

This is the peer reviewed version of the following article:

A class of singular first order differential equations with applications in reaction-diffusion / Enguiça, Ricardo; Gavioli, Andrea; Sanchez, Luis. - In: DISCRETE AND CONTINUOUS DYNAMICAL SYSTEMS. SERIES B. - ISSN 1553-524X. - STAMPA. - 33:1(2013), pp. 173-191. [10.3934/dcds.2013.33.173]

Terms of use:

The terms and conditions for the reuse of this version of the manuscript are specified in the publishing policy. For all terms of use and more information see the publisher's website.

18/12/2025 23:39

A class of singular first order differential equations with applications in reaction-diffusion

Ricardo Enguiça*, Andrea Gavioli⁺, Luís Sanchez[#]

* Área Departamental de Matemática, Instituto Superior de Engenharia de Lisboa,
Rua Conselheiro Emídio Navarro, 1 - 1950-062 Lisboa, Portugal.
E-mail : rroque@dec.isel.ipl.pt

⁺ Dipartimento di Matematica Pura ed Applicata, Univ. di Modena e Reggio Emilia,
Via Campi, 213b, 41100 Modena, Italy. E-mail : gavioli@unimore.it

[#] Faculdade de Ciências da Universidade de Lisboa, CMAF
Avenida Professor Gama Pinto 2, 1649-003 Lisboa, Portugal.
E-mail : sanchez@ptmat.fc.ul.pt

Abstract

We study positive solutions $y(u)$ for the first order differential equation

$$y' = q(cy_+^{\frac{1}{p}} - f(u))$$

where $c > 0$ is a parameter, $p > 1$ and $q > 1$ are conjugate numbers and f is a continuous function in $[0, 1]$ such that $f(0) = 0 = f(1)$. We shall be particularly concerned with positive solutions $y(u)$ such that $y(0) = 0 = y(1)$. Our motivation lies in the fact that this problem provides a model for the existence of travelling wave solutions for analogues of the FKPP equation in one spacial dimension, where diffusion is represented by the p -Laplacian operator. We obtain a theory of admissible velocities and some other features that generalize classical and recent results, established for $p = 2$.

Key words: p -Laplacian, FKPP equation, heteroclinic, travelling wave, critical speed, sharp solution.

AMS Subject Classification: 34B18, 34C37, 35K57.

1 Introduction

In this paper we study some features of positive solutions to ordinary differential equations of the form

$$y' = q(cy_+^{\frac{1}{p}} - f(u)), \quad 0 \leq u \leq 1 \tag{1.1}$$

where $y_+ = \max(y, 0)$. We look for certain positive solutions $y = y(u)$ of (1.1) that vanish at one or both endpoints of the interval $[0, 1]$. Here p, q are positive numbers such that

$$\frac{1}{p} + \frac{1}{q} = 1,$$

$c > 0$ is a parameter, and $f : [0, 1] \rightarrow \mathbb{R}$ is a continuous function of types A, B or C, by which we mean

(Type A) $f(0) = f(1) = 0$ and $f(u) > 0$ if $u \in (0, 1)$.

(Type B) $f(0) = f(1) = 0$ and there exists $\alpha \in (0, 1)$ such that $f(u) = 0$ if $u \in [0, \alpha]$ and $f(u) > 0$ if $u \in (\alpha, 1)$.

(Type C) $f(0) = f(1) = 0$ and there exists $\alpha \in (0, 1)$ such that $f(u) < 0$ if $u \in (0, \alpha)$ and $f(u) > 0$ if $u \in (\alpha, 1)$.

This terminology was introduced by Berestycki and Nirenberg ([4]).

The motivation for considering (1.1) is the following. Consider the partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left[D(u) \left| \frac{\partial u}{\partial x} \right|^{p-2} \frac{\partial u}{\partial x} \right] + g(u), \quad (1.2)$$

which, in case $p = 2$, provides a model for a large variety of biological and chemical phenomena. We refer the reader to [3], [4], [11], [17], just to cite a few, and their references. In this equation g is a reaction term, while the first term in the right-hand side represents density dependent nonlinear diffusion in one-dimensional space. In the case $p = 2$ this is the well known FKPP equation (without convection term). Recently, models where the p -Laplacian operator replaces the usual Laplacian have been considered in the literature (e.g. [15], [8]).

When g is of one of the types A, B or C, $u = 0$ and $u = 1$ are two equilibrium solutions. An important problem related to this equation is that of finding travelling wave solutions, that is, solutions of the form $u(t, x) = U(x - ct)$ for some $c > 0$. Here c is the propagation speed of the wave. It is in addition required that the wave front $U(s)$ is defined in $(-\infty, +\infty)$ and satisfies $U(-\infty) = 1$, $U(+\infty) = 0$. This amounts to look for the solutions of the second order ordinary differential equation

$$(D(u)|u'|^{p-2}u')' + cu' + g(u) = 0 \quad (1.3)$$

satisfying the limit conditions

$$u(-\infty) = 1, \quad u(+\infty) = 0. \quad (1.4)$$

For certain values of the parameter c , such solutions are in addition monotone, with $u' < 0$ in their whole domain. Now if we set

$$-v := D(u)|u'|^{p-2}u',$$

for such monotone decreasing solutions, v may be seen as function of u . For simplicity, assume that $D(u) > 0$ for all $u \in (0, 1)$. Then a simple calculation shows that $v = v(u)$ must satisfy

$$\frac{1}{qD(u)^{q-1}} \frac{d}{du} v^q - c \left(\frac{v}{D(u)} \right)^{q-1} + g(u) = 0$$

and therefore **if we define**

$$y(u) = v(u)^q$$

the function y will solve (1.1) with $f(u) = D(u)^{q-1}g(u)$.

Moreover, the conditions (1.4) for monotone solutions defined in the real line imply that

$$u'(-\infty) = 0, \quad u'(+\infty) = 0.$$

This fact is well known in case $p = 2$ (see for instance [5]) and the argument easily carries out to the general case, as we show later for completeness. In terms of y this translates into

$$y(0) = 0, \quad y(1) = 0, \tag{1.5}$$

thus motivating the study of the existence of solutions of (1.1)-(1.5).

The study of admissible speeds for the problem (1.3) in case $p = 2$ has a long and rich history, starting with the seminal paper by Kolmogorov, Petrovski and Piscounov [10], including the in-depth approach by Aronson and Weinberger [3], and many recent contributions that the reader may find in the references. In Gilding and Kersner [6] and Malaguti and Marcelli [13] a singular integral equation technique has been used in the investigation of (1.3) and analogue equations for $p = 2$.

In this paper we propose, alternatively, to study the singular differential equation (1.1), thus constructing a first order model for admissible speeds and asymptotic behaviour (a method already used in [5]). This in turn provides information for (1.3) that is the counterpart of classic and recent results which have been obtained along years by many authors in case $p = 2$. In particular, we consider the differences between problems with functions of type A, on one hand, and with types B and C, on the other [3, 4]; we acknowledge the occurrence of sharp solutions, which were found in [16] and later systematized in [14]; we deal with sign change in diffusion density [11, 12] and with negative density diffusion [17].

The first order theory is developed in sections 2 to 4 and the applications to second order equations are given in section 6.

Notation. Let us introduce some notation and basic conditions to be used in the next sections. **For $c > 0$, we consider the function $\phi_c : [0, \infty) \rightarrow \mathbb{R}$** defined by

$$\phi_c(z) = cz^{1/p} - z. \tag{1.6}$$

We remark that ϕ_c vanishes at the two points 0 and c^q , is positive if and only if $0 < z < c^q$ and ϕ_c attains its absolute maximum M_c at a point $\omega_c \in (0, c^q)$. Namely: $\omega_c = (c/p)^q$, $M_c = \omega_c(p-1)$. **Now for any $x \in [0, M_c)$,** the function ϕ_c takes the value x at exactly two points, let us say $\omega_c^-(x) \in [0, \omega_c)$ and $\omega_c^+(x) \in (\omega_c, c^q]$: in particular, we set

$$J_c(x) = [\omega_c^-(x), \omega_c^+(x)].$$

If $x \geq M_c$ we set, by definition, $\omega_c^-(x) = \omega_c^+(x) = \omega_c$.

Also, we shall be dealing with functions f of type A satisfying

$$\sup_{u \in (0,1)} \frac{f(u)}{u^{q-1}} = \mu < +\infty \quad (1.7)$$

or the stronger property

$$\lim_{u \rightarrow 0^+} \frac{f(u)}{u^{q-1}} = \lambda < +\infty. \quad (1.8)$$

2 Functions of type A: existence of solutions and admissible speeds

Consider the boundary value problem

$$\begin{cases} y'(u) = q(c y_+(u)^{\frac{1}{p}} - f(u)), & 0 \leq u \leq 1 \\ y(0) = y(1) = 0, \end{cases} \quad (2.1)$$

for which we look for positive solutions in the interval $(0, 1)$. By a solution we mean a function $y \in C^1[0, 1]$ satisfying the equation above and the boundary conditions. In what follows *positive solution* means a solution y such that $y(u) > 0 \forall u \in (0, 1)$.

The following proposition allows us to conclude that the existence of a positive lower solution satisfying a strict inequality in the interval $(0, 1]$ is enough to get existence and uniqueness for (2.1).

Proposition 2.1. *Let f be of type A. Suppose that $s(u)$ is a C^1 -function in $[0, 1]$ such that $s(0) = 0$, $s(u) > 0$ if $u \in (0, 1)$ and for all $u \in (0, 1]$,*

$$s'(u) \leq q \left(c s(u)^{\frac{1}{p}} - f(u) \right). \quad (2.2)$$

Then (2.1) has a unique positive solution.

Proof. By a well known argument, (2.2) implies that there exists a solution $y(u)$ of the differential equation in (2.1) with $y(0) = 0$ such that $s(u) \leq y(u)$. Now consider the solution \bar{y} of the initial value problem

$$\bar{y}' = q \left(c \bar{y}_+^{\frac{1}{p}} - f(u) \right), \quad \bar{y}(1) = 0. \quad (2.3)$$

(In fact this problem enjoys uniqueness in $[0, 1]$ because the right-hand side of the equation is nondecreasing in the dependent variable.) It is easy to see that $\bar{y} \geq 0$ in $[0, 1]$. Moreover $0 < \bar{y}(u) < y(u)$ for all $u \in (0, 1)$. For, if u_0 is a zero of \bar{y} in $(0, 1)$, then the differential equation implies $\bar{y}'(u_0) < 0$, which is a contradiction with the fact that $\bar{y} \geq 0$; and if there exists $u_1 \in (0, 1)$ such that $\bar{y}(u_1) = y(u_1)$, then by uniqueness of solution we would have $\bar{y} = y$, which contradicts the fact that $\bar{y}(1) = 0$. By continuity we have $\bar{y}(0) = 0$. The fact that the solution is unique is a direct consequence of uniqueness for (2.3). \square

Remark 2.2. One could also have invoked the fact that the functions s and 0 are lower and upper solutions with respect to the periodic problem for (1.1) in $[0, 1]$.

Proposition 2.3. Assume that f is a function of type A in $[0, 1]$ satisfying (1.7). Then there exists a constant $c^* > 0$ (depending on f and p) such that (2.1) admits a unique positive solution if and only if $c \geq c^*$. Moreover we have the estimate $c^* \leq q^{\frac{1}{q}} p^{\frac{1}{p}} \mu^{\frac{1}{q}}$.

Proof. It is obvious that for c large enough, the inequality $\phi_c(\beta) \geq \mu$ has positive solutions β if and only if $\omega_c(p-1) \geq \mu$, that is,

$$c \geq q^{\frac{1}{q}} p^{\frac{1}{p}} \mu^{\frac{1}{q}}.$$

For one such β , let $s(u) = \beta u^q$. Then, for all $u \in (0, 1]$, we have

$$s'(u) = q\beta u^{q-1} \leq \left(c\beta^{\frac{1}{p}} - \mu\right) q u^{q-1} \leq q \left(c s(u)^{\frac{1}{p}} - f(u)\right).$$

The previous proposition allows us to conclude that for such value c , the boundary value problem (2.1) has a unique positive solution.

Now let c^* be the infimum of the values $c > 0$ such that problem (2.1) has a unique positive solution. The estimate for c^* follows from what we have just seen. Let us prove that for all $c > c^*$, problem (2.1) has a solution. Given $c_1 > c^*$, let us consider a value \tilde{c} such that (2.1) has a positive solution $y_{\tilde{c}}$ and $c^* < \tilde{c} < c_1$. For all $u \in (0, 1]$ we have

$$y'_{\tilde{c}} = q \left(\tilde{c} y_{\tilde{c}}^{\frac{1}{p}} - f(u) \right) < q \left(c_1 y_{\tilde{c}}^{\frac{1}{p}} - f(u) \right),$$

so $y_{\tilde{c}}$ is a lower solution for the problem with $c = c_1$ and by the previous proposition, we conclude the solvability for (2.1) with $c = c_1$.

To prove the solvability for $c = c^*$, consider a decreasing sequence c_n tending to c^* and the correspondent positive solutions y_n . First, the argument we used at the end of the proof of Proposition 2.1 shows that $y_n \leq y_1 \forall n$. It is also easy to conclude that the sequence (y_n) is uniformly bounded in $C^1([0, 1])$. By the Ascoli-Arzelà theorem, there exists a continuous function y^* such that $y_n \rightarrow y^*$ uniformly in $[0, 1]$. It turns out that y^* satisfies (2.1) with $c = c^*$ and $y^* > 0$ in $(0, 1)$.

Let us now prove that $c^* > 0$. If $c^* = 0$, using the above notation we would obtain $y^*(u) = -q \int_0^u f(s) ds$ meaning that $y^* < 0$ in $(0, 1]$, a contradiction. \square

3 Behaviour of the solutions near the origin

Throughout this section we suppose, as in the previous one, that f is a type A function, and try to get more detailed information about the behaviour of the solutions of equation (1.1) near 0, under the only condition $y(0) = 0$. To this end, we need conditions (1.7) or (1.8) on f . We are going to show that, whenever (1.8) holds:

$$(a) \lim_{u \rightarrow 0} \frac{y(u)}{u^q} = \omega_c^-(\lambda) \quad \text{or} \quad (b) \lim_{u \rightarrow 0} \frac{y(u)}{u^q} = \omega_c^+(\lambda). \quad (3.1)$$

In particular we point out that, if the stronger condition

$$\sup_{u \in (0,1)} \frac{f(u)}{u^q} < +\infty \quad (3.2)$$

holds, then $\lambda = 0$, so that $\omega_c^-(\lambda) = 0$. In this case, actually, we can say more than (3.1a), namely:

$$\sup_{0 < u \leq 1} \frac{y(u)}{u^{pq}} < +\infty. \quad (3.3)$$

In order to show the properties above, we need some preliminary results. To this end, let $y \in \mathcal{S}_c$ be fixed, and put

$$\gamma(u) = \frac{y'(u)}{qu^{q-1}}, \quad z(u) = \frac{y(u)}{u^q}, \quad \lambda(u) = \frac{f(u)}{u^{q-1}}. \quad (3.4)$$

Let us denote respectively by γ^- , l^- and λ^- the lower limits of $\gamma(u)$, $z(u)$ and $\lambda(u)$ as $u \rightarrow 0$. Similarly, the scripts γ^+ , l^+ and λ^+ will stand for the corresponding upper limits. From

$$y' = q(cy^{\frac{1}{p}} - f(u)) \quad (3.5)$$

it is easy to see that $z(u)$, at least for small values of u , solves the following differential equation:

$$z' = \frac{q}{u}(\phi_c(z) - \lambda(u)). \quad (3.6)$$

Lemma 3.1. *Let f satisfy (1.7), $y \in \mathcal{S}_c$, λ^\pm and l^\pm be defined as above. Then:*

- (a) $M_c \geq \lambda^-$, that is: $c \geq (\lambda^- q)^{1/q} p^{1/p}$. In particular: $c^* \geq (\lambda^- q)^{1/q} p^{1/p}$.
- (b) $l^+ \in J_c(\lambda^-)$, $l^- \notin J_c(\lambda^+)^{\circ}$.
- (c) $\lambda^+ < M_c \Rightarrow \omega_c \notin (l^-, l^+)$.
- (d) (1.8) \Rightarrow (3.1).

Proof. (a) We remark that $\gamma^- \leq l^- \leq l^+ \leq \gamma^+$, as follows easily from Cauchy's theorem. Furthermore the function y , as long as it is positive, solves equation (3.5). If we divide both its sides by qu^{q-1} , and recall that $(q-1)p = q$, we get:

$$\gamma(u) = cz(u)^{1/p} - \lambda(u). \quad (3.7)$$

But now (3.7) implies that $\gamma^+ \leq c(l^+)^{1/p} - \lambda^-$, as we can argue on taking the upper limit of both sides as $u \rightarrow 0$. On the other hand, since $\gamma^+ \geq l^+$, we actually get $\lambda^- \leq \phi_c(l^+) \leq M_c$.

(b) It is enough to remark that $\phi_c(l^+) \geq \lambda^-$ and $\phi_c(l^-) \leq \lambda^+$. Indeed, the first inequality was shown in the previous step. As regards the latter, it can be achieved in a similar way, when taking in (3.7) a lower limit instead than an upper limit.

(c) Suppose, by contradiction, $l^- < \omega_c < l^+$: in particular, the maximum among the three values λ^+ , $\phi_c(l^-)$, $\phi_c(l^+)$, say h , is less than M_c . Now, let $j, m \in (h, M_c)$ such that $j < m$, and put $I = [\rho^-, \rho^+] := \phi_c^{-1}([m, M_c])$, so that $I \subseteq (l^-, l^+)$. According to the definitions of λ^+ , l^- and l^+ we can find $\delta > 0$ such that $\lambda(u) \leq j$ for $0 < u \leq \delta$, and two points u^- and $u^+ \in (0, \delta]$ such that $u^- < u^+$, $z(u^-) = \rho^+$, $z(u^+) = \rho^-$ and

$\rho^+ \leq z(u) \leq \rho^-$ for $u^- \leq u \leq u^+$. In particular, the interval $[u^-, u^+]$ must contain a point θ at which $z' < 0$. On the other hand $\phi_c(z(\theta)) \geq m$, so that (3.6) would yield the contradiction $z'(\theta) \geq (q/\theta)(m - j) > 0$. Hence, actually, $\omega_c \notin (l^-, l^+)$.

(d) Since $\lambda^- = \lambda^+ = \lambda$, from (a) we get $\lambda \leq M_c$, and applying claim (c) we infer that l^- and l^+ lie on the same side with respect to ω_c . On the other hand, let us replace λ^- and λ^+ in claim (b) by their common value λ : according to whether, respectively, l^- and l^+ lie to the left or to the right of ω_c , we infer what follows: either $l^- \leq \omega_c^-(\lambda) \leq l^+ \leq \omega_c$ or $\omega_c \leq l^- \leq \omega_c^+(\lambda) \leq l^+$. In both cases it is enough to show that $l^- = l^+$. As regards the former, let us suppose, by contradiction, that $l^- < l^+$: then both values l^- and l^+ can be approximated along a sequence of local extrema of z , which are, in particular, critical values. More precisely, we can find points a_i and b_i ($i \in \mathbf{Z}^+$) at which z' vanishes, in such a way that $a_i \rightarrow 0$, $b_i \rightarrow 0$ and the sequences $(z(a_i))_i$ and $(z(b_i))_i$ converge respectively to l^- and l^+ . Since $z'(a_i) = z'(b_i) = 0$, (3.6) entails $\phi_c(z(a_i)) = \lambda(a_i)$ and $\phi_c(z(b_i)) = \lambda(b_i)$. Now, let us first suppose $l^+ < \omega_c$: then both equalities $z(a_i) = \omega_c^-(\lambda(a_i))$ and $z(b_i) = \omega_c^-(\lambda(b_i))$ hold for large values of i : since ω_c^- is continuous, and $\lambda(u) \rightarrow \lambda$ as $u \rightarrow 0$, from the previous relations we get, as $i \rightarrow +\infty$, the contradiction $l^- = l^+ = \omega_c^-(\lambda)$. Now assume let $l^+ = \omega_c$: then possibly $z(b_i) = \omega_c^+(\lambda(b_i))$ for infinitely many values of i : in this case, however, $\omega_c = l^+ = \omega_c^+(\lambda)$, so that, actually, $\lambda = M_c$. Then we can write again $l^+ = \omega_c^-(\lambda)$, and get the same contradiction as before. Finally in the case $\omega_c \leq l^- \leq \omega_c^+(\lambda) \leq l^+$ the conclusion is straightforward by virtue of (b). \square

Corollary 3.2. *If (1.7), (1.8) hold and $\mu = \lambda$, then $c^* = q^{\frac{1}{q}} p^{\frac{1}{p}} \lambda^{\frac{1}{q}}$.*

Proof. It suffices to combine the Proposition 2.3 with Lemma 3.1 (a). \square

This generalizes the well known result for the case $p = 2$, where $\lambda = f'(0)$, for which $M = f'(0)$ implies $c^* = 2\sqrt{f'(0)}$.

Now, let $r, A, c > 0$ be fixed. For any function $y \in C([0, r])$ we denote by $N(y)$ the supremum of $|y(u)|/u^q$ for $0 < u \leq r$: then it is easy to check that the subspace V of $C([0, r])$ where $N(y) < +\infty$ is a Banach space with respect to the norm $\|y\| := N(y)$. Now we define a closed subset E of V and a map $T : E \rightarrow V$ as follows:

$$E = \{y \in V : y(u) \geq Au^q, 0 \leq u \leq r\}, \quad (3.8)$$

$$[T(y)](u) = q \int_0^u (cy(s)^{1/p} - f(s))ds, \quad y \in E, \quad 0 \leq u \leq r. \quad (3.9)$$

Lemma 3.3. *Let f fulfil (1.8), $\nu = \sup \{f(u)/u^{q-1}; 0 < u \leq r\}$. Then the following properties hold.*

- (a) $T(E) \subseteq V$.
- (b) If $A > \omega_c$, $T : E \rightarrow V$ is a contraction with respect to $\|\cdot\|$.
- (c) If $\phi_c(A) \geq \nu$, then $T(E) \subseteq E$. In particular, if $\omega_c < A \leq \omega_c^+(\nu)$, T is a contraction of E into itself.

Proof. (a) If $y \in E$, then obviously $w := T(y) \in C([0, r])$. In order to prove that $\|w\| < +\infty$ we only need to divide both sides of the following inequality by u^q , and take the supremum for $0 < u \leq r$.

$$w(u) \leq qc \int_0^u y(s)^{1/p} ds \leq qc \int_0^u (\|y\| s^q)^{1/p} ds = c\|y\|^{1/p} u^q. \quad (3.10)$$

(b) We notice that, for any $\alpha > 0$, the function $y^{1/p}$ admits, on the half-line $[\alpha, +\infty)$, the Lipschitz constant $L(\alpha) = (p\alpha^{1/q})^{-1}$. Now, for $i = 1, 2$, let $y_i \in E$, $w_i = T(y_i)$. Then:

$$\begin{aligned} |w_2(u) - w_1(u)| &\leq cq \int_0^u |y_2(s)^{1/p} - y_1(s)^{1/p}| ds \leq \\ &\leq cq \int_0^u L(As^q) |y_2(s) - y_1(s)| ds \leq \\ &\leq \frac{cq}{p} \int_0^u (As^q)^{-1/q} \|y_2 - y_1\| s^q ds = \frac{c}{p} A^{-1/q} u^q \|y_2 - y_1\|. \end{aligned} \quad (3.11)$$

Also here we can divide the extreme sides by u^q and take the supremum for $0 < u \leq r$, so as to infer that $k = (c/p)A^{-1/q}$ is a Lipschitz constant for T with respect to $\|\cdot\|$. But the condition $A > \omega_c$ is just equivalent to $k < 1$.

(c) If $y \in E$ and $w = T(y)$, then $w(u) \geq q \int_0^u [c(As^q)^{1/p} - \nu s^{q-1}] ds$, where the right-hand side is precisely $(cA^{1/p} - \nu)u^q$. Hence $w(u) \geq Au^q$ if and only if $\phi_c(A) \geq \nu$. As regards the last claim, it is enough to remark that the two conditions $\phi_c(A) \geq \nu$ and $A > \omega_c$ hold together if and only if $\omega_c < A \leq \omega_c^+(\nu)$. \square

Remark 3.4. The condition $\phi_c(A) \geq \nu$ in Lemma 3.3 is equivalent to state that the function Au^q is a subsolution of (1.1) on $[0, r]$.

Proposition 3.5. *Let f satisfy (1.8), set $\bar{c} := (\lambda q)^{1/q} p^{1/p}$, and assume $c > \bar{c}$. Then the following properties hold true.*

- (a) \mathcal{S}_c contains exactly one function y which verifies (3.1b), say $y =: \psi_c$.
- (b) If $y \in \mathcal{S}_c$, $y \neq \psi_c$, then (3.1a) holds.
- (c) If $y \in \mathcal{S}_c$, $y \neq \psi_c$, then $\psi_c(u) > y(u)$ for any $u \in (0, 1]$.
- (d) If w is a subsolution of (1.1) and $w(0) = 0$, then $\psi_c \geq w$ on $[0, 1]$.
- (e) If $\theta > c$ then $\psi_\theta(u) > \psi_c(u)$ for any $u \in (0, 1]$.
- (f) $\sup \{|\psi_\theta(u) - \psi_c(u)|/u^q; u \in (0, 1]\} \rightarrow 0$ as $\theta \rightarrow c$.
- (g) If $y \in \mathcal{S}_c$, $y \neq \psi_c$ and (3.2) holds, then y satisfies (3.3).

Proof. (a) From our condition on c and Lemma 3.1-(a) we get $M_c > \lambda$: therefore, if $r > 0$ is suitably small, the number ν which appears in Lemma 3.3 lies below M_c as well, that is $\omega_c < \omega_c^+(\nu)$. So, let $\omega_c < A < \omega_c^+(\nu)$, and define E and T as in (3.8), (3.9). Since E is a closed subset of the Banach space $(V, \|\cdot\|)$, Lemma 3.2-(c) and Banach's contraction

principle ensure that T admits a unique fixed point y , which obviously solves (1.1) on $[0, r]$ and fulfils the condition $y(0) = 0$. In particular, the extension of y to the whole interval $[0, 1]$ (as a solution of (1.1)) belongs to \mathcal{S}_c . On the other hand, since $y(u) \geq Au^q$ on $[0, r]$, of (3.1a) and (3.1b) only the latter can hold. As regards uniqueness, let $\tilde{y} \in \mathcal{S}_c$ fulfil (3.1b): then \tilde{y} belongs to the same space E as before, and is again a fixed point for T , so that, necessarily, $\tilde{y} = y$.

(b) It follows at once from Lemma 3.1-(d).

(c) Since y and ψ_c satisfy respectively (3.1(a)) and (3.1(b)), and $\omega_c^-(\lambda) < \omega_c^+(\lambda)$, the inequality $\psi_c(u) > y(u)$ surely holds in a right neighbourhood of 0, say $(0, \rho]$. By contradiction, let $\sigma \in (\rho, 1]$ be the first point at which the function $z = \psi_c - y$ vanishes: since $z' \geq 0$ on $[0, \sigma]$ and $z(0) = z(\sigma) = 0$, we should get the contradiction $\psi_c \equiv y$ on $[0, \sigma]$.

(d) By virtue of the previous claim, the inequality $\psi_c \geq y$ holds true for any $y \in \mathcal{S}_c$. On the other hand, since w is a subsolution of (1.1), we can find $y \geq w$ such that $y(0) = 0$ and (1.1) holds: then $w \leq y \leq \psi_c$.

(e) If $\theta > c$, then $\phi_\theta > \phi_c$ and therefore $\omega_c^+ < \omega_\theta^+$. Hence $\psi_c < \psi_\theta$ in a right neighborhood of 0, by virtue of (3.1b). Then the inequality in fact holds in $(0, 1]$.

(f) Let r, ν and A be again as in the previous steps. Since M_c, ω_c and $\omega_c^+(\nu)$ depend continuously on c , let $\alpha \in (\bar{c}, c)$, $\beta > c$ such that $M_\alpha > \nu$ and $\omega_\beta < A < \omega_\alpha^+(\nu)$. For any $\theta \in U := (\alpha, \beta)$ put $c = \theta$ in (3.9), denote by T_θ the corresponding map and by ψ_θ^r the restriction of ψ_θ to $[0, r]$, which can be characterized as the unique fixed point of T_θ . We point out that the maps T_θ , for $\theta \in U$, are defined on the same set E we introduced in the proof of claim (a), a set which does not depend on θ . Furthermore, the map $(\theta, y) \mapsto T_\theta(y)$ is continuous, and $k = (\beta/p)A^{-1/q} < 1$ is a Lipschitz constant, with respect to the norm of V , for all maps T_θ , $\theta \in U$. Then it is easy to show that the fixed point of T_θ depends continuously on θ . More precisely: the map $\theta \mapsto \psi_\theta^r$ is continuous from U to $(E, \|\cdot\|)$, and the same we can say, as a consequence, for the map $\theta \mapsto \psi_\theta(r)$ from U to \mathbb{R} . Then well-known results about the dependence on initial data of the solution of a Cauchy problem entail that, as $\theta \rightarrow c$, $\psi_\theta \rightarrow \psi_c$, uniformly on $[r, 1]$. Now, let us put $\Delta(\theta) = \|\psi_\theta^r - \psi_c^r\|$, and denote by $S(\theta)$ the supremum of $|\psi_\theta - \psi_c|$ over $[r, 1]$: according to the previous arguments, both $\Delta(\theta)$ and $S(\theta)$ converge to 0 as $\theta \rightarrow c$. On the other hand, the supremum which appears in our claim does not exceed $\max(\Delta(\theta), S(\theta)/r^q)$.

(g) Let $K < +\infty$ be the supremum in (3.2): in particular, as we already pointed out, (1.8) holds true with $\lambda = 0$. Since we are dealing with a function y which does not fulfil (3.1b), and $\omega_c^-(0) = 0$, from (3.1a) we argue that $y(u)/u^q$ converges to 0 as $u \rightarrow 0$, and the same we can say of $y(u)^{1/q}/u$. Now, let us suppose, by contradiction, that (3.3) is not satisfied: actually, in this case, the ratio $y(u)/u^{pq}$ is not bounded from above on any right neighbourhood of 0, and the same we can say of $y(u)^{1/p}/u^q$. By combining the two previous remarks, we easily find $\varepsilon > 0$ such that

$$\frac{y(\varepsilon)^{1/p}}{\varepsilon^q} \left(c - p \frac{y(\varepsilon)^{1/q}}{\varepsilon} \right) \geq K. \quad (3.12)$$

Let $h = y(\varepsilon)/\varepsilon^{pq}$: then the function $w(u) = hu^{pq}$ satisfies the conditions

$$(a) \ w(\varepsilon) = y(\varepsilon), \quad (b) \ w'(u) \leq q(cw(u)^{1/p} - f(u)), \quad 0 \leq u \leq \varepsilon. \quad (3.13)$$

Indeed, (3.13a) is obviously due to our choice of h , while (3.13b) can be proved as follows:

$$\begin{aligned} f(u) &\leq Ku^q \leq [c(y(\varepsilon)^{1/p}/\varepsilon^q) - p\varepsilon^{p-1}(y(\varepsilon)/\varepsilon^{pq})]u^q = \\ &= (ch^{1/p} - ph\varepsilon^{p-1})u^q \leq ch^{1/p}u^q - phu^{pq-1} = cw(u)^{1/p} - (w'(u)/q). \end{aligned} \quad (3.14)$$

In particular: the second inequality in (3.14) follows from (3.12). The inequality of the second line of (3.14) comes from $u \leq \varepsilon$ and $p + q = pq$. Then (3.13b) holds true as well, so that w is a subsolution of (1.1), and (3.13a) implies $y \leq w$ on $[0, \varepsilon]$. But now, from the expression of w , we conclude that (3.3) holds, in contrast with our initial assumption. \square

Remark 3.6. Let us consider the estimate from below which is given by Lemma 3.1(a) on the critical value c^* , and combine it with the final claim of Prop. 2.3: under the assumption (1.8), and according to the notation we introduced in Proposition 3.5, we can write $\bar{c} = q^{\frac{1}{q}}p^{\frac{1}{p}}\lambda^{\frac{1}{q}} \leq c^* \leq q^{\frac{1}{q}}p^{\frac{1}{p}}\mu^{\frac{1}{q}}$. We also point out that, in the limit case $c = \bar{c}$, the maximum value M_c of (1.6) is λ , so that $\omega_c^-(\lambda) = \omega_c^+(\lambda) = \omega_c$: hence (3.1) has no meaning for $c = \bar{c}$.

Theorem 3.7. *Let f satisfy (1.8), $c \geq c^*$ and, according to Proposition 2.3, let y be the only solution of (1.1) such that $y(0) = y(1) = 0$. Then:*

- (a) $c > c^* \Rightarrow y$ satisfies (3.1a).
- (b) $c = c^* \Rightarrow y$ satisfies (3.1b).

Proof. (a) Let $c^* < \theta < c$, put $c = \theta$ in (1.1) and call \tilde{y} the solution of the corresponding equation such that $\tilde{y}(0) = \tilde{y}(1) = 0$. Suppose, by contradiction, that (3.1a) does not hold, and exchange the roles of θ and c in Prop. 3.5e, so as to get $y = \psi_c > \psi_\theta \geq \tilde{y}$ on $(0, 1]$. In particular, we get the contradiction $0 = y(1) > \tilde{y}(1)$.

(b) It is enough to prove that (3.1a) $\Rightarrow c > c^*$. So, let us suppose that $y \in \mathcal{S}_c$, $y \neq \psi_c$: then, from Prop. 3.3c, we infer that $\psi_c(1) > y(1) = 0$. On the other hand, Prop. 3.5f ensures, in particular, that the map $c \mapsto \psi_c(1)$ is continuous, so that $\psi_\theta(1) > 0$ for some $\theta < c$. Now, let us put $c = \theta$ in (1.1), and denote by \tilde{y} the solution of the corresponding equation which fulfils the condition $\tilde{y}(1) = 0$. By the same arguments as in the previous section, we get $\tilde{y}(0) = 0$ as well: hence $\theta \geq c^*$, so that $c > c^*$. \square

The reader may find related and complementary results in [9] and in [1].

4 Functions of types B and C: existence of solutions

Let us now consider the cases where f is a type B or type C function.

Lemma 4.1. *Assume f is continuous in $[0, 1]$, $f(0) = 0$ and*

$$\liminf_{u \rightarrow 0} \frac{f(u)}{u^{q-1}} > -\infty. \quad (4.1)$$

Then any solution of

$$y' = q(cy_+^{\frac{1}{p}} - f(u)), \quad y(0) = 0, \quad (4.2)$$

positive in a neighborhood of 0, satisfies

$$\sup_{u \in (0,1]} \frac{y(u)}{u^q} < +\infty.$$

Proof.

Claim: Given $k > 0$, there exists $M > 0$ such that any solution of

$$z' = q(c z_+^{\frac{1}{p}} + k u^{q-1}), \quad z(0) = 0 \quad (4.3)$$

positive in a neighborhood of 0, satisfies $z(u) \leq M u^q$, $0 \leq u \leq 1$.

If we set $w(u) = z(u)^{1/q}$, we have $w' = c + k(\frac{u}{w})^{q-1}$, $w(0) = 0$. Defining

$$l = \liminf_{u \rightarrow 0} \frac{w(u)}{u}, \quad L = \limsup_{u \rightarrow 0} \frac{w(u)}{u}$$

and

$$l' = \liminf_{u \rightarrow 0} w'(u), \quad L' = \limsup_{u \rightarrow 0} w'(u)$$

we obtain $c \leq l' \leq l \leq L \leq L' = c + \frac{k}{l^{q-1}} < +\infty$ and our Claim follows.

Now choose $k > 0$ so that $-f(u) < k u^{q-1}$ if $0 < u \leq 1$. For each $h > 0$ consider the solution z_h of

$$z'_h = q(c z_{h+}^{\frac{1}{p}} + k u^{q-1}), \quad z_h(0) = h. \quad (4.4)$$

Then z_h converges, as $h \rightarrow 0$, to the maximal solution of (4.3). On the other hand if we pick a solution y of (4.2) it is clear that $y < z_h$. Using the Claim, we obtain the conclusion of the lemma.

Theorem 4.2. *Let f be a type B or a type C function. In the latter case assume $\int_0^1 f(s) ds > 0$ and (4.1) holds. Then there exists a number $\hat{c} > 0$ such that the boundary value problem $y' = q(c y_+^{\frac{1}{p}} - f(u))$, $0 \leq u \leq 1$, $y(0) = y(1) = 0$ has a positive solution if and only if $c = \hat{c}$.*

Proof. By the hypothesis there exists $\alpha \in (0, 1)$ so that $f > 0$ in $(\alpha, 1)$ and either $f \equiv 0$ or $f < 0$ in $(0, \alpha)$. For $c \geq 0$ consider the Cauchy problem

$$y' = q(c y_+^{\frac{1}{p}} - f(u)), \quad y(1) = 0, \quad (4.5)$$

which, as we have already remarked, has a unique solution y_c in $[0, 1]$. Also, the usual compactness argument shows that y_c depends continuously on $c \in [0, +\infty)$ in the norm of $C([0, 1])$. Clearly, $y_c(u) \geq 0$ at least for $u \in (\alpha, 1)$. In particular, by our assumptions, $y_0(u) = q \int_u^1 f(s) ds > 0$ for all $u \in [0, 1)$.

Step 1: solutions decrease with c . Given $c_1 < c_2$, the corresponding solutions $y_1 \equiv y_{c_1}$ and $y_2 \equiv y_{c_2}$ are such that $y_1(u) > y_2(u)$ whenever $y_1(u) > 0$. In fact we cannot have $y_1 < y_2$ in any open subinterval of $(\alpha, 1)$, otherwise $y_1 - y_2$ would be decreasing in that interval, contradicting the fact that it must reach the value 0.

Set $\hat{c} = \sup\{c > 0 \mid y_c(u) > 0 \forall u \in (0, 1)\}$.

Step 2: $0 < \hat{c} < +\infty$. It is obvious that $\hat{c} > 0$. If $\hat{c} = +\infty$, there exists $c_n \rightarrow +\infty$ with $y_n \equiv y_{c_n} > 0$ in $(0, 1)$. If f is type B, then

$$y_n(u)^{\frac{1}{q}} = y_n(\alpha)^{\frac{1}{q}} - c_n(\alpha - u) \quad (4.6)$$

for $u \in (0, \alpha)$. Note also that $y_n(\alpha) \leq q \int_{\alpha}^1 f(s) ds$. Hence y_n must become negative for n large, a contradiction. If f is type C the same argument applies because then the solution in $(0, \alpha)$ must stay below the function given by the expression in the righthandside of (4.6).

Step 3: $y_{\hat{c}}(0) = 0$ and $y_{\hat{c}}(u) > 0 \forall u \in (0, 1)$. By definition of \hat{c} and continuous dependence on c , $y_{\hat{c}}$ must vanish in $[0, \alpha]$. Let $\gamma \in [0, \alpha]$ be its largest zero. If $\gamma > 0$, then for $c < \hat{c}$ and $u \leq \gamma$

$$y_c(u)^{\frac{1}{q}} \leq y_c(\gamma)^{\frac{1}{q}} - c(\gamma - u)$$

and since $y_c(\gamma) \rightarrow 0$ as $c \rightarrow \hat{c}$, if $\hat{c} - c$ is sufficiently small y_c must vanish in $(0, \gamma)$, contradicting the definition of \hat{c} .

Step 4: If $c > \hat{c}$ and f is of type B, then $y_c \equiv 0$ in some interval $[0, \gamma]$, $0 < \gamma \leq \alpha$. By Step 1, $0 < y_c(\alpha) < y_{\hat{c}}(\alpha)$. The graph of y_c cannot meet the graph of $y_{\hat{c}}$ in $[0, \alpha]$ if $y_c > 0$ in $(0, \alpha)$. Hence there exists $\gamma \in (0, \alpha]$ such that $y_c(\gamma) = 0$ and the claim follows.

Step 5: If $c > \hat{c}$ and f is of type C, then $y_c(0) < 0$. As in the previous step, $0 < y_c(\alpha) < y_{\hat{c}}(\alpha)$. Since $f > 0$ in $(0, \alpha)$ we easily obtain $y_c(u) < y_{\hat{c}}(u) \forall u \in [0, \alpha]$.

Step 6: If $c < \hat{c}$, then $y_c(0) > 0$. Suppose to the contrary that $y_c(0) = 0$. By the previous arguments $y_c > y_{\hat{c}}$ on $(0, 1)$.

Case 1: f is of type B. By separation of variables, the only solution of $y' = q d y_+^{\frac{1}{p}}$ satisfying $y(0) = 0$ and positive in a neighborhood of zero is the function $y_0(u) \equiv d^q u^q$. Hence we obtain $c^q u^q > \hat{c}^q u^q$ in $[0, \alpha]$, a contradiction.

Case 2: f is of type C. By a lower solution argument, (4.2) with $c = \hat{c}$ has a solution $z(u)$ such that $z(0) = 0$ and $z > y_{\hat{c}}$ in $[0, \alpha]$. But we now show that such solutions must coincide, obtaining a contradiction. Let z, w be two solutions of (4.2). If $z \neq w$ it is easily seen that they are ordered, say $z < w$ in $(0, \alpha)$. By the preceeding Lemma there exists a constant $M > 0$ so that, with a computation similar to that of (3.11),

$$0 \leq w(u) - z(u) \leq qc \int_0^u \frac{w(s) - z(s)}{p(c^q s^q)^{1/q}} ds \leq \frac{M}{p} u^q, \quad 0 \leq u < \alpha.$$

Iterating this argument we obtain

$$0 \leq w(u) - z(u) \leq \frac{M}{p^2} u^q, \quad 0 \leq u < \alpha$$

and in fact

$$0 \leq w(u) - z(u) \leq \frac{M}{p^k} u^q, \quad 0 \leq u < \alpha$$

for all integers $k \in \mathbb{N}$. We conclude that $z \equiv w$ in $[0, \alpha]$. □

Let f be a type A function such that $\sup_{0 < u < 1} \frac{f(u)}{u^{q-1}} < +\infty$. It is easy to see that there exists a decreasing sequence of positive values ϵ_n tending to zero such that the

corresponding sequence of type B functions

$$f_n(u) = \begin{cases} 0, & u \in [0, \epsilon_{n+1}] \\ \min(l_n(u), f(u)), & u \in [\epsilon_{n+1}, \epsilon_n] \\ f(u), & u \in [\epsilon_n, 1], \end{cases}$$

where $l_n(u) = f(\epsilon_n) \frac{u - \epsilon_{n+1}}{\epsilon_n - \epsilon_{n+1}}$, is increasing and tends uniformly to $f(u)$. Let $\hat{c}(f_n)$ be the unique value such that the boundary value problem (2.1) with $f(u) = f_n(u)$ has a positive solution. The following theorem uses this fact to give a new characterization of the critical speed c^* introduced in section 2. Results of this type may be also found in [4, 9].

Theorem 4.3. *Consider a type A function f with $\sup_{u \in (0,1)} \frac{f(u)}{u^{q-1}} < +\infty$ and a sequence of type B functions f_n in the conditions mentioned above. Then $\hat{c}(f_n)$ is an increasing sequence and $\lim \hat{c}(f_n) = c^*$ where c^* is associated to f in Proposition 2.3.*

Proof. Consider two arbitrary consecutive elements f_n and f_{n+1} of the sequence of type B functions considered above. These two functions are different in some interval $(\epsilon_{n+2}, b) \subset (0, \epsilon_n)$, where $f_n(u) < f_{n+1}(u)$, having the same values outside the interval $(\epsilon_{n+2}, \epsilon_n)$. Consider the problem

$$y' = q(c y_+^{\frac{1}{p}} - f_{n+1}(u)), \quad y(0) = 0, \quad (4.7)$$

and let $\hat{c}(f_{n+1})$ be the unique value such that there exists a solution of (4.7) satisfying $y(1) = 0$. It is easy to see that this solution $y_{n+1}(u)$ satisfies

$$y_{n+1}'(u) \leq q(\hat{c}(f_{n+1}) (y_{n+1})_+(u)^{\frac{1}{p}} - f_n(u)), \quad \forall u \in [0, 1]$$

with strict inequality for $u \in (\epsilon_{n+2}, b)$. so the equation

$$y'(u) = q(\hat{c}(f_{n+1}) y_+(u)^{\frac{1}{p}} - f_n(u))$$

has a solution $z_n(u)$ such that $z_n(0) = 0$, $z_n(u) > y_{n+1}(u)$ for $u \in (\epsilon_{n+2}, 1]$ and in fact $z_n(u) > y_{n+1}(u)$ for $u \in (0, 1]$ (since we see that $z_n - y_{n+1}$ increases). Since the positive solution starting from $(0, 0)$ is unique, the solution $w_n(u)$ of the same equation with $w_n(1) = 0$ must vanish at some point $\gamma_n \in (0, \epsilon_n]$ (see the argument in Step 4 of the preceeding proof). Hence by the construction of \hat{c}_n we have $\hat{c}(f_n) < \hat{c}(f_{n+1})$. This allows us to conclude that $\hat{c}(f_n)$ is an increasing sequence.

Now let $c \geq c^*$ and consider the unique solution $z(t)$ of (2.1). Then we have $z' < q(c z_+^{\frac{1}{p}} - f_n(u))$ for $u \in (0, \epsilon_{n+1})$ and $z' \leq q(c z_+^{\frac{1}{p}} - f_n(u))$ for $u \in [0, 1]$, The same argument as above allows us to conclude that $\hat{c}(f_n) < c$ and consequently, the sequence is bounded from above by c^* . A simple application of Ascoli-Arzelà's lemma allows us to conclude that the solutions y_n tend to a solution of (2.1). This implies that $\hat{c}(f_n) \rightarrow l \geq c^*$ and consequently we conclude that $\hat{c}(f_n) \nearrow c^*$ \square

5 Behaviour near $u = 1$

Lemma 5.1. *Let $c > 0$, and let y be a positive solution of (1.1) in some interval $(a, 1)$ such that $y(1) = 0$. Suppose that*

$$m := \lim_{u \rightarrow 1^-} \frac{f(u)}{(1-u)^{q-1}} < \infty \quad \text{exists.} \quad (5.1)$$

Then $\lim_{u \rightarrow 1^-} \frac{y(u)}{(1-u)^q}$ exists and is the root α of $\alpha + c\alpha^{1/p} = m$.

Proof. *Claim:* $\sup_{u \in (a, 1)} \frac{y(u)}{(1-u)^q} < \infty$. Just take $m_1 > m$ and integrate the inequality $y'(u) \geq -qm_1(1-u)^{q-1}$ in a suitable interval of the form $(b, 1)$.

Let us complete the proof, setting $y(u) = z(u)(1-u)^q$. Then z is a solution of the differential equation

$$z' = \frac{q}{1-u} (z + cz^{\frac{1}{p}} - \mu(u)), \quad (5.2)$$

where $\mu(u) = \frac{f(u)}{(1-u)^{q-1}}$. Arguing as in the proof of Lemma 3.1-(d) it is easy to see that $\liminf_{u \rightarrow 1} z(u)$ and $\limsup_{u \rightarrow 1} z(u)$ must be equal and must coincide with the root of $\alpha + c\alpha^{1/p} = m$.

6 Some applications to the second order problem

In this section we consider the problem (1.3)-(1.4) with several assumptions on D and g .

Lemma 6.1. *Let g be a function of type A. The derivative of a nonincreasing solution u of (1.3) with $0 < u(x) < 1$ does not vanish. If the interval where u is defined extends to $+\infty$, we have $\lim_{t \rightarrow +\infty} D(u)|u'|^{p-1} = 0$. A similar statement holds with $-\infty$ replacing $+\infty$.*

Proof. If there exists x_0 such that $u'(x_0) = 0$ and $0 < u(x_0) < 1$, using the differential equation we would have $\left(D(u)|u'|^{p-2}u'\right)'|_{(x=x_0)} < 0$, which contradicts the fact that $D(u)|u'|^{p-2}u'$ attains a maximum at $x = x_0$.

Concerning the statement on the limit we will only consider $+\infty$, the case of $-\infty$ being similar. Suppose towards a contradiction that $\liminf_{x \rightarrow +\infty} D(u)|u'|^{p-1} = \delta > 0$. We can take two sequences t_n and s_n tending to $+\infty$ such that $u'(t_n) \rightarrow 0$ and $D(u(s_n))|u'|^{p-1}(s_n) \rightarrow \delta$. Integrating the differential equation in $[0, t_n]$, we easily conclude that the sequence $\int_0^{t_n} g(u(x)) dx$ is bounded and therefore $\int_0^{+\infty} g(u(x)) dx$ is convergent. Consequently we have

$$\begin{aligned} 0 &= \int_{t_n}^{s_n} \left(D(u)|u'|^{p-2}u'\right)' + cu' + g(u) dx = \\ &= D(u(s_n))|u'(s_n)|^{p-2}u'(s_n) - D(u(t_n))|u'(t_n)|^{p-2}u'(t_n) + c(u(s_n) - u(t_n)) + \int_{t_n}^{s_n} g(u) dx \end{aligned}$$

and making $n \rightarrow \infty$ we would get the contradiction $-\delta = 0$. \square

We set

$$f(u) = D(u)^{q-1}g(u), \quad u \in [0, 1]. \quad (6.1)$$

and we assume

$$(D1) \ D \in C^1[0, 1] \text{ and } D > 0 \text{ in } (0, 1).$$

$$(G1) \ g \text{ is a function of type A.}$$

Clearly, f given by (6.1) is of type A.

Proposition 6.2. *We have that $u(t)$ is a monotone solution of (1.3) in some interval (a, b) such that $0 < u(t) < 1 \ \forall t \in (a, b)$ and*

$$\lim_{t \rightarrow a^+} u(t) = 1, \quad \lim_{t \rightarrow b^-} u(t) = 0, \quad \lim_{t \rightarrow a^+} D(u)|u'|^{p-1} = \lim_{t \rightarrow b^-} D(u)|u'|^{p-1} = 0 \quad (6.2)$$

if and only if $y = v^q$ where $v = D(u)|u'|^{p-1}$ is a positive solution of (1.1)-(1.5).

Proof. The necessary condition was essentially proved in the introduction. Conversely, given a positive solution $y(u)$ of (1.1)-(1.5), we recover a solution of (1.3) by solving the Cauchy problem

$$u' = -\frac{y(u)^{\frac{1}{p}}}{D(u)^{q-1}}, \quad u(0) = \frac{1}{2}. \quad (6.3)$$

The solution of (6.3) exists in (t_-, t_+) , where

$$t_- = -\int_{1/2}^1 \frac{D(u)^{q-1} du}{y(u)^{\frac{1}{p}}}, \quad t_+ = \int_0^{1/2} \frac{D(u)^{q-1} du}{y(u)^{\frac{1}{p}}}. \quad (6.4)$$

□

Consider the assumptions

$$\sup_{u \in (0,1)} \frac{g(u)}{u^{q-1}} < +\infty. \quad (6.5)$$

$$\sup_{u \in (0,1)} \frac{g(u)}{(1-u)^{p-1}} < +\infty. \quad (6.6)$$

and the following strengthened form of (D1)

$$(D1') \ D \in C^1[0, 1] \text{ and } D > 0 \text{ in } [0, 1].$$

Under the conditions (D1), (G1), (6.5), Proposition 2.3 is applicable to $f = D^{q-1}g$ and a positive number c^* is associated to f . This number plays a central role in the following theorem, which in case $p = 2$ corresponds to well known results, see [10, 3, 13] and references. Note that according to the results in sections 2 and 3 we have the estimate

$$q^{\frac{1}{q}} p^{\frac{1}{p}} \left(\liminf_{u \rightarrow 0} \frac{D(u)^{q-1} g(u)}{u^{q-1}} \right)^{\frac{1}{q}} \leq c^* \leq q^{\frac{1}{q}} p^{\frac{1}{p}} \left(\sup_{u \in (0,1)} \frac{D(u)^{q-1} g(u)}{u^{q-1}} \right)^{\frac{1}{q}}. \quad (6.7)$$

Theorem 6.3. Suppose that $(D1')$, $(G1)$, (6.5) and (6.6) are satisfied and let $1 < p \leq 2$. Then (1.3)-(1.4) has a decreasing solution $u(t)$ taking values in $(0, 1)$ if and only if $c \geq c^*$. That solution is unique up to translation.

If, further, $g^*(0) \equiv \lim_{u \rightarrow 0^+} \frac{g(u)}{u^{q-1}}$ exists then

$$\lim_{t \rightarrow +\infty} \frac{u'(t)}{u(t)^{q-1}} = \begin{cases} -\frac{\omega_c^- (D(0)^{q-1} g^*(0))^{1/p}}{D(0)^{q-1}}, & c > c^* \\ -\frac{\omega_{c^*}^+ (D(0)^{q-1} g^*(0))^{1/p}}{D(0)^{q-1}}, & c = c^* \end{cases}$$

Proof. Let $y(u)$ be a solution of (1.1)-(1.5) for some $c \geq c^*$. Consider the Cauchy problem (6.3). The solution of (6.3) exists in (t_-, t_+) , given by (6.4). Since (1.1) implies $y' \leq qcy^{1/p}$, it turns out that

$$\sup_{u \in (0,1)} \frac{y(u)}{u^q} < +\infty, \quad (6.8)$$

and it is clear that $t_+ = +\infty$, since $q \geq 2$. Similarly the estimate we get from (6.5) and $(D1')$ on

$$\frac{f(u)}{(1-u)^{p-1}}$$

implies $y(u) \leq C(1-u)^p$ for some constant C and therefore $t_- = -\infty$. The solution of (6.3) satisfies (1.3)-(1.4).

On the other hand, since we can write $\frac{u'(t)}{u(t)^{q-1}} = -\left(\frac{y(u)}{u^q}\right)^{\frac{1}{p}} \frac{1}{D(u)^{q-1}}$, the last statement follows easily from Theorem 3.7. \square

Next we shall consider the case where D is “degenerate” in the sense that

$$(D2) \ D \in C^1[0, 1], \ D > 0 \text{ in } (0, 1], \ D(0) = 0 \text{ and } D'(0) > 0.$$

The following theorem corresponds to results given in [16, 13] for $p = 2$.

Theorem 6.4. Suppose that $(D2)$, $(G1)$, (6.5) and (6.6) are satisfied. Let $1 < p \leq 2$. Then

(i) Problem (1.3)-(1.4) has a decreasing solution $u(t)$ taking values in $(0, 1)$ if and only if $c > c^*$.

(ii) If $c = c^*$ (1.3) has a decreasing solution defined in $(-\infty, 0]$ with $u(-\infty) = 1$, $u(0) = 0$ and

$$u'(0) = -\left(\frac{c^*}{D'(0)}\right)^{q-1}. \quad (6.9)$$

Those solutions are unique up to translation.

(iii) If $c < c^*$, problem (1.3) has no decreasing solution in any interval $(-\infty, b)$ with $\lim_{t \rightarrow -\infty} u(t) = 1$, $\lim_{t \rightarrow b^-} u(t) = 0$.

The solutions considered in (ii) are called *sharp solutions*.

Proof. Proceeding as in the preceeding proof, we consider (6.3). Under the assumptions of the theorem, we have $f^*(0) = 0$. To compute the limit of the right-hand side of (6.3) we write

$$\lim_{u \rightarrow 0^+} \frac{y(u)^{\frac{1}{p}}}{D(u)^{q-1}} = \lim_{u \rightarrow 0^+} \left(\frac{y(u)}{u^q} \right)^{\frac{1}{p}} \left(\frac{u}{D(u)} \right)^{q-1}. \quad (6.10)$$

Noting that

$$\omega_c^-(0) = 0, \quad \omega_c^+(0) = c^q$$

and using Theorem 3.7 we may conclude:

$$\lim_{u \rightarrow 0^+} \frac{y(u)^{\frac{1}{p}}}{D(u)^{q-1}} = \begin{cases} 0, & c > c^* \\ \left(\frac{c^*}{D'(0)} \right)^{q-1}, & c = c^* \end{cases} \quad (6.11)$$

Moreover if $c > c^*$ and since f satisfies (3.2b), Proposition 3.5-(g) implies $t_+ = +\infty$. The fact that $t_- = -\infty$ follows as in the proof of the preceeding theorem.

If $c = c^*$ it is clear that the solution of (6.3) can remain positive for $t \geq 0$ only in some finite interval $[0, b)$ so that $\lim_{t \rightarrow b^-} u(t) = 0$ and $\lim_{t \rightarrow b^-} u'(t) = - \left(\frac{c^*}{D'(0)} \right)^{q-1}$. \square

We next consider a case of negative diffusion (see [17]), considering the assumption (D3) $D(u) < 0 \forall u \in (0, 1]$; $D(0) = 0$ and $D'(0) < 0$.

and introducing the conditions

$$\sup_{u \in (0,1)} \frac{g(u)}{(1-u)^{q-1}} < +\infty, \quad \sup_{u \in (0,1)} \frac{g(u)}{u^p} < +\infty. \quad (6.12)$$

In [17] the authors consider (1.3)-(1.4) with $D < 0$ for $p = 2$ and reduce this problem to a non-singular system, assuming (D3) and additional regularity assumptions.

It is easily seen that the change of variables

$$u(-t) = 1 - z(t)$$

yields an equivalence between (1.3)-(1.4) and

$$(E(z)|z'|^{p-2}z')' + cz' + h(z) = 0, \quad z(-\infty) = 1, \quad z(+\infty) = 0 \quad (6.13)$$

where

$$E(z) = -D(1-z), \quad h(z) = g(1-z), \quad 0 \leq z \leq 1. \quad (6.14)$$

We have the following result that contains some statements made in Proposition 3 of [17] for $p = 2$.

Theorem 6.5. *Let $1 < p \leq 2$. Suppose that (D3), (G1) and (6.12) are satisfied. Then there exists $c^* > 0$ such that (1.3)-(1.4) has a decreasing solution $u(t)$ taking values in $(0, 1)$ if and only if $c \geq c^*$. Moreover*

$$q^{\frac{1}{q}} p^{\frac{1}{p}} \left(\liminf_{u \rightarrow 1} \frac{|D(u)|^{q-1} g(u)}{(1-u)^{q-1}} \right)^{\frac{1}{q}} \leq c^* \leq q^{\frac{1}{q}} p^{\frac{1}{p}} \left(\sup_{u \in (0,1)} \frac{|D(u)|^{q-1} g(u)}{(1-u)^{q-1}} \right)^{\frac{1}{q}}. \quad (6.15)$$

Those solutions are unique up to translation.

Proof. Let us start with problem (6.13). We consider the associated first order problem

$$y' = q(c y_+^{\frac{1}{p}} - E(z)^{q-1} h(z)), \quad 0 \leq u \leq 1, \quad y(0) = y(1) = 0. \quad (6.16)$$

There exists c^* such that this problem has a positive solution if and only if $c \geq c^*$ and c^* satisfies the desired estimates. We recover the solution of (6.13) via the differential equation

$$z' = -\frac{y(z)^{\frac{1}{p}}}{E(z)^{q-1}}, \quad z(0) = 1/2.$$

As in the proof of Theorem 6.3 we see, using the first condition (6.12), that the solution is defined in an interval (t_-, t_+) where $t_+ = +\infty$. The second condition (6.12) and (D3) imply, as is easily seen, that $y(z) \leq K(1-z)^{p+q}$ for some constant K . Combining this with (D3), it turns out that the integrand in the expression of t_- is bounded below by some multiple of $\frac{1}{1-u}$ and therefore $t_- = -\infty$. Setting $u(t) = 1 - z(-t)$ we obtain the desired solutions of (1.3). \square

In a similar way we are able to deal with the analogue of a model considered in [18].

Theorem 6.6. *Let $1 < p \leq 2$. Assume $-g$ is a function of type A, (D2), (6.5) and (6.6) hold. Then there exists $-c^* < 0$ such that (1.3)-(1.4) has a decreasing solution $u(t)$ taking values in $(0, 1)$ if and only if $c \leq -c^*$. Moreover*

$$q^{\frac{1}{q}} p^{\frac{1}{p}} \left(\liminf_{u \rightarrow 1} \frac{D(u)^{q-1} |g(u)|}{(1-u)^{q-1}} \right)^{\frac{1}{q}} \leq c^* \leq q^{\frac{1}{q}} p^{\frac{1}{p}} \left(\sup_{u \in (0,1)} \frac{D(u)^{q-1} |g(u)|}{(1-u)^{q-1}} \right)^{\frac{1}{q}}. \quad (6.17)$$

Those solutions are unique up to translation.

Proof. We use the change of variables $u(-t) = 1 - z(t)$ again. Then the problem (1.3)-(1.4) turns into

$$(F(z)|z'|^{p-2}z')' - cz' + h(z) = 0, \quad z(-\infty) = 1, \quad z(+\infty) = 0, \quad (6.18)$$

where

$$F(z) = D(1-z), \quad h(z) = -g(1-z), \quad 0 \leq z \leq 1. \quad (6.19)$$

\square

Finally we examine a situation where both D and g change sign. A problem of this type has been studied in [11] in the case $p = 2$.

We introduce the assumptions:

(G2) $g(0) = g(1) = 0$ and there exists $\alpha \in (0, 1)$ such that $(u - \alpha)g(u) > 0 \forall u \in (0, 1) \setminus \{\alpha\}$.

(D4) There exists $\beta \in (0, 1)$ such that $(u - \beta)D(u) < 0 \forall u \in (0, 1) \setminus \{\beta\}$.

(GD0)

$$\liminf_{h \rightarrow 0} \frac{D(h)^{q-1}g(h)}{h^{q-1}} > -\infty, \quad \int_0^\beta \left(\frac{D(h)}{h}\right)^{q-1} dh = +\infty.$$

(GD1)

$$\limsup_{h \rightarrow 1} \frac{|D(h)|^{q-1}g(h)}{(1-h)^{q-1}} < +\infty, \quad \int_\beta^1 \left(\frac{|D(h)|}{1-h}\right)^{q-1} dh = +\infty.$$

(GD2) $\alpha < \beta$ and $\int_0^\beta D(u)^{q-1}g(u) du > 0$.

Let us consider the problem in the interval $[0, \beta]$

$$y' = q(cy^{\frac{1}{p}} - D(u)^{q-1}g(u)), \quad 0 \leq u \leq \beta \quad (6.20)$$

Note that $D(u)^{q-1}g(u)$ is of type C on $[0, \beta]$. According to Theorem 4.2 we know that under the first condition in (GD0) and (GD2) there exists a (unique) number $\hat{c} > 0$ such that (6.20) has a positive solution satisfying

$$y(0) = 0 = y(\beta). \quad (6.21)$$

On the other hand, the problem in the interval $[\beta, 1]$

$$y' = q(cy^{\frac{1}{p}} - \tilde{D}(z)^{q-1}\tilde{g}(z)), \quad \beta \leq z \leq 1 \quad (6.22)$$

where

$$\tilde{D}(z) = |D(1 + \beta - z)|, \quad \tilde{g}(z) = g(1 + \beta - z); \quad \beta \leq z \leq 1 \quad (6.23)$$

involves a function of type A (in $[\beta, 1]$, of course) and, since the first condition in (GD1) holds, there exists a number c^* such that (6.22) has a positive solution satisfying

$$y(\beta) = 0 = y(1) \quad (6.24)$$

if and only if $c \geq c^*$. We are now in a position to state the following

Theorem 6.7. *Let $1 < p \leq 2$. Assume (G2), (D4), (GD0), (GD1), (GD2), $D'(\beta) < 0$. Then if $\hat{c} \geq c^*$, the problem (1.3)-(1.4) has a solution (unique up to translation) for $c = \hat{c}$.*

Proof. *Step 1: connection between 0 and β .* We fix $c = \hat{c}$ and consider the positive solution of (6.20) satisfying (6.21). A corresponding solution of (1.3) is obtained via

$$u' = -\frac{y(u)^{\frac{1}{p}}}{D(u)^{q-1}}, \quad u(0) = \frac{\beta}{2}. \quad (6.25)$$

Using assumption $D'(\beta) < 0$ and Lemma 5.1 we easily compute

$$\lim_{u \rightarrow \beta^-} \frac{y(u)^{\frac{1}{p}}}{D(u)^{q-1}} = \frac{\alpha^{\frac{1}{p}}}{|D'(\beta)|^{q-1}} \quad (6.26)$$

where $\alpha + \hat{c}\alpha^{\frac{1}{p}} = |D'(\beta)|^{q-1}g(\beta)$. Using the first part of (GD0) and Lemma 4.1 (where $f = |D|^{q-1}g$) we see that $\frac{y(u)}{u^q}$ is bounded in $(0, \beta)$. Hence we have obtained a solution of (1.3) that satisfies $0 < u(t) \leq \beta$ and (using the second part of (GD0))

$$u(t_1) = \beta, \quad u'(t_1) = -\frac{\alpha^{\frac{1}{p}}}{|D'(\beta)|^{q-1}}, \quad u(+\infty) = 0 \quad (6.27)$$

for some $t_1 > -\infty$.

Step 2: connection between β and 1. The change of variable

$$u(t) = 1 + \beta - z(-t)$$

defines a new second order problem

$$(\tilde{D}(z)|z'|^{p-2}z')' + cz' + \tilde{g}(z) = 0 \quad (6.28)$$

with \tilde{D} and \tilde{g} given by (6.23), in such a way that $u(t)$ takes values in $(\beta, 1)$ and solves (1.3) if and only if $z(t)$ takes values in $(\beta, 1)$ and solves (6.28). Now since $\hat{c} \geq c^*$ the problem (6.22) with $c = \hat{c}$ has a positive solution $y(z)$ that satisfies (6.24). This originates a solution of

$$z' = -\frac{y(z)^{\frac{1}{p}}}{\tilde{D}(z)^{q-1}}, \quad z(0) = \frac{1 - \beta}{2}. \quad (6.29)$$

Now $\lim_{z \rightarrow 1^-} \frac{\tilde{D}(z)^{q-1}\tilde{g}(z)}{(1-z)^{q-1}} = |D'(\beta)|^{q-1}g(\beta)$. hence, as in Step 1, we compute

$$\lim_{z \rightarrow 1^-} \frac{y(z)^{\frac{1}{p}}}{\tilde{D}(z)^{q-1}} = \frac{\alpha^{\frac{1}{p}}}{|D'(\beta)|^{q-1}} \quad (6.30)$$

with the same meaning of α . Now from Lemma 3.1-(d) (where $f = |\tilde{D}|^{q-1}\tilde{g}$, β playing the role of 0) and the first part of (GD1) we know that $\frac{y(z)}{(z-\beta)^q}$ is bounded in $(\beta, 1)$. Taking also the second part of (GD1) into account, we have shown that the solution of (6.29) is defined in some interval $(-t_2, +\infty)$ with $t_2 < +\infty$, $\beta < z(t) \leq 1$ and

$$z(-t_2) = 1, \quad z'(-t_2) = -\frac{\alpha^{\frac{1}{p}}}{|D'(\beta)|^{q-1}}, \quad z(+\infty) = \beta. \quad (6.31)$$

Accordingly, the function $u(t) = 1 + \beta - z(-t)$ satisfies $\beta < u(t) \leq 1$ and

$$u(t_2) = \beta, \quad u'(t_2) = -\frac{\alpha^{\frac{1}{p}}}{|D'(\beta)|^{q-1}}, \quad u(-\infty) = 1. \quad (6.32)$$

Step 3: conclusion. Comparing (6.27) with (6.32) we see that, after a translation of one of the solutions thus defined, we obtain the desired connection between 0 and 1. \square

Remark 6.8. (a) As in [11] for $p = 2$, it can be shown that the solution of (1.3)-(1.4) exists only if $c = \hat{c}$.

(b) Under the conditions of Theorem 6.7 we must have

$$\hat{c} \geq \left\{ \liminf_{h \rightarrow 0} \left[\left(\frac{|D(1-h)|}{h} \right)^{q-1} g(1-h) \right] q \right\}^{1/q} p^{1/p}.$$

Remark 6.9. If we modify condition (D4) so as to extend the strict inequality to the endpoints 0 and 1, then the second part of assumptions (GD0) and (GD1) of Theorem 6.7 can obviously be dropped, since it follows from the inequality $q \geq 2$.

Remark 6.10. In the above theorems we have obtained heteroclinic solutions taking values strictly between 0 and 1. If we let $p > 2$ the same procedure yields heteroclinics that are possible *finite* in the sense that they become constant outside a finite interval.

Acknowledgements. The first and third authors were supported by Fundação para a Ciência e a Tecnologia via Financiamento Base ISFL-1-209 and also (first author) Project PTDC/MAT/113383/2009 and (third author) Project PTDC/MAT/115168/2009. The second author was supported by MIUR (the Italian Ministry of University and Research) inside the national project “Equazioni differenziali ordinarie ed applicazioni”.

References

- [1] Arias, Margarita; Campos, Juan; Marcelli, Cristina *Fastness and continuous dependence in front propagation in Fisher-KPP equations*, Discrete Contin. Dyn. Syst. Ser. B 11 (2009), no. 1, 11-30
- [2] M. Arias, J. Campos, A. Robles Pérez, L. Sanchez, *Fast and heteroclinic solutions for a second order ODE related to the Fisher-Kolmogorov's equation*, Calculus of Variations and Partial Differential Equations, **21** (2004), 319-334.
- [3] D. G. Aronson and H. F. Weinberger, *Multidimensional nonlinear diffusion arising in population genetics*, Adv. in Math. **30** (1978), no. 1, 33-76.
- [4] H. Berestycki, L. Nirenberg, *Travelling fronts in cylinders*, Annales de l'Institut Henri Poincaré - Analyse non linéaire, Vol. 9 **5** (1992), 497-572

- [5] D. Bonheure, L. Sanchez, *Heteroclinic orbits for some classes of second and fourth order differential equations*, Handbook of Differential Equations: Ordinary Differential Equations, vol. 3, A. Căi, $\frac{1}{2}$ ada, P. Drabek, a. Fonda, editors, Elsevier (2006).
- [6] Gilding, Brian H.; Kersner, Robert *Travelling waves in nonlinear diffusion-convection reaction*, Progress in Nonlinear Differential Equations and their Applications, 60. Birkhauser Verlag, Basel, 2004. x+209 pp. ISBN: 3-7643-7071-8
- [7] Gilding, B. H.; Kersner, R. *A Fisher/KPP-type equation with density-dependent diffusion and convection: travelling-wave solutions*, J. Phys. A 38 (2005), no. 15, 3367-3379.
- [8] Hamydy, A. *Travelling wave for absorption-convection-diffusion equations*, Electronic Journal of Diff. Eq. Vol.2006 (2006), no. 86, 1-9.
- [9] Hou, Xiaojie; Li, Yi; Meyer, Kenneth R. *Traveling wave solutions for a reaction diffusion equation with double degenerate nonlinearities*, Discrete Contin. Dyn. Syst. 26 (2010), no. 1, 265-290.
- [10] A. Kolmogorov, I. Petrovski, and N. Piscounov, *Etude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique*, Bull. Univ. Moskou Ser. Internat. Sec. A **1** (1937), 1-25.
- [11] Maini, Philip K.; Malaguti, Luisa; Marcelli, Cristina; Matucci, Serena *Diffusion-aggregation processes with mono-stable reaction terms*, Discrete Contin. Dyn. Syst. Ser. B 6 (2006), no. 5, 1175-1189
- [12] Maini, Philip K.; Malaguti, Luisa; Marcelli, Cristina; Matucci, Serena *Aggregative movement and front propagation for bi-stable population models*, Math. Models Methods Appl. Sci. 17 (2007), no. 9, 1351-1368.
- [13] L. Malaguti, C. Marcelli *Travelling wavefronts in reaction-diffusion equations with convection effects and non-regular terms*, Math. Nachr., **242** (2002), 148-164.
- [14] L. Malaguti, C. Marcelli *Sharp Profiles in degenerate and doubly degenerate Fisher-KPP equations*, Journal of Differential Equations, **195** (2003), 471-496.
- [15] P. Pang, Y. Wang, J. Yin *Periodic solutions for a class of reaction-diffusion equations with p -Laplacian*, Nonlinear Analysis: Real World Applications Vol.11, (2010), 323-331.
- [16] Sánchez-Garduño, Faustino; Maini, Philip K. *Existence and uniqueness of a sharp travelling wave in degenerate non-linear diffusion Fisher-KPP equations*, Journal of Mathematical Biology (1994), no. 33, 163 - 192.
- [17] Sánchez-Garduño, Faustino; Maini, Philip K.; Pérez-Velázquez, Judith *A non-linear degenerate equation for direct aggregation and traveling wave dynamics*, Discrete Contin. Dyn. Syst. Ser. B 13 (2010), no. 2, 455 - 487.
- [18] Sánchez-Valdés, Ariel; Hernández-Bermejo, Benito *New travelling wave solutions for the Fisher-KPP equation with general exponents*, Appl. Math. Lett. 18 (2005), no. 11, 1281-1285