3.7])

$$m(w, \iota(w)) = m(\iota(w), w) = 0 \qquad \text{for every } w \in W. \tag{4.17}$$

Moreover, by exploiting (4.4) and arguing exactly as in the proof of [4, Corollary 3.10], we recognize that the map

$$\tau_\alpha : \mathbb{R}^N \longrightarrow \mathbb{R}^N, \qquad \tau_\alpha(y) := \Lambda(\alpha, y) = \alpha * y,$$

is a local diffeomorphism in $\mathbb{R}^N$, hence is open (for every fixed $\alpha \in \mathbb{R}^N$). From this, by using the associativity of $*$, we can write

$$\mathbb{R}^N = \bigcup_{n \in \mathbb{N}} \Big\{ w_1 * \cdots * w_n \mid w_1, \ldots, w_n \in W \Big\}. \tag{4.18}$$

Now, by combining (4.17) with the fact that $\Lambda \equiv m$ on $\mathbb{R}^N \times W$ (since W is convex), it can be deduced from (4.18) that both the equations

$$x * y = 0 \qquad \text{and} \qquad y * x = 0,$$

have the same unique solution $y = \mathrm{i}(x)$: indeed, if we write (by (4.18))

$$x = w_1 * \cdots * w_n,$$

for suitable $w_1, \ldots, w_n \in W$ (not necessarily unique), we have

$$\mathrm{i}(x) = \iota(w_n) * \cdots * \iota(w_1).$$

As a consequence, the map i does not actually depend on $w_1, \ldots, w_n$ and it defines a global inversion map for $*$. Furthermore, by the Implicit Function Theorem, it readily follows that i is smooth on $\mathbb{R}^N$.

Summing up, $\mathbb{G} = (\mathbb{R}^N, *)$ is a Lie group on $\mathbb{R}^N$, with neutral element $e = 0$ and inversion map given by i; moreover, by (4.9), $\mathrm{Lie}(\mathbb{G}) = \mathfrak{g}$.

5 Appendix: A BCH Theorem for ODEs

We give the proof of Theorem 1.6, of which we tacitly inherit the notation.

Proof of Theorem 1.6 The existence of ε can be obtained as in Dynkin's classical result [15], since $\mathfrak{g}$ is finite-dimensional by (D) (hence equipped with some norm turning it into a Banach algebra).

Let now $X, Y \in \mathfrak{g}$ satisfy $\|X\|, \|Y\| < \varepsilon$; we also fix any $x \in \mathbb{R}^N$. For $t \in [0, 1]$ we consider the functions

$$F(t) := \exp(tY)(\exp(X)(x)), \quad G(t) := \exp(Z(X, tY))(x).$$

They are well-posed due to assumption (C) (as $Z(X, tY)$ converges and is an element of $\mathfrak{g}$); moreover they are smooth. We claim that $F \equiv G$ on $[0, 1]$; the identity $F(1) = G(1)$ gives the desired (1.12).

Hence we are left with the claimed equality of F and G on $[0, 1]$: since $F(0) = G(0) = \exp(X)(x)$, by uniqueness results for Cauchy problems, all that remains to prove is that F and G satisfy the same ODE. By the very definition of an integral curve, one has $F'(t) = Y_{F(t)}$, for any $t \in [0, 1]$. Thus we aim to prove that

$$G'(t) = Y_{G(t)}, \quad t \in [0, 1]. \tag{5.1}$$

This is less trivial to prove. We argue as follows. Let us fix any linear basis $J_1, \dots, J_N$ of $\mathfrak{g}$; for $\xi \in \mathbb{R}^N$ we use the notation $\xi \cdot J$ for $\sum_{j=1}^N \xi_j J_j$. Then there exists a smooth $\mathbb{R}^N$-valued map $\xi(t)$ such that $Z(X, tY) = \xi(t) \cdot J$ on $[0, 1]$. Since $\mathfrak{g}$ is finite dimensional, it is simple to calculate the differential of the function $\xi \mapsto \Psi_t^{\xi \cdot J}(x)$. In the sequel we denote by $\mathrm{d}_x f$ the differential of a smooth map f at the point x. We have the following computation

$$G'(t) = \mathrm{d}_x\big(\Psi_1^{\xi(t)\cdot J}\big) \left[\frac{e^{\mathrm{ad}\,(\xi(t)\cdot J)} - 1}{\mathrm{ad}\,(\xi(t) \cdot J)} \Big(\sum_{j=1}^N \xi_j'(t) J_j \Big) \right]_x . \tag{5.2}$$

On the other hand, by the results in [3, Th. 3.1] we have

$$\sum_{j=1}^N \xi_j'(t) J_j = \frac{\mathrm{d}}{\mathrm{d}t} Z(X, tY) = \frac{-\mathrm{ad}\, Z(X, tY)}{e^{-\mathrm{ad}\,(Z(X,tY))} - 1}\, Y.$$

Observe the use of an analog of Poincaré's identity (1.3). Inserting this in (5.2), and since $Z(X, tY) = \xi(t) \cdot J$, we get

$$\begin{aligned} G'(t) &= \mathrm{d}_x\big(\Psi_1^{\xi(t)\cdot J}\big)\Big[e^{\mathrm{ad}\,(\xi(t)\cdot J)}Y\Big]_x = \mathrm{d}_x\Psi_1^{\xi(t)\cdot J}\Big[\mathrm{d}\Psi_{-1}^{\xi(t)\cdot J}Y\Big]_x \\ &= \mathrm{d}_x\big(\Psi_1^{\xi(t)\cdot J}\big)\Big(\mathrm{d}_{\Psi_1^{\xi(t)\cdot J}(x)}\Psi_{-1}^{\xi(t)\cdot J}\Big(Y_{\Psi_1^{\xi(t)\cdot J}(x)}\Big)\Big) \\ &= \mathrm{d}_{\Psi_1^{\xi(t)\cdot J}(x)}\Big(\Psi_1^{\xi(t)\cdot J}\circ\Psi_{-1}^{\xi(t)\cdot J}\Big)\Big(Y_{\Psi_1^{\xi(t)\cdot J}(x)}\Big) = Y_{\Psi_1^{\xi(t)\cdot J}(x)}. \end{aligned} \tag{5.3}$$

In the second equality we used the identity $\mathrm{d}\Psi_{-t}^X Y = e^{\mathrm{ad}\,(tX)}(Y)$ (valid for any t and any $X, Y \in \mathfrak{g}$), another simple consequence of assumptions (C), (D). Since $\xi(t) \cdot J = Z(X, tY)$, we have $\Psi_1^{\xi(t)\cdot J}(x) = G(t)$, so that (5.3) gives $G'(t) = Y_{G(t)}$ and the proof of (5.1) is complete. □

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