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A class of singular first order differential equations with applications in reaction-diffusion

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Abstract

We study positive solutions $y(u)$ for the first order differential equation

$$y' = q(cy^{\frac{1}{p}} - f(u))$$

where $c > 0$ is a parameter, $p > 1$ and $q > 1$ are conjugate numbers and f is a continuous function in $[0, 1]$ such that $f(0) = 0 = f(1)$. We shall be particularly concerned with positive solutions $y(u)$ such that $y(0) = 0 = y(1)$. Our motivation lies in the fact that this problem provides a model for the existence of travelling wave solutions for analogues of the FKPP equation in one spacial dimension, where diffusion is represented by the p -Laplacian operator. We obtain a theory of admissible velocities and some other features that generalize classical and recent results, established for $p = 2$.

Key words: p -Laplacian, FKPP equation, heteroclinic, travelling wave, critical speed, sharp solution.

AMS Subject Classification: 34B18, 34C37, 35K57.

1 Introduction

In this paper we study some features of positive solutions to ordinary differential equations of the form

$$y' = q(cy_+^{\frac{1}{p}} - f(u)), \quad 0 \leq u \leq 1 \tag{1.1}$$

where $y_+ = \max(y, 0)$. We look for certain positive solutions $y = y(u)$ of (1.1) that vanish at one or both endpoints of the interval $[0, 1]$. Here p, q are positive numbers such that

$$\frac{1}{p} + \frac{1}{q} = 1,$$

$c > 0$ is a parameter, and $f : [0, 1] \rightarrow \mathbb{R}$ is a continuous function of types A, B or C, by which we mean

(Type A) $f(0) = f(1) = 0$ and $f(u) > 0$ if $u \in (0, 1)$.

(Type B) $f(0) = f(1) = 0$ and there exists $\alpha \in (0, 1)$ such that $f(u) = 0$ if $u \in [0, \alpha]$ and $f(u) > 0$ if $u \in (\alpha, 1)$.

(Type C) $f(0) = f(1) = 0$ and there exists $\alpha \in (0, 1)$ such that $f(u) < 0$ if $u \in (0, \alpha)$ and $f(u) > 0$ if $u \in (\alpha, 1)$.

This terminology was introduced by Berestycki and Nirenberg ([4]).

The motivation for considering (1.1) is the following. Consider the partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left[D(u) \left| \frac{\partial u}{\partial x} \right|^{p-2} \frac{\partial u}{\partial x} \right] + g(u), \quad (1.2)$$

which, in case $p = 2$, provides a model for a large variety of biological and chemical phenomena. We refer the reader to [3], [4],[11], [17], just to cite a few, and their references. In this equation g is a reaction term, while the first term in the right-hand side represents density dependent nonlinear diffusion in one-dimensional space. In the case $p = 2$ this is the well known FKPP equation (without convection term). Recently, models where the p -Laplacian operator replaces the usual Laplacian have been considered in the literature (e.g. [15], [8]).

When g is of one of the types A, B or C, $u = 0$ and $u = 1$ are two equilibrium solutions. An important problem related to this equation is that of finding travelling wave solutions, that is, solutions of the form $u(t, x) = U(x - ct)$ for some $c > 0$. Here c is the propagation speed of the wave. It is in addition required that the wave front $U(s)$ is defined in $(-\infty, +\infty)$ and satisfies $U(-\infty) = 1$, $U(+\infty) = 0$. This amounts to look for the solutions of the second order ordinary differential equation

$$(D(u)|u'|^{p-2}u')' + cu' + g(u) = 0 \quad (1.3)$$

satisfying the limit conditions

$$u(-\infty) = 1, \quad u(+\infty) = 0. \quad (1.4)$$

For certain values of the parameter c , such solutions are in addition monotone, with $u' < 0$ in their whole domain. Now if we set

$$-v := D(u)|u'|^{p-2}u',$$

for such monotone decreasing solutions, v may be seen as function of u . For simplicity, assume that $D(u) > 0$ for all $u \in (0, 1)$. Then a simple calculation shows that $v = v(u)$ must satisfy

$$\frac{1}{qD(u)^{q-1}} \frac{d}{du} v^q - c \left(\frac{v}{D(u)} \right)^{q-1} + g(u) = 0$$

and therefore **if we define**

$$y(u) = v(u)^q$$

the function y will solve (1.1) with $f(u) = D(u)^{q-1}g(u)$.

Moreover, the conditions (1.4) for monotone solutions defined in the real line imply that

$$u'(-\infty) = 0, \quad u'(+\infty) = 0.$$

This fact is well known in case $p = 2$ (see for instance [5]) and the argument easily carries out to the general case, as we show later for completeness. In terms of y this translates into

$$y(0) = 0, \quad y(1) = 0, \tag{1.5}$$

thus motivating the study of the existence of solutions of (1.1)-(1.5).

The study of admissible speeds for the problem (1.3) in case $p = 2$ has a long and rich history, starting with the seminal paper by Kolmogorov, Petrovski and Piscounov [10], including the in-depth approach by Aronson and Weinberger [3], and many recent contributions that the reader may find in the references. In Gilding and Kersner [6] and Malaguti and Marcelli [13] a singular integral equation technique has been used in the investigation of (1.3) and analogue equations for $p = 2$.

In this paper we propose, alternatively, to study the singular differential equation (1.1), thus constructing a first order model for admissible speeds and asymptotic behaviour (a method already used in [5]). This in turn provides information for (1.3) that is the counterpart of classic and recent results which have been obtained along years by many authors in case $p = 2$. In particular, we consider the differences between problems with functions of type A, on one hand, and with types B and C, on the other [3, 4]; we acknowledge the occurrence of sharp solutions, which were found in [16] and later systematized in [14]; we deal with sign change in diffusion density [11, 12] and with negative density diffusion [17].

The first order theory is developed in sections 2 to 4 and the applications to second order equations are given in section 6.

Notation. Let us introduce some notation and basic conditions to be used in the next sections. **For $c > 0$, we consider the function $\phi_c : [0, \infty) \rightarrow \mathbb{R}$** defined by

$$\phi_c(z) = cz^{1/p} - z. \tag{1.6}$$

We remark that ϕ_c vanishes at the two points 0 and c^q , is positive if and only if $0 < z < c^q$ and ϕ_c attains its absolute maximum M_c at a point $\omega_c \in (0, c^q)$. Namely: $\omega_c = (c/p)^q$, $M_c = \omega_c(p-1)$. **Now for any $x \in [0, M_c)$** , the function ϕ_c takes the value x at exactly two points, let us say $\omega_c^-(x) \in [0, \omega_c)$ and $\omega_c^+(x) \in (\omega_c, c^q)$: in particular, we set

$$J_c(x) = [\omega_c^-(x), \omega_c^+(x)].$$

If $x \geq M_c$ we set, by definition, $\omega_c^-(x) = \omega_c^+(x) = \omega_c$.

Also, we shall be dealing with functions f of type A satisfying

$$\sup_{u \in (0,1)} \frac{f(u)}{u^{q-1}} = \mu < +\infty \quad (1.7)$$

or the stronger property

$$\lim_{u \rightarrow 0^+} \frac{f(u)}{u^{q-1}} = \lambda < +\infty. \quad (1.8)$$

2 Functions of type A: existence of solutions and admissible speeds

Consider the boundary value problem

$$\begin{cases} y'(u) = q(c y_+(u)^{\frac{1}{p}} - f(u)), & 0 \leq u \leq 1 \\ y(0) = y(1) = 0, \end{cases} \quad (2.1)$$

for which we look for positive solutions in the interval $(0, 1)$. By a solution we mean a function $y \in C^1[0, 1]$ satisfying the equation above and the boundary conditions. In what follows *positive solution* means a solution y such that $y(u) > 0 \forall u \in (0, 1)$.

The following proposition allows us to conclude that the existence of a positive lower solution satisfying a strict inequality in the interval $(0, 1]$ is enough to get existence and uniqueness for (2.1).

Proposition 2.1. *Let f be of type A. Suppose that $s(u)$ is a C^1 -function in $[0, 1]$ such that $s(0) = 0$, $s(u) > 0$ if $u \in (0, 1)$ and for all $u \in (0, 1]$,*

$$s'(u) \leq q \left(c s(u)^{\frac{1}{p}} - f(u) \right). \quad (2.2)$$

Then (2.1) has a unique positive solution.

Proof. *By a well known argument,* (2.2) implies that there exists a solution $y(u)$ of the differential equation in (2.1) with $y(0) = 0$ such that $s(u) \leq y(u)$. Now consider the solution \bar{y} of the initial value problem

$$\bar{y}' = q \left(c \bar{y}_+^{\frac{1}{p}} - f(u) \right), \quad \bar{y}(1) = 0. \quad (2.3)$$

(In fact this problem enjoys uniqueness in $[0, 1]$ because the right-hand side of the equation is nondecreasing in the dependent variable.) It is easy to see that $\bar{y} \geq 0$ in $[0, 1]$. Moreover $0 < \bar{y}(u) < y(u)$ for all $u \in (0, 1)$. For, if u_0 is a zero of \bar{y} in $(0, 1)$, then the differential equation implies $\bar{y}'(u_0) < 0$, which is a contradiction with the fact that $\bar{y} \geq 0$; and if there exists $u_1 \in (0, 1)$ such that $\bar{y}(u_1) = y(u_1)$, then by uniqueness of solution we would have $\bar{y} = y$, which contradicts the fact that $\bar{y}(1) = 0$. By continuity we have $\bar{y}(0) = 0$. The fact that the solution is unique is a direct consequence of uniqueness for (2.3). \square

Remark 2.2. One could also have invoked the fact that the functions s and 0 are lower and upper solutions with respect to the periodic problem for (1.1) in $[0, 1]$.

Proposition 2.3. Assume that f is a function of type A in $[0, 1]$ satisfying (1.7). Then there exists a constant $c^* > 0$ (depending on f and p) such that (2.1) admits a unique positive solution if and only if $c \geq c^*$. Moreover we have the estimate $c^* \leq q^{\frac{1}{q}} p^{\frac{1}{p}} \mu^{\frac{1}{q}}$.

Proof. It is obvious that for c large enough, the inequality $\phi_c(\beta) \geq \mu$ has positive solutions β if and only if $\omega_c(p-1) \geq \mu$, that is,

$$c \geq q^{\frac{1}{q}} p^{\frac{1}{p}} \mu^{\frac{1}{q}}.$$

For one such β , let $s(u) = \beta u^q$. Then, for all $u \in (0, 1]$, we have

$$s'(u) = q\beta u^{q-1} \leq \left(c\beta^{\frac{1}{p}} - \mu\right) q u^{q-1} \leq q \left(c s(u)^{\frac{1}{p}} - f(u)\right).$$

The previous proposition allows us to conclude that for such value c , the boundary value problem (2.1) has a unique positive solution.

Now let c^* be the infimum of the values $c > 0$ such that problem (2.1) has a unique positive solution. The estimate for c^* follows from what we have just seen. Let us prove that for all $c > c^*$, problem (2.1) has a solution. Given $c_1 > c^*$, let us consider a value \tilde{c} such that (2.1) has a positive solution $y_{\tilde{c}}$ and $c^* < \tilde{c} < c_1$. For all $u \in (0, 1]$ we have

$$y'_{\tilde{c}} = q \left(\tilde{c} y_{\tilde{c}}^{\frac{1}{p}} - f(u) \right) < q \left(c_1 y_{\tilde{c}}^{\frac{1}{p}} - f(u) \right),$$

so $y_{\tilde{c}}$ is a lower solution for the problem with $c = c_1$ and by the previous proposition, we conclude the solvability for (2.1) with $c = c_1$.

To prove the solvability for $c = c^*$, consider a decreasing sequence c_n tending to c^* and the correspondent positive solutions y_n . First, the argument we used at the end of the proof of Proposition 2.1 shows that $y_n \leq y_1 \forall n$. It is also easy to conclude that the sequence (y_n) is uniformly bounded in $C^1([0, 1])$. By the Ascoli-Arzelà theorem, there exists a continuous function y^* such that $y_n \rightarrow y^*$ uniformly in $[0, 1]$. It turns out that y^* satisfies (2.1) with $c = c^*$ and $y^* > 0$ in $(0, 1)$.

Let us now prove that $c^* > 0$. If $c^* = 0$, using the above notation we would obtain $y^*(u) = -q \int_0^u f(s) ds$ meaning that $y^* < 0$ in $(0, 1]$, a contradiction. \square

3 Behaviour of the solutions near the origin

Throughout this section we suppose, as in the previous one, that f is a type A function, and try to get more detailed information about the behaviour of the solutions of equation (1.1) near 0 , under the only condition $y(0) = 0$. To this end, we need conditions (1.7) or (1.8) on f . We are going to show that, whenever (1.8) holds:

$$(a) \lim_{u \rightarrow 0} \frac{y(u)}{u^q} = \omega_c^-(\lambda) \quad \text{or} \quad (b) \lim_{u \rightarrow 0} \frac{y(u)}{u^q} = \omega_c^+(\lambda). \quad (3.1)$$

In particular we point out that, if the stronger condition

$$\sup_{u \in (0,1)} \frac{f(u)}{u^q} < +\infty \quad (3.2)$$

holds, then $\lambda = 0$, so that $\omega_c^-(\lambda) = 0$. In this case, actually, we can say more than (3.1a), namely:

$$\sup_{0 < u \leq 1} \frac{y(u)}{u^{pq}} < +\infty. \quad (3.3)$$

In order to show the properties above, we need some preliminary results. To this end, let $y \in \mathcal{S}_c$ be fixed, and put

$$\gamma(u) = \frac{y'(u)}{qu^{q-1}}, \quad z(u) = \frac{y(u)}{u^q}, \quad \lambda(u) = \frac{f(u)}{u^{q-1}}. \quad (3.4)$$

Let us denote respectively by γ^- , l^- and λ^- the lower limits of $\gamma(u)$, $z(u)$ and $\lambda(u)$ as $u \rightarrow 0$. Similarly, the scripts γ^+ , l^+ and λ^+ will stand for the corresponding upper limits. From

$$y' = q(cy^{\frac{1}{p}} - f(u)) \quad (3.5)$$

it is easy to see that $z(u)$, at least for small values of u , solves the following differential equation:

$$z' = \frac{q}{u}(\phi_c(z) - \lambda(u)). \quad (3.6)$$

Lemma 3.1. *Let f satisfy (1.7), $y \in \mathcal{S}_c$, λ^\pm and l^\pm be defined as above. Then:*

- (a) $M_c \geq \lambda^-$, that is: $c \geq (\lambda^- q)^{1/q} p^{1/p}$. In particular: $c^* \geq (\lambda^- q)^{1/q} p^{1/p}$.
- (b) $l^+ \in J_c(\lambda^-)$, $l^- \notin J_c(\lambda^+)$.
- (c) $\lambda^+ < M_c \Rightarrow \omega_c \notin (l^-, l^+)$.
- (d) (1.8) \Rightarrow (3.1).

Proof. (a) We remark that $\gamma^- \leq l^- \leq l^+ \leq \gamma^+$, as follows easily from Cauchy's theorem. Furthermore the function y , as long as it is positive, solves equation (3.5). If we divide both its sides by qu^{q-1} , and recall that $(q-1)p = q$, we get:

$$\gamma(u) = cz(u)^{1/p} - \lambda(u). \quad (3.7)$$

But now (3.7) implies that $\gamma^+ \leq c(l^+)^{1/p} - \lambda^-$, as we can argue on taking the upper limit of both sides as $u \rightarrow 0$. On the other hand, since $\gamma^+ \geq l^+$, we actually get $\lambda^- \leq \phi_c(l^+) \leq M_c$.

(b) It is enough to remark that $\phi_c(l^+) \geq \lambda^-$ and $\phi_c(l^-) \leq \lambda^+$. Indeed, the first inequality was shown in the previous step. As regards the latter, it can be achieved in a similar way, when taking in (3.7) a lower limit instead than an upper limit.

(c) Suppose, by contradiction, $l^- < \omega_c < l^+$: in particular, the maximum among the three values λ^+ , $\phi_c(l^-)$, $\phi_c(l^+)$, say h , is less than M_c . Now, let $j, m \in (h, M_c)$ such that $j < m$, and put $I = [\rho^-, \rho^+] := \phi_c^{-1}([m, M_c])$, so that $I \subseteq (l^-, l^+)$. According to the definitions of λ^+ , l^- and l^+ we can find $\delta > 0$ such that $\lambda(u) \leq j$ for $0 < u \leq \delta$, and two points u^- and $u^+ \in (0, \delta]$ such that $u^- < u^+$, $z(u^-) = \rho^+$, $z(u^+) = \rho^-$ and

$\rho^+ \leq z(u) \leq \rho^-$ for $u^- \leq u \leq u^+$. In particular, the interval $[u^-, u^+]$ must contain a point θ at which $z' < 0$. On the other hand $\phi_c(z(\theta)) \geq m$, so that (3.6) would yield the contradiction $z'(\theta) \geq (q/\theta)(m - j) > 0$. Hence, actually, $\omega_c \notin (l^-, l^+)$.

(d) Since $\lambda^- = \lambda^+ = \lambda$, from (a) we get $\lambda \leq M_c$, and applying claim (c) we infer that l^- and l^+ lie on the same side with respect to ω_c . On the other hand, let us replace λ^- and λ^+ in claim (b) by their common value λ : according to whether, respectively, l^- and l^+ lie to the left or to the right of ω_c , we infer what follows: either $l^- \leq \omega_c^-(\lambda) \leq l^+ \leq \omega_c$ or $\omega_c \leq l^- \leq \omega_c^+(\lambda) \leq l^+$. In both cases it is enough to show that $l^- = l^+$. As regards the former, let us suppose, by contradiction, that $l^- < l^+$: then both values l^- and l^+ can be approximated along a sequence of local extrema of z , which are, in particular, critical values. More precisely, we can find points a_i and b_i ($i \in \mathbf{Z}^+$) at which z' vanishes, in such a way that $a_i \rightarrow 0$, $b_i \rightarrow 0$ and the sequences $(z(a_i))_i$ and $(z(b_i))_i$ converge respectively to l^- and l^+ . Since $z'(a_i) = z'(b_i) = 0$, (3.6) entails $\phi_c(z(a_i)) = \lambda(a_i)$ and $\phi_c(z(b_i)) = \lambda(b_i)$. Now, let us first suppose $l^+ < \omega_c$: then both equalities $z(a_i) = \omega_c^-(\lambda(a_i))$ and $z(b_i) = \omega_c^-(\lambda(b_i))$ hold for large values of i : since ω_c^- is continuous, and $\lambda(u) \rightarrow \lambda$ as $u \rightarrow 0$, from the previous relations we get, as $i \rightarrow +\infty$, the contradiction $l^- = l^+ = \omega_c^-(\lambda)$. Now assume let $l^+ = \omega_c$: then possibly $z(b_i) = \omega_c^+(\lambda(b_i))$ for infinitely many values of i : in this case, however, $\omega_c = l^+ = \omega_c^+(\lambda)$, so that, actually, $\lambda = M_c$. Then we can write again $l^+ = \omega_c^-(\lambda)$, and get the same contradiction as before. Finally in the case $\omega_c \leq l^- \leq \omega_c^+(\lambda) \leq l^+$ the conclusion is straightforward by virtue of (b). \square

Corollary 3.2. *If (1.7), (1.8) hold and $\mu = \lambda$, then $c^* = q^{\frac{1}{q}} p^{\frac{1}{p}} \lambda^{\frac{1}{q}}$.*

Proof. It suffices to combine the Proposition 2.3 with Lemma 3.1 (a). \square

This generalizes the well known result for the case $p = 2$, where $\lambda = f'(0)$, for which $M = f'(0)$ implies $c^* = 2\sqrt{f'(0)}$.

Now, let $r, A, c > 0$ be fixed. For any function $y \in C([0, r])$ we denote by $N(y)$ the supremum of $|y(u)|/u^q$ for $0 < u \leq r$: then it is easy to check that the subspace V of $C([0, r])$ where $N(y) < +\infty$ is a Banach space with respect to the norm $\|y\| := N(y)$. Now we define a closed subset E of V and a map $T : E \rightarrow V$ as follows:

$$E = \{y \in V : y(u) \geq Au^q, 0 \leq u \leq r\}, \quad (3.8)$$

$$[T(y)](u) = q \int_0^u (cy(s)^{1/p} - f(s)) ds, \quad y \in E, \quad 0 \leq u \leq r. \quad (3.9)$$

Lemma 3.3. *Let f fulfil (1.8), $\nu = \sup \{f(u)/u^{q-1}; 0 < u \leq r\}$. Then the following properties hold.*

- (a) $T(E) \subseteq V$.
- (b) If $A > \omega_c$, $T : E \rightarrow V$ is a contraction with respect to $\|\cdot\|$.
- (c) If $\phi_c(A) \geq \nu$, then $T(E) \subseteq E$. In particular, if $\omega_c < A \leq \omega_c^+(\nu)$, T is a contraction of E into itself.

Proof. (a) If $y \in E$, then obviously $w := T(y) \in C([0, r])$. In order to prove that $\|w\| < +\infty$ we only need to divide both sides of the following inequality by u^q , and take the supremum for $0 < u \leq r$.

$$w(u) \leq qc \int_0^u y(s)^{1/p} ds \leq qc \int_0^u (\|y\|s^q)^{1/p} ds = c\|y\|^{1/p}u^q. \quad (3.10)$$

(b) We notice that, for any $\alpha > 0$, the function $y^{1/p}$ admits, on the half-line $[\alpha, +\infty)$, the Lipschitz constant $L(\alpha) = (p\alpha^{1/q})^{-1}$. Now, for $i = 1, 2$, let $y_i \in E$, $w_i = T(y_i)$. Then:

$$\begin{aligned} |w_2(u) - w_1(u)| &\leq cq \int_0^u |y_2(s)^{1/p} - y_1(s)^{1/p}| ds \leq \\ &\leq cq \int_0^u L(As^q) |y_2(s) - y_1(s)| ds \leq \\ &\leq \frac{cq}{p} \int_0^u (As^q)^{-1/q} \|y_2 - y_1\| s^q ds = \frac{c}{p} A^{-1/q} u^q \|y_2 - y_1\|. \end{aligned} \quad (3.11)$$

Also here we can divide the extreme sides by u^q and take the supremum for $0 < u \leq r$, so as to infer that $k = (c/p)A^{-1/q}$ is a Lipschitz constant for T with respect to $\|\cdot\|$. But the condition $A > \omega_c$ is just equivalent to $k < 1$.

(c) If $y \in E$ and $w = T(y)$, then $w(u) \geq q \int_0^u [c(As^q)^{1/p} - \nu s^{q-1}] ds$, where the right-hand side is precisely $(cA^{1/p} - \nu)u^q$. Hence $w(u) \geq Au^q$ if and only if $\phi_c(A) \geq \nu$. As regards the last claim, it is enough to remark that the two conditions $\phi_c(A) \geq \nu$ and $A > \omega_c$ hold together if and only if $\omega_c < A \leq \omega_c^+(\nu)$. \square

Remark 3.4. The condition $\phi_c(A) \geq \nu$ in Lemma 3.3 is equivalent to state that the function Au^q is a subsolution of (1.1) on $[0, r]$.

Proposition 3.5. *Let f satisfy (1.8), set $\bar{c} := (\lambda q)^{1/q} p^{1/p}$, and assume $c > \bar{c}$. Then the following properties hold true.*

- (a) \mathcal{S}_c contains exactly one function y which verifies (3.1b), say $y =: \psi_c$.
- (b) If $y \in \mathcal{S}_c$, $y \neq \psi_c$, then (3.1a) holds.
- (c) If $y \in \mathcal{S}_c$, $y \neq \psi_c$, then $\psi_c(u) > y(u)$ for any $u \in (0, 1]$.
- (d) If w is a subsolution of (1.1) and $w(0) = 0$, then $\psi_c \geq w$ on $[0, 1]$.
- (e) If $\theta > c$ then $\psi_\theta(u) > \psi_c(u)$ for any $u \in (0, 1]$.
- (f) $\sup \{ |\psi_\theta(u) - \psi_c(u)| / u^q; u \in (0, 1] \} \rightarrow 0$ as $\theta \rightarrow c$.
- (g) If $y \in \mathcal{S}_c$, $y \neq \psi_c$ and (3.2) holds, then y satisfies (3.3).

Proof. (a) From our condition on c and Lemma 3.1-(a) we get $M_c > \lambda$: therefore, if $r > 0$ is suitably small, the number ν which appears in Lemma 3.3 lies below M_c as well, that is $\omega_c < \omega_c^+(\nu)$. So, let $\omega_c < A < \omega_c^+(\nu)$, and define E and T as in (3.8), (3.9). Since E is a closed subset of the Banach space $(V, \|\cdot\|)$, Lemma 3.2-(c) and Banach's contraction

principle ensure that T admits a unique fixed point y , which obviously solves (1.1) on $[0, r]$ and fulfils the condition $y(0) = 0$. In particular, the extension of y to the whole interval $[0, 1]$ (as a solution of (1.1)) belongs to \mathcal{S}_c . On the other hand, since $y(u) \geq Au^q$ on $[0, r]$, of (3.1a) and (3.1b) only the latter can hold. As regards uniqueness, let $\tilde{y} \in \mathcal{S}_c$ fulfil (3.1b): then \tilde{y} belongs to the same space E as before, and is again a fixed point for T , so that, necessarily, $\tilde{y} = y$.

(b) It follows at once from Lemma 3.1-(d).

(c) Since y and ψ_c satisfy respectively (3.1(a)) and (3.1(b)), and $\omega_c^-(\lambda) < \omega_c^+(\lambda)$, the inequality $\psi_c(u) > y(u)$ surely holds in a right neighbourhood of 0, say $(0, \rho]$. By contradiction, let $\sigma \in (\rho, 1]$ be the first point at which the function $z = \psi_c - y$ vanishes: since $z' \geq 0$ on $[0, \sigma]$ and $z(0) = z(\sigma) = 0$, we should get the contradiction $\psi_c \equiv y$ on $[0, \sigma]$.

(d) By virtue of the previous claim, the inequality $\psi_c \geq y$ holds true for any $y \in \mathcal{S}_c$. On the other hand, since w is a subsolution of (1.1), we can find $y \geq w$ such that $y(0) = 0$ and (1.1) holds: then $w \leq y \leq \psi_c$.

(e) If $\theta > c$, then $\phi_\theta > \phi_c$ and therefore $\omega_c^+ < \omega_\theta^+$. Hence $\psi_c < \psi_\theta$ in a right neighborhood of 0, by virtue of (3.1b). Then the inequality in fact holds in $(0, 1]$.

(f) Let r, ν and A be again as in the previous steps. Since M_c, ω_c and $\omega_c^+(\nu)$ depend continuously on c , let $\alpha \in (\bar{c}, c), \beta > c$ such that $M_\alpha > \nu$ and $\omega_\beta < A < \omega_\alpha^+(\nu)$. For any $\theta \in U := (\alpha, \beta)$ put $c = \theta$ in (3.9), denote by T_θ the corresponding map and by ψ_θ^r the restriction of ψ_θ to $[0, r]$, which can be characterized as the unique fixed point of T_θ . We point out that the maps T_θ , for $\theta \in U$, are defined on the same set E we introduced in the proof of claim (a), a set which does not depend on θ . Furthermore, the map $(\theta, y) \mapsto T_\theta(y)$ is continuous, and $k = (\beta/p)A^{-1/q} < 1$ is a Lipschitz constant, with respect to the norm of V , for all maps $T_\theta, \theta \in U$. Then it is easy to show that the fixed point of T_θ depends continuously on θ . More precisely: the map $\theta \mapsto \psi_\theta^r$ is continuous from U to $(E, \|\cdot\|)$, and the same we can say, as a consequence, for the map $\theta \mapsto \psi_\theta(r)$ from U to \mathbb{R} . Then well-known results about the dependence on initial data of the solution of a Cauchy problem entail that, as $\theta \rightarrow c, \psi_\theta \rightarrow \psi_c$, uniformly on $[r, 1]$. Now, let us put $\Delta(\theta) = \|\psi_\theta^r - \psi_c^r\|$, and denote by $S(\theta)$ the supremum of $|\psi_\theta - \psi_c|$ over $[r, 1]$: according to the previous arguments, both $\Delta(\theta)$ and $S(\theta)$ converge to 0 as $\theta \rightarrow c$. On the other hand, the supremum which appears in our claim does not exceed $\max(\Delta(\theta), S(\theta)/r^q)$.

(g) Let $K < +\infty$ be the supremum in (3.2): in particular, as we already pointed out, (1.8) holds true with $\lambda = 0$. Since we are dealing with a function y which does not fulfil (3.1b), and $\omega_c^-(0) = 0$, from (3.1a) we argue that $y(u)/u^q$ converges to 0 as $u \rightarrow 0$, and the same we can say of $y(u)^{1/q}/u$. Now, let us suppose, by contradiction, that (3.3) is not satisfied: actually, in this case, the ratio $y(u)/u^{pq}$ is not bounded from above on any right neighbourhood of 0, and the same we can say of $y(u)^{1/p}/u^q$. By combining the two previous remarks, we easily find $\varepsilon > 0$ such that

$$\frac{y(\varepsilon)^{1/p}}{\varepsilon^q} \left(c - p \frac{y(\varepsilon)^{1/q}}{\varepsilon} \right) \geq K. \quad (3.12)$$

Let $h = y(\varepsilon)/\varepsilon^{pq}$: then the function $w(u) = hu^{pq}$ satisfies the conditions

$$(a) \ w(\varepsilon) = y(\varepsilon), \quad (b) \ w'(u) \leq q(cw(u)^{1/p} - f(u)), \quad 0 \leq u \leq \varepsilon. \quad (3.13)$$

Indeed, (3.13a) is obviously due to our choice of h , while (3.13b) can be proved as follows:

$$\begin{aligned} f(u) &\leq Ku^q \leq [c(y(\varepsilon)^{1/p}/\varepsilon^q) - p\varepsilon^{p-1}(y(\varepsilon)/\varepsilon^{pq})]u^q = \\ &= (ch^{1/p} - ph\varepsilon^{p-1})u^q \leq ch^{1/p}u^q - phu^{pq-1} = cw(u)^{1/p} - (w'(u)/q). \end{aligned} \quad (3.14)$$

In particular: the second inequality in (3.14) follows from (3.12). The inequality of the second line of (3.14) comes from $u \leq \varepsilon$ and $p + q = pq$. Then (3.13b) holds true as well, so that w is a subsolution of (1.1), and (3.13a) implies $y \leq w$ on $[0, \varepsilon]$. But now, from the expression of w , we conclude that (3.3) holds, in contrast with our initial assumption. \square

Remark 3.6. Let us consider the estimate from below which is given by Lemma 3.1(a) on the critical value c^* , and combine it with the final claim of Prop. 2.3: under the assumption (1.8), and according to the notation we introduced in Proposition 3.5, we can write $\bar{c} = q^{\frac{1}{q}}p^{\frac{1}{p}}\lambda^{\frac{1}{q}} \leq c^* \leq q^{\frac{1}{q}}p^{\frac{1}{p}}\mu^{\frac{1}{q}}$. We also point out that, in the limit case $c = \bar{c}$, the maximum value M_c of (1.6) is λ , so that $\omega_c^-(\lambda) = \omega_c^+(\lambda) = \omega_c$: hence (3.1) has no meaning for $c = \bar{c}$.

Theorem 3.7. *Let f satisfy (1.8), $c \geq c^*$ and, according to Proposition 2.3, let y be the only solution of (1.1) such that $y(0) = y(1) = 0$. Then:*

- (a) $c > c^* \Rightarrow y$ satisfies (3.1a).
- (b) $c = c^* \Rightarrow y$ satisfies (3.1b).

Proof. (a) Let $c^* < \theta < c$, put $c = \theta$ in (1.1) and call \tilde{y} the solution of the corresponding equation such that $\tilde{y}(0) = \tilde{y}(1) = 0$. Suppose, by contradiction, that (3.1a) does not hold, and exchange the roles of θ and c in Prop. 3.5e, so as to get $y = \psi_c > \psi_\theta \geq \tilde{y}$ on $(0, 1]$. In particular, we get the contradiction $0 = y(1) > \tilde{y}(1)$.

(b) It is enough to prove that (3.1a) $\Rightarrow c > c^*$. So, let us suppose that $y \in \mathcal{S}_c$, $y \neq \psi_c$: then, from Prop. 3.3c, we infer that $\psi_c(1) > y(1) = 0$. On the other hand, Prop. 3.5f ensures, in particular, that the map $c \mapsto \psi_c(1)$ is continuous, so that $\psi_\theta(1) > 0$ for some $\theta < c$. Now, let us put $c = \theta$ in (1.1), and denote by \tilde{y} the solution of the corresponding equation which fulfils the condition $\tilde{y}(1) = 0$. By the same arguments as in the previous section, we get $\tilde{y}(0) = 0$ as well: hence $\theta \geq c^*$, so that $c > c^*$. \square

The reader may find related and complementary results in [9] and in [1].

4 Functions of types B and C: existence of solutions

Let us now consider the cases where f is a type B or type C function.

Lemma 4.1. *Assume f is continuous in $[0, 1]$, $f(0) = 0$ and*

$$\liminf_{u \rightarrow 0} \frac{f(u)}{u^{q-1}} > -\infty. \quad (4.1)$$

Then any solution of

$$y' = q(cy_+^{\frac{1}{p}} - f(u)), \quad y(0) = 0, \quad (4.2)$$

positive in a neighborhood of 0, satisfies

$$\sup_{u \in (0,1]} \frac{y(u)}{u^q} < +\infty.$$

Proof.

Claim: Given $k > 0$, there exists $M > 0$ such that any solution of

$$z' = q(c z_+^{\frac{1}{p}} + k u^{q-1}), \quad z(0) = 0 \quad (4.3)$$

positive in a neighborhood of 0, satisfies $z(u) \leq M u^q$, $0 \leq u \leq 1$.

If we set $w(u) = z(u)^{1/q}$, we have $w' = c + k(\frac{u}{w})^{q-1}$, $w(0) = 0$. Defining

$$l = \liminf_{u \rightarrow 0} \frac{w(u)}{u}, \quad L = \limsup_{u \rightarrow 0} \frac{w(u)}{u}$$

and

$$l' = \liminf_{u \rightarrow 0} w'(u), \quad L' = \limsup_{u \rightarrow 0} w'(u)$$

we obtain $c \leq l' \leq l \leq L \leq L' = c + \frac{k}{l^{q-1}} < +\infty$ and our Claim follows.

Now choose $k > 0$ so that $-f(u) < k u^{q-1}$ if $0 < u \leq 1$. For each $h > 0$ consider the solution z_h of

$$z'_h = q(c z_{h+}^{\frac{1}{p}} + k u^{q-1}), \quad z_h(0) = h. \quad (4.4)$$

Then z_h converges, as $h \rightarrow 0$, to the maximal solution of (4.3). On the other hand if we pick a solution y of (4.2) it is clear that $y < z_h$. Using the Claim, we obtain the conclusion of the lemma.

Theorem 4.2. *Let f be a type B or a type C function. In the latter case assume $\int_0^1 f(s) ds > 0$ and (4.1) holds. Then there exists a number $\hat{c} > 0$ such that the boundary value problem $y' = q(c y_+^{\frac{1}{p}} - f(u))$, $0 \leq u \leq 1$, $y(0) = y(1) = 0$ has a positive solution if and only if $c = \hat{c}$.*

Proof. By the hypothesis there exists $\alpha \in (0, 1)$ so that $f > 0$ in $(\alpha, 1)$ and either $f \equiv 0$ or $f < 0$ in $(0, \alpha)$. For $c \geq 0$ consider the Cauchy problem

$$y' = q(c y_+^{\frac{1}{p}} - f(u)), \quad y(1) = 0, \quad (4.5)$$

which, as we have already remarked, has a unique solution y_c in $[0, 1]$. Also, the usual compactness argument shows that y_c depends continuously on $c \in [0, +\infty)$ in the norm of $C([0, 1])$. Clearly, $y_c(u) \geq 0$ at least for $u \in (\alpha, 1)$. In particular, by our assumptions, $y_0(u) = q \int_u^1 f(s) ds > 0$ for all $u \in [0, 1)$.

Step 1: solutions decrease with c . Given $c_1 < c_2$, the corresponding solutions $y_1 \equiv y_{c_1}$ and $y_2 \equiv y_{c_2}$ are such that $y_1(u) > y_2(u)$ whenever $y_1(u) > 0$. In fact we cannot have $y_1 < y_2$ in any open subinterval of $(\alpha, 1)$, otherwise $y_1 - y_2$ would be decreasing in that interval, contradicting the fact that it must reach the value 0.

Set $\hat{c} = \sup\{c > 0 \mid y_c(u) > 0 \forall u \in (0, 1)\}$.

Step 2: $0 < \hat{c} < +\infty$. It is obvious that $\hat{c} > 0$. If $\hat{c} = +\infty$, there exists $c_n \rightarrow +\infty$ with $y_n \equiv y_{c_n} > 0$ in $(0, 1)$. If f is type B, then

$$y_n(u)^{\frac{1}{q}} = y_n(\alpha)^{\frac{1}{q}} - c_n(\alpha - u) \quad (4.6)$$

for $u \in (0, \alpha)$. Note also that $y_n(\alpha) \leq q \int_{\alpha}^1 f(s) ds$. Hence y_n must become negative for n large, a contradiction. If f is type C the same argument applies because then the solution in $(0, \alpha)$ must stay below the function given by the expression in the righthandside of (4.6).

Step 3: $y_{\hat{c}}(0) = 0$ and $y_{\hat{c}}(u) > 0 \forall u \in (0, 1)$. By definition of \hat{c} and continuous dependence on c , $y_{\hat{c}}$ must vanish in $[0, \alpha]$. Let $\gamma \in [0, \alpha]$ be its largest zero. If $\gamma > 0$, then for $c < \hat{c}$ and $u \leq \gamma$

$$y_c(u)^{\frac{1}{q}} \leq y_c(\gamma)^{\frac{1}{q}} - c(\gamma - u)$$

and since $y_c(\gamma) \rightarrow 0$ as $c \rightarrow \hat{c}$, if $\hat{c} - c$ is sufficiently small y_c must vanish in $(0, \gamma)$, contradicting the definition of \hat{c} .

Step 4: If $c > \hat{c}$ and f is of type B, then $y_c \equiv 0$ in some interval $[0, \gamma]$, $0 < \gamma \leq \alpha$. By Step 1, $0 < y_c(\alpha) < y_{\hat{c}}(\alpha)$. The graph of y_c cannot meet the graph of $y_{\hat{c}}$ in $[0, \alpha]$ if $y_c > 0$ in $(0, \alpha)$. Hence there exists $\gamma \in (0, \alpha]$ such that $y_c(\gamma) = 0$ and the claim follows.

Step 5: If $c > \hat{c}$ and f is of type C, then $y_c(0) < 0$. As in the previous step, $0 < y_c(\alpha) < y_{\hat{c}}(\alpha)$. Since $f > 0$ in $(0, \alpha)$ we easily obtain $y_c(u) < y_{\hat{c}}(u) \forall u \in [0, \alpha]$.

Step 6: If $c < \hat{c}$, then $y_c(0) > 0$. Suppose to the contrary that $y_c(0) = 0$. By the previous arguments $y_c > y_{\hat{c}}$ on $(0, 1)$.

Case 1: f is of type B. By separation of variables, the only solution of $y' = q d y_+^{\frac{1}{p}}$ satisfying $y(0) = 0$ and positive in a neighborhood of zero is the function $y_0(u) \equiv d^q u^q$. Hence we obtain $c^q u^q > \hat{c}^q u^q$ in $[0, \alpha]$, a contradiction.

Case 2: f is of type C. By a lower solution argument, (4.2) with $c = \hat{c}$ has a solution $z(u)$ such that $z(0) = 0$ and $z > y_{\hat{c}}$ in $[0, \alpha]$. But we now show that such solutions must coincide, obtaining a contradiction. Let z, w be two solutions of (4.2). If $z \neq w$ it is easily seen that they are ordered, say $z < w$ in $(0, \alpha)$. By the preceding Lemma there exists a constant $M > 0$ so that, with a computation similar to that of (3.11),

$$0 \leq w(u) - z(u) \leq qc \int_0^u \frac{w(s) - z(s)}{p(c^q s^q)^{1/q}} ds \leq \frac{M}{p} u^q, \quad 0 \leq u < \alpha.$$

Iterating this argument we obtain

$$0 \leq w(u) - z(u) \leq \frac{M}{p^2} u^q, \quad 0 \leq u < \alpha$$

and in fact

$$0 \leq w(u) - z(u) \leq \frac{M}{p^k} u^q, \quad 0 \leq u < \alpha$$

for all integers $k \in \mathbb{N}$. We conclude that $z \equiv w$ in $[0, \alpha]$. □

Let f be a type A function such that $\sup_{0 < u < 1} \frac{f(u)}{u^{q-1}} < +\infty$. It is easy to see that there exists a decreasing sequence of positive values ϵ_n tending to zero such that the

corresponding sequence of type B functions

$$f_n(u) = \begin{cases} 0, & u \in [0, \epsilon_{n+1}] \\ \min(l_n(u), f(u)), & u \in [\epsilon_{n+1}, \epsilon_n] \\ f(u), & u \in [\epsilon_n, 1], \end{cases}$$

where $l_n(u) = f(\epsilon_n) \frac{u - \epsilon_{n+1}}{\epsilon_n - \epsilon_{n+1}}$, is increasing and tends uniformly to $f(u)$. Let $\hat{c}(f_n)$ be the unique value such that the boundary value problem (2.1) with $f(u) = f_n(u)$ has a positive solution. The following theorem uses this fact to give a new characterization of the critical speed c^* introduced in section 2. Results of this type may be also found in [4, 9].

Theorem 4.3. *Consider a type A function f with $\sup_{u \in (0,1)} \frac{f(u)}{u^{q-1}} < +\infty$ and a sequence of type B functions f_n in the conditions mentioned above. Then $\hat{c}(f_n)$ is an increasing sequence and $\lim \hat{c}(f_n) = c^*$ where c^* is associated to f in Proposition 2.3.*

Proof. Consider two arbitrary consecutive elements f_n and f_{n+1} of the sequence of type B functions considered above. These two functions are different in some interval $(\epsilon_{n+2}, b) \subset (0, \epsilon_n)$, where $f_n(u) < f_{n+1}(u)$, having the same values outside the interval $(\epsilon_{n+2}, \epsilon_n)$. Consider the problem

$$y' = q(cy_+^{\frac{1}{p}} - f_{n+1}(u)), \quad y(0) = 0, \quad (4.7)$$

and let $\hat{c}(f_{n+1})$ be the unique value such that there exists a solution of (4.7) satisfying $y(1) = 0$. It is easy to see that this solution $y_{n+1}(u)$ satisfies

$$y_{n+1}'(u) \leq q(\hat{c}(f_{n+1}) (y_{n+1})_+(u)^{\frac{1}{p}} - f_n(u)), \quad \forall u \in [0, 1]$$

with strict inequality for $u \in (\epsilon_{n+2}, b)$. so the equation

$$y'(u) = q(\hat{c}(f_{n+1}) y_+(u)^{\frac{1}{p}} - f_n(u))$$

has a solution $z_n(u)$ such that $z_n(0) = 0$, $z_n(u) > y_{n+1}(u)$ for $u \in (\epsilon_{n+2}, 1]$ and in fact $z_n(u) > y_{n+1}(u)$ for $u \in (0, 1]$ (since we see that $z_n - y_{n+1}$ increases). Since the positive solution starting from $(0, 0)$ is unique, the solution $w_n(u)$ of the same equation with $w_n(1) = 0$ must vanish at some point $\gamma_n \in (0, \epsilon_n]$ (see the argument in Step 4 of the preceding proof). Hence by the construction of \hat{c}_n we have $\hat{c}(f_n) < \hat{c}(f_{n+1})$. This allows us to conclude that $\hat{c}(f_n)$ is an increasing sequence.

Now let $c \geq c^*$ and consider the unique solution $z(t)$ of (2.1). Then we have $z' < q(cz_+^{\frac{1}{p}} - f_n(u))$ for $u \in (0, \epsilon_{n+1})$ and $z' \leq q(cz_+^{\frac{1}{p}} - f_n(u))$ for $u \in [0, 1]$, The same argument as above allows us to conclude that $\hat{c}(f_n) < c$ and consequently, the sequence is bounded from above by c^* . A simple application of Ascoli-Arzelà's lemma allows us to conclude that the solutions y_n tend to a solution of (2.1). This implies that $\hat{c}(f_n) \rightarrow l \geq c^*$ and consequently we conclude that $\hat{c}(f_n) \nearrow c^*$ \square

5 Behaviour near $u = 1$

Lemma 5.1. *Let $c > 0$, and let y be a positive solution of (1.1) in some interval $(a, 1)$ such that $y(1) = 0$. Suppose that*

$$m := \lim_{u \rightarrow 1^-} \frac{f(u)}{(1-u)^{q-1}} < \infty \quad \text{exists.} \quad (5.1)$$

Then $\lim_{u \rightarrow 1^-} \frac{y(u)}{(1-u)^q}$ exists and is the root α of $\alpha + c\alpha^{1/p} = m$.

Proof. *Claim:* $\sup_{u \in (a, 1)} \frac{y(u)}{(1-u)^q} < \infty$. Just take $m_1 > m$ and integrate the inequality $y'(u) \geq -qm_1(1-u)^{q-1}$ in a suitable interval of the form $(b, 1)$.

Let us complete the proof, setting $y(u) = z(u)(1-u)^q$. Then z is a solution of the differential equation

$$z' = \frac{q}{1-u} (z + cz^{\frac{1}{p}} - \mu(u)), \quad (5.2)$$

where $\mu(u) = \frac{f(u)}{(1-u)^{q-1}}$. Arguing as in the proof of Lemma 3.1-(d) it is easy to see that $\liminf_{u \rightarrow 1} z(u)$ and $\limsup_{u \rightarrow 1} z(u)$ must be equal and must coincide with the root of $\alpha + c\alpha^{1/p} = m$.

6 Some applications to the second order problem

In this section we consider the problem (1.3)-(1.4) with several assumptions on D and g .

Lemma 6.1. *Let g be a function of type A. The derivative of a nonincreasing solution u of (1.3) with $0 < u(x) < 1$ does not vanish. If the interval where u is defined extends to $+\infty$, we have $\lim_{t \rightarrow +\infty} D(u)|u'|^{p-1} = 0$. A similar statement holds with $-\infty$ replacing $+\infty$.*

Proof. If there exists x_0 such that $u'(x_0) = 0$ and $0 < u(x_0) < 1$, using the differential equation we would have $(D(u)|u'|^{p-2}u')'|_{(x=x_0)} < 0$, which contradicts the fact that $D(u)|u'|^{p-2}u'$ attains a maximum at $x = x_0$.

Concerning the statement on the limit we will only consider $+\infty$, the case of $-\infty$ being similar. Suppose towards a contradiction that $\liminf_{x \rightarrow +\infty} D(u)|u'|^{p-1} = \delta > 0$. We can take two sequences t_n and s_n tending to $+\infty$ such that $u'(t_n) \rightarrow 0$ and $D(u(s_n))|u'|^{p-1}(s_n) \rightarrow \delta$. Integrating the differential equation in $[0, t_n]$, we easily conclude that the sequence $\int_0^{t_n} g(u(x)) dx$ is bounded and therefore $\int_0^{+\infty} g(u(x)) dx$ is convergent. Consequently we have

$$\begin{aligned} 0 &= \int_{t_n}^{s_n} \left(D(u)|u'|^{p-2}u' \right)' + cu' + g(u) dx = \\ &= D(u(s_n))|u'(s_n)|^{p-2}u'(s_n) - D(u(t_n))|u'(t_n)|^{p-2}u'(t_n) + c(u(s_n) - u(t_n)) + \int_{t_n}^{s_n} g(u) dx \end{aligned}$$

and making $n \rightarrow \infty$ we would get the contradiction $-\delta = 0$. \square

We set

$$f(u) = D(u)^{q-1}g(u), \quad u \in [0, 1]. \quad (6.1)$$

and we assume

$$(D1) \quad D \in C^1[0, 1] \text{ and } D > 0 \text{ in } (0, 1).$$

$$(G1) \quad g \text{ is a function of type A.}$$

Clearly, f given by (6.1) is of type A.

Proposition 6.2. *We have that $u(t)$ is a monotone solution of (1.3) in some interval (a, b) such that $0 < u(t) < 1 \forall t \in (a, b)$ and*

$$\lim_{t \rightarrow a^+} u(t) = 1, \quad \lim_{t \rightarrow b^-} u(t) = 0, \quad \lim_{t \rightarrow a^+} D(u)|u'|^{p-1} = \lim_{t \rightarrow b^-} D(u)|u'|^{p-1} = 0 \quad (6.2)$$

if and only if $y = v^q$ where $v = D(u)|u'|^{p-1}$ is a positive solution of (1.1)-(1.5).

Proof. The necessary condition was essentially proved in the introduction. Conversely, given a positive solution $y(u)$ of (1.1)-(1.5), we recover a solution of (1.3) by solving the Cauchy problem

$$u' = -\frac{y(u)^{\frac{1}{p}}}{D(u)^{q-1}}, \quad u(0) = \frac{1}{2}. \quad (6.3)$$

The solution of (6.3) exists in (t_-, t_+) , where

$$t_- = -\int_{1/2}^1 \frac{D(u)^{q-1} du}{y(u)^{\frac{1}{p}}}, \quad t_+ = \int_0^{1/2} \frac{D(u)^{q-1} du}{y(u)^{\frac{1}{p}}}. \quad (6.4)$$

□

Consider the assumptions

$$\sup_{u \in (0,1)} \frac{g(u)}{u^{q-1}} < +\infty. \quad (6.5)$$

$$\sup_{u \in (0,1)} \frac{g(u)}{(1-u)^{p-1}} < +\infty. \quad (6.6)$$

and the following strengthened form of (D1)

$$(D1') \quad D \in C^1[0, 1] \text{ and } D > 0 \text{ in } [0, 1].$$

Under the conditions (D1), (G1), (6.5), Proposition 2.3 is applicable to $f = D^{q-1}g$ and a positive number c^* is associated to f . This number plays a central role in the following theorem, which in case $p = 2$ corresponds to well known results, see [10, 3, 13] and references. Note that according to the results in sections 2 and 3 we have the estimate

$$q^{\frac{1}{q}} p^{\frac{1}{p}} \left(\liminf_{u \rightarrow 0} \frac{D(u)^{q-1} g(u)}{u^{q-1}} \right)^{\frac{1}{q}} \leq c^* \leq q^{\frac{1}{q}} p^{\frac{1}{p}} \left(\sup_{u \in (0,1)} \frac{D(u)^{q-1} g(u)}{u^{q-1}} \right)^{\frac{1}{q}}. \quad (6.7)$$

Theorem 6.3. *Suppose that (D1'), (G1), (6.5) and (6.6) are satisfied and let $1 < p \leq 2$. Then (1.3)-(1.4) has a decreasing solution $u(t)$ taking values in $(0, 1)$ if and only if $c \geq c^*$. That solution is unique up to translation.*

If, further, $g^(0) \equiv \lim_{u \rightarrow 0^+} \frac{g(u)}{u^{q-1}}$ exists then*

$$\lim_{t \rightarrow +\infty} \frac{u'(t)}{u(t)^{q-1}} = \begin{cases} -\frac{\omega_c^- (D(0)^{q-1} g^*(0))^{1/p}}{D(0)^{q-1}}, & c > c^* \\ -\frac{\omega_{c^*}^+ (D(0)^{q-1} g^*(0))^{1/p}}{D(0)^{q-1}}, & c = c^* \end{cases}$$

Proof. Let $y(u)$ be a solution of (1.1)-(1.5) for some $c \geq c^*$. Consider the Cauchy problem (6.3). The solution of (6.3) exists in (t_-, t_+) , given by (6.4). Since (1.1) implies $y' \leq qcy^{1/p}$, it turns out that

$$\sup_{u \in (0,1)} \frac{y(u)}{u^q} < +\infty, \quad (6.8)$$

and it is clear that $t_+ = +\infty$, since $q \geq 2$. Similarly the estimate we get from (6.5) and (D1') on

$$\frac{f(u)}{(1-u)^{p-1}}$$

implies $y(u) \leq C(1-u)^p$ for some constant C and therefore $t_- = -\infty$. The solution of (6.3) satisfies (1.3)-(1.4).

On the other hand, since we can write $\frac{u'(t)}{u(t)^{q-1}} = -\left(\frac{y(u)}{u^q}\right)^{\frac{1}{p}} \frac{1}{D(u)^{q-1}}$, the last statement follows easily from Theorem 3.7. \square

Next we shall consider the case where D is “degenerate” in the sense that

$$(D2) \quad D \in C^1[0, 1], \quad D > 0 \text{ in } (0, 1], \quad D(0) = 0 \text{ and } D'(0) > 0.$$

The following theorem corresponds to results given in [16, 13] for $p = 2$.

Theorem 6.4. *Suppose that (D2), (G1), (6.5) and (6.6) are satisfied. Let $1 < p \leq 2$. Then*

(i) *Problem (1.3)-(1.4) has a decreasing solution $u(t)$ taking values in $(0, 1)$ if and only if $c > c^*$.*

(ii) *If $c = c^*$ (1.3) has a decreasing solution defined in $(-\infty, 0]$ with $u(-\infty) = 1$, $u(0) = 0$ and*

$$u'(0) = -\left(\frac{c^*}{D'(0)}\right)^{q-1}. \quad (6.9)$$

Those solutions are unique up to translation.

(iii) *If $c < c^*$, problem (1.3) has no decreasing solution in any interval $(-\infty, b)$ with $\lim_{t \rightarrow -\infty} u(t) = 1$, $\lim_{t \rightarrow b^-} u(t) = 0$.*

The solutions considered in (ii) are called *sharp solutions*.

Proof. Proceeding as in the preceding proof, we consider (6.3). Under the assumptions of the theorem, we have $f^*(0) = 0$. To compute the limit of the right-hand side of (6.3) we write

$$\lim_{u \rightarrow 0^+} \frac{y(u)^{\frac{1}{p}}}{D(u)^{q-1}} = \lim_{u \rightarrow 0^+} \left(\frac{y(u)}{u^q} \right)^{\frac{1}{p}} \left(\frac{u}{D(u)} \right)^{q-1}. \quad (6.10)$$

Noting that

$$\omega_c^-(0) = 0, \quad \omega_c^+(0) = c^q$$

and using Theorem 3.7 we may conclude:

$$\lim_{u \rightarrow 0^+} \frac{y(u)^{\frac{1}{p}}}{D(u)^{q-1}} = \begin{cases} 0, & c > c^* \\ \left(\frac{c^*}{D'(0)} \right)^{q-1}, & c = c^* \end{cases} \quad (6.11)$$

Moreover if $c > c^*$ and since f satisfies (3.2b), Proposition 3.5-(g) implies $t_+ = +\infty$. The fact that $t_- = -\infty$ follows as in the proof of the preceding theorem.

If $c = c^*$ it is clear that the solution of (6.3) can remain positive for $t \geq 0$ only in some finite interval $[0, b)$ so that $\lim_{t \rightarrow b^-} u(t) = 0$ and $\lim_{t \rightarrow b^-} u'(t) = -\left(\frac{c^*}{D'(0)} \right)^{q-1}$. \square

We next consider a case of negative diffusion (see [17]), considering the assumption (D3) $D(u) < 0 \forall u \in (0, 1]$; $D(0) = 0$ and $D'(0) < 0$.

and introducing the conditions

$$\sup_{u \in (0,1)} \frac{g(u)}{(1-u)^{q-1}} < +\infty, \quad \sup_{u \in (0,1)} \frac{g(u)}{u^p} < +\infty. \quad (6.12)$$

In [17] the authors consider (1.3)-(1.4) with $D < 0$ for $p = 2$ and reduce this problem to a non-singular system, assuming (D3) and additional regularity assumptions.

It is easily seen that the change of variables

$$u(-t) = 1 - z(t)$$

yields an equivalence between (1.3)-(1.4) and

$$(E(z)|z'|^{p-2}z')' + cz' + h(z) = 0, \quad z(-\infty) = 1, \quad z(+\infty) = 0 \quad (6.13)$$

where

$$E(z) = -D(1-z), \quad h(z) = g(1-z), \quad 0 \leq z \leq 1. \quad (6.14)$$

We have the following result that contains some statements made in Proposition 3 of [17] for $p = 2$.

Theorem 6.5. *Let $1 < p \leq 2$. Suppose that (D3), (G1) and (6.12) are satisfied. Then there exists $c^* > 0$ such that (1.3)-(1.4) has a decreasing solution $u(t)$ taking values in $(0, 1)$ if and only if $c \geq c^*$. Moreover*

$$q^{\frac{1}{q}} p^{\frac{1}{p}} \left(\liminf_{u \rightarrow 1} \frac{|D(u)|^{q-1} g(u)}{(1-u)^{q-1}} \right)^{\frac{1}{q}} \leq c^* \leq q^{\frac{1}{q}} p^{\frac{1}{p}} \left(\sup_{u \in (0,1)} \frac{|D(u)|^{q-1} g(u)}{(1-u)^{q-1}} \right)^{\frac{1}{q}}. \quad (6.15)$$

Those solutions are unique up to translation.

Proof. Let us start with problem (6.13). We consider the associated first order problem

$$y' = q(c y_+^{\frac{1}{p}} - E(z)^{q-1} h(z)), \quad 0 \leq u \leq 1, \quad y(0) = y(1) = 0. \quad (6.16)$$

There exists c^* such that this problem has a positive solution if and only if $c \geq c^*$ and c^* satisfies the desired estimates. We recover the solution of (6.13) via the differential equation

$$z' = -\frac{y(z)^{\frac{1}{p}}}{E(z)^{q-1}}, \quad z(0) = 1/2.$$

As in the proof of Theorem 6.3 we see, using the first condition (6.12), that the solution is defined in an interval (t_-, t_+) where $t_+ = +\infty$. The second condition (6.12) and (D3) imply, as is easily seen, that $y(z) \leq K(1-z)^{p+q}$ for some constant K . Combining this with (D3), it turns out that the integrand in the expression of t_- is bounded below by some multiple of $\frac{1}{1-u}$ and therefore $t_- = -\infty$. Setting $u(t) = 1 - z(-t)$ we obtain the desired solutions of (1.3). \square

In a similar way we are able to deal with the analogue of a model considered in [18].

Theorem 6.6. *Let $1 < p \leq 2$. Assume $-g$ is a function of type A, (D2), (6.5) and (6.6) hold. Then there exists $-c^* < 0$ such that (1.3)-(1.4) has a decreasing solution $u(t)$ taking values in $(0, 1)$ if and only if $c \leq -c^*$. Moreover*

$$q^{\frac{1}{q}} p^{\frac{1}{p}} \left(\liminf_{u \rightarrow 1} \frac{D(u)^{q-1} |g(u)|}{(1-u)^{q-1}} \right)^{\frac{1}{q}} \leq c^* \leq q^{\frac{1}{q}} p^{\frac{1}{p}} \left(\sup_{u \in (0,1)} \frac{D(u)^{q-1} |g(u)|}{(1-u)^{q-1}} \right)^{\frac{1}{q}}. \quad (6.17)$$

Those solutions are unique up to translation.

Proof. We use the change of variables $u(-t) = 1 - z(t)$ again. Then the problem (1.3)-(1.4) turns into

$$(F(z)|z'|^{p-2}z')' - cz' + h(z) = 0, \quad z(-\infty) = 1, \quad z(+\infty) = 0, \quad (6.18)$$

where

$$F(z) = D(1-z), \quad h(z) = -g(1-z), \quad 0 \leq z \leq 1. \quad (6.19)$$

\square

Finally we examine a situation where both D and g change sign. A problem of this type has been studied in [11] in the case $p = 2$.

We introduce the assumptions:

(G2) $g(0) = g(1) = 0$ and there exists $\alpha \in (0, 1)$ such that $(u - \alpha)g(u) > 0 \forall u \in (0, 1) \setminus \{\alpha\}$.

(D4) There exists $\beta \in (0, 1)$ such that $(u - \beta)D(u) < 0 \forall u \in (0, 1) \setminus \{\beta\}$.

(GD0)

$$\liminf_{h \rightarrow 0} \frac{D(h)^{q-1}g(h)}{h^{q-1}} > -\infty, \quad \int_0^\beta \left(\frac{D(h)}{h}\right)^{q-1} dh = +\infty.$$

(GD1)

$$\limsup_{h \rightarrow 1} \frac{|D(h)|^{q-1}g(h)}{(1-h)^{q-1}} < +\infty, \quad \int_\beta^1 \left(\frac{|D(h)|}{1-h}\right)^{q-1} dh = +\infty.$$

(GD2) $\alpha < \beta$ and $\int_0^\beta D(u)^{q-1}g(u) du > 0$.

Let us consider the problem in the interval $[0, \beta]$

$$y' = q(cy^{\frac{1}{p}} - D(u)^{q-1}g(u)), \quad 0 \leq u \leq \beta \quad (6.20)$$

Note that $D(u)^{q-1}g(u)$ is of type C on $[0, \beta]$. According to Theorem 4.2 we know that under the first condition in (GD0) and (GD2) there exists a (unique) number $\hat{c} > 0$ such that (6.20) has a positive solution satisfying

$$y(0) = 0 = y(\beta). \quad (6.21)$$

On the other hand, the problem in the interval $[\beta, 1]$

$$y' = q(cy^{\frac{1}{p}} - \tilde{D}(z)^{q-1}\tilde{g}(z)), \quad \beta \leq z \leq 1 \quad (6.22)$$

where

$$\tilde{D}(z) = |D(1 + \beta - z)|, \quad \tilde{g}(z) = g(1 + \beta - z); \quad \beta \leq z \leq 1 \quad (6.23)$$

involves a function of type A (in $[\beta, 1]$, of course) and, since the first condition in (GD1) holds, there exists a number c^* such that (6.22) has a positive solution satisfying

$$y(\beta) = 0 = y(1) \quad (6.24)$$

if and only if $c \geq c^*$. We are now in a position to state the following

Theorem 6.7. *Let $1 < p \leq 2$. Assume (G2), (D4), (GD0), (GD1), (GD2), $D'(\beta) < 0$. Then if $\hat{c} \geq c^*$, the problem (1.3)-(1.4) has a solution (unique up to translation) for $c = \hat{c}$.*

Proof. *Step 1: connection between 0 and β .* We fix $c = \hat{c}$ and consider the positive solution of (6.20) satisfying (6.21). A corresponding solution of (1.3) is obtained via

$$u' = -\frac{y(u)^{\frac{1}{p}}}{D(u)^{q-1}}, \quad u(0) = \frac{\beta}{2}. \quad (6.25)$$

Using assumption $D'(\beta) < 0$ and Lemma 5.1 we easily compute

$$\lim_{u \rightarrow \beta^-} \frac{y(u)^{\frac{1}{p}}}{D(u)^{q-1}} = \frac{\alpha^{\frac{1}{p}}}{|D'(\beta)|^{q-1}} \quad (6.26)$$

where $\alpha + \hat{c}\alpha^{\frac{1}{p}} = |D'(\beta)|^{q-1}g(\beta)$. Using the first part of (GD0) and Lemma 4.1 (where $f = |D|^{q-1}g$) we see that $\frac{y(u)}{u^q}$ is bounded in $(0, \beta)$. Hence we have obtained a solution of (1.3) that satisfies $0 < u(t) \leq \beta$ and (using the second part of (GD0))

$$u(t_1) = \beta, \quad u'(t_1) = -\frac{\alpha^{\frac{1}{p}}}{|D'(\beta)|^{q-1}}, \quad u(+\infty) = 0 \quad (6.27)$$

for some $t_1 > -\infty$.

Step 2: connection between β and 1. The change of variable

$$u(t) = 1 + \beta - z(-t)$$

defines a new second order problem

$$(\tilde{D}(z)|z'|^{p-2}z')' + cz' + \tilde{g}(z) = 0 \quad (6.28)$$

with \tilde{D} and \tilde{g} given by (6.23), in such a way that $u(t)$ takes values in $(\beta, 1)$ and solves (1.3) if and only if $z(t)$ takes values in $(\beta, 1)$ and solves (6.28). Now since $\hat{c} \geq c^*$ the problem (6.22) with $c = \hat{c}$ has a positive solution $y(z)$ that satisfies (6.24). This originates a solution of

$$z' = -\frac{y(z)^{\frac{1}{p}}}{\tilde{D}(z)^{q-1}}, \quad z(0) = \frac{1 - \beta}{2}. \quad (6.29)$$

Now $\lim_{z \rightarrow 1^-} \frac{\tilde{D}(z)^{q-1}\tilde{g}(z)}{(1-z)^{q-1}} = |D'(\beta)|^{q-1}g(\beta)$. hence, as in Step 1, we compute

$$\lim_{z \rightarrow 1^-} \frac{y(z)^{\frac{1}{p}}}{\tilde{D}(z)^{q-1}} = \frac{\alpha^{\frac{1}{p}}}{|D'(\beta)|^{q-1}} \quad (6.30)$$

with the same meaning of α . Now from Lemma 3.1-(d) (where $f = |\tilde{D}|^{q-1}\tilde{g}$, β playing the role of 0) and the first part of (GD1) we know that $\frac{y(z)}{(z-\beta)^q}$ is bounded in $(\beta, 1)$. Taking also the second part of (GD1) into account, we have shown that the solution of (6.29) is defined in some interval $(-t_2, +\infty)$ with $t_2 < +\infty$, $\beta < z(t) \leq 1$ and

$$z(-t_2) = 1, \quad z'(-t_2) = -\frac{\alpha^{\frac{1}{p}}}{|D'(\beta)|^{q-1}}, \quad z(+\infty) = \beta. \quad (6.31)$$

Accordingly, the function $u(t) = 1 + \beta - z(-t)$ satisfies $\beta < u(t) \leq 1$ and

$$u(t_2) = \beta, \quad u'(t_2) = -\frac{\alpha^{\frac{1}{p}}}{|D'(\beta)|^{q-1}}, \quad u(-\infty) = 1. \quad (6.32)$$

Step 3: conclusion. Comparing (6.27) with (6.32) we see that, after a translation of one of the solutions thus defined, we obtain the desired connection between 0 and 1. \square

Remark 6.8. (a) As in [11] for $p = 2$, it can be shown that the solution of (1.3)-(1.4) exists only if $c = \hat{c}$.

(b) Under the conditions of Theorem 6.7 we must have

$$\hat{c} \geq \left\{ \liminf_{h \rightarrow 0} \left[\left(\frac{|D(1-h)|}{h} \right)^{q-1} g(1-h) \right] q \right\}^{1/q} p^{1/p}.$$

Remark 6.9. If we modify condition (D4) so as to extend the strict inequality to the endpoints 0 and 1, then the second part of assumptions (GD0) and (GD1) of Theorem 6.7 can obviously be dropped, since it follows from the inequality $q \geq 2$.

Remark 6.10. In the above theorems we have obtained heteroclinic solutions taking values strictly between 0 and 1. If we let $p > 2$ the same procedure yields heteroclinics that are possible *finite* in the sense that they become constant outside a finite interval.

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