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Abstract. This paper shows that the Internal-Rate-of-Return (IRR) approach is unreliable, and that the recently introduced Average-Internal-Rate-of-Return (AIRR) model constitutes the basis for an alternative theoretical paradigm of rate of return. To this end, we divide the paper into two parts: a *pars destruens* and a *pars construens*. In the "destructive" part, we present a compendium of eighteen flaws associated with the IRR approach. In the "constructive" part, we construct the alternative approach from four (independent) economic intuitions and put the paradigm to the test by showing that it does not suffer from any of the flaws previously investigated. We also show how the IRR, as a rate of return, is absorbed into the new approach.

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1. Introduction

The determination of a project's economic profitability is a matter of central importance in economics, finance and accounting for both ex ante decision-making and ex post evaluation. Generally speaking, there are two camps, one of which favors Internal Rate of Return (IRR) and one which favors Net Present Value (NPV). Some serious weaknesses of IRR have been debated in academia long since; corporate finance textbooks (e.g. Brealey et al. 2011; Ross et al. 2011) and some engineering economy textbooks (e.g. Hartman 2007; Blank and Tarquin 2012) warn against its use, endorsing the Net Present Value (NPV) as a correct decision criterion; yet, despite some evidence of an increasing use of NPV, a substantial majority of practitioners still favor IRR (Remer et al. 1993; Burns and Walker 1997; Graham and Harvey 2001; Ryan and Ryan 2002). One of the reasons seems to lie with the greater intuitive appeal of a relative (i.e. percentage) measure as opposed to an absolute (i.e. money) measure (Evans and Forbes, 1993; Yung and Sherman 1995). Perhaps, given that a relative measure (cost of capital) is necessary for assessing wealth creation, a comparison between two relative measures (IRR and cost of capital) seems more intuitive to many than the computation of an absolute amount (NPV) which depends on a relative measure (cost of capital). No doubt, IRR is a fascinating metric and, as such, the academic literature on the IRR never ceases to produce new contributions and some important results have been found in recent years, especially dealing with multiple IRRs and complex-valued IRRs (Hazen 2003; Hartman and Schafrick 2004; Pierru 2010; Osborne 2010).

This paper is divided into two parts: a *pars destruens* and a *pars construens*. In the first part, we discuss eighteen flaws associated with the IRR approach, which leads to a refutation of the IRR approach as a general approach to rate of return. In the second part, we build upon Magni (2010) to constructively show that the Average-Internal-Rate-of-Return (AIRR) model is the basis of an economically significant paradigm of rate of return, which is intuitive and computationally simple, and it is not subject to the flaws previously investigated.

The remainder of the paper is structured as follows. In section 2 we present some notational conventions and definitions. In section 3, the *pars destruens*, we discuss the eighteen flaws of the IRR (most of them new in the literature) and, for each of them, we provide at least one illustrative example. This section naturally leads to a refutation of the IRR model as a general approach to rate of return. The remaining sections make up the *pars construens*. In particular, section 4 presents the alternative theory in a simple way, showing that it can be derived from any of four economic intuitions; in this section, we also introduce the "economic AIRR", a rate of return directly derived from market valuation theory. In section 5, we put the new theory to the test and show how it overcomes each and every fallacy. Section 6 underscores the connections between the IRR and the AIRR approach, showing when and how IRR can be used as an appropriate rate of return. Some concluding remarks end the paper.

2. Notational conventions and definitions

In general, let $\boldsymbol{f} = (f_0, f_1, f_2, \dots, f_n)$ with $f_t \in \mathbb{R}$ be the cash-flow stream of a project P, and let r be the cost of capital (COC), also known as the required rate of return or the minimum attractive rate of return. The COC should reflect the expected rate of return of an equivalent-risk asset traded in the market. The project's net present value is $NPV = \sum_{t=0}^{n} f_t\, v^t$, $v \coloneqq (1+r)^{-1}$.

We will use the terms "return", "interest" and "income" as synonyms. The relation linking interest, capital and cash flow is a most important one in economics and finance:

$$f_t - I_t = c_{t-1} - c_t \tag{1a}$$

where I_t is the return and c_t denotes capital, $t = 1, 2 \dots, n$, with $c_0 \coloneqq -f_0$ being the initial investment. We assume, as usual, that cash flows and returns are generated at the end of the period. Equation ($1a$), which we call the "fundamental economic relation", can be framed, recursively, as

$$c_t = c_{t-1} + I_t - f_t. \tag{1b}$$

Essentially, the end-of-period capital value is equal to the beginning-of-period capital c_{t-1}, increased by the return I_t and decreased (increased) by the cash flow generated by (injected in) the project. If $c_t \neq 0$ for every $t < n$, the project's period rates of return i_t are well defined:

$$i_t = \frac{I_t}{c_{t-1}} = \frac{f_t + c_t - c_{t-1}}{c_{t-1}}, \qquad t = 1,2, \dots, n \tag{2}$$

so that ($1b$) becomes

$$c_t = c_{t-1}(1 + i_t) - f_t. \tag{3}$$

A project is economically profitable (and, therefore, acceptable) if wealth is created for the investors. Wealth creation occurs if and only if $NPV > 0$. Among a bundle of competing projects, the one with the highest NPV will be preferred, for NPV maximization means wealth maximization.

We will also make use of the net present value of the capital stream $\boldsymbol{c} = (c_0, c_1, \dots, c_{n-1})$:

$$C \coloneqq \sum_{t=1}^{n} c_{t-1} v^{t-1} \tag{4}$$

Eq. (4) represents the overall capital invested in the project.

An IRR is a solution of the equation $\sum_{t=0}^{n} f_t \, (1 + k)^{-t} = 0$. We will call the latter the 'IRR equation'. (Abusing notation, k will denote either the unknown of the equation or a solution, depending on the context.) The recursively defined amount

$$c_t(k) = c_{t-1}(k)(1 + k) - f_t \tag{5}$$

is here called the "internal value". One can also define the internal value prospectively as $c_t(k) = \sum_{h=t+1}^{n} f_h (1 + k)^{t-k}$. The internal value is also known as "Hotelling" value in the accounting literature, though Hotelling (1925) used the cost of capital to discount cash flows, not the IRR: "It might be more appropriate to refer to Hotelling depreciation as Preinreich depreciation in recognition of the fact that it appears to be Preinreich (1938) who suggested that the present value of the future cash flows should, in general, be calculated using the rate of profit [=IRR], rather than the cost of capital as in Hotelling (1925)." (Stark 1989, p. 75, endnote 1). We denote as $\boldsymbol{c}(k) = (c_0, c_1(k), \dots, c_{n-1}(k))$ the vector of internal capitals. Note that (5) is only a particular case of (3): it represents the interim capital invested in the project *under the* *assumption that capital increases, within each period, at a constant growth rate equal to* k. Likewise, the amount $C(k) := \sum_{t=1}^{n} c_{t-1}(k) \cdot v^{t-1}$ is a particular case of (4).

According to the IRR criterion, a project is economically profitable if $k > r$ and, among competing projects, the one with the greatest IRR should be preferred (IRR maximization).

If cost of capital is variable across periods, the NPV becomes $NPV = \sum_{t=0}^{n} f_t \, v^{t,0}$, where $v^{t,0} := \prod_{k=1}^{t} (1 + r_k)^{-1}$, $v^{0,0} := 1$ and r_k is the market (forward) rate holding in the k-th period (between time $k - 1$ and time k). Throughout the paper, we assume that the cost of capital (either constant or variable) is fixed exogenously.

Profitability index (PI) is the ratio of NPV to initial investment:

$$PI = \frac{NPV}{c_0}. \tag{6}$$

It gives expression to wealth increase for each unit of initial investment. It is then a "bang for the buck" measure. The benefit-cost ratio (BC) is the ratio of the discounted value of total inflow $F^{+} = \sum_{t:f_t>0} f_t \, v^t$ to the discounted value of total outflow $F^{-} = -\sum_{t:f_t<0} f_t \, v^t$ (henceforth, total investment cash flow):

$$BC = \frac{F^{+}}{F^{-}}. \tag{7}$$

It provides a measure of efficiency of total invested cash flow. Benefit-cost ratio and profitability index are sometimes exchanged for one another or considered synonyms; some other times, slightly different definitions are given. For example, Blank and Tarquin (2012) define the PI as $PI = \frac{\sum_{t=1}^{n} f_t v^t}{c_0} = 1 + \frac{NPV}{c_0}$; Kellison (2009) uses the term "profitability index" for both eq. (6) and eq. (7); Ross et al. (2011) use both expressions for eq. (6), as well as Hartman (2007), but the latter reserves the term "benefit-cost ratio" to investment made by government entities. Rao (1992) calls eq. (7) "profitability index". Park (2011) defines profitability index as the ratio of benefits net of operating costs to capital expenditure (see also Newnan et al., 2009, Brealey et al. 2011). We explicitly distinguish the two indexes, as they convey different pieces of information.

Excess return (ER), also known as residual income, is defined as the difference between the actual return in a period and the return that would accrue if the investor invested the beginning-of-period capital at the return rate r_t. It therefore measures the wealth increase in a given period:

$$ER_t(c_{t-1}, i_t) = c_{t-1} \cdot (i_t - r_t). \tag{8}$$

3. *Pars destruens* — Refutation of the IRR approach

Refutation of the IRR approach is based on the following eighteen fallacies (labeled F1-F18).

F1 — Multiple rates of return

A project may have multiple real-valued IRRs. While multiple IRRs seem to be uncommon (see Ben-Horin and Kroll 2012 for some results), engineering projects may have a considerable length and several changes in sign may occur in the cash-flow stream. This occurs, for example, in investments with disposal/remediation costs, phased expansion, natural-resource extraction (see Hartman 2007). The multiple-IRR problem may also be encountered when ex-post economic performance is assessed for an ongoing activity (such as a firm or a business unit) in a given interval of time, if dividends and new investments alternate. A third case is given by investment funds, where the investor's choices about deposits and withdrawals can determine several changes in sign. Further, multiple IRRs can easily arise even in the most regular circumstances, when a levered project is studied or a portfolio of investments and borrowings is considered.

Regardless of the frequency of multiple IRRs, the issue is relevant, for they signal a theoretical anomaly of the IRR approach. At first sight, none of the IRRs can be given a clear role. However, after scholars painstakingly strived for decades to find a solution to this conundrum, Hazen (2003) shed an important light on the issue, showing that any IRR can be used for assessing value creation: NPV can be framed as

$$NPV = C(k) \cdot \frac{(k-r)}{1+r} \tag{9}$$

so the project is interpretable as an overall investment of $C(k)$ dollars (or an overall borrowing of the same amount if the discounted sum is negative) at the excess rate of return $k-r$. This implies that a project is acceptable if and only if

$$k > r \quad iff \quad C(k) > 0 \tag{10}$$

(see Hazen 2003, Theorem 4). Eq. (10) can be applied to any one of m multiple IRRs, the decision being the same.

This solution sweeps away the old (technical) multiple-IRR problem, but raises a new (economic) multiple-IRR problem: if, technically, any one out of the m IRRs can be used for assessing value creation, which one of them is, economically, the "correct" rate of return? Which IRR should be used for ranking projects or for ranking managers' performances? Which one should be used by a regulator to set prices and tariffs? Which one should be used by a policy maker to impose a tax on returns? Which one for compensation plans based on rates of return? (See Carey 2012, on this problem.) And which one should be used for determining the amount of capital invested in the project? In all such cases, the use of a rate of return as opposed to another one may change the analysis and the related decisions, so one may not rely on *any* IRR. Put differently: the cardinal value as well as the ordinal value of a rate of return is at stake, so the determination of the correct capital is essential.

Example 1. Consider $\boldsymbol{f} = (-100, 390, -503, 214.5)$ with a COC equal to $r = 14\%$. The IRRs are $k(1) = 10\%, k(2) = 30\%, k(3) = 50\%$ (see Park 2011, p. 350) and the NPV is negative ($NPV = -0.156$) so the project should be rejected. Any IRR can be used for accept/reject decision: for example, $k(1) = 10\%$ is a rate of return on $C(k(1)) = 4.43$ (investment), so the project is not acceptable since $10\% < 14\%$. Conversely, 30% is a rate of cost on $C(k(2)) = -1.11$ (borrowing), which leads to the same decision conclusion, owing to (9) (analogously with 50%, which is associated with $C(k(3)) = -0.49$). But eq. (10) gives no clue on which one IRR should be used by an evaluator to rank $\boldsymbol{f}$ among other projects, or which IRR should be used to reward a manager, or which IRR a regulator should use to impose a tax. Finally, consider that the given cash-flow vector can be the result of a very conventional situation. Suppose the investor invests in a project whose initial outlay is $-1300 and subsequent cash flows are equal to \$1000, \$107, \$214.5 at time 1, 2, 3 respectively. To undertake the project, the investor borrows \$1200, repayable with two level installments equal to \$610. The resulting equity cash-flow vector is just $\boldsymbol{f}$ (and there are infinite such levered projects which result in $\boldsymbol{f}$).

F2 — No rate of return

As widely known, the IRR may not exist if a project ends with an outlay. In this case, the evaluator has no information to use. Hazen (2003) proposed to heal the flaw by taking account of the real parts of the complex-valued interim capitals. However, although complex-valued rates may be given economic significance (Pierru 2010), the appeal of real-valued rates in real-life applications is overwhelming, suggesting that there is a strong need of an approach which always provides reliable *real-valued* rates of return. As low as the frequency of no-IRR projects can be in real life, the consequences are devastating when such cases occur. Also, a theoretical paradigm which does not guarantee existence of a real-valued rate of return should be treated with suspicion. (Osborne 2010 interestingly shows that the absolute value of the NPV per unit of dollar is the product of the absolute values of all the roots, real-valued and complex-valued, of the polynomial).

Example 2. Consider a transaction whereby a person makes a payments of \$10 immediately and \$25 at the end of two years, in exchange for a payment of \$30 at the end of one year. The cash-flow vector is $\boldsymbol{f} = (-10, 30, -25)$. A real-valued IRR does not exist and the complex-valued solutions $0.5 \pm 0.5i$ are unappealing to managers and practitioners.

Note that this problem is typical of common situations such as performance dynamic analysis. Consider an investment of c_0 dollars in an asset and consider a dynamic analysis such that, at every date t, the investor measures performance in the interval $[0, t]$. If $f_\tau = 0$ for $\tau = 1,2, \dots, t$, the IRR is does not exist, for the cash-flow stream is $(f_0, 0, \dots, 0)$. (The evaluator can try to overcome the problem by selecting a terminal value for the project at time t, but this does not prevent him from incurring the fallacy of intertemporal inconsistency. See F14).

F3 — Varying costs of capital

If the COC varies across periods (e.g. the term structure of interest rates is not flat), the IRR rule breaks down, for there is no clear way of comparing a project's rate of return with a sequence $(r_1, r_2, \dots, r_n)$ of costs of capital. This situation can be encountered in several circumstances. For example, a notable one is the case of investment portfolio management: the cost of capital is usually given by the rate of return of a benchmark fund, which is variable over time.

Example 3. An investor deposits \$10,000 in a fund at time 0, then withdraws \$2,000 after one period, injects an additional \$3,000 at time 2 and liquidates the investment at time 3. The fund's beginning-of-period market values (after due consideration for withdrawals and deposits) are \$6,000 at time 1, \$13,000 at time 2. The market value of the investment at time 3 is \$18,000. This implies that the investment's cash flows vector is $\boldsymbol{f} = (-10{,}000; 2{,}000; -3{,}000; 18{,}000)$. To measure wealth creation, suppose that the fund's performance is set against a benchmark whose period rates of return are $r_1 = 22\%, r_2 = 8\%, r_3 = 15\%$. The fund's performance (in thousands) is

$$NPV = -10{,}000 + \frac{2{,}000}{1.22} + \frac{-3{,}000}{(1.22)(1.08)} + \frac{18{,}000}{(1.22)(1.08)(1.15)} = 1{,}241.78$$

which suggests a profitable investment. The unique IRR is $k = 20\%$ and is not clearly comparable to any of the COCs to signal wealth creation.

F4 — Arbitrage strategies

An arbitrage strategy is formally represented as a sequence of cash flows $f = (f_0, f_1, \dots, f_n)$ such that $f_t \geq 0$ for all t and there exists some t such that $f_t > 0$. The IRR is not capable of measuring the rate of return of an arbitrage strategy, for a necessary condition for IRR to exist is that there are at least two cash flows different in sign (we assume $k > -1$).

Example 4. A bank is able to borrow \$10,000 for one year at 5% and lend \$9,375 to a client at 12%. The vector of net cash flows is $\boldsymbol{f} = (625{,}0)$. The IRR of this arbitrage strategy cannot be computed, since $625 + 0/(1+k) \neq 0$ for every $k \in \mathbb{R}$.

F5 — Mutually exclusive projects and project ranking

The IRR criterion ignores investment scale. No rational investor would prefer an investment of 10 dollars at 100% to an investment of 10,000 dollars at 10% if the appropriate cost of capital is 5%. Yet, according to the IRR criterion, the former is the preferable one. The reason why the approach fails is that any rate of return, taken in isolation, is uninformative about wealth creation. As Hazen (2003, p. 46) puts it, "The magnitude of the internal rate by itself carries no further information … So the fact that one internal rate has greater magnitude than another … indicates that the corresponding net investment [=invested capital] has smaller magnitude. There are no other economic implications". In order to provide information about wealth created, a standardized rate of return allowing for scale is needed. But the IRR approach cannot provide such an index.

Consider a bundle of $p \geq 2$ projects, whose cash-flow vectors are denoted as $\boldsymbol{f^j} = \left(f_0^j, f_1^j, \dots, f_{n_j}^j\right)$, $j = 1,2,\dots,p$. If every project has a unique IRR equal to k^j, the IRR ranking is not consistent, in general, with the NPV ranking; the reason is just that the various IRRs refer to different investment scales, so to compare them is like comparing apples and oranges. On the other hand, if some project has multiple IRRs, it is not clear which one among the multiple IRRs should be used for ranking purposes or for choosing between mutually exclusive projects (choice between mutually exclusive alternatives is equivalent to the ranking of $p = 2$ projects).

The use of incremental IRR is sometimes advocated in these cases (e.g., Blank and Tarquin 2012), but incremental IRR may not exist, in which case, the choice between two alternatives cannot be accomplished (multiple incremental IRRs may exist as well, but Hazen's rule, eq. (10), can be used in such cases). Also, in favorable cases, the procedure is tedious and time-consuming if the number of projects is high (the maximum number of pairwise comparisons in a bundle of m projects is $m(m-1)/2$). Furthermore, even when ranking is possible and m is low, this method gives no information about the magnitude of the relative wealth created by one project with respect to the other ones.

Example 5. Consider $\boldsymbol{f^1} = (-1000, -1500, 3070)$, $\boldsymbol{f^2} = (-900, -300, 600{,}1140)$, $\boldsymbol{f^3} = (-800, 300, -100, 1000)$, $\boldsymbol{f^4} = (-1200, 1397)$ and $r = 5\%$ so that $NPV_1 = 356$, $NPV_2 = 343.3$, $NPV_3 = 258.8$, $NPV_4 = 130.5$. The NPV ranking is $1 \succ 2 \succ 3 \succ 4$. The IRRs are, respectively,

$k^1 = 15.59\%$, $k^2 = 16.73\%$, $k^3 = 17.46\%$, $k^4 = 16.42\%$, so the IRR ranking is incorrect: $3 \succ 2 \succ 4 \succ 1$. The aggregate investments in the four projects are, respectively, $C(k^1) = 3529.4$, $C(k^2) = 3072.1$, $C(k^3) = 2181.4$, $C(k^4) = 1200$ and they are the capital bases to which the IRRs are applied.

F6 — Rate of return on initial capital (or total investment cash flow)

The IRR does not measure a return per unit of initial investment; nor does it inform about the return per unit of total outflow. The IRR measures the rate of return on the overall capital $C(k)$ (see eq. (9)). From this point of view, the profitability index (PI) and the benefit-cost ratio (BC) do a better job, in a rather explicit way (see eq. (6) and (7)).

Example 6. Consider $\boldsymbol{f} = (-100,\ 80, -40,\ 90, 30, -120, 70, 90)$ with $r = 10\%$. An IRR is 22.4%, which is the rate of return on an overall investment of $C = 344.3$; it is not a rate of return on the initial investment of c_0 =100 nor on the total investment cash flow of F^- =207.6. (The profitability index is $PI = 0.39$, the benefit-cost ratio is $BC = 1.19$).

F7 — Framing effects: present value vs. future value

A rational decision maker abides by the principle of description invariance, according to which evaluation of a given situation and the related decision do not change if the same situation is presented in a different (but logically equivalent) format. Violations of description invariance are known as “framing effects” (Tversky and Kahneman 1981; Kahneman and Tversky 1984).

The IRR equation is a present-value equation, for all elements of the equation are referred to time 0. However, one expects that the result should not change if the elements are referred to any future date $T > 0$. Multiplying both sides of $\sum_{t=0}^{n} f_t\,(1+k)^{-t} = 0$ by $(1+k)^T$ one gets the future-value equation $\sum_{t=0}^{n} f_t\,(1+k)^{T-t} = 0$; if $T > n$, the solution $k = -1$ holds irrespective of the cash flows, which absurdly signals the loss of 100% of the invested capital for *any* project. Further, if $f_0 < 0$ and $f_t = 0$ for $t = 1, 2, \ldots, n$ (i.e., the investor loses the entire capital initially invested) the present-format equation becomes $f_0 = 0$, which has no solution. As a result, IRR either *never* or *always* signals a loss of the entire capital, depending on how the equation is framed.

Example 7. Consider the four-year project $\boldsymbol{f} = (-200,\ 0,\ 0, 266.2, 0)$. In time-0 terms, the IRR is 10%, but, in time-4 terms, the IRR is -100%. If, instead, $\boldsymbol{f} = (-200,\ 0,\ 0, 0, 0)$, the IRR will either signal or not signal the loss of the invested capital, depending on whether the future-value- or the present-value-format is used.

F8 — Framing effects: expected value of stochastic IRR vs. IRR of expected investment

Consider an investment with stochastic cash flow-stream $\tilde{f} = (f_0, \tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_n)$ $\tilde{f}_t$, $t = 1, 2, \ldots, n$; let $E(\tilde{f}) = (f_0, E(\tilde{f}_1), E(\tilde{f}_2), \ldots, E(\tilde{f}_n))$ be the expected investment and denote its IRR as $k(E(\tilde{f}))$. Consider now $k(\tilde{f})$, the stochastic IRR of $\tilde{f}$, and let $E(k(\tilde{f}))$ be its expected value. It is easy to see that, in general,

$$k\left(E(\tilde{f})\right) \neq E\left(k(\tilde{f})\right).$$

Which is the right order to compute the investment's IRR? Should one compute expectations of cash flows and then compute the IRR of the expected investment or should one compute the IRR for each possible cash-flow stream and compute the expected value of the stochastic IRR? The different framings lead to different IRRs.

Example 8. Table 1 collects the stochastic cash flows of a project, assuming three states of the world (optimistic, base, pessimistic), and presents the resulting IRRs and the expected cash flows. We have $E(\tilde{f}) = (-100, 52, 49, 55)$ so that $k(E(\tilde{f})) = 25.75\%$, whereas $k(\tilde{f}) = (59.5\%,\ 23.4\%,\ -70\%)$ so that $E\left(k(\tilde{f})\right) = 0.3(59.5\%) + 0.5(23.4\%) + 0.2(-70\%) = 15.5\%$.

Table 1. IRR with stochastic cash flows

Time	0	1	2	3	Prob	IRR
Case						
Optimistic	−100	70	80	100	0.3	59.5%
Base	−100	50	50	50	0.5	23.4%
Pessimistic	−100	30	0	0	0.2	−70.0%
Expected value	−100	52	49	55		25.75%

However, it is not clear which one is the "correct" project's IRR: 25.75% or 15.5%?

In the above case, assuming $r = 1\%$, the NPV is 52.9, and both IRRs exceed the COC, but it may well occur that the COC lies between the two values. For example, assume that the cash-flow stream is $(-100, 80, 120, 150)$ in the optimistic scenario, $(-100,\ 80, 50, 0)$ in the base scenario, and $(-100, 60,\ 0, 0)$ in the pessimistic scenario. We have

$$E\left(k(\tilde{f})\right) = -9.8\% < 1\% < k\left(E(\tilde{f})\right) = 12.1\%,$$

so the accept-reject decision itself is impaired.

F9 — Framing effects: value additivity

One of the most important ideas in finance is *value additivity* (Brealey et al., 2011, ch. 34, give a list of the seven most important ideas in finance: value additivity is the fourth one of the list). If a portfolio consists of a bundle of assets whose values amount to \$100, then the portfolio's value must be \$100; in general, value additivity means that "[t]he value of the whole is equal to the sum of the values of the parts." (Brealey et al. 2011, p. 901).

Value additivity is not preserved by the IRR approach. Consider p projects labeled $1,2,\dots,p$ with cash-flow vectors $\boldsymbol{f^j} = \left(f_0^j, f_1^j, \dots, f_{n_j}^j\right), j = 1,2,\dots,p$ and consider a portfolio $J = \sum_{j=1}^{p} j$ of those p projects. By additivity, the resulting cash-flow vector is

$$\boldsymbol{f^J} := \sum_{j=1}^{p} \boldsymbol{f^j} = (f_0^J, f_1^J, \dots, f_N^J)$$

where $f_t^J := \sum_{j=1}^{p} f_t^j$, $t = 0,1,\dots,N$ and $N = \max[n_1, n_2, \dots, n_p]$. Denoting as k^j the (assumed unique) IRR of project j and as k^J the (assumed unique) IRR of the portfolio, the internal value of project j is $c_t^j(k^j) = c_{t-1}^j(k^j)(1+k^j) - f_t^j$, while the internal value of the portfolio is $c_t^J(k^J) = c_{t-1}^J(k^J)(1+k^J) - f_t^J$. However, the sum of the projects' internal values at time t is $\sum_{j=1}^p c_t^j(k^j)$. It is easy to see that, in general, $c_t^J(k^J) \neq \sum_{j=1}^p c_t^j(k^j)$. This implies that, if one makes use of the IRR notion, the interim value of the portfolio changes depending on the way it is computed: (i) first, by summing the project's cash flows and then computing the internal value or (ii) first, by computing the projects' internal values and then summing them. Obviously, the infringement of value additivity occurs at the aggregate level as well: $C^j(k^j) = \sum_{t=1}^{n_j} c_{t-1}^j(k^j)v^{t-1}$ is the IRR-implied aggregate value of project $j = 1,2,\dots,p$. The IRR-implied aggregate value of the portfolio depends on the way it is computed:

$$C^J(k^J) = \sum_{t=1}^{N} c_{t-1}^J(k^J)v^{t-1} \neq \sum_{j=1}^{p}\sum_{t=1}^{n_j} c_{t-1}^j(k^j)v^{t-1} = \sum_{j=1}^{p} C^j(k^j)$$

Example 9. Consider $\boldsymbol{f^1} = (-8, 2, 2, 2, 2), \boldsymbol{f^2} = (-2,5), \boldsymbol{f^3} = (-4, 0, 2, 4)$. The three IRRs are $k^1 = 0\%$, $k^2 = 150\%$, $k^3 = 16.5\%$, and the portfolio's IRR is $k^{1+2+3} = 15.8\%$ It is straightforward that

$$\boldsymbol{c^1}(0\%) + \boldsymbol{c^2}(150\%) + \boldsymbol{c^3}(16.5\%) = (14\,, 10.7\,, 7.4\,, 2) \neq (14\,, 9.2\,, 6.7\,, 1.7) = \boldsymbol{c^{1+2+3}}(15.8\%)$$

and, assuming $r = 2\%$,

$$C^1(0\%) + C^2(150\%) + C^3(16.5\%) = 33.48 \neq 31.07 = C^{1+2+3}(15.8\%).$$

F10 — Project's operating life

The IRR approach neglects the whole operating and economic life of a project. To see how, suppose two firms are incorporated to undertake a project: firm A operates until time n, then ceases operations and liquidates; firm B continues to operate until time $n+m$. Suppose the two projects have the same cash flows up to time n; from time $n+1$ on, firm B generates an income which is equal to the change in capital value: that is, $I_t = c_t - c_{t-1}$. This implies that cash flows are zero for $t = n+1, \dots, n+m$. The respective cash-flow vectors are $f^A = (f_0, f_1, \dots, f_n)$ and $f^B = (f_0, f_1, \dots, f_n, \underbrace{0, 0, \dots, 0}_{m \text{ zeros}})$. The IRR approach disregards the operating activity of firm B in the last m periods; implicitly, it is assumed that the operating activity of B has ceased at time n and nothing economically significant occurs in the interval $[n, n+m]$: allegedly, no capital is invested and no income is generated, for the mere fact that cash flow is zero. In other words, the IRR forgets, so to speak, that, as a matter of fact, firm B continues to use materials, plants, equipment, working capital, human capital, and operates for other m periods during which net incomes continue to accrue. In simple terms: the IRR is a rate of return which considers actual cash flows but neglects actual returns which derive from operations and economic transactions.

Example 10. Suppose $f^A = (-100, 65, 60)$ has interim capitals equal to \$100, \$70, \$0 at time 0, 1, 2 respectively, and $f^B = (-100, 65, 60, 0, 0, 0)$ has interim capitals equal to \$100, \$70, \$40, \$50, \$70, \$0 at time 0, 1, 2, 3, 4, 5 respectively. Under an IRR perspective, it is as if firm B did not operate in the last three periods, since investors do not receive nor invest any more cash flow. Therefore, the rate of return would be k =16.5% for both firms. However, in the interval

[2,5], firm B does operate by deploying economic resources; using (1a), the returns are \$10 at time 3, \$20 at time 4, \$−70 at time 5.

F11 — Concocted capital

Scholars often interpret internal value (sometimes called "project balance") as the capital which is invested in the project (e.g. Spies 1983; Lohmann 1988; Hazen 2003; Crean 2005; Blank and Tarquin 2012). This value is derived in an automatic way: the solution of the IRR equation is put into equation (5), which is solved for $c_t(k)$ recursively. As anticipated, (5) is only a particular case of (3) where it is assumed that, in every period, the capital grows at a constant rate of return. However, this assumption is, in general, false, so internal capitals do not represent the capital employed in the project: they are not consistent with the economic resources actually deployed by the investor and bear no relation with either market values or accounting values or estimated capitals or any other pattern of interim values having some recognizable economic referent. Therefore, internal values are *concocted* (see also Altshuler and Magni 2012). As a result, the IRR can be conceived of (at the very best) as a period return repeatedly applied to concocted interim capitals which misrepresent true capitals.

Example 11a. Consider the simple case of an investor who has invested \$100,000 in an investment fund. Suppose the portfolio's market value has increased to \$130,000 at time 1 and to \$180,000 at time 2, then decreased to \$133,100 at time 3, when the investment is liquidated. Assuming zero interim cash flows and a COC equal to $r = 2\%$, the investment has created value ($NPV = 25{,}423.1$). The IRR is 10% and internal values at time 1 and 2 are $c_1(10\%) = 110{,}000$, $c_2(10\%) = 121{,}000$. Yet, the actual invested capitals at the beginning of the second and third period are $c_1 = 130{,}000$ and $c_2 = 180{,}000$, respectively. Therefore, the internal values do not adequately represent the economic milieu faced by the investor; they only represent an automatic byproduct of the IRR procedure itself, which picks a solution of a polynomial equation (10%) and puts it into a recurrence equation where the capital is implicitly assumed to increase, within every period, at a constant growth rate just equal to the solution itself (10%).

Example 11b. Consider a real estate investment of \$1,500,000: a building is purchased at time 0 and rents will be collected for seven periods, after which the asset will be sent. Estimated rents and selling price are collected in Table 2, which also presents the estimated marketplace values of the building. With a 3% COC, the NPV is \$1,507,451.7. The IRR is 14.46% and the corresponding internal values are evidently biased against the estimated market values.

Example 11c. A company has the opportunity of undertaking a five-year project which entails a capital expenditure of \$5,000,000 and an investment in working capital of \$4,000,000. Pro forma income statements and balance sheets are constructed where estimation of incremental sales, costs, invested capitals are shown year by year (a 33% tax rate is assumed). Table 3a collects all the relevant data, among which are the estimated invested capital, the Net Operating Profit After Taxes (NOPAT), and the Return on Investment (ROI). Table 3b converts the estimated accounting data into free cash flow (FCF) by making use of the fundamental economic relation (1). The IRR is $k = 25.5\%$ and the NPV is \$5,830,581 ($r = 3\%$ is assumed). The internal values, which are collected in Table 3b, manifestly contradict the estimated invested capitals made by the company, on the basis of which the prospective cash flows have been computed. Also, the IRR manifestly contradicts the fact that estimated capital does not increase at a constant force of return (see the ROI line in Table 3a).

Table 2. Real estate investment				
Time	Project's cash flows	Proceeds from sale	internal value ($k=14.46\%$)	Estimated market values
0	−1,500,000		1,500,000	1,500,000
1	50,000		1,666,969	1,550,000
2	45,000		1,863,091	1,600,000
3	42,000		2,090,580	1,800,000
4	70,000		2,322,976	2,000,000
5	60,000		2,598,986	2,500,000
6	50,000		2,924,920	3,000,000
7	48,000	3,300,000		

Table 3a. Pro forma balance sheets and income statements*						
Time	0	1	2	3	4	5
Balance Sheet						
Gross fixed assets	5,000	5,000	5,000	5,000	5,000	5,000
-cumulative depreciation	0	-1,000	-2,000	-3,000	-4,000	-5,000
Net fixed assets	5,000	4,000	3,000	2,000	1,000	0
WCR	4,000	1,000	2,000	800	2,000	0
Invested capital	**9,000**	**5,000**	**5,000**	**2,800**	**3,000**	**0**
Income Statement						
Sales		4,000	6,000	4,900	5,500	4,200
Cost of sales		1,600	1,700	1,800	1,900	2,000
Depreciation		1,000	1,000	1,000	1,000	1,000
EBIT (Earnings Before Interest and Taxes)		1,400	3,300	2,100	2,600	1,200
Taxes		462	1,089	693	858	396
NOPAT		938	2,211	1,407	1,742	804
ROI		10.42%	44.22%	28.14%	62.21%	26.8%

*Numbers in thousands

Table 3b. Converting accounting constructs into cash flows*						
NOPAT		**938**	**2,211**	**1,407**	**1,742**	**804**
−Capital expenditures	−5,000	0	0	0	0	0
+Depreciation		1,000	1,000	1,000	1,000	1,000
−change in Working Capital	−4,000	3,000	−1,000	1,200	−1,200	2,000
FCF	**−9,000**	**4,938**	**2,211**	**3,607**	**1,542**	**3,804**
IRR	**25.54%**					
internal value	9,000.0	6,360.6	5,774.2	3,641.9	3,030.1	0.0

*Numbers in thousands

F12 — Ad hoc consistency with NPV

Hazen (2003) makes use of the internal values to establish consistency of IRR with NPV (see eq. (10) above). However, consider the capital defined as $\mathbb{C}:=\frac{NPV(1+r)}{k-r}$. Owing to (9), one gets $C(k)=\mathbb{C}$ and, owing to (10), this implies that

$$k>r \quad iff \quad \mathbb{C}>0. \tag{10'}$$

Eq. (10′) is equivalent to (10), but the latter makes it clear that IRR can detect wealth creation with *no need of making recourse to the internal values*. In other words, formal consistency of IRR with NPV does not depend on the internal sequence $\boldsymbol{c}(k)$. In fact, the IRR is univocally associated with the aggregate capital $\mathbb{C}$, not with a specific sequence of interim capitals. This implies that, to ensure formal consistency of IRR and NPV, one does not need assume that capital grows at k in each period: consistency is necessarily obtained from $\mathbb{C}$, not from $\boldsymbol{c}(k)$. This seems to be good news, given that, as we have just seen, internal values do not represent, in general, the actual invested capital. Unfortunately, not even $\mathbb{C}$ has anything to do with the actual overall invested capital, for it bears no relation to the actual economic milieu met by the investor. Indeed, one can impose formal consistency of any arbitrary number γ with NPV. Consider, for any fixed δ, r, and NPV, the consistency equation

$$NPV=\frac{x\cdot(\delta-r)}{1+r}; \tag{11}$$

if one plugs the fictitious capital $x=(1+r)NPV/(\delta-r)$ into it, one obtains a rate of return which is formally consistent with NPV. But this evidently does not warrant labeling δ as a meaningful economic rate of return nor considering x the actual capital invested. Choosing $\delta=k$, one just finds $x=\mathbb{C}$; as a result, $\mathbb{C}$ is only a plug which bears no relation to anything an economist would recognize as meaningful values of economic resources. Therefore, the IRR is an *ad hoc* rate of return on a distorted capital base which is forced to enjoy NPV consistency.

Example 12. Consider example 11a. The IRR is 10%. The overall invested capital is $C=100{,}000+130{,}000(1.02)^{-1}+180{,}000(1.02)^{-2}=400{,}461.36$ (in thousands), but this capital does not guarantee consistency of the IRR with the NPV: $400{,}461.36\cdot(0.1-0.02)(1.02)^{-1}=31.408\neq 25{,}423.1=NPV$. If one distorts the actual capital by imposing the equality $25{,}423.1=\mathbb{C}(0.1-0.02)(1.02)^{-1}$, one finds $\mathbb{C}=324{,}144.56$; thus, the IRR can only be given formal consistency with NPV if one plugs $\mathbb{C}$ into the consistency equation, setting aside the actual capital C. Evidently, with this very line of reasoning, we might conclude that, say, $\delta=25\%$ is the correct project's rate of return, associated with $x=112{,}745.93$, just by invoking the consistency equality $25{,}423.1=112{,}745.93\cdot(0.25-0.02)(1.02)^{-1}$.

F13 — Multiple project balances, multiple excess returns

The IRR-implied capital $\mathbb{C}$ is *uniquely* associated with the IRR. However, there are infinitely many capital sequences $\mathbb{c}=(\mathbb{c}_0,\mathbb{c}_1,\dots,\mathbb{c}_{n-1})$ such that $\mathbb{c}_0=-f_0$ and $\sum_{t=1}^{n}\mathbb{c}_{t-1}v^{t-1}=\mathbb{C}$, the internal sequence $\boldsymbol{c}(k)$ being only one of them. Any such vector gives rise to a sequence of holding period rates k_t and the mean of such rates weighted by the discounted interim capitals as a proportion of the IRR-implied aggregate capital is just the IRR itself:

$$k=w_1k_1+w_2k_2+\cdots+w_nk_n \qquad w_t=\frac{\mathbb{c}_{t-1}v^{t-1}}{\mathbb{C}}, \qquad k_t=\frac{f_t+\mathbb{c}_t-\mathbb{c}_{t-1}}{\mathbb{c}_{t-1}} \tag{12}$$

(Magni 2010, Theorem 3). Therefore, $\boldsymbol{c}(k)$ is sufficient to generate the IRR (by assuming constant force of interest) but not necessary: any sequence $\mathbb{c}$ generates the IRR. Hence, the IRR is *not a period rate of return but a weighted arithmetic mean of period rates* and the IRR approach does not even supply an unambiguous stream of project balances. An implication of this result is that the equality

$$NPV = \frac{\mathbb{C}(k-r)}{1+r} = \frac{\sum_{t=1}^{n} \mathbb{c}_{t-1}(k_t - r)}{1+r} \tag{13}$$

does not unambiguously identify the project's excess returns (residual incomes): for every t, $ER(\mathbb{c}_{t-1}, k_t) = \mathbb{c}_{t-1}(k_t - r_t)$ is not unique. As a result, the IRR cannot provide a reliable economic analysis of the project, for the use of the IRR always leads to a problem of multiple interim capitals and multiple excess returns.

Example 13. Consider Example 11c, where $r = 3\%$ is assumed. Table 4a arbitrarily selects four (from infinite) IRR-implied streams of project balances, each associated with its own sequence of holding period rates $k_t(j)$, such that the net present value for each sequence is $\mathbb{C} = 26{,}643.2$, which implies that the weighted average of the holding period rates is just $k = 25.54\%$ no matter what capital stream is selected. Table 4b presents the corresponding excess returns obtained by application of eq. (8). An IRR approach obviously cannot tell us which sequence of interim capitals and corresponding excess returns should be appropriate for economic analysis.

Table 4a. Multiple project balances

t	f_t	$\mathbb{c}_t(1)$ $(k_t(1))$	$\mathbb{c}_t(2)$ $(k_t(2))$	$\mathbb{c}_t(3)$ $(k_t(3))$	$\mathbb{c}_t(4)$ $(k_t(4))$
0	−9,000	9,000	9,000	9,000	9,000
1	4,938	6,360.6 (25.54%)	5,549.0 (16.52%)	6,787.5 (30.28%)	10,032.8 (66.34%)
2	2,211	5,774.2 (25.54%)	8,287.0 (89.19%)	5.500.2 (13.61%)	4,120.5 (−36.89%)
3	3,607	3,641.9 (25.54%)	2,429.4 (−27.16%)	3,500.5 (29.22%)	2,125.9 (39.13%)
4	1,542	3,030.1 (25.54%)	2,500.0 (66.38%)	3,000.0 (29.75%)	2,333.3 (82.29%)
5	3,804	0 (25.54%)	0 (52.16%)	0 (26.80%)	0 (63.03%)
NPV	5,830.58	26,643.18	26,643.18	26,643.18	26,643.18
IRR (weighted average of k_t's)		**25.54%**	**25.54%**	**25.54%**	**25.54%**

Table 4b. Multiple excess returns

	$ER_t(\mathbb{c}_{t-1}(1), k_t(1))$	$ER_t(\mathbb{c}_{t-1}(2), k_t(2))$	$ER_t(\mathbb{c}_{t-1}(3), k_t(3))$	$ER_t(\mathbb{c}_{t-1}(4), k_t(4))$
1	2,028.6	2,028.6	2,028.6	2,028.6
2	1,433.7	1,250.8	1,529.9	2,261.4
3	1,301.5	1,867.9	1,239.8	928.8
4	820.9	547.6	789.0	479.2
5	683.0	563.5	676.2	525.9
NPV	5,830.58	5,830.58	5,830.58	5,830.58

F14 — Intertemporal inconsistency

Consider an investment of c_0 dollars and suppose a performance dynamic analysis is carried out such that, at every date t, the investor computes the rate of return in the past interval $[0, t]$. If $f_\tau = 0$ for $\tau = 1,2, \dots, t$, the IRR does not exist in the given interval (see F2), so neither the one-period rates k_τ, $\tau = 1,2, \dots, t$ nor the associated capitals $\mathbb{c}_\tau$ exist. Suppose that, at some time l, a cash flow f_l is released. The IRR in $[0, l]$ is equal to $k = (f_l/c_0)^{1/l} - 1$ which is the average of the internal period rates k_τ, $\tau = 1,2, \dots, t, t+1, \dots, l$, associated with the interim capitals $\mathbb{c}_\tau$. But this contradicts the fact that, at every date $t < l$, it was established that such rates and relative capitals did not exist.

To overcome this awkward result, one may reasonably consider, for each date t, a terminal capital c_t, exogenously selected. Hence, the rate of return in $[0, t]$ can be computed as $k^{[0,t]} = (c_t/c_0)^{1/t} - 1$, $t = 1,2, \dots, l-1$. At time l, one finds that the IRR is $k = ((f_l + c_l)/c_0)^{1/l} - 1$, which implies that the interim capital at time t is $\mathbb{c}_t$, $t = 1,2, \dots, l-1$. But, in general, $\mathbb{c}_t \neq c_t$, so the evaluation of the capital at time t changes depending on the time when the evaluation is made. This means that the IRR equation accomplishes a retrospective revision of the interim capitals. Such a retrospective evaluation has no economic rationale, yet, it is an inevitable byproduct of the IRR equation.

Example 14. An IRR-minded investor purchases a bond at time 0 at a price of \$100 and aims to accomplish a performance dynamic analysis. At time 1, the investor makes recourse to the market value of the asset to overcome the no-IRR problem. Assume the market value is $c_1 = 160$ so that $k^{[0,1]} = 60\%$. At time 2, the market value is 121 and the bond is sold. Assuming a 2% COC, the NPV is 16.3. The investor's IRR is $k = 10\%$, which necessarily implies that the internal capital at time 1 is $\mathbb{c}_1 = 110$ and the period rate is $k_1 = 10\%$, which contradict the previous choice of $c_1 = 160$ and $k^{[0,1]} = 60\%$, respectively.

F15 — Accounting variables

Despite considerable efforts of economists and accounting scholars, it is well-known that a firm's IRR is not capable of summarizing accounting information provided by accounting variables, in particular by accounting rates of return (Kay 1976; Peasnell 1982a,b; Fisher and McGowan 1983; Franks and Hodges 1984; Peasnell 1996). Surprisingly, accounting scholars interpret this fact as a flaw of accounting rates of return rather than a flaw of the internal rate of return. In fact, the former are obtained from capital values which consist of recognizable resources and transactions, whereas the IRR-implied capital $\mathbb{C}$ has no empirical referents. In particular, in corporate investment decisions, cash flows are often estimated on the basis of pro forma financial statements and then converted to forecasts of cash flows (e.g., Brealey et al. 2011; Titman and Martin 2011). This implies that cash flows are second-order variables, whereas accounting constructs are first-order variables: "accounting variables are the 'independent' variables and net dividends the 'dependent' variable, not the other way around" (Brief 1996, p. 28). Formally, let $WC_t =$ Working capital, $NFA_t =$ Net Fixed Assets, $NOPAT_t =$ Net Operating Profit After Taxes, and let Δ denotes variation. The fundamental economic relation boils down to

$$\overbrace{FCF_t}^{f_t} = \overbrace{NOPAT}^{I_t} - \overbrace{(\Delta WC_t + \Delta NFA_t)}^{c_t - c_{t-1}} \tag{14}$$

which implies that the IRR equation boils down to

$$\sum_{t=1}^{n} \frac{NOPAT_t - \Delta WC_t - \Delta NFA_t}{(1+k)^t} = 0 \tag{15a}$$

or

$$\sum_{t=1}^{n} \frac{ROI_t \cdot (NFA_{t-1} + WC_{t-1}) - \Delta WC_t - \Delta NFA_t}{(1+k)^t} = 0 \qquad (15b)$$

where $ROI_t = NOPAT_t/(NFA_{t-1} + WC_{t-1})$ is the well-known 'Return On Investment' (ROI) and $(NFA_{t-1} + WC_{t-1})$ represents the beginning-of-period invested capital ($= c_{t-1}$). Equation (15b) shows that the IRR is a function of the ROIs and the invested capitals. In general, if the estimation of accounting constructs changes, the IRR changes as well. So, those scholars who consider the IRR a correct measure of economic profitability, while considering ROI an incorrect measure, are trapped in a paradox: if one considers IRR economically significant, one cannot consider the estimated ROIs as incorrect, for the former is just computed on the basis of the latter.

One would rather expect that a suitable aggregation of the estimated ROIs should result in a significant index of wealth creation. Decades of research in accounting have shown that the IRR is not such an index (see Brief and Peasnell 1996) and we have just seen that the IRR is an arithmetic capital-weighted mean of concocted period rates k_t which have nothing to do with the estimated ROIs. This implies that the way the IRR equation aggregates ROIs is not the appropriate one.

Example 15. Consider again Example 11c. Note that the IRR is derived from a stream of FCFs which have been derived from a vector of estimated NOPATs and a vector of estimated invested capitals (see Tables $3a$ and $3b$). In particular, the vector of invested capitals (in thousands) is $\boldsymbol{c} = (9{,}000;\ 5{,}000;\ 5{,}000;\ 2{,}800;\ 3{,}000)$, which is different from any IRR-implied sequence of project balances $\mathbb{c}$. The vector of estimated ROIs is different from any IRR-implied sequence of holding period rates k_t, and the estimated aggregate invested capital is

$$C = 9{,}000 + 5{,}000[(1.03)^{-1} + (1.03)^{-2}] + 2{,}800(1.03)^{-3} + 3{,}000(1.03)^{-4} = 23{,}795.2,$$

whereas the IRR-implied invested capital is $\mathbb{C} = (1.03) \cdot \frac{5{,}830.581}{0.2554 - 0.03} = 26{,}643.18$. Hence, the IRR is the result of an aggregation of estimated accounting variables (see eq. (15)), but the IRR approach aggregates them in an incorrect way: it implicitly devises an artificial capital $\mathbb{C}$ which denies the estimations made by the investor. Such a capital $\mathbb{C}$ is related to incorrect (and ambiguous) period rates that contravene the very ROIs (and therefore, the very interim capitals) from which the IRR itself is derived.

F16 — Makeham's formula

In the nineteenth century, the actuarian W. Makeham devised a formula which divides the value of a loan into two parts: the value of interest and the value of principal repayments. In a loan, interest is obtained as $I_t = i_t c_{t-1}$ where c_{t-1} is the principal outstanding of the loan (economically, it represents the capital invested by the lender at the beginning of the period) and i_t denotes the interest rate holding in the t-th period. The principal repayments are $P_t \coloneqq c_{t-1} - c_t$. The value of interest is defined as the present value of interest payments: $\mathcal{I} = \sum_{t=1}^{n} I_t v^t$. The value of principal repayments is analogously defined as the present value of principal repayments: $\mathcal{P} = \sum_{t=1}^{n} P_t\, v^t$. The cash flows for the lender are $f_t = P_t + I_t$, $t = 1,2,\ldots,n$. Assume now that the loan's interest rate is constant across periods. This means that $i_1 = i_2 = \cdots = i_n = k$ is the lender's IRR, since the terminal condition $c_n = 0$ guarantees that

$c_0 = \sum_{t=1}^{n} f_t(1+k)^{-t}$ is satisfied, where $c_0 = -f_0$ is the amount committed by the lender. Makeham's formula can be written

$$\mathcal{I} = \frac{k}{r}(c_0 - \mathcal{P}) \tag{16}$$

(Makeham 1874; Glen 1893; Broverman 2008). While introduced for actuarial purposes, and never used by engineering economists nor financial theorists for capital investment analysis, the formula evidently holds for any sequence of cash flows. Dealing with capital investment projects, $\mathcal{I}$ is interpretable as the interest income, $\mathcal{P}$ represents that part of the committed capital c_0 which is recovered by the project. Therefore, the difference $c_0 - \mathcal{P}$ represents an unrecovered capital. Intuitively, a project is worth undertaking if and only if the overall return $\mathcal{I}$ generated by the project exceeds the unrecovered capital: $\mathcal{I} > c_0 - \mathcal{P}$. This is confirmed by the relation $NPV = \mathcal{I} + (\mathcal{P} - c_0)$ which stems from the definition of $\mathcal{I}$ and $\mathcal{P}$. Also, by (16),

$$NPV = (c_0 - \mathcal{P})\left(\frac{k}{r} - 1\right). \tag{17}$$

We can then introduce the following

Makeham's criterion. *A project is acceptable if and only if*

$$\frac{k}{r} > 1 \quad iff \quad c_0 > \mathcal{P}.$$

Makeham's criterion is NPV-consistent by (17). Further, it *formally* (but not *economically*) solves the multiple-IRR problem. To see it, let $k(1) \neq k(2)$ be any two IRR of a project and let $\mathcal{P}_{k(1)} \neq \mathcal{P}_{k(2)}$ the corresponding recovered capitals. From (17),

$$\left(c_0 - \mathcal{P}_{k(1)}\right)\left(\frac{k(1)}{r} - 1\right) = \left(c_0 - \mathcal{P}_{k(2)}\right)\left(\frac{k(2)}{r} - 1\right),$$

which implies that any one IRR can be used for making the correct accept-reject decision. This criterion is logically equivalent to Hazen's (2003) in eq. (10).

If the loan has varying interest rates, then the ratio of the IRR to the valuation rate r is no longer adequate to express the project's relative profitability and Makeham's formula fails to capture the value of interest: $\mathcal{I} \neq \frac{k}{r}(c_0 - \mathcal{P})$ (more so if the valuation rate r itself is varying across periods, in which case it is not clear how to replace the constant r). The reason for this failure is that IRR does not adequately summarize the information derived by the varying interest rates.

Example 16. Consider a loan of \$154.8, with varying interest rates, repayable with four installments (see Table 5). Assuming $r = 2\%$, the value of interest is $\mathcal{I} = 70.03$ and the IRR is $k = 10.38\%$; equation (16) is not fulfilled:

$$\frac{10.38\%}{2\%}(154.8 - 142.08) = 66.02 \neq 70.03.$$

Table 5. Makeham's formula and varying interest rates					
Time	f_t	c_t	I_t	P_t	i_t
0	−154.8	154.8			
1	20	140.99	6.19	13.81	4%
2	25	127.27	11.28	13.72	8%
3	15	125	12.73	2.27	10%
4	20	125	20	0	16%
5	150	0	25	125	20%
Present values	$NPV = 57.31$	$C = 648.63$	$\mathcal{I} = 70.03$	$\mathcal{P} = 142.08$	

F17 — Changes in capital

Just because IRR is a cash-flow-based measure which neglects the actual operations involved in a project, it remains constant under changes in capital. That is, any two projects with the same cash flows have the same IRR, no matter what the invested capital actually is.

Economically speaking, the same vector of cash flows can result from different economic policies, which result in different capitals invested and different returns. For example, consider two firms, A and B, each investing c_0 in an n-period investment. Given the different way the investments are managed, the projects are economically different, with different returns $\boldsymbol{I^j} = (I_1^j, I_2^j, \ldots, I_n^j)$ and different interim capitals $\boldsymbol{c^j} = (c_0, c_1^j, c_2^j, \ldots, c_{n-1}^j)$, $j = A, B$. Suppose that the difference in returns is compensated by the difference in capital depreciation:

$$I_t^A - I_t^B = [c_{t-1}^B - c_t^B] - [c_{t-1}^A - c_t^A]$$

so that the cash-flow streams coincide: $f_t^A = f_t^B = f_t$ for $t = 0,1, \ldots, n$. The two projects are empirically characterized by different economic activities, which result in different invested capitals and returns: a proper approach to rate of return should acknowledge this difference, whereas the IRR forces both projects to share the fictitious capital $\mathbb{C}$, despite the empirical evidence against $\mathbb{C}$. As a result, the two different projects are made to collapse into the same asset, whence the same rate of return is obtained.

Example 17. Consider the two projects described in Table 6 and assume $r = 6\%$.

Table 6. Different projects, equal IRR							
	Project A			Project B			Cash flows
Time	Invested capital	Income and rate of return		Invested capital	Income and rate of return		
0	150			150			−150
1	80	−20	(13.33%)	50	−50	(33.33%)	50
2	40	−5	(6.25%)	30	15	(30%)	35
3	40	30	(75%)	50	50	(166.6%)	30
4	0	50	(125%)	0	40	(80%)	90

The cash flows are the same but the projects are *economically* different: they derive from different operations and the magnitude and the distribution of incomes and interim capitals are different. With different invested capitals, different incomes, and different period return rates, one would expect that the overall rate of return of the two projects should be different. Yet, the IRR approach cancels out any economic difference between the two projects, and the rate of return is made to coincide: the IRR is 12.31% and is the return on an inputted capital equal to $\mathbb{C} = 416.32$, which is incorrect, since the overall capitals invested is different: respectively, $C^A = 150 + 80(1.06)^{-1} + 40(1.06)^{-2} + 40(1.06)^{-3} = 294.66$ and $C^B = 150 + 50(1.06)^{-1} + 30(1.06)^{-2} + 50(1.06)^{-3} = 265.85$. To employ the IRR approach means to disregard the fact that, while wealth creation depends on cash flows, rate of return depends on the capital base.

F18 — Computational issues

While advanced software solving polynomial equations is in principle available to anyone (Matlab, Mathematica etc.), it is well-known that the average practitioner is not willing or is not capable of engaging in difficult-to-use and costly tools for computing a rate of return. Managers, professionals, financial advisors need a handy tool for measuring rate of return. The use of spreadsheet functions (most notably, IRR and XIRR in Excel) is ubiquitous and overcomes the problem of manually computing an IRR. However, Excel makes use of an iterative procedure whose starting point depends on a choice left to the evaluator. Once a solution has been found, Excel stops and multiple IRRs are therefore not detected. (Strictly speaking, this may not be considered a fallacy, but only a practical limitation).

Example 18. Consider Example 1. With no guess, Excel individuates 10% as the IRR (it is just the default value for Excel), but with a different guess, either 30% or 50% are detected, depending on the guess, in a rather irregular way (see Table 7).

Table 7. Computation of IRR with Excel for $\boldsymbol{f = (-1000, 3900, -5030, 2145)}$

Guess of the user	Computed IRR
No guess	10%
10%	10%
18%	10%
19%	50%
20%	30%
30%	30%
40%	50%
50%	50%
60%	50%

4. *Pars construens*: A simple genesis for an alternative theory of rate of return

What is a rate of return? One can answer the question by resting on four simple intuitions.

Intuition 1: A rate of return $i^{(1)}$ of any asset is, by definition, an amount of interest earned per unit of capital invested in the asset. (A rate is a "fixed relation (as of quantity, amount or degree) between two things … a quantity, amount or degree of something measured per unit of something else" (Webster's Third New International Dictionary, 1961). In a rate of return, "something" is return and "something else" is capital). For example, if capital invested is 150 and return is 15, then the rate of return is 15/150=0.1. Therefore, a project's rate of return should be formalized as a ratio:

$$i^{(1)} = \frac{I}{C} \tag{18a}$$

with $I =$ return, $C =$ invested capital. Intuitively, in a multiperiod project, the invested capital C is the sum of all interim capitals invested, and the return I is the sum of all period returns. Therefore, taking into account the time-value of money, $C = \sum_{t=1}^{n} c_{t-1}v^{t-1}$ and $I = \sum_{t=1}^{n} I_t v^{t-1}$ so that (18a) boils down to

$$i^{(1)} = \frac{\sum_{t=1}^{n} I_t v^{t-1}}{\sum_{t=1}^{n} c_{t-1}v^{t-1}}. \tag{18b}$$

Intuition 2: A rate of return $i^{(2)}$ should summarize (and therefore be compatible with) information which is already included in the asset's period return rates i_t. Therefore, one would expect that the overall rate of return should be a suitable weighted average of the period rates of return:

$$i^{(2)} = \alpha_1 i_1 + \alpha_2 i_2 + \cdots + \alpha_n i_n. \tag{19}$$

The weights α_t should not only fulfill the coherence condition $\sum_{t=1}^{n} \alpha_t = 1$ but also be economically meaningful: α_t should represent the capital invested in the t-th period as a proportion of the total capital invested. The choice $\alpha_t := c_{t-1}v^{t-1}/C$ just fulfills this condition as well as the coherence condition.

Intuition 3. The project's *NPV* represents the investors' incremental wealth with respect to the minimum attractive alternative. Therefore, the ratio $\frac{NPV}{C}$ represents the incremental wealth per unit of overall capital invested over and above the cost of capital. This index might then be conceived of as an interest rate which marks up the cost of capital to the project's rate of return. Denoting the latter as $i^{(3)}$:

$$(1+r)\cdot\left(1+\frac{NPV}{C}\right) = 1 + i^{(3)}$$

so that

$$i^{(3)} = (1+r)\cdot\left(1+\frac{NPV}{C}\right) - 1. \tag{20}$$

In this view, the project's rate of return is therefore obtained by "grossing-up" the cost of capital by the relative incremental wealth.

Intuition 4. Given that the value of a project is the sum of the NPV and the initial capital invested c_0, one might define the *overall* value of a project as the sum of the NPV and the *overall* capital invested. Let $W_0 \coloneqq NPV + C$ denote such a value and let W_1 be the project's overall market value at time 1: the investors invest overall capital C and receive, after one period, an asset whose overall market value is W_1. The rate of return is then

$$i^{(4)} = \frac{W_1 - C}{C}. \tag{21}$$

As it turns out, the four rate intuitions give rise to the same rate of return. Owing to the definition of i_t and the choice $\alpha_t := c_{t-1}v^{t-1}/C$, one finds $i^{(1)} = i^{(2)}$. As for (20), just consider that $NPV(1+r) = \sum_{t=1}^{n}(I_t - rc_{t-1})v^t = (I - rC)$ so that, applying (18), the equality $i^{(1)} = i^{(3)}$ is obtained. Finally, $W_1 = W_0(1+r)$ so that $i^{(4)} = \frac{(NPV+C)(1+r)-C}{C} = i^{(3)}$.

Equations (18)-(21) make it clear that a project rate of return is a capital-based notion, not a cash-flow-based one: wealth creation does depend on cash flows, but a rate of return depends on capital. We denote as i the project's rate of return on capital C. Given that $i = i^{(j)}$ for every $j = 1, 2, 3, 4$, we can state the following

Definition 1. *The rate of return of a project is alternatively defined as*

- *(i) Return on invested capital – eq. (18)*
- *(ii) Weighted average of holding period rates – eq. (19)*
- *(iii) "Grossed-up" cost of capital – eq. (20).*
- *(iv) Market-determined interest per unit of overall invested capital– eq. (21)*

The rate of return i is called Average Internal Rate of Return (AIRR). The expression "Average Internal Rate of Return" stems from the fact that the holding period rate i_t can be viewed as a one-period IRR for the project $(-c_{t-1}, f_t + c_t)$. The AIRR is the weighted average of such IRRs.

Although equation (20) refers to overall invested capital C, one can use the same schema for drawing other pieces of information, namely the return on any desired capital base. To this end, consider the 'AIRR function' $i(x) = (1+r)\cdot\left(1+\frac{NPV}{x}\right) - 1$ which can be written as

$$i(x) = r + \frac{NPV(1+r)}{x}. \tag{22}$$

Equation (22) represents the rate of return corresponding to a capital base of x dollars (if $x = C$, then $i(C) = i$); geometrically, it is an indifference curve which supplies the same NPV for any combination (x, i) of capital and rate. For this reason, we call such a curve the *iso-value line*. In such a way, one can draw economic information about return on overall capital invested $(x = C)$, return on initial investment $(x = c_0)$, return on total investment cash flow $(x = F^-)$, return on average capital $(x = \bar{C}$, where $\bar{C}$ is any desired average of interim capitals) or any other economically meaningful capital base (see Figure 1). Equation (22) implies that any $i(x)$ can be directly derived from the four economic intuitions above described by using x as the invested capital.

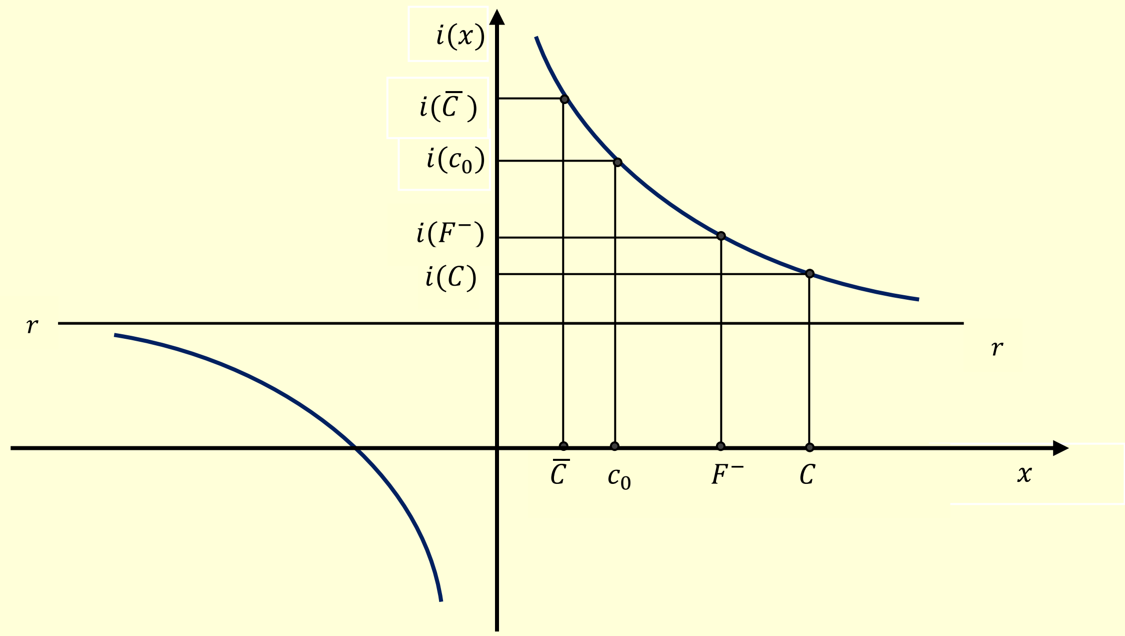

Figure 1. The iso-value line of a positive-NPV project. To each capital base there corresponds a unique rate of return. (If some capital base were negative, the left arm of the hyperbola would be involved as well).

The AIRR decision criterion is then straightforward.

Accept/reject decisions. *For any capital base x, a project is acceptable if and only if*

$$i(x) > r \quad iff \quad x > 0$$

(see also Magni 2010, Theorem 2).

A common idea is that a project P should be defined as a vector of cash flows. However, to any vector of cash flows there corresponds a set R of rates of return (for nonzero-NPV projects, $R = \mathbb{R} - \{r\}$). This indeterminacy is solved if the usual conception is changed by introducing a more appropriate definition of project.

Definition 2. A project P is a vector of cash flows and capitals:

$$P = (\boldsymbol{f}, \boldsymbol{c}) = (f_0, f_1, \dots, f_n; c_0, c_1, \dots, c_{n-1}) \qquad c_0 = -f_0$$

where n represents the end of the operating activity.

Definition 2 does not give any problem of indeterminacy, for, in order to determine a rate of return, the evaluator has to fix capitals as well as cash flows. The vector $(c_0, c_1, \dots, c_{n-1})$ should represent the economic resources actually deployed in the project. The estimation of the appropriate capital base is an empirical matter, not a mathematical one, just as is the estimation of cash flows. And it depends on the purpose of the analysis, on the economic milieu met by the investor, and on the information available.

In particular, the type of projects is relevant in choosing the appropriate $\boldsymbol{c}$. For example,

Type of project	Appropriate capital
Loan	Principal outstanding
Security or financial portfolio	Market value
Capital investment (industrial, engineering project)	Estimated value of net assets
Real estate investment	Marketplace value (selling price)

The choice of the capital base also depends on the purpose of the analysis. For example, if the investor is willing to know the rate of return on initial investment or total investment cash flow, then the capital base will be c_0 or F^-, respectively.

What if information on interim capitals is not available and the evaluator is not willing or not capable of engaging in estimation? In these cases, a natural and compelling choice is the *economic value* V_t, which is defined as the price at which the project (or a replicating portfolio) would be traded in an efficient market where r is the equilibrium rate of return:

$$V_t = \sum_{h=t+1}^{n} \frac{f_h}{(1+r)^{h-t}} \qquad t = 0,1,2,\dots,n-1 \tag{23}$$

whence

$$C = c_0 + \sum_{t=1}^{n} V_t \cdot v^t = c_0 + \sum_{t=1}^{n} \sum_{h=t+1}^{n} \frac{f_h}{(1+r)^{h-t}} \cdot v^t.$$

Therefore, if only cash flows are available, the project is

$$P = (\boldsymbol{f}, \boldsymbol{c}) = (f_0, f_1, \dots, f_n; c_0, V_1, \dots, V_{n-1}) \qquad c_0 = -f_0.$$

The rate of return, which we call "economic AIRR", is

$$i^E = r + \frac{NPV(1+r)}{c_0 + \sum_{t=1}^{n} V_t \cdot v^t}. \tag{24}$$

Practically, equation (24) is a fast and frugal procedure for computing a rate of return. Theoretically, its significance derives from finance theory and can be gleaned by noting that, in an efficient market, whenever a firm undertakes a project which is positively evaluated by the market, the firm finds itself in a temporary state of disequilibrium. Consequently, the share's price increases so that shareholder's wealth is generated as a windfall gain equal to $NPV = V_0 - c_0$. A project creates wealth for shareholders if and only if $p' > p$, where p is the share's equilibrium price before acceptance of the project and p' is the revised share's equilibrium price after acceptance of the project. For example, assuming the project is financed with new equity, the equilibrium firm values before and after acceptance of the project are, respectively, $V = s \cdot p$ and $V' = (s' + s) \cdot p'$, where s is the numer of firm's shares before acceptance of the project, and s' is the number of additional shares issued at price p'. The net present value of the project is $NPV = V_0 - c_0 = V' - V - c_0$. Given that $p' \cdot s' = c_0$, then $NPV = s(p' - p)$, which means that price increases if and only if the NPV of the project is positive (see also Rubinstein 1973, footnote 10, and Magni 2009, Appendix 1). In such a way, equilibrium is established again. Assuming all expectations are fulfilled, the value of the firm at the end of the first period will be V_1, which implies that shareholder's rate of return in the first period is $i_1 = (f_1 + V_1 - c_0)/c_0$. Note that $c_0(i_1 - r) = ER_0(c_0, i_1)$ represents shareholder value creation; in the following periods, the firm will be in equilibrium and only the equilibrium return rate r will be earned by shareholders: $i_t = (f_t + V_t - V_{t-1})/V_{t-1} = r$ for $t > 1$ (shareholder value creation is zero). Therefore, the economic AIRR is a weighted average of i_1 and r:

$$i^E = w \cdot i_1 + (1 - w) \cdot r \qquad \text{where } w := \frac{c_0}{c_0 + \sum_{t=1}^{n} V_t \cdot v^t}.$$

Note that $i(c_0) = (f_1 + V_1 - c_0)/c_0$ (Magni, 2010, pp. 167-169) so that $i^E = w \cdot i(c_0) + (1 - w) \cdot r$: the rate of return on initial capital represents the shareholder's rate of return in the first period, when disequilibrium has occurred. This confirms the economic significance of $i(c_0)$.

The difference between $i(x)$ and r is the project's excess AIRR function, which we denote as $\xi(x) = i(x) - r = \frac{NPV(1+r)}{x}$. Obviously, $\xi(x)$ has the same (opposite) sign as the NPV if $x > 0$ (< 0), so the comparison between $i(x)$ and r boils down to checking the sign of $\xi(x)$.

Relations with PI and BC ratio. Using (6) and (22) one gets

$$PI = \frac{i(c_0) - r}{1 + r} = \frac{\xi(c_0)}{1 + r} \quad \Leftrightarrow \quad \xi(c_0) = PI(1 + r) \tag{25}$$

so the profitability index is but a discounted excess AIRR on *initial investment*. Using (7) and (22) one gets to

$$BC - 1 = \frac{i(F^-) - r}{1 + r} = \frac{\xi(F^-)}{1 + r} \quad \Leftrightarrow \quad BC = \frac{1 + i(F^-)}{1 + r} \tag{26}$$

so the benefit-cost ratio essentially captures the (discounted) excess AIRR on *total investment cash flow*. In other words, $1 + i(F^-)$ is the ending value of one dollar of total investment cash flow and the BC ratio is the corresponding economic value. If one defined the benefit-cost ratio as $BC = \frac{NPV}{F^-} = \frac{F^+ - F^-}{F^-}$ (i.e. as a profitability index as expressed per unit of total investment cash flow), then one would get $BC = \frac{i(F^-) - r}{1+r}$.

Owing to (25), one can also link economic AIRR and profitability index:

$$i^E = r + w \cdot PI(1 + r).$$

Variable COCs. To extend the accept/reject criterion to the case of variable COCs, consider that $NPV = \sum_{t=1}^{n}(I_t - r_t c_{t-1}) v^{t,0}$. Some algebraic manipulations lead to the following adjustment for AIRR and COC:

$$\bar{\imath}(\mathcal{C}) = \sum_{t=1}^{n} w_t i_t \qquad \bar{r}(\mathcal{C}) = \sum_{t=1}^{n} w_t r_t \tag{27}$$

where $w_t := \frac{c_{t-1}v^{t,0}}{\mathcal{C}}$, $\mathcal{C} := c_0 v^{1,0} + c_1 v^{2,0} + \cdots + c_{n-1} v^{n,0}$. We have $NPV = \mathcal{C} \cdot (\bar{\imath}(\mathcal{C}) - \bar{r}(\mathcal{C}))$ whence

$$\bar{\imath}(\mathcal{C}) = \bar{r}(\mathcal{C}) + \frac{NPV}{\mathcal{C}}. \tag{28}$$

In such a way, the accept/reject criterion is generalized by replacing the triplet (C, i, r) with the triplet $(\mathcal{C}, \bar{\imath}(\mathcal{C}), \bar{r}(\mathcal{C}))$.

The criterion for choosing between mutually exclusive alternatives is simple. Consider project P_1 and project P_2 (in the sense of Definition 2). We have $P_1 = P_2 + (P_1 - P_2)$, so, a comparison between the two projects boils down to accepting or rejecting the incremental alternative $P_1 - P_2$.

Choice between mutually exclusive alternatives. *Project P_1 is preferred to project P_2 if $P_1 - P_2$ is acceptable.*

Choice between mutually exclusive alternatives is, actually, a particular case of project ranking. The latter task can then be accomplished by a pairwise application of the above criterion but, as previously noted, the task is cumbersome and can be time-consuming. The project's AIRRs as such cannot be contrasted, for they refer to different capital bases. However, we can compute a standardized AIRR for each project. The AIRR function is such that

$$NPV = x \cdot (i(x) - r) = C \cdot (i - r)$$

for every x. Let B be the benchmark capital which is to be used to standardize the AIRR. Then, there exists a unique rate of return, denoted as $i(B)$, that would result from employing B capital that is on the same iso-value line:

$$i(B) = r + \frac{C}{B}(i - r)\,. \tag{29}$$

Project ranking. *Ranking projects with the standardized AIRR is equivalent to ranking projects with the NPV.*

From a computational point of view, there is no need of computing the project's AIRRs: equation (29) can be evidently replaced by

$$i(B) = r + \frac{NPV(1 + r)}{B}. \tag{30}$$

It is worth noting that the excess standardized AIRRs $\xi(B) = i(B) - r$ supply correct information about the relative magnitude of the wealth created, given that

$$\frac{\xi^j(B)}{\xi^l(B)} = \frac{NPV^j}{NPV^l}$$

for any pair of projects P^j and P^l and for any benchmark capital B.

5. *Part costruens*: Corroboration of the theory

We now put the AIRR paradigm to the test and show that it defies each and every flaw associated with the IRR approach.

F1 — Multiple rates of return
In example 1, the project's cash-flow vector is $\boldsymbol{f} = (-100, 390, -503, 214.5)$ with $r = 14\%$. As seen in Definition 2, any project should be linked with an appropriate capital base. Having no information on capital, the appropriate choice is the economic values determined by the market: $c_1 = V_1 = -276.18$, $c_2 = V_2 = 188.16$, so that $C = 100 - 276.18(1.14)^{-1} + 188.16(1.14)^{-2} = 2.521$ is the overall invested capital. Hence, the project is

$$P = (\boldsymbol{f}, \boldsymbol{c}) = (-100, 390, -503, 214.5; 100, -276.18, 188.16).$$

Using (20), one gets the economic AIRR: $i^E = 0.14 + \frac{-0.156(1.14)}{2.52} = 6.97\%$. The project is equivalent to an investment of 2.521 at a 6.967% return; the project is not acceptable since $6.967\% < 14\%$ (note that 2.521·(6.967%−14%)/(1+0.14)= -0.156= NPV. The multiple IRRs arise because, in the IRR approach, a project's rate of return is not found by exogenously fixing the capital base but, rather, by solving a polynomial equation which gives rise to multiple solutions and, therefore, to multiple invested capitals. Given that different rates of return are related to different capital bases (see the iso-value line), then different IRRs are reflected by different points on the iso-value line.

F2 — No rate of return
Consider example 2 where we assume $r = 20\%$. To get the rate of return, one must fix a capital base. Assume, as before, that no information is available on capital. Eq. (23) leads to $V_1 = -20.83$ so that $C = 10 - 20.83(1.2)^{-1} = -7.36 < 0$. Hence, the economic AIRR is $i^E = 0.2 + \frac{-2.36(1.2)}{-7.36} = 58.49\%$ (the project is not worth undertaking).

F3 — Varying costs of capital
As seen, the overall cost of capital is itself an arithmetic mean of the varying COCs. Consider example 3. Using (27), one finds $\bar{r}(\mathcal{C}) = 16.2\%$, with $\mathcal{C} = 21{,}329.95$. Using (28), one gets $\bar{\iota}(\mathcal{C}) = 0.162 + \frac{1{,}241.78}{21{,}329.95} = 22.02\%$. The excess AIRR is $\xi(\mathcal{C}) = \bar{\iota}(\mathcal{C}) - \bar{r}(\mathcal{C}) = 5.82\% > 0$, so the project is worth undertaking. Applying the excess AIRR to the invested capital, the NPV is found as $5.82\% \cdot (21{,}329.9) = 1{,}241.7$.

F4 — Arbitrage strategy
Arbitrage strategies are naturally encompassed in the AIRR approach. Consider example 4 where $\boldsymbol{f} = (625, 0)$. The NPV is 625 regardless of the COC. It is rather natural to set $x = c_0 = -625$ so that the project is

$$P(\boldsymbol{f}, \boldsymbol{c}) = (625, 0; -625, 0)$$

and $i(-625) = r + \frac{625(1+r)}{-625} = -100\%$. The cash-flow stream represents a net borrowing of \$625 whereby the borrower (the bank) pays interest at a negative rate of -100%. Which means that the borrower has made money out of a borrowing (it has earned the entire borrowed capital).

F5 —Mutually exclusive projects and project ranking

Consider example 5. Wealth creation depends on both rate and capital, so wealth maximization cannot be captured by rate of return alone, just because, by definition, it neutralizes the investment scale. Therefore, to compare AIRRs as such is like comparing apples and oranges. As seen, the AIRR model enables one to standardize the AIRRs. Consider the average investment cash flow as the benchmark capital B. The total investment cash flows are, respectively, 2,428.6 for the first project, 1,185.7 for the second one, 890.7 for the third one, and 1,200 for the fourth one. The (simple) average is then $B = 1{,}426.2$. Equation (30) provides the standardized AIRRs. Denoting as $i^j(B)$ the standardized AIRR of project j, one gets the correct ranking:

$$i^1(1{,}426.2) = 0.05 + \frac{356(1.05)}{1{,}426.2} = 31.21\%$$

$$i^2(1{,}426.2) = 0.05 + \frac{343.3(1.05)}{1{,}426.2} = 30.27\%$$

$$i^3(1{,}426.2) = 0.05 + \frac{258.8(1.05)}{1{,}426.2} = 24.06\%$$

$$i^4(1{,}426.2) = 0.05 + \frac{130.5(1.05)}{1{,}426.2} = 14.61\%.$$

We can use the excess standardized AIRRs to disclose information about the relative wealth created. For example,

$$\frac{\xi^2(1426.2)}{\xi^1(1426.2)} = \frac{30.27\% - 5\%}{31.21\% - 5\%} = 96.42\%$$

means that project 2 supplies 96.42% of the wealth created by project 1. Analogously, $\xi^3(1426.2)/\xi^1(1426.2) = 72.7\%$ means that project 3 supplies 72.7% of the wealth created by project 1 (for project 4, one finds 36.65%). (It is worth reminding that the choice of B does not affect the relative wealth created).

F6 — Rate of return on initial capital (total investment cash flow)

From the AIRR function, it is easy to get the rate-of-return counterparts of PI and BC ratio. Consider Example 6, such that $c_0 = 100$ and $F^- = 207.57$. Picking $x = 100$ in the AIRR function, one gets the rate of return on initial investment: $i(100) = 52.86\%$ (note that $PI = \frac{0.5286-0.1}{1.1} = 0.39$). Picking $x = 207.57$ in the AIRR function, one gets the rate of return on total investment cash flow: $i(207.57) = 30.65\%$ (note that $BC = 1.3065/1.1$=1.19).

F7 — Framing effects: present value vs. future value

Consider the basic definition of AIRR in eq. (18). Referring the elements of the ratio to any future time T, the ratio does not change, for

$$\frac{\sum_{t=1}^{n} I_t v^{t-1}}{\sum_{t=1}^{n} c_{t-1} v^{t-1}} = \frac{\sum_{t=1}^{n} I_t v^{t-1-T}}{\sum_{t=1}^{n} c_{t-1} v^{t-1-T}} \qquad \text{for every } T \in \mathbb{R}.$$

Consider $\boldsymbol{f} = (-200, 0, 0, 266.2, 0)$ in Example 7. Assume, for instance, that $r = 6\%$ and no information is given about capitals, so that economic values are employed. One finds $c_1 = V_1 = 236.92$, $c_2 = V_2 = 251.13$, $c_3 = V_3 = 0$, as well as $I_1 = 36.92$, $I_2 = 14.22$, $I_3 = 15.07$, $I_4 = 0$. Applying (18) and using $v = 1/1.06$, one gets

$$i^E = \frac{36.92 + 14.22 \cdot v^{1-T} + 15.07 \cdot v^{2-T} + 0 \cdot v^{3-T}}{200 + 236.92 \cdot v^{1-T} + 251.13 \cdot v^{2-T} + 0 \cdot v^{3-T}} = 9.85\% \qquad \text{for every } T.$$

With $\boldsymbol{f} = (-200, 0, 0, 0, 0)$, picking $x = c_0 = 200$ and considering $c_t = 0$ for $t > 0$, so that $I_1 = -200$, $I_2 = I_3 = I_4 = 0$, one gets

$$i = \frac{-200 + 0 \cdot v^{-T} + 0 \cdot v^{2-T} + 0 \cdot v^{3-T}}{200} = -100\% \qquad \text{for every } T.$$

The AIRR correctly signals the entire loss of the capital.

F8 — Framing effects: expected value of stochastic IRR vs. IRR of expected investment

Let $i(\tilde{\boldsymbol{f}}) = r + NPV(\tilde{\boldsymbol{f}})(1 + r)/x$ be the project's stochastic AIRR, referred to a capital base of x dollars. It is easy to see that the expected AIRR is unambiguously defined:

$$i\left(E(\tilde{\boldsymbol{f}})\right) = E\left(i(\tilde{\boldsymbol{F}})\right).$$

For example, consider again Table 1 and assume the capitals are estimated at $c_1 = 66$, $c_2 = 33$. This implies

$$i\left(E(\tilde{\boldsymbol{f}})\right) = r + \frac{NPV\left(E(\tilde{\boldsymbol{f}})\right)(1 + r)}{x} = 0.01 + \frac{52.9(1.01)}{197.7} = 28.03\%.$$

Also, $i(\tilde{\boldsymbol{f}}) = (75\%, 25\%, -34.9\%)$, which implies $E\left(i(\tilde{\boldsymbol{f}})\right) = 0.3(75\%) + 0.5(25\%) + 0.2(-34.9\%) = 28.03\%$.

As for the second example in F8, assuming the decision maker requires the rate of return on initial capital, it is easy to check that $E\left(i(\tilde{\boldsymbol{f}})\right) = i\left(E(\tilde{\boldsymbol{f}})\right) = 19.8\%$.

F9 — Framing effects: value additivity

The capital invested in a portfolio of m projects is, by value additivity, equal to the sum of the capitals invested in the projects: $C^{\sum_{j=1}^{m} j} = \sum_{j=1}^{m} C^j$. The pitfall of the IRR lies in the fact that the portfolio's invested capital is obtained endogenously (i.e. automatically) from the portfolio's cash-flow vector, rather than exogenously from the projects' capital streams. The AIRR approach does not incur this problem, just because capital is fixed exogenously. In Example 9, consider the economic values for $t = 1,2,3$: using (23) for the three projects and the portfolio, one gets

Time	Project 1	Project 2	Project 3	Portfolio
1	5.77	0.00	5.81	11.58
2	3.88	0.00	3.92	7.8
3	1.96	0.00	0.00	1.96

The sums of discounted values at $r = 2\%$ are $C^1 = 19.23$, $C^2 = 2$, $C^3 = 13.46$ and the portfolio's overall capital is just $C^{1+2+3} = 34.69$. The latter is also equal to $C^1 + C^2 + C^3$.

F10 — Project's operating life

The existence of a project is evidenced empirically by the existence of an on-going operating structure (plants, equipment, working capital, human capital), not by the appearance of cash flows. From operations, incomes and capitals are generated and the new definition of project (Definition 2) just takes account of this.

Let us back to example 10, and let us assume $r = 10\%$. Firm A's invested capital is $C^A = 100 + \frac{70}{1.1} = 163.64$ and the return is $I^A = 35 - \frac{10}{1.1} = 25.91$, so the return on capital is

$$i^A = \frac{25.91}{163.64} = 15.83\%.$$

The firm B's income is $I^B = 35 + \frac{30}{1.1} + \frac{10}{1.1^2} + \frac{20}{1.1^3} + \frac{-70}{1.1^4} = 37.75$. The invested capital is $C^B = 100 + \frac{70}{1.1} + \frac{40}{1.1^2} + \frac{50}{1.1^3} + \frac{70}{1.1^4} = 282.07$, so

$$i^B = \frac{37.75}{282.07} = 13.38\%.$$

The smaller rate of return is easily explained by the fact that the prosecution of the operations causes firm B to earn an additional income equal to $I^B - I^A = 11.84$. To produce this incremental return, firm B utilizes a greater capital $C^B - C^A = 119.13$. The resulting incremental return on capital is 11.84/118.4 = 10%. Firm B's performance is a weighted average of firm A's performance and the incremental performance, so the resulting rate of return is smaller than firm A's. In general,

$$\alpha \cdot i^A + (1 - \alpha) \cdot i^{A-B} = i^B$$

where $\alpha \coloneqq \frac{C^A}{C^B}$ and $i^{A-B} = r$. In this case, $0.58 \cdot 15.83\% + 0.42 \cdot 10\% = 13.38\%$.

It may be of some interest to note the fact that, whereas the AIRRs are different, the firms' NPVs are equal. That is, NPV disregards temporal information. In contrast, the AIRR notion is a measure of worth which differentiates on length of time (AIRR depends on the invested capital, which in turn depends on the project's operating life). For a given NPV, it seems more natural to prefer the project which generates that NPV in a shorter time. If so, AIRR maximization might be an appropriate decision criterion for ranking equal-NPV projects in several circumstances. The conditions under which this rule works deserve to be investigated (thanks are due to Joseph Hartman for this insight).

F11 — Concocted capital

The AIRR paradigm is based on the idea that the correct rate of return depends on an exogenous choice of the appropriate interim capitals, which are then summed so as to supply the overall capital C. In Table 8, we collect the overall invested capitals and the corresponding AIRRs based on the input data provided in Examples 11a-c. In the same table we also derive the distortion of the invested capital made by the IRR approach, which in turn produces a distortion of the projects' rate of return. In Example 12a, the IRR approach overestimates the invested capital, which induces an underestimation of the project's rate of return. In Examples 12b-c the reverse holds: an underestimation of capital leads to an overestimation of the rate of return. In terms of the iso-value line, the IRR is identified by the point $(\mathbb{C}, k)$ on the iso-value line, which is biased with respect to the point (C, i) (see Figure 2 in the next section).

Table 8. Under- and over-estimation of invested capital by IRR approach

	Example 11a	Example 11b	Example 11c
Invested capital (C)	400,461.36	12,606,211.64	23,795,206.27
AIRR (i)	8.48%	15.32%	28.24%
IRR-implied capital error ($C - \mathbb{C}$)	76,316.8	−936,942.33	−2,847,979.77
IRR-implied return error ($i - k$)	−1.52%	0.85%	2.7%

F12 — Ad hoc consistency with NPV

As seen, in the IRR approach, the capital is obtained by picking $i(x) = k$ and reverse-engineering the relation $i(x) = r + NPV(1 + r)/x$. With this approach, the capital is made to depend on the rate of return, which is incorrect: it is the rate of return which depends on capital. Consider Example 12 (= Example 11a): the point (324,144.56; 10%) on the iso-value line is detected by solving the IRR equation and then computing $i^{-1}(10\%) = 324{,}144.56$. But the point $(324{,}144.56, 10\%)$ has the same economic status as, say, the point (112,745.93; 25%), which is found by arbitrarily imposing that 25% is the rate of return. An economically meaningful point on the AIRR function is (400,461.36; 8.48%), as seen in Table 8, which is found by exogenously choosing the actual invested capital: $8.48\% = i(400{,}461.36)$.

F13 — Multiple project balances, multiple excess returns

Multiple project balances associated with any IRR arise owing to the reverse engineering procedure which generates $\mathbb{C}$; in contrast, the AIRR procedure exogenously starts from one single vector of interim capitals (which is intrinsic in the new definition of project), so ambiguity is not possible:

AIRR	IRR
$\boldsymbol{c} \Longrightarrow C \Longrightarrow i(C)$	$k \Longrightarrow i^{-1}(k) = \mathbb{C} \Longrightarrow$ infinitely many $\mathbb{c}$

Consider Example 13 (= Example 11c). Table 3 collects a unique stream of interim capitals: the project is

$$P = (\boldsymbol{f}, \boldsymbol{c}) = (-9{,}000;\ 4{,}938; 2{,}211; 3{,}607; 1{,}542; 3{,}804;\ 9{,}000; 5{,}000; 5{,}000; 2{,}800; 3{,}000).$$

Applying eq. (8) with $c_{t-1} = NFA_t + WC_t$ and $i_t = ROI_t = NOPAT_t/(NFA_t + WC_t)$ one gets the unique excess return

$$ER_t(c_{t-1}, i_t) = NOPAT_t - r(NFA_{t-1} + WC_{t-1}) = (NFA_{t-1} + WC_{t-1})(ROI_t - r)$$

(see Table 9). In this case, the COC is the well-known *Weighted Average Cost of Capital.*

Table 9. Unique excess returns*

	$ER_t(c_{t-1}, i_t)$
1	668
2	2,061
3	1,257
4	1,658
5	714
NPV	5,830.6

*Numbers in thousands

F14 — Intertemporal inconsistency
In the AIRR approach, an evaluator is not forced to retrospectively revise the capitals: the latter can be estimated in a timely fashion and those values can be frozen. In such a way, the evaluator can use, at time l, the very data which have been used at any time $t < l$. This avoids intertemporal inconsistency. Consider example 14. At time 1, the evaluator fixes the interim capital at $c_1 = 160$ and the AIRR is 60% (in one period, the AIRR coincides with the holding period rate). At time 2, the analyst does not revise the evaluation and makes use of the same interim capital to compute the AIRR of the entire investment:

$$i = \frac{0.6(100) + (-24.375\%)(160)(1.02)^{-1}}{100 + 160(1.02)^{-1}} = 8.47\%.$$

F15 — Accounting variables
The AIRR approach is capable of aggregating accounting variables in an appropriate way, which testifies of the major role accounting constructs can play in expressing economic profitability. The overall capital is equal to the present value of the estimated interim capitals, and the AIRR is equal to the estimated average ROI:

$$i = \overline{ROI} = \frac{\sum_{t=1}^{n} ROI_t \cdot (NFA_{t-1} + WC_{t-1})v^{t-1}}{\sum_{t=1}^{n}(NFA_{t-1} + WC_{t-1})v^{t-1}}$$

which can be written as

$$\overline{ROI} = r + \frac{NPV(1 + r)}{\sum_{t=1}^{n}(NFA_{t-1} + WC_{t-1})v^{t-1}}.$$

In particular, in the example depicted in Table 3, one gets $\overline{ROI} = 3\% + \frac{5{,}830.58(1.03)}{23{,}795.21} = 28.24\%$, which is the project's rate of return. The project should be accepted (28.24%>3%) and $NPV = 23{,}795.21(28.24\% - 3\%)(1.03)^{-1} = 5{,}830.58$.

F16 — Makeham's formula

With varying interest rates, Makeham's formula does hold if i replaces k. To show it, we first show that $C = \frac{c_0 - \mathcal{P}}{r}(1+r)$. From $P_t = c_{t-1} - c_t$ it follows $c_t = P_{t+1} + \cdots + P_n$, whence

$$C = (P_1 + P_2 + \cdots + P_n) + (P_2 + .. + P_n)v + (P_3 + \cdots + P_n)v^2 + \cdots + P_n v^{n-1}.$$

Manipulating algebraically,

$$C = P_1 + (P_2 + P_2 v) + (P_3 + P_3 v + P_3 v^2) + \cdots + (P_n + P_n v + \cdots + P_n v^{n-1})$$

which means

$$C = P_1 \cdot \frac{1-v}{1-v} + P_2 \cdot \frac{1-v^2}{1-v} + \cdots + P_n \cdot \frac{1-v^n}{1-v}.$$

As $1 - v = r/(1+r)$,

$$C = P_1 \cdot \frac{1-v}{r}(1+r) + P_2 \cdot \frac{1-v^2}{r}(1+r) + \cdots + P_n \cdot \frac{1-v^n}{r}(1+r),$$

whence, finally,

$$C = \frac{P_1 + P_2 + \cdots + P_n - (P_1 v + P_2 v^2 + \cdots + P_n v^n)}{r}(1+r) = \frac{c_0 - \mathcal{P}}{r}(1+r).$$

Using this result and the equality $NPV(1+r) = C \cdot (i - r)$, one gets $NPV = \frac{c_0 - \mathcal{P}}{r}(i - r)$. As $NPV = \mathcal{I} + \mathcal{P} - c_0$,

$$\mathcal{I} = \frac{i}{r}(c_0 - \mathcal{P}). \tag{31}$$

We call (31) the *generalized Makeham's formula* (Makeham's formula being only one particular case of it). We can then introduce the following NPV-consistent decision criterion.

Generalized Makeham's criterion. *A project is acceptable if and only*

$$i > r \quad iff \quad c_0 > \mathcal{P}.$$

The sign of the unrecovered capital $c_0 - \mathcal{P}$ signals whether the asset is an investment or a borrowing: in case of a borrowing, the recovered capital $\mathcal{P}$ exceeds the committed capital c_0 so the AIRR is a borrowing rate and wealth creation is generated if $i < r$.

The example presented in Table 5 can now be easily coped with. The AIRR is $i(648.63) = 11.01\%$ and

$$\mathcal{I} = 70.03 = \frac{11.01\%}{2\%}(154.8 - 142.08).$$

F17 — Changes in capital

The actual economic unfolding of the project and the actions made by investors, summarized in income and capital, are relevant in computing the rate of return. So, projects with the same cash flows have, in general, different rates of return. In example 17 (see Table 6) all data are available to compute the AIRR (average ROI) for the two projects. One gets $\overline{ROI}^A = 14.92\%$ and $\overline{ROI}^B = 15.89\%$. The cash flows are the same, but the projects are *empirically* different.

F18 — Computational issues
As seen, the AIRR has no computational problems for no equation is involved. Indeed, there are essentially three ways to compute it which can be used depending on the data available: incomes and capitals (eq. (18)); period rates and capitals (eq. (19)); cash flows and capital base (eq. (20) or eq. (21)). Even in those occasions where interim capitals are not available and the evaluator is not willing to engage in estimation of interim capitals, the problem is easily overcome by picking the economic values V_t, which are easily computed from prospective cash flows. There is no need for advanced software: a pocket calculator will suffice for any situation.

6. Relations with IRR

Refutation of the IRR approach as a general approach to rate of return extends to any modification of such an approach which computes interim values automatically, without any value judgment. For example, XIRR, which is a financial function canned in Excel, , is the solution of an IRR equation where the coefficients represent daily cash flows (if, at day t, project generates no cash flow, then the coefficient of the tth-degree term in the polynomial is zero). The same applies to the well-known Modified Internal Rate of Return (MIRR): its interim values are concocted values of a modified project.

Refutation of IRR approach does not mean refutation of IRR as a rate of return. The set of a project's IRRs is a subset of R (the class of all project's AIRRs). Any IRR is then graphically identified by a point on the iso-value line. However, in the IRR approach, the causation effect is reversed: rather than requiring an exogenous estimation of the capital in order to determine the rate of return, the IRR approach requires that the rate of return is first determined via the IRR equation; the related capital is automatically and implicitly obtained as the IRR's counterimage: $\mathbb{C} = i^{-1}(k)$ where $i^{-1}(\cdot)$ denotes inverse function (see Figure 2).

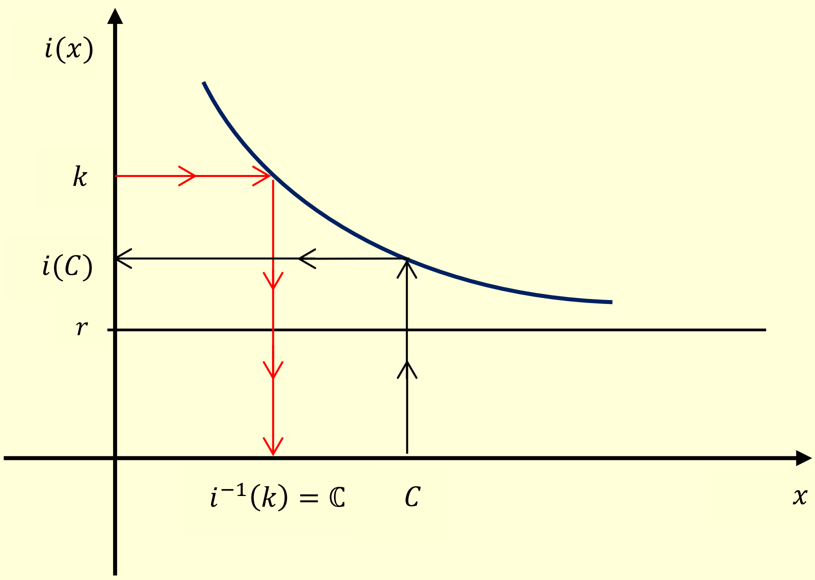

Figure 2. Any IRR belongs to the set R of the project's AIRRs, and, as such, it lies on the iso-value line. (The figure represents the case of a positive-NPV project with unique IRR and positive IRR-implied aggregate value.)

This reversed procedure imposes its own concocted values, so preventing the evaluator to make value judgments, which are essential in establishing whether a given AIRR is indeed the appropriate rate of return for the project at hand. However, just because the real-valued IRR (or IRRs) of a project represents an element of R, it can be an appropriate rate of return in those situations where it is *economically* meaningful. There may be some cases where $\mathbb{C}$ is the appropriate capital: for example, in a loan with constant interest rate, it is rather natural to set the principal outstanding as the capital invested by the lender, which implies that the invested capital grows at a constant force of return, which in turn means that $\mathbb{C}$ is the appropriate capital and, therefore, the appropriate AIRR is exactly the IRR: $i(\mathbb{C})$. The same applies for those financial securities where the interest rate is constant.

Technically, the AIRR paradigm does help IRR in several senses:

Project ranking. To compare projects' IRRs is illegitimate (this is not a flaw of the IRR, but a flaw of those scholars who insist in comparing rates of return referred to different capital bases). However, one can compute an adjusted IRR allowing for scale: letting B the benchmark capital,

$$k(B) = k + \frac{\mathbb{C}}{B}(k - r). \tag{32}$$

Maximization of $k(B)$ is equivalent to NPV maximization, so the adjusted IRR is a correct relative measure to be used in ranking projects.

No rate of return. If a project has no IRR, a *quasi-IRR* can nonetheless be computed, which is *quasi* an IRR of the project in the sense that it is the IRR of the (twin) project which most closely resembles the original project. Such a quasi-IRR represents an AIRR of the original project, associated with the IRR-implied capital of the twin project (see Pressacco et al. 2011).

Reinvestment assumption. The IRR has sometimes been criticized as it allegedly implies the assumption of reinvestment of cash flows at the IRR. We have shown that the IRR is a weighted average of holding period rates applied to the invested capitals $\mathbb{c}_t$, not to reinvested cash flows: reinvestment plays no role in the AIRR paradigm.

Varying costs of capital. One can use (27) along with any sequence $\mathbb{c}$ to cope with varying COCs. The project is acceptable if and only if the IRR exceeds an appropriate average COC, which is itself an AIRR of an equivalent-risk asset:

$$k > \bar{r} = w_1 r_1 + w_2 r_2 + \cdots + w_n r_n \tag{33}$$

where $w_t = \mathbb{c}_{t-1} v^{t,0} / C$ with $C = \mathbb{c}_0 v^{1,0} + \mathbb{c}_1 v^{2,0} + \cdots + \mathbb{c}_{n-1} v^{n,0}$.

Use of IRR in conjunction with other AIRRs. In those cases where some interim capital values are known and some others are not, IRR can be incorporated in the computation of the appropriate AIRR. Consider some convenient unit of time; let $T_m = \{t_1, t_2, \ldots, t_m\} \subset [0, n]$ be the set of dates where the investment's values c_{t_s} are available. The estimated interim value at time $\tau \in (t_s, t_{s+1})$ can be computed as the economic value:

$$V_\tau = \sum_{h=\tau+1}^{t_{s+1}} f_h (1+r)^{\tau-h} + c_{t_{s+1}} (1+r)^{\tau - t_{s+1}} \tag{34}$$

where r is the COC referred to the unit of time selected. Alternatively, if there is some reason supporting the assumption of constant growth between the dates t_s and t_{s+1}, one may replace COC with IRR to get the internal values

$$c_\tau(k) = \sum_{h=\tau+1}^{t_{s+1}} f_h(1+k)^{\tau-h} + c_{t_{s+1}}(1+k)^{\tau-t_{s+1}} \tag{35}$$

where k now denotes the internal rate of return of the cash-flow stream $\left(-c_{t_s}, f_{t_s+1}, \ldots, f_{t_{s+1}-1}, f_{t_{s+1}} + c_{t_{s+1}}\right)$ and where $-c_{t_s}$ and c_{t_s+1} represents a constructive purchase and a constructive sale, respectively. That is,

$$-c_{t_{s+1}} + \sum_{h=t_s+1}^{t_{s+1}} f_h(1+k)^{t_s-h} + c_{t_{s+1}}(1+k)^{\tau-t_{s+1}} = 0$$

For example, consider $-\$100$ at time 0, \$40 after six months, \$20 after one year, \$70 after one year and a half, so that $f = (-100, 40, 20, 70)$. Suppose the interim value after one year is $c_2 = 45$, while c_1 (interim value after half a year) is unknown. One can use either (34) or (35) to compute the missing value. For example, if (35) is used, we have $-100 + \frac{40}{1+k} + \frac{20+45}{(1+k)^2}$ whence $k = 3.07\%$. Therefore, the internal value is $c_1(3.07\%) = \frac{65}{1.0307} = 63.066$, so that $P(f, c) = (-100, 40, 20, 70; 100, 63.07, 45)$. Assuming $r = 8\%$, the NPV of the capitals is $C = 100 + \frac{63.07}{1.08} + \frac{45}{1.08^2} = 196.97$. Hence, the project rate of return: $i = 13.35\% = 8\% + 9.75(1.08)/196.97$. Evidently, if there is more than one segment where interim values are unavailable, one can even use a different AIRR for each segment (provided there is an economic rationale for each choice).

It is worth stressing that the IRR cannot be retrieved acritically and cannot be detached from the AIRR approach: (i) the use of the adjusted IRR just denies the IRR approach, because $k(B) = i(B)$ and because it implies the exogenous choice of the appropriate benchmark capital base (average capital? Average total investment cash flows? Other bases?); (ii) the choice of the quasi-IRR is just the choice of a project's AIRR, (iii) given that (33) holds for any $\mathbb{c}$, the evaluator needs choose one among infinitely many possible $\mathbb{c}$'s in order to single out the appropriate average COC; (iv) if an IRR is selected whenever interim capitals are not available, then there must be some good reason to prefer IRR over economic AIRR (or another AIRR); that is, there must be some good reason to believe that the project's internal values are more economically meaningful, as invested capitals, than the project's economic values.

Therefore, in all those cases where IRR can be retrieved as a legitimate rate of return, (capital) value judgments are necessary, which means that the IRR can be legitimately used only if it is embraced in the AIRR realm.

As a matter of fact, IRR is not the only rate of return that can be retrieved: the AIRR paradigm is capable of rescuing any other approach to rate of return, either appeared in the past literature or to appear in the future: letting κ be any possible rate of return derived by any model, it will represent the AIRR which corresponds to an invested capital of $C = (1 + r)NPV/(\kappa - r)$. Therefore, any (past or future) rate of return is some AIRR, and, for this reason, it is both technically reliable and NPV-compatible. As with IRR, the problem is whether the rate of return advocated is *economically* meaningful, that is, whether it indeed expresses the relative growth of the capital actually invested in the project. The answer to this question is domain-specific and must be supported by sound economic reasoning and empirical evidence capable of providing the rate of return with the appropriate economic informational content.

Conclusions

This paper is a refutation of the IRR approach and a corroboration of the (recently introduced) AIRR approach as a general approach to investment decisions and economic profitability. As such, it calls for a paradigm shift in the conceptualization of the rate-of-return notion.

In the *pars destruens*, the paper discusses eighteen fallacies of the IRR approach:

- F1. Multiple IRRs may occur
- F2. No IRR may occur
- F3. IRR cannot cope with varying COCs
- F4. IRR cannot measure the rate of return of an arbitrage strategy
- F5. Choice between mutually exclusive projects and project ranking are not consistent with the NPV rule
- F6. IRR cannot measure the rate of return on initial investment or on total investment cash flow
- F7. IRR either *never* or *always* signals a loss of the entire capital, depending on how the equation is framed
- F8. IRR of the expected investment is not equal to the expected IRR of the investment
- F9. The IRR approach does not fulfill value additivity: the value of a portfolio of projects is not equal to the sum of the project's values
- F10. The IRR neglects the project's operating life
- F11. The interim capitals implied by the assumption of constant force of return (internal values) have nothing to do with the actual capital values, based on empirical available data
- F12. The IRR-implied overall capital invested is a plug which imposes formal consistency with the NPV rather than deriving it from actual economic referents
- F13. The IRR-implied interim capitals are not unique; the IRR-implied excess returns are, therefore, not unique
- F14. If an on-going analysis of ex post performance is accomplished, the IRR equation generates a retrospective revision of the interim capitals fixed at preceding dates, so that intertemporal inconsistency cannot be avoided
- F15. The IRR, which is based on cash flows, is not capable of summarizing accounting information provided by those very accounting variables which the investor uses to estimate cash flows
- F16. The use of the IRR in Makeham's formula makes the latter incorrect in those cases where interest rate is variable over time (e.g., capital investments, variable-interest loans, investment funds)
- F17. The IRR is constant under changes in capital, so it does not signal that any two projects are empirically different
- F18. Ordinary spreadsheet functions such as IRR/XIRR in Excel do not inform investors about multiple IRRs. Excel's choice of an IRR depends on the investor's guess.

In the *pars construens*, we constructively show that a project's rate of return can be alternatively obtained by (i) dividing return by invested capital, (ii) computing the capital-weighted arithmetic mean of period rates, (iii) grossing-up the cost of capital by the investors' relative wealth increase, (iv) calculating the market-based interest per unit of overall capital invested. These definitions make the AIRR-based paradigm economically significant (it explicitly links wealth increase, return and capital), intuitive (it stems from simple intuitions), user-friendly (no equation is involved, only basic arithmetic operations), empirically-driven (choice of the capital is domain-specific, based on available economic referents), consistent with corporate financial theory and, in particular, with the net-present-value notion (a direct formal and conceptual link with NPV is provided), inclusive of other financial indexes (profitability index, benefit-cost ratio and IRR itself). We put the AIRR approach to the test and show that it is devoid of each and every flaw that plagues IRR.

Why does the AIRR paradigm succeed where the IRR approach fails? The reason just lies in the founding idea (admittedly, a tautological one) that a rate of return is an amount of return per unit of capital invested and that a project is more properly described as a stream of cash flows *and* capitals. Epistemologically, this implies that no rate of return for a project can be singled out without selecting, implicitly or explicitly, the capital base. The IRR approach lets an automatic procedure implicitly select an artificial capital which misrepresents the project's actual capital. In contrast, the AIRR model lets the evaluator exogenously select the proper capital base on the basis of the type of projects, the available economic information and the piece of information required (rate of return on overall capital, on initial investment, on total investment cash flow etc.).

This paper also introduces a new rate of return, the "economic AIRR", related to the *economic values* of the project. This rate of return has two favorable features. On one hand, it is a fast and frugal computational shortcut, for it can be computed even if interim capitals are not available. On the other hand, it is based on information derived from capital markets: it stems from basic principles of corporate financial theory and from the assumption of a well-functioning market where profitable investment opportunities are soon arbitraged away.

It should be clear that the refutation of the IRR *approach* does not imply that the IRR as a *rate of return* is refuted. The AIRR-based model encompasses the IRR (as well as any conceivable rate of return): the latter belongs to the AIRR family (it lies on the iso-value line). In those cases where there are empirical reasons which *exogenously* support the assumption that the interim capital grows at a constant force of return (typically, a loan with constant interest rate or a financial security whose value increases at a constant pace), the IRR-implied capital base is validated and IRR is indeed the appropriate rate of return. If, instead, no economic evidence is given that the IRR-implied capital represents the value of the economic resources deployed in the project, the IRR is interpretable as just one (out of infinitely many) inappropriate AIRRs.

The fundamental issue is: to choose one among infinitely many AIRRs. This is not a mathematical issue, but an economic and philosophical one. Indeed, it has to do with the so-called *problem of underdetermination of theory by data*, a well-known problem in the philosophy of science. We have shown that this indeterminacy problem can be solved, for a rate of return, by making value judgments. Only value judgments will provide a sound economic measure of worth; any reluctance to make explicit value judgments (i.e., the acritical acceptance of ineffective mathematical procedures) will prevent the achievement of the appropriate/correct economic rate of return.

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